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Algebraic Geometry / *Géométrie algébrique*

Indecomposable K_1 classes on a Surface and Membrane Integrals

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Abstract. Let X be a projective algebraic surface. We recall the K-group $K_{1,\mathrm{ind}}^{(2)}(X)$ of indecomposables and provide evidence that membrane integrals are sufficient to detect these indecomposable classes.

Résumé. Soit X une surface algébrique projective. Nous rappelons le groupe K, $K_{1,\mathrm{ind}}^{(2)}(X)$ indécomposables et apporter la preuve que les intégrales membranaires sont suffisantes pour détecter ces classes indécomposables.

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1. Introduction

Let $X/\mathbb{C}$ be a smooth projective surface, and consider a class $\{\xi\} \in K_1^{(2)}(X)$ which can be represented in the form

$$\xi = \sum_{j=1}^{N} (f_j, D_j), \quad f_j \in \mathbb{C}(D_j)^{\times}, \quad \sum_{j=1}^{N} \mathrm{div}_{D_j}(f_j) = 0 \text{ in } X,$$

and where D_j is irreducible, with $\mathrm{codim}_X D_j = 1$. ξ is said to be decomposable if $f_j \in \mathbb{C}^{\times}$ for $j = 1, \ldots, N$. A class $\{\xi\}$ is said to be indecomposable if, modulo the tame symbol image $T\big(K_2(\mathbb{C}(X))\big)$, ξ is not decomposable. The quotient space of indecomposables is denoted by $K_{1,\mathrm{ind}}^{(2)}(X)$. There is a Betti cycle class map $\mathrm{cl}_{2,1} : K_{1,\mathrm{ind}}^{(2)}(X) \to H^3(X, \mathbb{Z}(2))$, evidently with torsion image (due to Hodge theory), and whose kernel is denoted by $K_{1,\mathrm{ind}}^{(2)}(X)^{\circ}$. The map $\mathrm{cl}_{2,1}$ is defined as follows. Put $\gamma_j := f_j^{-1}[-\infty, 0]$ and $\gamma := \sum_{j=1}^{N} \gamma_j$. Then we have $\partial\gamma = \sum_{j=1}^{N} \mathrm{div}_{D_j}(f_j) = 0$, hence γ defines a

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class $\{\gamma\} \in H_1(X,\mathbb{Z}) \simeq H^3(X,\mathbb{Z}(2))$ (Poincaré duality). We now assume that $\xi \in K_{1,\mathrm{ind}}^{(2)}(X)^\circ$. Then γ bounds a real 2-chain ζ. There is the *integrally defined* transcendental Abel–Jacobi map,

$$\underline{\Phi} : K_{1,\mathrm{ind}}^{(2)}(X)^\circ \to \frac{H^{2,0}(X)^\vee}{H_2(X,\mathbb{Z})}, \quad \underline{\Phi}(\xi)(\omega) = \int_\zeta \omega.$$

It is our belief that $\underline{\Phi}$ is injective. This is the subject matter of our paper where we provide some evidence in support of this. Our main results are stated in Theorem 4 and Corollary 5.

2. Notation

We assume that the reader is familiar with mixed Hodge structures (MHS). Let V be a $\mathbb{Z}$-MHS and $\mathbb{Z}(r)$ the Tate twist. We put

$$\Gamma V = \hom_{\mathrm{MHS}}\big(\mathbb{Z}(0), V\big), \text{ and } JV = \mathrm{Ext}_{\mathrm{MHS}}\big(\mathbb{Z}(0), V\big).$$

3. The full Abel–Jacobi map on indecomposables

Let $H^2_{\mathrm{tr}}(X,\mathbb{Z}) := H^2(X,\mathbb{Z})\big/\mathrm{NS}(X)$ be transcendental cohomology, where NS stands for the Neron–Severi group. There is a the full Abel–Jacobi map [3] on indecomposables,

$$\Phi : K_{1,\mathrm{ind}}^{(2)}(X)^\circ \to J\big(H^2_{\mathrm{tr}}(X,\mathbb{Z}(2))\big) = \frac{\big(H^{2,0}(X) \oplus H^{1,1}_{\mathrm{tr}}(X)\big)^\vee}{H_2(X,\mathbb{Z})},$$

given by (for $\omega \in H^{2,0}(X) \oplus H^{1,1}_{\mathrm{tr}}(X)$):

$$\{\xi\} \mapsto \Phi(\xi)(\omega) := \frac{1}{2\pi \mathrm{i}}\left(\sum_{j=1}^N \int_{D_j} \log(f_j)\omega \; - 2\pi \mathrm{i} \int_\zeta \omega\right),$$

where log has the principal branch. It turns out that we can say something about this map. For this we recall the (limit) Betti cycle class map

$$K_2(\mathbb{C}(X)) \xrightarrow{\mathrm{d}\log_2} \Gamma H^2(\mathbb{C}(X),\mathbb{Z}(2)),\ \{f,g\} \mapsto \mathrm{d}\log f \wedge \mathrm{d}\log g,$$

where

$$\Gamma H^2(\mathbb{C}(X),\mathbb{Z}(2)) = \varinjlim_{U} \Gamma H^2(U,\mathbb{Z}(2)),\ U \subset X \text{ Zariski open.}$$

Theorem 1. Φ *is injective iff* $\mathrm{d}\log_2$ *is surjective.*

Proof. See [2, Corollary 6.5], where it should also be pointed out that $H^2(\mathbb{C}(X),\mathbb{Z}(2))$ is torsion free. □

The following conjecture seems to have survived critical examination (see [2] and the references cited there).

