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Norm-Controlled Inversion of Banach algebras of infinite matrices

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Abstract. In this paper we provide a polynomial norm-controlled inversion of Baskakov–Gohberg–Sjöstrand Banach algebra in a Banach algebra $\mathscr{B}(\ell^q)$, $1 \le q \le \infty$, which is not a symmetric $*-$ Banach algebra.

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1. Introduction

N. Wiener in [19] proved that if a periodic function with absolutely convergent Fourier series never vanishes, then it also has an absolutely convergent Fourier series.

A Banach subalgebra $\mathscr{A}$ of a Banach algebra $\mathscr{B}$ having a common identity is called *inverse-closed* in $\mathscr{B}$ if $A \in \mathscr{A}$ with $A^{-1} \in \mathscr{B}$ implies $A^{-1} \in \mathscr{A}$. For a Banach subalgebra $\mathscr{A}$ which is inverse-closed in $\mathscr{B}$, we say that $\mathscr{A}$ admits a *norm-controlled inversion* in $\mathscr{B}$ if there exists a function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|A^{-1}\|_{\mathscr{A}} \le h(\|A^{-1}\|_{\mathscr{B}}, \|A\|_{\mathscr{A}})$$

for all $A \in \mathscr{A}$ with an inverse A^{-1} in $\mathscr{B}$, where $\|\cdot\|_{\mathscr{A}}$ and $\|\cdot\|_{\mathscr{B}}$ are norms on $\mathscr{A}$ and $\mathscr{B}$, respectively.

N. Nikolski in [9] showed that the algebra of absolutely convergent Fourier series does not admit norm-controlled inversion in the algebra of continuous periodic functions.

Let a discrete set $\Lambda \subset \mathbb{R}^d$ be relatively-separated, that is,

$$R(\Lambda) = \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{\lambda + [0,1)^d}(x) < \infty. \tag{1}$$

The set Λ may not form a group. Our prime models are paraboloids $\{(x,y,z): z = ax^2+by^2, x,y \in \mathbb{Z}\}$, and elliptical hyperboloids $\{(x,y,z): z^2 = ax^2+by^2, x,y \in \mathbb{Z}\}$, where $a,b>0$, and the set $\{k+\delta_k : k \in \mathbb{Z}^d, \delta_k \in [0,1)^d\}$.

For $1 \le p \le \infty$ and $r \ge 0$, define the Baskakov–Gohberg–Sjöstrand(BGS) class $\mathscr{C}_{p,r}(\Lambda)$ by

$$\mathscr{C}_{p,r}(\Lambda) = \{A = (a(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda} : \|A\|_{\mathscr{C}_{p,r}(\Lambda)} < \infty\}$$

where for $1 \le p < \infty$,

$$\|A\|_{\mathscr{C}_{p,r}(\Lambda)} = \left(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} |a(\lambda,\lambda')|^p (1+|\lambda-\lambda'|)^{pr} \chi_{k+[0,1)^d}(\lambda-\lambda')\right)^{1/p}, \tag{2}$$

and for $p = \infty$,

$$\|A\|_{\mathscr{C}_{\infty,r}(\Lambda)} = \sup_{\lambda,\lambda'\in\Lambda} |a(\lambda,\lambda')|(1+|\lambda-\lambda'|)^r, \tag{3}$$

where for $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $|x| = \max(|x_1|,\dots,|x_d|)$. The above classes of infinite matrices form Banach algebras. In particular, when $p = \infty$, $\mathscr{C}_{\infty,r}(\Lambda)$ is called a Jaffard algebra and written as $\mathscr{J}_r(\Lambda)$ with the norm $\|\cdot\|_{\mathscr{J}_r(\Lambda)}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be Banach $*$-algebras with common identity and involution. If $\mathscr{B}$ is a symmetric algebra (see [4]) and $\mathscr{A}$ is a differential subalgebra of $\mathscr{B}$, then $\mathscr{A}$ admits norm-controlled inversion in $\mathscr{B}$ (see [6, 7, 12]). Several algebras of infinite matrices with certain off-diagonal decay including Gröchenig–Schur algebra, Baskakov–Gohberg–Sjöstrand algebra and Jaffard algebra are shown to be differential *-subalgebra of $\mathscr{B}(\ell^2(\mathbb{Z}^d))$ (see [3,5–8,10–13,15–18]), where for $1 \le q \le \infty$, $\mathscr{B}(\ell^q(V))$ denotes the space of all bounded linear operators on $\ell^q(V)$ with the norm $\|\cdot\|_{\mathscr{B}(\ell^q)}$ and $\ell^q(V)$ is the set of all q-summable sequences on V with the norm $\|\cdot\|_q$.

