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On the Bohr inequality for the Cesáro operator

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Abstract. We investigate an analog of Bohr's results for the Cesáro operator acting on the space of holomorphic functions defined on the unit disk. The asymptotical behaviour of the corresponding Bohr sum is also estimated.

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1. Introduction and Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and \mathcal{B} denote the class of all holomorphic functions in \mathbb{D} such that $|f(z)| \leq 1$ in \mathbb{D} . Then the classical Bohr theorem in its final form [3] asserts that if $f \in \mathcal{B}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n$ is the associated majorant series of $f(z)$, then

$$M_f(r) \leq 1 \tag{1}$$

for $r \leq 1/3$, and the constant $1/3$ cannot be improved. More generally, one can consider

$$m(r) = \sup \frac{M_f(r)}{\|f\|_{\infty}}, \quad \text{for } r \in [0, 1),$$

where the supremum is taken over all bounded analytic functions $f \neq 0$ with associated sup norm $\|f\|_{\infty}$. Computing $m(r)$ is of relevance for understanding the rate of growth of bounded analytic functions, which is not an easy task. The behaviour of $m(r)$ as $r \rightarrow 1$ was studied first by

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Djakov and Ramanujan [7] although a precise formula for $m(r)$ for all values of $r \in [0, 1)$ remains unknown. In [4], Bombieri has proved that

$$m(r) = \frac{3 - \sqrt{8(1 - r^2)}}{r}, \quad \text{for } 1/3 \leq r \leq 1/\sqrt{2}$$

(see also [13] an alternate proof of it). We have the inequality $m(r) \leq \frac{1}{\sqrt{1-r^2}}$, which is an immediate consequence of the Cauchy–Bunyakovskii inequality, namely,

$$M_f(r) \leq \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} r^{2k} \right)^{1/2} \leq \frac{\|f\|_{\infty}}{\sqrt{1-r^2}}.$$

Bombieri and Bourgain [5] proved that for $r > 1/\sqrt{2}$,

$$m(r) < \frac{1}{\sqrt{1-r^2}}.$$

A lower bound for $m(r)$ as $r \rightarrow 1$ was also obtained in [5]. In fact, with a help of number of lemmas and a delicate analysis of exponential sums, they constructed a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with specific coefficients a_n and proved that for $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ depending on $\varepsilon > 0$ such that

$$m(r) \geq (1 - r^2)^{-1/2} - \left(c \log \frac{1}{1-r} \right)^{3/2+\varepsilon} \quad \text{as } r \rightarrow 1.$$

In the recent years, there has been a great deal of research activity including refinements, ramifications and extensions of Bohr-type theorems in different settings. See [2, 10–12] and the survey chapters from [1, 8], and the references therein. Interestingly, Dixon [6] used Bohr’s phenomena in operator theory in connection with the long-standing problem of characterization of Banach algebras that satisfy the von Neumann inequality [14].

Motivated by these results, we obtain in this paper an operator counterpart of Bohr’s inequality in the case of Cesáro operator on the space of holomorphic functions of the unit disk. Cesáro operator and its various generalizations are well established topic, for example, in the study of boundedness and compactness of them on various function spaces. In the classical setting, for a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on \mathbb{D} , the Cesáro operator is defined by [9] (see also [15])

$$\mathcal{C}f(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n = \int_0^1 \frac{f(tz)}{1-tz} dt.$$

If f is bounded by 1, then as in the case of Bohr, we can replace Bohr’s sum (1) by

$$\mathcal{C}_f(r) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n |a_k| \right) r^n.$$

It is worth pointing out that for each $|z| = r \in [0, 1)$, the integral representation for the Cesáro operator quickly yields the following sharp inequality for $f \in \mathcal{B}$:

$$|\mathcal{C}f(z)| \leq \frac{1}{r} \log \frac{1}{1-r}.$$

The main aim of this article is to show that a similar fact is also true for $\mathcal{C}_f(r)$, but the question is to find sharp value of R for the truth of the last inequality with $\mathcal{C}_f(r)$ in place of $|\mathcal{C}f(z)|$ for all $r \leq R$ —a situation similar to Bohr’s inequality.

Theorem 1. *If $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$\mathcal{C}_f(r) \leq \frac{1}{r} \log \frac{1}{1-r} \tag{2}$$

for $r \leq R$, where $R = 0.5335\dots$ is the positive root of the equation

$$2x = 3(1-x) \log \frac{1}{1-x}.$$

The number R is best possible and cannot be improved.

