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"Elementary" Number Theory / *Théorie « élémentaire » des nombres*

The relative growth rate for the digits in Lüroth expansions

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Abstract. In this note, the rate of growth of digits in the Lüroth expansion of an irrational number is studied relative to the rate of approximation of the number by its convergents. The Hausdorff dimension of exceptional sets of points with a given relative growth rate is established.

Keywords. Lüroth expansion, Hausdorff dimension, relative growth rate.

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1. Introduction

Let us consider the Lüroth expansion of a real number x from the interval $(0,1]$ given by

$$x=\frac{1}{d_1}+\frac{1}{d_1(d_1-1)d_2}+\cdots+\frac{1}{d_1(d_1-1)d_2\ldots d_{n-1}(d_{n-1}-1)d_n}+\cdots \tag{1}$$

for some $d_i\geq 2(i\in\mathbb{N})$, which was introduced in 1883 by Lüroth [8]. Each irrational number has a unique Lüroth expansion and each rational number has either a finite expansion or a periodic one. We denote the Lüroth expansion of $x\in(0,1]$ by $x=[d_1(x),d_2(x),\cdots]$ for short.

The Lüroth expansion can be given by the map $T:(0,1]\to(0,1]$, which is defined by

$$T(x):=d_1(x)(d_1(x)-1)\left(x-\frac{1}{d_1(x)}\right),\quad \text{where}\quad d_1(x)=\left[\frac{1}{x}\right]+1,$$

and

$$d_n(x)=d_1\left(T^{n-1}(x)\right),\quad n\geq 1,$$

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where T^n denotes the n^{th} iterate of $T(T^0 = Id_{(0,1]})$. The Lebesgue measure is T-invariant and the map T is ergodic (see [4]). For more details about the Lüroth expansion, one is referred to [4].

The properties of digits in the Lüroth expansion have been intensely studied in recent years. The behavior of the digits in Lüroth expansion were investigated in [1] and [6]. More recently, Liao and Rams [7] studied the increasing rate of the Birkhoff sums in the infinite iterated function systems, which include Lüroth expansion and continued fraction expansion as special cases.

For any $x \in (0,1]$, write $L_n(x) = \max\{d_1(x), d_2(x), \ldots, d_n(x)\}$ to be the maximal digit among the first n terms in the Lüroth expansion of x. The growth rates of $L_n(x)$ were studied by Shen et al. [10] and Song et al. [11] from the viewpoint of multifractal analysis.

Let the n^{th} convergent $P_n(x)/Q_n(x)$ of x in the Lüroth expansion be defined as the partial sum of the first n terms of the series (1), i.e.

$$\frac{P_n(x)}{Q_n(x)} = \sum_{j=1}^{n} \frac{1}{d_1(x)(d_1(x)-1)\ldots d_{j-1}(x)(d_{j-1}(x)-1)d_j(x)}.$$

From the ergodicity of T and Birkhoff's individual ergodic Theorem, we can obtain the following result:

$$\lim_{n\to\infty} -\frac{1}{n}\log\left|x - \frac{P_n(x)}{Q_n(x)}\right| = d, \quad \text{a.e.} \quad \text{where} \quad d \approx 2.03. \tag{2}$$

Here and in the following a.e. will be with respect to Lebesgue measure.

In [14], they investigated the growth rate of the maximal digit relative to that of approximation of the number by its convergents. More precisely, they proved

Theorem 1 (cf. [14]). *The set*

$$\left\{ x \in (0,1] : \lim_{n\to\infty} \frac{L_n(x)\log\log n}{-\log\left|x - \frac{P_n(x)}{Q_n(x)}\right|} = \alpha \right\}$$

is of Hausdorff dimension 1, for any $\alpha \geq 0$.

We use $\dim_H$ to denote the Hausdorff dimension. For more information about Hausdorff dimension, see the book [3].

In this note, we are interested in finding what happens when the relative growth rate of the digits relative to that of approximation of the number by its convergents is a given number. For any $z \geq 0$, let

$$F(z) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{\log d_{n+1}(x)}{-\log\left|x - \frac{P_n(x)}{Q_n(x)}\right|} = z \right\},$$

we obtain the following result.