Conjecture 2 (Beilinson–Milnor–Hodge conjecture). $\mathrm{d}\log_2$ *is surjective.*

Remark 3. As argued in [2], this conjecture is *equivalent* to the corresponding conjecture with $\mathbb{Q}$-coefficients.

Let $X = \mathscr{X}_0 := \rho^{-1}(0)$ be a very general[1] member of a family of surfaces $\rho : \mathscr{X} \to S$, $0 \in S$. Here ρ is a smooth and proper morphism of smooth quasi-projective varieties. We recall the Kodaira–Spencer map:

$$\kappa : T_0(S) \to H^1(X,\Theta_X),$$

and put $H^1_{\mathrm{alg}}(X,\Theta_X) := \kappa(T_0(S))$. We prove the following:

[1] Very general in this context means outside of a countable union of proper analytic subsets.

Theorem 4. *Assume that $H^1_{\mathrm{alg}}(X,\Theta_X)\otimes H^{2,0}(X)\xrightarrow{\cup}H^{1,1}_{\mathrm{tr}}(X)$ is surjective. Then the correspondence $\Phi(\xi)\mapsto\underline{\Phi}(\xi)$ is injective.*

Proof. This proof takes inspiration from [1]. First of all, the assumptions in the theorem do not change if we shrink S and/or replace S by a finite cover $S'\to S$, which will be unramified over $0\in S$ by Sard's lemma, together with 0 a very general point of S. Indeed a very general point of S' will map to a very general point of S, and we can just as easily work over S'. However we can assume $0\in S$ corresponds to a very general point of S' and instead work over a polydisk neighbourhood of $0\in S$. So for simplicity, we will replace S by a polydisk, and we will assume this. Thus a cycle $\xi\in K^{(2)}_{1,\mathrm{ind}}(X)^\circ$ lifts to a relative spread cycle $\widetilde{\xi}\in K^{(2)}_1(\mathscr{X}/S)^\circ$, and corresponding normal function $\nu_{\tilde{\xi}}$, where $\nu_{\tilde{\xi}}(0)=\Phi(\xi)$. Let $\nabla=\partial\otimes 1$ be the Gauss–Manin connection associated to the Hodge bundle $\mathscr{H}:=\mathscr{O}_S\otimes R^2\rho_*\mathbb{C}$, $\mu\in H^0(S,\Theta_S)$ a linear differential operator, which we identify with the corresponding operator ∇_μ on $\mathscr{H}$. If $\omega\in H^{2,0}(X)$, there is a variational $\widetilde{\omega}\in\mathscr{K}(\mathscr{X}/S)$ (relative canonical sheaf), with $\widetilde{\omega}_0=\omega$. Now suppose that $\Phi(\xi)\neq 0$, and yet $\langle\nu_{\tilde{\xi}},\widetilde{\omega}\rangle=0$ for all $\widetilde{\omega}\in\mathscr{K}(\mathscr{X}/S)$. This translates to saying that $\langle\nu_{\tilde{\xi}},\widetilde{\omega}\rangle=\langle\gamma,\widetilde{\omega}\rangle$, for some period $\gamma\in H^0(S,R^2\rho_*\mathbb{Z}(2))$. Now for all $\mu\in H^0(S,\Theta_S)$, we arrive at:

$$\langle\gamma,\nabla_\mu\widetilde{\omega}\rangle=\mu\langle\gamma,\widetilde{\omega}\rangle=\mu\langle\nu_{\tilde{\xi}},\widetilde{\omega}\rangle=\langle\nabla_\mu\nu_{\tilde{\xi}},\widetilde{\omega}\rangle+\langle\nu_{\tilde{\xi}},\nabla_\mu\widetilde{\omega}\rangle=\langle\nu_{\tilde{\xi}},\nabla_\mu\widetilde{\omega}\rangle,$$

where we use the well-known fact that $\nu_{\tilde{\xi}}$ is quasi-horizontal, implying that $\langle\nabla_\mu\nu_{\tilde{\xi}},\widetilde{\omega}\rangle=0$ for Hodge type reasons. By our assumption on the Kodaira–Spencer map, we have $\nu_{\tilde{\xi}}\equiv 0$, a fortiori $\Phi(\xi)=0$, a contradiction. This tells us that $\underline{\Phi}(\widetilde{\xi}_t)\neq 0$ for very general $t\in S$. Since $0\in\Delta$ already corresponds to a very general $X=\mathscr{X}_0$, we can assume that $t=0$, and hence $\Phi(\xi)\neq 0\Rightarrow\underline{\Phi}(\xi)\neq 0$. □

Corollary 5. *Same assumptions as given in the above theorem. Further, let us assume Conjecture 2. Then the correspondence $\Phi\mapsto\underline{\Phi}$ is injective. In particular, $\underline{\Phi}$ is injective.*

Proof. Beilinson rigidity and Conjecture 2, imply that $K^{(2)}_{1,\mathrm{ind}}(X)^\circ$ is countable (Voisin's conjecture). Let $\xi\in K^{(2)}_{1,\mathrm{ind}}(X)^\circ$. A spread of ξ will involve a finite cover $S'_\xi\to S$ and the very general points of S'_ξ map to a dense subset S_ξ of S, being a countable intersection of open dense subsets (Baire). But again by Baire, the countable intersection

$$\bigcap_{\xi\in K^{(2)}_{1,\mathrm{ind}}(X)^\circ}S_\xi$$

is likewise dense in S; indeed and more explicitly, it amounts to the complement of a countable union of proper analytic subsets of S. Since $0\in S$ is already very general, it belongs to that intersection. Therefore if $\Phi\mapsto\underline{\Phi}$ were not injective, then it would fail to be injective for some ξ as well. Now apply the above theorem. □

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