Using the commutator trick and the partition of the identity, J. Sjöstrand in [14] showed Wiener's lemma for $\mathscr{C}_{1,0}(\mathbb{Z}^d)$. The polynomial norm-controlled inversion is studied in [6] for a differential subalgebra of a symmetric Banach algebra and in [7] for matrices in Besov algebras, Bessel algebras, Dales–Davie algebras, Baskakov–Gohberg–Sjöstrand algebras and Jaffard algebras. A. G. Baskakov in [1,2] depending on Bochner–Phillips theorem proved that Jaffard algebras and Baskakov–Gohberg–Sjöstrand algebras with $p = 1$ admit norm-controlled inversion in $B(\ell^2)$. E. Samei and V. Shepelska in [11] showed that the convolution algebras as a subalgebra of a C^*-algebra admits an inversion controlled by a subexponential function. In [13], it is shown that a Beurling algebra admits a polynomial norm-controlled inversion in a symmetric Banach algebra $\mathscr{B}(\ell^2(V))$, where V, E are the sets of vertices and edges in the graph $\mathscr{G} = (V,E)$, respectively, which has a complicated structure to prove the norm-controlled inversion.

In some applications in the field of mathematics and engineering, widespread-used algebras $\mathscr{B}$ of infinite matrices are Banach algebras $\mathscr{B}(\ell^p)$ for $p \in [1,\infty]$, while they are symmetric only when $p = 2$. The results in [1,2,6,7,11,13] deal with the norm-controlled inversions in symmetric algebras, on the other hand, we provide the norm-controlled inversion in a nonsymmetric algebra. In this paper, for $1 \le p,q \le \infty$, $r > d(1-1/p)$ and a relatively-separated subset Λ of $\mathbb{R}^d$, we give the simple proof for the norm-controlled inversion of the Baskakov–Gohberg–Sjöstrand subalgebra $\mathscr{C}_{p,r}(\Lambda)$ of a nonsymmetric Banach algebra $\mathscr{B}(\ell^q(\Lambda))$. We expect that the method in this paper can be applied to algebras of infinite matrices having off-diagonal decay with different weights from polynomial functions. The proof of the main theorem is based on commutator trick and the partition of the identity in [14].

For $a = (a_1,\dots,a_d) \in \mathbb{R}$, we write $[a] = ([a_1],\dots,[a_d])$, where for $b \in \mathbb{R}$, $[b]$ denotes the largest preceding integer of b.

2. Norm-Controlled Inversion

To state our result on norm-controlled inversion for localized infinite matrices, we recall some concepts. For a relatively-separated subset Λ of $\mathbb{R}^d$ satisfying (1), we define Schur norm of an infinite matrix $A=(a(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$ by

$$\|A\|_{\mathscr{S}(\Lambda)}=\max\left(\sup_{\lambda'\in\Lambda}\sum_{\lambda\in\Lambda}|a(\lambda,\lambda')|,\ \sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}|a(\lambda,\lambda')|\right). \tag{4}$$

For any $1\le q\le\infty$, one can show that the Schur class $\mathscr{S}(\Lambda)$ is a subalgebra of the Banach algebra $\mathscr{B}(\ell^q(\Lambda))$ and

$$\|A\|_{\mathscr{B}(\ell^q(\Lambda))}\le\|A\|_{\mathscr{S}(\Lambda)}. \tag{5}$$

Let $A=(a(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$ be an infinite matrix in a BGS algebra, we define its approximation matrices $A_N, N\ge1$, with finite bandwidth by

$$A_N:=\left(a(\lambda,\lambda')\chi_{[0,1]}(|\lambda-\lambda'|/N)\right)_{\lambda,\lambda'\in\Lambda}. \tag{6}$$

We have the following properties of the algebra $\mathscr{C}_{p,r}(\Lambda)$ for $1\le p\le\infty$ and $r>0$.