Theorem 2. *For any* $0 < z \leq 1$, $\dim_H F(z) = \frac{1-z}{2}$. $F(0)$ *is of Lebesgue measure 1. If* z *is not in* $[0,1]$, $F(z)$ *is empty.*

It is worth pointing out that the kinds of relative growth rate of partial quotients for regular continued fraction expansion have been attacked by Hass [5], Sun and Wu [12] and Tan and Zhou [13] in recent years.

2. Preliminaries

In this section, we briefly recall some basic properties and results of Lüroth expansion.

By [4], any sequence of integers $\{d_n\}_{n\geq 1}$ with $d_n \geq 2$ for $n \geq 1$, is admissible, i.e., there exists some $x \in (0,1]$ whose Lüroth expansion satisfying $d_n(x) = d_n$ for $n \geq 1$.

For any $d_1, d_2, \ldots, d_n \in \mathbb{N}\setminus\{0,1\}$, let $I_n(d_1, d_2, \ldots, d_n)$ be the set of numbers in $(0,1]$ whose Lüroth expansion begins by $d_1, d_2, \ldots, d_n$ and called a rank$-n$ basic interval. It is clear that its length is given by the following formula.

Lemma 3 (cf. [4]). $|I_n(d_1, d_2, \ldots, d_n)| = \left(\prod_{j=1}^n d_j(d_j-1)\right)^{-1}$.

Lemma 4 (cf. [7]). *Let $\{s_n\}_{n\geq 1}$ be a sequence of positive integers tending to infinity, then for any positive number $N \geq 2$, let*

$$\Lambda = \left\{x \in (0,1] : \, s_n \leq d_n(x) < N s_n \;\; \textit{for} \;\; n \geq 1\right\},$$

then

$$\dim_H \Lambda \; = \; \liminf_{n\to\infty} \frac{\log(s_1 s_2 \ldots s_n)}{2\log(s_1 s_2 \ldots s_n) + \log s_{n+1}}.$$

Lemma 5 (cf. [9]). *For any $a > 1$ and $b > 1$,*

$$\dim_H \left\{x \in (0,1] : \, d_n(x) \geq a^{b^n}, \, \textit{for}\, n \geq 1\right\} = \frac{1}{b+1}.$$

Lemma 6 (cf. [2]). *For any $\nu \geq 2$,*

$$\dim_H \left\{x \in (0,1] : \, |x - P_n(x)/Q_n(x)| < Q_n(x)^{-\nu} \,\textit{for infinitely many}\; n\right\} = \frac{1}{\nu}.$$

To end this section, we borrow a result from [9]. It tells us that the Hausdorff dimension will be same if we change the restrictions on the first finite digits. Namely, let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two sequences of nonempty subsets of $\mathbb{N}\setminus\{0,1\}$ with $A_n = B_n$ when n is large. Set

$$\mathscr{A} = \{x \in (0,1] : \, d_n(x) \in A_n \;\; \text{for} \;\; n \geq 1\},$$

$$\mathscr{B} = \{x \in (0,1] : \, d_n(x) \in B_n \;\; \text{for} \;\; n \geq 1\}.$$

Then we have

Lemma 7 (cf. [9]). $\dim_H \mathscr{A} = \dim_H \mathscr{B}$.

3. The proof of Theorem 2

Our proof starts with the dimension result. For this purpose, we will transform the target set $F(z)$ to another set. For any $\beta \geq 1$, let

$$G(\beta) = \left\{ x \in (0,1] : \, \lim_{n\to\infty} \frac{-\log\left|x - \frac{P_n(x)}{Q_n(x)}\right|}{\log d_{n+1}} = \beta \right\}.$$

It follows immediately that

Lemma 8. *For any $0 < z \leq 1$, $F(z) = G(1/z)$.*

Consequently, we are expected to determine Hausdorff dimension of the set $G(\beta)$. To do this, we will determine the lower bound and upper bound of its dimension separately.