Proof. Let $|a_0| = a < 1$. First we observe that

$$\mathcal{C}_f(r) = \frac{a}{r} \log \frac{1}{1-r} + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^n |a_k| \right) r^n. \tag{3}$$

Next, by a well-known coefficient inequality for functions in \mathcal{B} , we have the inequalities $|a_n| \leq 1 - a^2$ for $n \geq 1$. Using this and (3), we easily get

$$\begin{aligned} \mathcal{C}_f(r) &\leq \frac{a}{r} \log \frac{1}{1-r} + (1-a^2) \sum_{n=0}^{\infty} \frac{n}{n+1} r^n \\ &= \frac{a}{r} \log \frac{1}{1-r} + (1-a^2) \left(\frac{1}{1-r} - \frac{1}{r} \log \frac{1}{1-r} \right). \end{aligned}$$

Let $A(a)$ denote the right-hand side of the latter equation. Note that $A''(a) \leq 0$ for all $a \in [0, 1]$ and all $r \in (0, 1)$. This means that

$$A'(a) \geq A'(1) = \frac{1}{r(1-r)} \left[-2r + 3(1-r) \log \frac{1}{1-r} \right] \geq 0$$

for $r \leq R$. Thus, $A(a)$ is an increasing function of a , for $r \leq R$. It follows that for all $a \in [0, 1]$,

$$A(a) \leq A(1) = \frac{1}{r} \log \frac{1}{1-r} \text{ for all } r \leq R.$$

The desired inequality (2) follows.

Now, let us show that R cannot be improved. In order to do this, we consider the function

$$\varphi_a(z) = \frac{z-a}{1-az} = -a + (1-a^2) \sum_{n=1}^{\infty} a^{n-1} z^n, \quad z \in \mathbb{D},$$

where $a \in [0, 1)$. Using (3), we organize the sum $\mathcal{C}_{\varphi_a(r)}$ as follows:

$$\begin{aligned} \mathcal{C}_{\varphi_a(r)} &= \frac{a}{r} \log \frac{1}{1-r} + (1-a^2) \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^n a^{k-1} \right) r^n \\ &= \frac{a}{r} \log \frac{1}{1-r} + (1+a) \sum_{n=1}^{\infty} \left(\frac{1}{n+1} (1-a^n) \right) r^n \\ &= \frac{a}{r} \log \frac{1}{1-r} + (1+a) \left[\frac{1}{r} \log \frac{1}{1-r} - \frac{1}{ar} \log \frac{1}{1-ar} \right] \\ &= \frac{1}{r} \log \frac{1}{1-r} + \frac{2a}{r} \log \frac{1}{1-r} - \frac{1+a}{ar} \log \frac{1}{1-ar} \\ &= \frac{1}{r} \log \frac{1}{1-r} + (1-a) \frac{3(1-r) \log(1-r) + 2r}{r(1-r)} + D_a(r), \end{aligned}$$

where

$$\begin{aligned} D_a(r) &= \frac{3-a}{r} \log \frac{1}{1-r} - \frac{2(1-a)}{1-r} - \frac{1+a}{ar} \log \frac{1}{1-ar} \\ &= \sum_{n=2}^{\infty} \left(\frac{3-a}{n} - 2(1-a) - \frac{(1+a)a^{n-1}}{n} \right) r^{n-1} \\ &= O((1-a)^2), \quad \text{as } a \rightarrow 1. \end{aligned}$$

Furthermore, it is easy to check that for $r > R$ the following inequality holds

$$\frac{3(1-r) \log(1-r) + 2r}{r(1-r)} > 0.$$

These two facts show that the number R cannot be increased. □

It is natural to conduct a similar research as Bombieri and Bourgain [5] did for the majorant series of $f \in \mathcal{B}$, considering the sum $\mathcal{C}_f(r)$, and to study the behaviour of $\mathcal{C}_f(r)$ as $r \rightarrow 1$.

Theorem 2. If $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then for $r \in [0, 1)$ the following inequality holds

$$\mathcal{C}_f(r) \leq \frac{1}{r} \sqrt{\frac{1+r}{1-r} \log(1+r) + \log(1-r)}. \quad (4)$$

Proof. Let us first represent the sum $\mathcal{C}_f(r)$ in a convenient equivalent form as

$$\mathcal{C}_f(r) = \sum_{k=0}^{\infty} |a_k| \phi_k(r),$$

where it is easy to see that

$$\phi_k(r) = \sum_{n=k}^{\infty} \frac{r^n}{n+1} = \frac{1}{r} \sum_{n=k}^{\infty} \int_0^r x^n dx = \frac{1}{r} \int_0^r \frac{x^k}{1-x} dx =: \frac{1}{r} \psi_k(r)$$

so that

$$\mathcal{C}_f(r) = \frac{1}{r} \sum_{k=0}^{\infty} |a_k| \psi_k(r). \quad (5)$$