Proposition 1. *Let $1\le p,q\le\infty$ and $r>d(1-1/p)$, and let Λ be a relatively-separated subset of $\mathbb{R}^d$ satisfying* (1). *Then the following statements hold.*

(1) *The BGS algebra $\mathscr{C}_{1,0}(\Lambda)$ is a subalgebra of Schur algebra $\mathscr{S}(\Lambda)$, and*

$$\|A\|_{\mathscr{S}(\Lambda)}\le 2R(\Lambda)\|A\|_{\mathscr{C}_{1,0}(\Lambda)}\ \text{for all } A\in\mathscr{C}_{1,0}(\Lambda). \tag{7}$$

(2) *The BGS algebra $\mathscr{C}_{1,0}(\Lambda)$ is a subalgebra of the Banach algebra $\mathscr{B}(\ell^q(\Lambda))$, and*

$$\|Ac\|_{\ell^q(\Lambda)}\le 2R(\Lambda)\|A\|_{\mathscr{C}_{1,0}(\Lambda)}\|c\|_{\ell^q(\Lambda)}\ \text{ for all } A\in\mathscr{C}_{1,0}(\Lambda) \text{ and } c\in\ell^q(\Lambda). \tag{8}$$

(3) *The BGS algebra $\mathscr{C}_{p,r}(\Lambda)$ is a subalgebra of the algebra $\mathscr{C}_{1,0}(\Lambda)$, and*

$$\|A\|_{\mathscr{C}_{1,0}(\Lambda)}\le\left(\frac{3^d r}{r-d(1-1/p)}\right)^{1-1/p}\|A\|_{\mathscr{C}_{p,r}}\ \text{for all } A\in\mathscr{C}_{p,r}(\Lambda). \tag{9}$$

(4) *The BGS algebra $\mathscr{C}_{p,r}(\Lambda)$ is a Banach algebra, and there exists a positive constant C_1 such that*

$$\|AB\|_{\mathscr{C}_{p,r}(\Lambda)}\le C_1\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\|B\|_{\mathscr{C}_{p,r}(\Lambda)}\ \text{for all } A,B\in\mathscr{C}_{p,r}(\Lambda). \tag{10}$$

(5) *A matrix A in $\mathscr{C}_{p,r}(\Lambda)$ is well approximated by its truncated matrix $A_N, N\ge1$, in the norm $\|\cdot\|_{\mathscr{C}_{1,0}(\Lambda)}$, and*

$$\|A-A_N\|_{\mathscr{C}_{1,0}(\Lambda)}\le\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\times\begin{cases}\left(\frac{d(1-1/p)}{r-d(1-1/p)}\right)^{1-1/p}N^{-r+d(1-1/p)} & \text{if } p\ne1,\\ N^{-r} & \text{if } p=1.\end{cases} \tag{11}$$

Proof.

(i) and (ii). Observing that for $\lambda\in\Lambda$,

$$\sup_{\lambda\in\Lambda}\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|\le R(\Lambda)\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|\chi_{k+[0,1)^d}(\lambda-\tilde\lambda) \tag{12}$$

and

$$\sup_{\tilde\lambda\in\Lambda}\sum_{\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|\le 2R(\Lambda)\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|\chi_{k+[0,1)^d}(\lambda-\tilde\lambda), \tag{13}$$

which imply (7) in (i). Combining (5) and (7), one can get (8) in (ii).

(iii) and (v). By direct computation, we obtain (iii) and (v).

(iv). Let $1 \le p < \infty$ and $r > d(1-1/p)$, and take the matrices $A = \big(a(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda} \in \mathscr{C}_{p,r}(\Lambda)$ and $B = \big(b(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda} \in \mathscr{C}_{p,r}(\Lambda)$. Then by the fact that $|a+b|^r \le 2^r(|a|^r + |b|^r)$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\|AB\|_{\mathscr{C}_{p,r}(\Lambda)} \le{}& 2^r \left(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} \left(\sum_{\lambda''\in\Lambda} |a(\lambda,\lambda'')|(1+|\lambda-\lambda''|)^r |b(\lambda'',\lambda')|\right)^p \chi_{k+[0,1)^d}(\lambda-\lambda')\right)^{1/p}\\ &+ 2^r \left(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} \left(\sum_{\lambda''\in\Lambda} |a(\lambda,\lambda'')||b(\lambda'',\lambda')|(1+|\lambda''-\lambda'|)^r\right)^p \chi_{k+[0,1)^d}(\lambda-\lambda')\right)^{1/p}\\ =:{}& J_1 + J_2. \end{aligned} \tag{14}$$