Proposition 9. *For any $\beta > 1$, $\dim_H G(\beta) \geq \frac{\beta-1}{2\beta}$.*

Proof. Let $\lambda = \frac{2}{\beta-1} + 1$ and

$$K(\lambda) = \left\{x \in (0,1] : \, \left[2^{\lambda^n}\right] \leq d_n(x) \leq 2\left[2^{\lambda^n}\right] \quad \text{for all} \quad n \geq 1\right\}.$$

Since

$$\left|x - \frac{P_n(x)}{Q_n(x)}\right| = \frac{T^n(x)}{d_1(x)(d_1(x)-1)\ldots d_n(x)(d_n(x)-1)}$$

and
$$\frac{1}{d_{n+1}(x)} < T^n x \le \frac{1}{d_{n+1}(x)-1},$$
we deduce that
$$\left(\prod_{j=1}^{n} d_j(x)(d_j(x)-1)d_{n+1}(x)\right)^{-1} \le \left|x-\frac{P_n(x)}{Q_n(x)}\right| \le \left(\prod_{j=1}^{n} d_j(x)(d_j(x)-1)(d_{n+1}(x)-1)\right)^{-1}. \qquad (3)$$
Then for any $x \in K(\lambda)$, we get
$$2^{-3n-2}2^{2\lambda^1}2^{2\lambda^2}\dots 2^{2\lambda^n}2^{\lambda^{n+1}} \le \left|x-\frac{P_n(x)}{Q_n(x)}\right|^{-1} \le 2^{2n+1}2^{2\lambda^1}2^{2\lambda^2}\dots 2^{2\lambda^n}2^{\lambda^{n+1}}.$$
Hence, it may be concluded that
$$\lim_{n\to\infty}\frac{-\log\left|x-\frac{P_n(x)}{Q_n(x)}\right|}{\log d_{n+1}} = \frac{2}{\lambda-1}+1=\beta,$$
which implies $K(\lambda)\subset G(\beta)$.

By Lemma 4, we see that
$$\begin{aligned}\dim_H G(\beta) \ge \dim_H K(\lambda) &= \liminf_{n\to\infty}\frac{\log\prod_{j=1}^n\left[2^{\lambda^j}\right]}{2\log\prod_{j=1}^n\left[2^{\lambda^j}\right]+\log\left[2^{\lambda^{n+1}}\right]}\\ &= \frac{1}{\lambda+1}=\frac{\beta-1}{2\beta},\end{aligned}$$
which is the assertion of the Proposition 9. □

We are now in a position to establish the upper bound of the Hausdorff dimension of $G(\beta)$. To do this, we will again transform it to another set. Let $\widetilde{Q}_n(x) = Q_n(x)(d_n(x)-1) = \prod_{j=1}^n d_j(x)(d_j(x)-1)$ and for any $\tau\ge 0$,
$$C(\tau)=\left\{x\in(0,1]:\ \lim_{n\to\infty}\frac{\log\widetilde{Q}_n(x)}{\log d_{n+1}}=\tau\right\}.$$
By definition of $\widetilde{Q}_n(x)$ and inequality (3), it is evident that

Lemma 10. *For any $\beta\ge 1$, $G(\beta)=C(\beta-1)$.*

If $\tau=0$, it is obvious that $\dim_H C(\tau)=0$ by Lemma 6. This implies that $\dim_H F(1)=\dim_H G(1)=0$. By virtue of the above Lemma 8, we are reduced to proving the following result.

Proposition 11. *For any $\tau>0$, $\dim_H C(\tau)\le\frac{\tau}{2\tau+2}$.*

Proof. Assume $\tau>0$. For any $0<\varepsilon<\frac{1}{\tau}$, choose $\delta>0$ small enough that
$$\left(\frac{1}{\tau}-\frac{\varepsilon}{2}\right)(\tau+\delta)<1,$$
and $n_0\in\mathbb{N}$ such that for any $n\ge n_0$,
$$\left[1-\left(\frac{1}{\tau}-\frac{\varepsilon}{2}\right)(\tau+\delta)\right]\left(\frac{1}{\tau}+1-\varepsilon\right)^n > \left(\frac{1}{\tau}-\frac{\varepsilon}{2}\right)(n+1)+1.$$
By definition of $\widetilde{Q}_n(x)$, it follows that
$$\widetilde{Q}_n(x)\frac{d_{n+1}(x)^2}{2}\le\widetilde{Q}_{n+1}(x)\le\widetilde{Q}_n(x)d_{n+1}(x)^2.$$
From this, for any $x\in C(\tau)$,
$$\lim_{n\to\infty}\frac{\log\widetilde{Q}_{n+1}(x)}{\log d_{n+1}}=\tau+2.$$

