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Algebraic Geometry / *Géométrie algébrique*

Note on quasi-polarized canonical Calabi–Yau threefolds

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Abstract. Let (X, L) be a quasi-polarized canonical Calabi–Yau threefold. In this note, we show that $|mL|$ is basepoint free for $m \geq 4$. Moreover, if the morphism $\Phi_{|4L|}$ is not birational onto its image and $h^0(X, L) \geq 2$, then $L^3 = 1$. As an application, if Y is an n-dimensional Fano manifold such that $-K_Y = (n-3)H$ for some ample divisor H, then $|mH|$ is basepoint free for $m \geq 4$ and if the morphism $\Phi_{|4H|}$ is not birational onto its image, then either Y is a weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1, \ldots, 1, 2, 5)$ or $h^0(Y, H) = n-2$.

Keywords. birationality, Calabi–Yau threefolds, Fano manifolds, freeness.

2020 Mathematics Subject Classification. 14E05, 14J30, 14J32, 14J45.

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1. Introduction

A normal projective complex threefold X is called a *canonical Calabi–Yau threefold* if $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, $h^1(X, \mathcal{O}_X) = 0$ and X has only canonical singularities. We say that X is a *minimal Calabi–Yau threefold*, if, in addition, X has only $\mathbb{Q}$-factorial terminal singularities. A pair of a normal projective variety X and a line bundle L is called a *polarized variety* if the line bundle L is ample, and a *quasi-polarized variety* if the line bundle L is nef and big. For a given quasi-polarized canonical Calabi–Yau threefold (X, L), the following questions naturally arise.

Question 1.

(1) *When is $\Phi_{|mL|}$ (the rational map defined by $|mL|$) birational onto its image?*
(2) *When is $|mL|$ basepoint free?*

These two questions have already been investigated by several mathematicians in various different settings [6, 13, 14] etc. Our first result in this note can be viewed as a generalization of [13, Theorem 1.1] and [14, Theorem 1].

Theorem 2. *Let (X, L) be a quasi-polarized canonical Calabi–Yau threefold. Then $|mL|$ is basepoint free for $m \geq 4$. Moreover, if $\Phi_{|4L|}$ is not birational onto its image, then either $L^3 = 1$ or $h^0(X, L) = 1$.*

The estimate is sharp as showed by a general weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,1,2,5)$. We remark also that we have always $h^0(X,L) \geq 1$ by [8, Proposition 4.1] and the morphism $\Phi_{|5L|}$ is always birational onto its image by [6, Theorem 1.7]. The basepoint freeness of $|4H|$ is an easy consequence of [12, Theorem 24] and the existence of semi-log canonical member in $|H|$ (cf. [8, Proposition 4.2]), and for the second part of the theorem, our proof basically goes along the line of [14, Theorem 1]. As the first application of Theorem 2, we generalize our previous result in [11, Theorem 1.7].

Corollary 3. *Let X be a weak Fano fourfold with at worst Gorenstein canonical singularities. Then*

(1) *the complete linear system $|-mK_X|$ is basepoint free for $m \geq 4$;*
(2) *the morphism $\Phi_{|-mK_X|}$ is birational onto its image for $m \geq 5$.*

As before, the estimates in Corollary 3 are both optimal as showed by a general weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,1,1,2,5)$. As the second application, in higher dimension, using the existence of good ladder on Fano manifolds with coindex four proved in [11] and the work of Fujita on polarized projective manifold with small Δ-genus and sectional genus (cf. [4]), we derive the following theorem which can also be viewed as a generalization of [13, Theorem 1.1] in higher dimension.

Theorem 4. *Let X be an n-dimensional Fano manifold such that $-K_X = (n-3)H$ for some ample divisor H. Then*

(1) *the complete linear system $|mH|$ is basepoint free when $m \geq 4$;*
(2) *the morphism $\Phi_{|mH|}$ is birational onto its image when $m \geq 5$.*

Moreover, if the morphism $\Phi_{|4H|}$ is not birational onto its image, then one of the following holds.

(i) *X is a weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,\dots,1,2,5)$.*
(ii) *$h^0(X,H) = n-2$.*

As in dimension four, the example given in Theorem 4 (i) guarantees that the estimates given in Theorem 4 are best possible. On the other hand, we have always $h^0(X,H) \geq n-2$ in Theorem 4 (cf. [11, Theorem 1.2]), and if X is a general weighted complete intersection of type (6,6) in the weighted projective space $\mathbb{P}(1,\dots,1,2,2,3,3)$ and $H \in |\mathcal{O}_X(1)|$, then X is a n-dimensional Fano manifold such that $-K_X = (n-3)H$ and $h^0(X,H) = n-1$. This leads us to ask the following natural question.