Observing from (12) that

$$\begin{aligned} J_1/2^r \le{}& \left(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} \Big(R(\Lambda) \sum_{\ell\in\mathbb{Z}^d} |a(\lambda,\widetilde{\lambda})|(1+|\lambda-\widetilde{\lambda}|)^r \chi_{k-\ell+(-1,1)^d}(\lambda-\widetilde{\lambda})\right.\\ &\left. \times |b(\widetilde{\lambda},\lambda')| \chi_{\ell+[0,1)^d}(\widetilde{\lambda}-\lambda')\Big)^p\right)^{1/p}\\ \le{}& R(\Lambda)\|A\|_{\mathscr{C}_{p,r}(\Lambda)} \|B\|_{\mathscr{C}_{1,0}(\Lambda)}, \end{aligned}$$

and similarly $J_2/2^r \le R(\Lambda)\|A\|_{\mathscr{C}_{1,0}(\Lambda)}\|B\|_{\mathscr{C}_{p,r}(\Lambda)}$, these together with (14) and (9) in (iii) imply (10) with $C_1 = 2^{r+1}R(\Lambda)\Big(\frac{3^d r}{r-d(1-1/p)}\Big)^{1-1/p}$ for $1 \le p < \infty$ and $r > d(1-1/p)$.

For $p = \infty$, we have

$$\begin{aligned} \|AB\|_{\mathscr{C}_{\infty,r}(\Lambda)} \le{}& 2^r \left(\sup_{\lambda,\lambda'\in\Lambda} \sum_{\lambda''\in\Lambda} |a(\lambda,\lambda'')|(1+|\lambda-\lambda''|)^r |b(\lambda'',\lambda')|\right)\\ &+ 2^r \left(\sup_{\lambda,\lambda'\in\Lambda} \sum_{\lambda''\in\Lambda} |a(\lambda,\lambda'')||b(\lambda'',\lambda')|(1+|\lambda''-\lambda'|)^r\right)\\ \le{}& 2^r \Big(\|B\|_{\mathscr{S}(\Lambda)}\|A\|_{\mathscr{C}_{\infty,r}(\Lambda)} + \|A\|_{\mathscr{S}(\Lambda)}\|B\|_{\mathscr{C}_{\infty,r}(\Lambda)}\Big). \end{aligned} \tag{15}$$

The desired result (10) follows from (7) and (15) for $p = \infty$. □

Let $h(t) := \min(\max(2-|t|,0),1)$ be the trapezoidal-shaped function. The function h is Lipschitz continuous.

For $1 \le q \le \infty$, a positive integer N and $A \in \mathscr{B}(\ell^q(\Lambda))$, define localization operators Ψ_i^N, χ_i^N and commutators $[\Psi_i^N, A]$, $i \in \mathbb{Z}^d$, by

$$\Psi_i^N c := \big(h(\lambda/N - i)c(\lambda)\big)_{\lambda\in\Lambda} \tag{16}$$

$$\chi_i^N c := \big(\chi_{[0,1]}(|i-\lambda|/N)c(\lambda)\big)_{\lambda\in\Lambda} \tag{17}$$

and

$$[\Psi_i^N, A]c = \Psi_i^N Ac - A\Psi_i^N c \quad \text{for } c := (c(\lambda))_{\lambda\in\Lambda} \in \ell^q(\Lambda),$$

where for a set I, $\chi_I(\cdot)$ denotes the characteristic function on I.

In the next theorem, we show the norm-controlled inversion of a Banach algebra $\|A\|_{\mathscr{C}_{p,r}(\Lambda)}$ in $\mathscr{B}(\ell^q)(\Lambda)$ which is not a symmetric *-Banach algebra.

Theorem 2. *Let $1 \le p, q \le \infty$, $r > d(1-1/p)$, Λ be a relatively-separated subset of $\mathbb{R}^d$ satisfying* (1), *and let $A \in \mathscr{C}_{p,r}(\Lambda)$ be invertible in $\mathscr{B}(\ell^q(\Lambda))$. Then there exists an absolute constant C, independent of A, such that*

$$\begin{aligned} \|A^{-1}\|_{\mathscr{C}_{p,r}(\Lambda)} \le{}& C\|A^{-1}\|_{\mathscr{B}(\ell^q)}\big(\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\big)^{(d/p+r)/\min(1,r-d(1-1/p))}\\ &\times \begin{cases} 1 & \text{if } r \ne d(1-1/p)+1,\\ \big(\ln(\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|A\|_{\mathscr{C}_{p,r}(\Lambda)})\big)^{(d/p+r)(1-1/p)} & \text{if } r = d(1-1/p)+1. \end{cases} \end{aligned} \tag{18}$$