Therefore

$$C(\tau)=\left\{x\in(0,1]:\ \lim_{n\to\infty}\frac{\log\widetilde{Q}_{n+1}(x)}{\log d_{n+1}}=\tau+2\right\}.$$

Writing

$$W(\tau,\delta,m)=\left\{x\in(0,1]:\ d_{n+1}(x)\geq\widetilde{Q}_{n+1}(x)^{\frac{1}{\tau+2+\delta}}\text{ for all }n\geq m\right\},$$

we can assert that

$$C(\tau)\subset\bigcup_{m=1}^{\infty}W(\tau,\delta,m).$$

Let

$$\mathscr{C}=\left\{x\in(0,1]:\ d_n(x)=\left[2^{\left(\frac{2}{\tau}+1-\varepsilon\right)^n}\right]+1\text{ for }1\leq n\leq n_0,\text{ and }d_{n+1}(x)\geq\widetilde{Q}_{n+1}(x)^{\frac{1}{\tau+2+\delta}}\text{ for all }n\geq n_0\right\}.$$

Due to Lemma 7, it gives that

$$\dim_H C(\tau)\leq\sup_{m\geq1}\dim_H W(\tau,\delta,m)=\dim_H\mathscr{C}.$$

Our next claim is that for any $x\in\mathscr{C}$ and for any $n\geq1$,

$$d_n(x)\geq 2^{\left(\frac{2}{\tau}+1-\varepsilon\right)^n}. \tag{4}$$

We proceed by induction. For any $1\leq n\leq n_0$, the inequality (4) holds by the definition of $\mathscr{C}$. When $n>n_0$, from the construction of $\mathscr{C}$,

$$d_n(x)\geq\widetilde{Q}_n(x)^{\frac{1}{\tau+2+\delta}}\geq\left(2^{-1}\widetilde{Q}_{n-1}(x)d_n(x)^2\right)^{\frac{1}{\tau+2+\delta}}.$$

Using induction on n, we obtain

$$d_n(x)^{1-\frac{2}{\tau+2+\delta}}\geq\left(2^{-n}d_1(x)^2d_2(x)^2\ldots d_{n-1}(x)^2\right)^{\frac{1}{\tau+2+\delta}}.$$

Thus

$$d_n(x)\geq\left(2^{-n}d_1(x)^2d_2(x)^2\ldots d_{n-1}(x)^2\right)^{\frac{1}{\tau+\delta}}\geq 2^{\frac{\left(\frac{2}{\tau}+1-\varepsilon\right)^n-1}{\left(\frac{1}{\tau}-\frac{\varepsilon}{2}\right)(\tau+\delta)}-\frac{n+1}{\tau+\delta}}.$$

By the choice of n_0, we deduce that

$$d_n(x)\geq 2^{\left(\frac{2}{\tau}+1-\varepsilon\right)^n}.$$

On account of Lemma 5, it follows that

$$\dim_H\mathscr{C}\leq\frac{1}{2+\frac{2}{\tau}-\varepsilon}.$$

Since ε is arbitrary, it is easily seen that

$$\dim_H C(\tau)\leq\dim_H\mathscr{C}\leq\frac{\tau}{2\tau+2}.$$ □

By Lemma 8, Lemma 10, Proposition 9 and Proposition 11, we can check at once that for $0<z<1$,

$$\dim_H F(z)=\dim_H G(1/z)=\dim_H C(1/z-1)=\frac{1-z}{2}.$$

Finally, what is left is to prove the last two statements. By [4, Corollary 6.6], it is obvious that

$$\lim_{n\to\infty}\frac{\log d_n(x)}{n}=0,\quad\text{a.e.}.$$

Combining this with the limit (2), the set $F(0)$ is of Lebesgue measure 1. According to the inequality (3), $F(z)$ is empty when z is not in $[0,1]$.

This finishes the proof of Theorem 2.

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