Question 5 (see [4, 2.14], [10, Problems 2.4]). *Is there an example of Fano n-fold X such that $-K_X = (n-3)H$ for some ample divisor H and $h^0(X,H) = n-2$?*

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2. Proof of the main results

Throughout the present paper, we work over the complex numbers and we adopt the standard notation in Kollár–Mori [9], and will freely use them. We start by selecting some results in minimal model program, and we shall use them in the sequel.

Lemma 6. *Let (X, L) be a quasi-polarized projective variety with at most canonical singularities.*

(1) *There exists a projective variety Y with only $\mathbb{Q}$-factorial terminal singularities and a proper surjective birational morphism $\nu\colon Y \to X$ such that $K_Y = \nu^* K_X$. Moreover, in this case, $M\colon = \nu^* L$ gives a quasi-polarization on Y.*

(2) *Assume moreover that $aL - K_X$ is nef and big for some positive integer a. Then $|mL|$ is basepoint free for any large m and gives a proper surjective birational morphism $\mu\colon X \to Z$ such that $L = \mu^* H$ for some ample line bundle H on Z.*

Proof. The assertion (1) is a consequence of [2, Corollary 1.4.4], and Y is called a terminal modification of X. The statement (2) is an easy corollary of the Basepoint-free theorem. In fact, applying Basepoint-free theorem (cf. [9, Theorem 3.3]), $|mL|$ is basepoint free for all large m and we define $\mu\colon X \to Z$ to be the Stein factorization of the morphism $\Phi_{|mL|}$. Clearly μ is independent of the choice of m. In particular, there exists two ample line bundles H_1 and H_2 on Z such that $mL = \mu^* H_1$ and $(m+1)L = \mu^* H_2$. Set $H = H_2 - H_1$. It follows that $L = \mu^* H$. □

Definition 7. *Let X be a reduced equi-dimensional algebraic scheme and B an effective $\mathbb{R}$-divisor on X. The pair (X, B) is said to be SLC (semi-log canonical) if the following conditions are satisfied.*

(1) *X satisfies the Serre condition S_2, and has only normal crossing singularities in codimension one.*
(2) *The singular locus of X does not contain any irreducible component of B.*
(3) *$K_X + B$ is an $\mathbb{R}$-Cartier divisor.*
(4) *For any birational morphism $\mu\colon Y \to X$ from a normal variety, if we write $K_Y + B_Y = \mu^*(K_X + B)$, then all the coefficients of B_Y are at most 1.*

Moreover, (X, B) is called a stable log pair if in addition

(5) *$K_X + B$ is ample.*

A stable variety is a stable log pair (X, B) with $B = 0$, and we will abbreviate it as X.

Definition 8. *Let (X, L) be an n-dimensional quasi-polarized projective manifold.*

(1) *The Δ-genus $\Delta(X, L)$ of (X, L) is defined to be $n + L^n - h^0(X, L)$.*
(2) *The sectional genus $g(X, L)$ of (X, L) is defined to be $\left(K_X \cdot L^{n-1} + (n-1)L^n\right)/2 + 1$.*

Now we give the proof of Theorem 2.

Proof of Theorem 2. Recall that canonical singularities are normal rational Cohen–Macaulay singularities. By Lemma 6 (2), there exists a proper surjective birational morphism $\mu\colon X \to Z$ such that $L = \mu^* H$ for some ample line bundle H on Z. Moreover, as $\mu_* K_X = K_Z$, we have $\mathscr{O}_Z(K_Z) = \mathscr{O}_Z$. In particular, Z has only canonical singularities. Thus, Z has only rational singularities and $R^i \mu_* \mathscr{O}_X = 0$ for $i > 0$. This implies $h^1(Z, \mathscr{O}_Z) = h^1(X, \mathscr{O}_X) = 0$. Hence (Z, H) is actually a polarized canonical Calabi–Yau threefold. On the other hand, using the projection formula, we get $\mu_* \mathscr{O}_X(mL) = \mathscr{O}_Z(mH)$ and $R^i \mu_* \mathscr{O}_X(mL) = 0$ for $i > 0$. This implies that the induced morphism $\mu^*\colon H^0(Z, mH) \to H^0(X, mL)$ is an isomorphism for all m. In particular, $|mL|$ is basepoint free if and only if $|mH|$ is basepoint free and $\Phi_{|mL|}$ is birational onto its image if and only if $\Phi_{|mH|}$ is birational onto its image. According to [8, Proposition 4.2], there exists a member $S \in |H|$ such that S is a stable surface with $K_S = H|_S$. Clearly the base locus of $|mH|$ is contained in S for any $m \geq 1$. By Kawamata–Viehweg vanishing theorem and our assumption, the natural restriction

$$H^0(Z, mH) \longrightarrow H^0(S, mH|_S)$$

is surjective for all $m \in \mathbb{Z}$. Thanks to [12, Theorem 24], $|mK_S|$ is basepoint free for all $m \geq 4$. Consequently, $|mH|$ is also basepoint free for all $m \geq 4$.