Proof. We follow the arguments in [18]. Let $1 \le q < \infty$ and $1 < p < \infty$. When $q = \infty$, $p = 1$ or $p = \infty$, we can follow the same proof. Write $p' = p/(p-1)$. Define the linear operator Φ_N on $\ell^q(\Lambda)$ by

$$\Phi_N c := \big(H(\lambda/N)c(\lambda)\big)_{\lambda\in\Lambda} \qquad \text{for } c := (c(\lambda))_{\lambda\in\Lambda} \in \ell^q(\Lambda),$$

where $H(t) = \big(\sum_{i\in\mathbb{Z}^d} h(t-i)\big)^{-1}$, $t \in \mathbb{R}^d$. By the invertibility of A, we have that

$$\begin{aligned}\|A^{-1}\|^{-1}_{\mathscr{B}(\ell^q)}\|\Psi_i^N c\|_q &\le \|\Psi_i^N Ac\|_q + \|[\Psi_i^N, A]c\|_q\\ &\le \|\Psi_i^N Ac\|_q + \|\chi_{iN}^{3N}[\Psi_i^N, A]c\|_q + \|(I-\chi_{iN}^{3N})A\chi_{iN}^{2N}\Psi_i^N c\|_q\\ &\le \|\Psi_i^N Ac\|_q + \sum_{j\in\mathbb{Z}^d}\|\chi_{iN}^{3N}[\Psi_i^N, A]\Phi_N\Psi_j^N c\|_q + \|A - A_N\|_{\mathscr{C}_{1,0}(\Lambda)}\|\Psi_i^N c\|_q. \end{aligned} \tag{19}$$

Choose N so large that

$$N \ge \left(2\left(\frac{d}{rp'-d}\right)^{1/p'}\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\right)^{1/(r-d/p')} \tag{20}$$

It follows from (11), (19) and (20) that

$$\|A^{-1}\|^{-1}_{\mathscr{B}(\ell^q)}\|\Psi_i^N c\|_q \le 2\|\Psi_i^N Ac\|_q + 2\sum_{j\in\mathbb{Z}^d}\|\chi_{iN}^{3N}[\Psi_i^N, A]\Phi_N\Psi_j^N c\|_q. \tag{21}$$

For $i, j \in \mathbb{Z}^d$ with $|i-j| \le 10$, we obtain from Lipschitz property of h that

$$\begin{aligned}&\|\chi_{iN}^{3N}[\Psi_i^N, A]\Phi_N\Psi_j^N c\|_q\\ &\quad\le \left(\sum_{\lambda\in\Lambda}\left(\sum_{\lambda'\in\Lambda}\chi_{iN}^{3N}(\lambda)|a(\lambda,\lambda')||h(\lambda/N-i)-h(\lambda'/N-i)|h(\lambda'/N-j)|c(\lambda')|\right)^q\right)^{1/q}\\ &\quad\le \left(\sum_{\lambda'\in\Lambda}\left(\sum_{|\lambda-\lambda'|\le 15N}\min(|\lambda-\lambda'|/N,1)|a(\lambda,\lambda')|h(\lambda'/N-j)||c(\lambda')|\right)^q\right)^{1/q}\\ &\quad\le R(\Lambda)\left(\sum_{|k|\le 15N}\min((|k|+1)/N,1)A_k\right)\|\Psi_j^N c\|_q, \end{aligned} \tag{22}$$

and for $i, j \in \mathbb{Z}^d$ with $|i-j| > 10$, we have that

$$\begin{aligned}\|\chi_{iN}^{3N}[\Psi_i^N, A]\Phi_N\Psi_j^N c\|_q &\le \left(\sum_{\lambda\in\Lambda}\left(\sum_{\lambda'\in\Lambda}\chi_{iN}^{3N}(\lambda)|a(\lambda,\lambda')|h(\lambda'/N-j)|c(\lambda')|\right)^q\right)^{1/q}\\ &\le \left(\sum_{\lambda\in\Lambda}\left(\sum_{(|i-j|-5)N\le|\lambda-\lambda'|\le(|i-j|+5)N}|a(\lambda,\lambda')|h(\lambda'/N-j)|c(\lambda')|\right)^q\right)^{1/q}\\ &\le 2R(\Lambda)\left(\sum_{(|i-j|-5)N\le|k|\le(|i-j|+5)N}A_k\right)\|\Psi_j^N c\|_q, \end{aligned} \tag{23}$$

where

$$A_k = \sup_{\lambda,\lambda'\in\Lambda}|a(\lambda,\lambda')|\chi_{k+[0,1)^d}(\lambda-\lambda').$$

We define a function

$$\widetilde{V}_{A,N}(i) = 2R(\Lambda)\times\begin{cases}\sum_{|k|\le 15N}\min((|k|+1)/N,1)A_k & \text{if } |i|\le 10,\\ \sum_{(|i|-5)N\le|k|\le(|i|+5)N}A_k & \text{if } |i|>10.\end{cases} \tag{24}$$

We have from (21), (22), (23) and (24) that

$$\|\Psi_i^N c\|_q \le 2\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|\Psi_i^N Ac\|_q + \sum_{j\in\mathbb{Z}^d}V_{A,N}(i,j)\|\Psi_j^N c\|_q, \tag{25}$$

where $V_{A,N} := (V_{A,N}(i,j))_{i,j\in\mathbb{Z}^d}$ and $V_{A,N}(i,j) = 2\|A^{-1}\|_{\mathscr{B}(\ell^q)}\widetilde{V}_{A,N}(i-j)$. We write

$$V^{\ell}_{A,N} = \big((V_{A,N})^{\ell}(i,j)\big)_{i,j\in\mathbb{Z}^d}. \tag{26}$$

We apply (25) repeatedly to have that

$$\begin{aligned}\|\Psi_i^N c\|_q \le 2\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|\Psi_i^N Ac\|_q + 2\|A^{-1}\|_{\mathscr{B}(\ell^q)}\sum_{\ell=1}^{n-1}\sum_{j\in\mathbb{Z}^d} V^{\ell}_{A,N}(i,j)\|\Psi_j^N Ac\|_q \\ + \sum_{j\in\mathbb{Z}^d} V^{n}_{A,N}(i,j)\|\Psi_j^N c\|_q.\end{aligned} \tag{27}$$

Note that

$$\|A\|_{\mathscr{C}_{p,r}(\Lambda)} \le 2^r\Big(\sum_{k\in\mathbb{Z}^d} A_k^p(1+|k|)^{pr}\Big)^{1/p} \tag{28}$$

and

$$\left(\sum_{k\in\mathbb{Z}^d} A_k^p(1+|k|)^{pr}\right)^{1/p} \le 2^r\|A\|_{\mathscr{C}_{p,r}(\Lambda)}. \tag{29}$$

Observing from (29) that

$$\begin{aligned}&\left(\sum_{|i|>10}\big(\widetilde{V}_{A,N}(i)\big)^p(1+|i|)^{pr}\right)^{1/p}\\ &\qquad\le 2R(\Lambda)(10N)^{d(p-1)/p}4^rN^{-r}\left(\sum_{|i|>10}\ \sum_{(|i|-5)N\le|k|\le(|i|+5)N} A_k^p(1+|k|)^{pr}\right)^{1/p}\\ &\qquad\le 10^d2^{3r+1}R(\Lambda)N^{-r+d/p'}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\left(\sum_{|i|\le10}\big(\widetilde{V}_{A,N}(i)\big)^p(1+|i|)^{pr}\right)^{1/p}\\ &\qquad\le 2R(\Lambda)\left(\sum_{|i|\le10}(1+|i|)^{pr}\left(\sum_{|k|\le15N}\min((|k|+1)/N,1)A_k\right)^p\right)^{1/p}\\ &\qquad\le 2R(\Lambda)11^{r+d/p}\Big(N^{-1}\sum_{|k|\le N-1}(|k|+1)A_k+\sum_{N\le|k|\le15N}A_k\Big)\\ &\qquad\le R(\Lambda)\,2^{r+2}11^{r+d/p}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\\ &\qquad\quad\times\begin{cases}\Big(\big(\frac{d}{rp'-d}\big)^{1/p'}+\big(\frac{2d+rp'-d}{|rp'-p'-d|}\big)^{1/p'}\Big)N^{-\min(1,r-d/p')} & \text{if } r\ne d/p'+1,\\ (2d)^{1/p'}N^{-1}(\ln(N+1))^{1/p'} & \text{if } r=d/p'+1,\end{cases}\end{aligned} \tag{31}$$

we have that

$$\|V_{A,N}\|_{\mathscr{C}_{p,r}(\mathbb{Z}^d)} \le D_{d,p,r}\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\times\begin{cases}N^{-\min(1,r-d/p')} & \text{if } r\ne d/p'+1,\\ N^{-1}(\ln(N+1))^{1/p'} & \text{if } r=d/p'+1,\end{cases} \tag{32}$$

where

$$D_{d,p,r} = 2^{3r+3}11^{d+r}R(\Lambda)\times\begin{cases}\big(\frac{d}{rp'-d}\big)^{1/p'}+\big(\frac{2d+rp'-d}{|rp'-p'-d|}\big)^{1/p'} & \text{if } r\ne d/p'+1,\\ (2d)^{1/p'} & \text{if } r=d/p'+1.\end{cases}$$