Next we consider the case where $\Phi_{|4L|}$ is not birational onto its image. By Lemma 6 (1), there exists a terminal modification $\nu\colon Y \to X$ such that (Y, M) is a quasi-polarized minimal Calabi–Yau threefold where $M = \nu^* L$. As above, we see that $L^3 = M^3$ and the induced morphism $\nu^*\colon H^0(X, mL) \to H^0(Y, mM)$ is an isomorphism for all m. In particular, $\Phi_{|mL|}$ is birational onto its image if and only if $\Phi_{|mM|}$ is birational onto its image. Thus, after replacing (X, L) by (Y, M), we may assume that (X, L) itself is a quasi-polarized minimal Calabi–Yau threefold. In particular, X is actually factorial by [7, Lemma 5.1]. As mentioned in the introduction, we have always $h^0(X, L) \geq 1$ by [8, Proposition 4.1]. Thus, to prove Theorem 2, we may assume that $h^0(X, L) \geq 2$ and we distinguish two cases according to whether $\dim \Phi_{|L|}(X) = 1$.

1st case. $\dim \Phi_{|L|}(X) \geq 2$**.** By Hironaka's resolution theorem, there exists a smooth projective threefold Y and a proper surjective birational morphism $\pi\colon Y \to X$ and a decomposition

$$|\pi^* L| = |F| + B$$

such that $|F|$ is basepoint free. Let $T \in |F|$ be a general smooth member. By the proof of [14, Theorem 1], $\Phi_{|(m+1)L|}$ is birational onto its image if $\Phi_{|\pi^* mL|_T + K_T|}$ is birational onto its image. Thus, if $(\pi^* L|_T)^2 \geq 2$, by [16, Theorem 1 (ii)], the complete linear system $|\pi^* mL|_T + K_T|$ is birational onto its image for $m \geq 3$. If $(\pi^* L|_T)^2 = 1$, by the projection formula, we get $L^2 \cdot \pi_* T = 1$ since T is a general member in the movable family $|F|$. Thanks to [14, Lemma 1.1 (4)], we see that $L^3 = 1$.

2nd case. $\dim \Phi_{|L|}(X) = 1$**.** Since $h^1(X, \mathscr{O}_X) = 0$, then $|L|$ is composed with a rational pencil of surfaces. Moreover, there exists a smooth projective threefold Y and a proper surjective birational morphism $\mu\colon Y \to X$ and a decomposition

$$|\mu^* L| = n|F| + B$$

such that $|F|$ is a free pencil. Let T be a general smooth element in $|F|$. Again by the proof of [14, Theorem 1], $\Phi_{|(m+1)L|}$ is birational onto its image if $\Phi_{|\pi^* mL|_T + K_T|}$ is birational onto its image. Using the same argument as in the 1st case, we obtain $L^3 = 1$ if $\Phi_{|4L|}$ is not birational onto its image. □

Corollary 3 is an immediate consequence of Theorem 2 and the existence of good divisor on weak Fano fourfolds established in [8, Theorem 5.2].

Proof of Corollary 3. The statement (2) was proved in [11, Theorem 1.7]. By Lemma 6 (2), there exists a surjective proper birational map $\mu\colon X \to Z$ and an ample line bundle H on Z such that $\mu^* H = -K_X$. Moreover, as $\mu_* K_X = K_Z$, it follows that $-K_Z = H$ and $\mu^* K_Z = K_X$. In particular, Z is a Fano foufold with at worst Gorenstein canonical singularities. According to [8, Theorem 5.2], there exists a member $Y \in |-K_Z|$ such that Y has only Gorenstein canonical singularities. Hence $(Y, -K_Z|_Y)$ is a polarized canonical Calabi–Yau threefold. Thanks to Kawamata–Viehweg vanishing theorem, the natural restriction map

$$H^0(Z, -mK_Z) \longrightarrow H^0(Y, -mK_Z|_Y)$$

is surjective for all $m \in \mathbb{Z}$. Then, by Theorem 2, we see that $|-mK_Z|$ is basepoint free for $m \geq 4$. On the other hand, it is easy to see that the induced morphism

$$\mu^*\colon H^0(Z, -mK_Z) \to H^0(X, -mK_X)$$

is an isomorphism for all m. Hence, $|-mK_X|$ is basepoint free for all $m \geq 4$. □

Finally we give the proof of Theorem 4.