Choose the smallest integer N_0 satisfying

$$1 \ge 2C_1D_{d,p,r}\|A^{-1}\|_{\mathscr{B}(\ell^q)}\|A\|_{\mathscr{C}_{p,r}(\Lambda)}\times\begin{cases}N_0^{-\min(1,r-d/p')} & \text{if } r\ne d/p'+1\\ N_0^{-1}(\ln N_0)^{1/p'} & \text{if } r=d/p'+1,\end{cases} \tag{33}$$

where C_1 is the constant in (10). Then N_0 satisfies (20), so (21) and (27) hold. From (33) there exist absolute constants C_2, C_3 such that for $r \neq d/p' + 1$,

$$N_0 \le C_2 (\|A^{-1}\|_{\mathscr{B}(\ell^q)} \|A\|_{\mathscr{C}_{p,r}(\Lambda)})^{1/\min(1, r-d/p')} \tag{34}$$

and for $r = d/p' + 1$

$$N_0 \le C_3 \|A^{-1}\|_{\mathscr{B}(\ell^q)} \|A\|_{\mathscr{C}_{p,r}(\Lambda)} \big(\ln(\|A^{-1}\|_{\mathscr{B}(\ell^q)} \|A\|_{\mathscr{C}_{p,r}(\Lambda)}\big)^{1/p'}. \tag{35}$$

It follows from (10), (32) and (33) that for $n \in \mathbb{N}$,

$$\|V_{A,N_0}^n\|_{\mathscr{C}_{p,r}(\mathbb{Z}^d)} \le 2^{-n}. \tag{36}$$

This combining with Proposition 1 (ii) and (iii) implies that

$$\lim_{n\to\infty} \left\| \left(\sum_{j\in\mathbb{Z}^d} V_{A,N_0}^n(i,j) \|\Psi_j^{N_0} c\|_q \right)_{i\in\mathbb{Z}^d} \right\|_q = 0,$$

so taking $n \to \infty$ in (27), we have that

$$\|\Psi_i^{N_0} c\|_q \le \|A^{-1}\|_{\mathscr{B}(\ell^q)} \sum_{j\in\mathbb{Z}^d} W_{A,N_0}(i,j) \|\Psi_j^{N_0} Ac\|_q, \tag{37}$$

where $W_{A,N_0}(i,j) = 2I + 2\sum_{\ell=1}^{\infty} V_{A,N_0}^{\ell}(i,j)$. It follows from (36) that

$$\|W_{A,N_0}\|_{\mathscr{C}_{p,r}(\mathbb{Z}^d)} \le 4. \tag{38}$$

We define an infinite matrix $H_{A,N_0} := (H_{A,N_0}(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$, where

$$H_{A,N_0}(\lambda,\lambda') = \sum_{|mN_0-\lambda|\le N_0} \sum_{|nN_0-\lambda'|\le 2N_0} W_{A,N_0}(m,n). \tag{39}$$

Since for $k \in \mathbb{Z}^d$,

$$\sup_{\lambda,\lambda'\in\Lambda} \chi_{k+[0,1)^d}(\lambda-\lambda') \sum_{|mN_0-\lambda|\le N_0} \sum_{|nN_0-\lambda'|\le 2N_0} W_{A,N_0}(m,n) \le \sum_{\varepsilon=\{-4,-3,\dots,3,4\}^d} \sum_{m-n=[k/N_0]+\varepsilon} W_{A,N_0}(m,n),$$

we have from (38) that

$$\|H_{A,N_0}\|_{\mathscr{C}_{p,r}(\Lambda)} \le 10^{d/p+r+d+1} N_0^{d/p+r}. \tag{40}$$

Let $A^{-1} := (a^{-1}(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}$, $a_{\lambda'}^{-1} = (a^{-1}(\lambda,\lambda'))_{\lambda\in\Lambda}$ and $a_{\lambda'}^{-1}(\lambda) = a^{-1}(\lambda,\lambda')$. Replace c by $a_{\lambda'}^{-1}$ in (37) to get that for $\lambda \in \Lambda$ and $m \in \mathbb{Z}^d$ with $|mN_0 - \lambda| \le N_0$

$$\begin{aligned} |a^{-1}(\lambda,\lambda')| &\le \|\Psi_m^{N_0} a_{\lambda'}^{-1}\|_q \\ &\le \|A^{-1}\|_{\mathscr{B}(\ell^q)} \sum_{|mN_0-\lambda|\le N_0} \sum_{|nN_0-\lambda'|\le 2N_0} W_{A,N_0}(m,n) \|\Psi_n^{N_0} A a_{\lambda'}^{-1}\|_q \\ &\le \|A^{-1}\|_{\mathscr{B}(\ell^q)} H_{A,N_0}(\lambda,\lambda'). \end{aligned} \tag{41}$$

It follows from (40) and (41) that

$$\|A^{-1}\|_{\mathscr{C}_{p,r}(\Lambda)} \le \|A^{-1}\|_{\mathscr{B}(\ell^q)} \|H_{A,N_0}\|_{\mathscr{C}_{p,r}(\Lambda)} \le 10^{d/p+r+d+1} N_0^{d/p+r} \|A^{-1}\|_{\mathscr{B}(\ell^q)}. \tag{42}$$

From (34), (35) and (42), (18) holds. □

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References

[1] A. G. Baskakov, "Wiener's theorem and asymptotic estimates for elements of inverse matrices", *Funkts. Anal. Prilozh.* **24** (1990), no. 3, p. 64-65, translation in *Funct. Anal. Appl.* **24** (1990), no. 3, p. 222-224.

[2] ———, "Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis", *Sib. Mat. Zh.* **38** (1997), no. 1, p. 14-28, translation in *Sib. Math. J.* **38** (1997), no. 1, p. 10-22.

[3] L. H. Brandenburg, "On identifying the maximal ideals in Banach Algebras", *J. Math. Anal. Appl.* **50** (1975), p. 489-510.

[4] K. Gröchenig, "Wiener's lemma: theme and variations, an introduction to spectral invariance and its applications", in *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2010.

[5] K. Gröchenig, A. Klotz, "Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices", *Constr. Approx.* **32** (2010), no. 3, p. 429-466.

[6] ———, "Norm-controlled inversion in smooth Banach algebras. I", *J. Lond. Math. Soc.* **88** (2013), no. 1, p. 49-64.

[7] ———, "Norm-controlled inversion in smooth Banach algebras. II", *Math. Nachr.* **287** (2014), no. 8-9, p. 917-937.

[8] S. Jaffard, "Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **7** (1990), no. 5, p. 461-476.

[9] N. Nikolski, "In search of the invisible spectrum", *Ann. Inst. Fourier* **49** (1999), no. 6, p. 1925-1966.

[10] K. S. Rim, C. E. Shin, Q. Sun, "Stability of localized integral operators on weighted L^p spaces", *Numer. Funct. Anal. Optim.* **33** (2012), no. 7-9, p. 1166-1193.

[11] E. Samei, V. Shepelska, "Norm-controlled inversion in weighted convolution algebras", *J. Fourier Anal. Appl.* **25** (2019), no. 6, p. 3018-3044.

[12] C. E. Shin, Q. Sun, "Differential subalgebras and norm-controlled inversion", submitted, https://arxiv.org/abs/1911.08679, 2019.

[13] ———, "Polynomial control on stability", *J. Funct. Anal.* **276** (2019), no. 1, p. 148-182.

[14] J. Sjöstrand, "Wiener type algebra of pseudodifferential operators", *Sémin. Équ. Dériv. Partielles* **1994-1995** (1995), article ID 4 (19 pages).

[15] Q. Sun, "Wiener's lemma for infinite matrices with polynomial off-diagonal decay", *C. R. Math. Acad. Sci. Paris* **340** (2005), p. 567-570.

[16] ———, "Wiener's lemma for infinite matrices", *Trans. Am. Math. Soc.* **359** (2007), no. 7, p. 3099-3123.

[17] ———, "Wiener's lemma for localized integral operators", *Appl. Comput. Harmon. Anal.* **25** (2008), no. 2, p. 148-167.

[18] ———, "Wiener's lemma for infinite matrices. II.", *Constr. Approx.* **34** (2011), no. 2, p. 209-235.

[19] N. Wiener, "Tauberian theorem", *Ann. Math.* **33** (1932), p. 1-100.