Proof of Theorem 4. By [11, Theorem 1.2] and [3, Theorem 1.1], there exists a descending sequence of subvarieties of X

$$X = X_n \supsetneq X_{n-1} \supsetneq \cdots \supsetneq X_3$$

such that $X_{i+1} \in |H|_{X_i}|$ and X_i has only Gorenstein canonical singularities. Moreover, it is easy to see that $(X_3, H|_{X_3})$ is a polarized canonical Calabi–Yau threefold and the base locus of $|H|$ is contained in X_3. Thanks to Theorem 2, $|mH|_{X_{n-3}}|$ is basepoint free if $m \geq 4$. By Kawamata–Viehweg vanishing theorem, it is easy to see that the natural restriction

$$H^0(X, mH) \longrightarrow H^0(X_3, mH|_{X_3})$$

is surjective for all $m \in \mathbb{Z}$. Thus $|mH|$ is basepoint free for $m \geq 4$. On the other hand, if $\Phi_{|4H|}$ is not birational onto its image, then we may assume that $\Phi_{|4H|_{X_3}|}$ is not birational onto its image since we can choose all X_i to be general (cf. [14, Lemma 1.3]). If $h^0(X, H) \neq n-2$, by [11, Theorem 1.2], we get $h^0(X, H) \geq n-1$. As a consequence, we obtain

$$h^0(X_3, H|_{X_3}) = h^0(X, H) - (n-3) \geq 2.$$

Then Theorem 2 implies $H^n = (H|_{X_3})^3 = 1$. Then, by definition, we have

$$g(X, H) := (K_X \cdot H^{n-1} + (n-1)H^n)/2 + 1 = H^n + 1 = 2,$$

and

$$\Delta(X, H) := H^n + n - h^0(X, H) \leq 1 + n - (n-1) = 2.$$

On the other hand, it is well-known that we have $\Delta(X, H) \geq 0$ with equality if and only if $g(X, H) = 0$ (cf. [5, Theorem 12.1]). This implies that $\Delta(X, H) = 1$ or 2 in our situation. According to [4, Proposition 2.3 and 2.4], X is isomorphic to either a weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1, \ldots, 1, 2, 5)$ or a weighted complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, \ldots, 1, 2, 2, 3, 3)$. However, if X is a weighted complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, \ldots, 1, 2, 2, 3, 3)$, then the group $H^0(X, mH)$ $(m \geq 3)$ contains the monomials

$$\{x_1 x_0^{m-1}, \ldots, x_{n-2} x_0^{m-1}, x_{n-1} x_0^{m-2}, x_n x_0^{m-2}, x_{n+1} x_0^{m-3}, x_{n+2} x_0^{m-3}\},$$

where x_i are the weighted homogeneous coordinates of $\mathbb{P}(1, \ldots, 1, 2, 2, 3, 3)$ in order. This shows that $\Phi_{|mH|}$ $(m \geq 3)$ is one-to-one on the non-empty Zariski open subset $\{x_0 \neq 0\} \cap X$ and is therefore birational, excluding this case. □

3. Further discussions

Let (X, L) be a quasi-polarized canonical Calabi–Yau threefold such that $h^0(X, L) = 1$. Let (Y, M) be a terminal modification of (X, L). Then Y is smooth in codimensioin two. Since canonical singularities are rational, by the Riemann–Roch formula and the projection formula, we obtain

$$\chi(X, tL) = \chi(Y, tM) = \frac{M^3}{6} t^3 + \frac{M \cdot c_2(Y)}{12} t + \chi(Y, \mathcal{O}_Y).$$

By Serre duality, we have $\chi(Y, \mathcal{O}_Y) = 0$. Thus, using Kawamata–Viehweg vanishing theorem, we obtain

$$1 = h^0(X, L) = h^0(Y, M) = \chi(Y, M) = \frac{1}{6} M^3 + \frac{1}{12} M \cdot c_2(Y).$$

Moreover, thanks to [15, Thereom 0.5], we have $M \cdot c_2(X) \geq 0$. It follows that

$$1 \leq L^3 = M^3 \leq 6.$$

On the other hand, a smooth ample divisor S on a 3-dimensional projective manifold X with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ is a minimal surface of general type. This simple observation yields a bridge between two important classes of algebraic varieties. In particular, the smooth ample divisor S is called a *rigid ample surface* if $h^0(X, \mathcal{O}_X(S)) = 1$. In this case, the geometric genus $p_g(S) := h^0(S, K_S)$ is zero and, by the Lefschetz theorem, the natural map $\pi_1(S) \to \pi_1(X)$ is an isomorphism. Thus, according to Theorem 2, it may be interesting to ask the following question.

Question 9. *Is there a smooth Calabi–Yau threefold X containing a rigid ample surface S?*

We remark that if we do not require the simple connectedness of X, such an example of (X, S) with the quaternion group of order 8 as its fundamental group, i.e. $\pi_1(X) = H_8$, was constructed by Beauville in [1].

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