Equation (5) shows, by the triangle inequality and the fact that $\sum_{k=0}^{\infty} |a_k|^2 \leq 1$,

$$\mathcal{C}_f(r) \leq \frac{1}{r} \sqrt{\sum_{k=0}^{\infty} |a_k|^2 \psi_k(r)} \leq \frac{1}{r} \sqrt{\Phi(r)}, \quad (6)$$

where $\Phi(r) = \sum_{k=0}^{\infty} \psi_k^2(r)$. To get the precise form of $\Phi(r)$, we take the derivative of $\Phi(r)$ and obtain

$$\begin{aligned} \Phi'(r) &= 2 \sum_{k=0}^{\infty} \psi_k(r) \psi_k'(r) = \frac{2}{1-r} \sum_{k=0}^{\infty} \int_0^r \frac{x^k r^k}{1-x} dx \\ &= \frac{2}{1-r} \int_0^r \frac{dx}{(1-x)(1-rx)} \\ &= \frac{2}{(1-r)^2} \log(1+r) = \log(1+r) \frac{d}{dr} \left(\frac{1+r}{1-r} \right), \end{aligned}$$

which by integration gives

$$\Phi(r) = \frac{1+r}{1-r} \log(1+r) + \log(1-r).$$

Using this in (6) completes the proof of Theorem 2. \square

Remark. Clearly, Theorem 2 shows that the upper estimate for $\mathcal{C}_f(r)$ behaves like $(\sqrt{2 \log 2})/\sqrt{1-r}$ as $r \rightarrow 1$.

It turns out that this estimate is close to be sharp. To show this, we consider the following example. Bombieri and Bourgain [5] constructed the function $g(z) = \sum_{n=0}^{\infty} b_n z^n$ with the following properties:

- (a) $|b_n| = \rho^n$, and
- (b) $\|g(z)\|_{\infty} < \frac{1}{\sqrt{1-\rho}}$.

For the function

$$W = \exp \left\{ -\log^+ \left(\sqrt{1-\rho^2} |g| \right) - iH \left[\log^+ \left(\sqrt{1-\rho^2} |g| \right) \right] \right\},$$

where H denotes the periodic Hilbert transform, it is proved in [5] that

$$\|1 - W\|_2 < \sqrt{1-\rho^2} \sqrt{\log \frac{1}{1-\rho}}.$$

For some fixed value of ρ , we define

$$h(z) = g(z) \cdot \sqrt{1-\rho^2} = \sum_{k=0}^{\infty} h_k z^k.$$

Therefore for $f = hW$ we have

$$\|f - g\|_2 < \frac{1}{\sqrt{1-\rho}} \cdot \sqrt{1-\rho^2} \|1 - W\|_2 < \sqrt{1-\rho^2} \sqrt{\log \frac{1}{1-\rho}}. \tag{7}$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. By (7) and recalling that $|h_k| = |b_k| \sqrt{1-\rho^2} = \rho^k \sqrt{1-\rho^2}$, we have

$$\begin{aligned} \mathcal{E}_f(r) &= \frac{1}{r} \sum_{k=0}^{\infty} |h_k - (h_k - a_k)| \psi_k(r) \\ &\geq \frac{1}{r} \left(\sum_{k=0}^{\infty} |h_k| \psi_k(r) - \sum_{k=0}^{\infty} |h_k - a_k| \psi_k(r) \right) \\ &= \frac{1}{r} \left(\sqrt{1-\rho^2} \sum_{k=0}^{\infty} \int_0^r \frac{\rho^k x^k}{1-x} dx - \sum_{k=0}^{\infty} |h_k - a_k| \psi_k(r) \right) \\ &\geq \frac{1}{r} \left(\sqrt{1-\rho^2} \int_0^r \frac{dx}{(1-x)(1-\rho x)} - \sqrt{\sum_{k=0}^{\infty} |h_k - a_k|^2} \cdot \sqrt{\sum_{k=0}^{\infty} \psi_k^2(r)} \right) \\ &\geq \frac{1}{r} \sqrt{\frac{1+\rho}{1-\rho}} \log \frac{1-r\rho}{1-r} \\ &\quad - \frac{1}{r} \sqrt{1-\rho^2} \sqrt{\log \frac{1}{1-\rho}} \cdot \sqrt{\frac{1+r}{1-r} \log(1+r) + \log(1-r)} \\ &= \frac{1}{r\sqrt{1-r}} \left(\sqrt{\frac{(1+\rho)(1-r)}{1-\rho}} \log \frac{1-r\rho}{1-r} \right. \\ &\quad \left. - \sqrt{1-\rho^2} \sqrt{\log \frac{1}{1-\rho}} \cdot \sqrt{(1+r) \log(1+r) + (1-r) \log(1-r)} \right). \end{aligned}$$

It can be easily seen that

$$\sqrt{1-\rho^2} \sqrt{\log \frac{1}{1-\rho}} \cdot \sqrt{(1+r) \log(1+r) + (1-r) \log(1-r)} \rightarrow 0$$

when $r, \rho \rightarrow 1$. Taking $\rho = r^\alpha$, we obtain

$$\mathcal{E}_f(r) \sim \frac{1}{\sqrt{1-r}} \left(\sqrt{\frac{(1+r^\alpha)(1-r)}{1-r^\alpha}} \log \frac{1-r^{\alpha+1}}{1-r} \right) \sim \frac{1}{\sqrt{1-r}} \cdot \frac{\sqrt{2} \log(1+\alpha)}{\sqrt{\alpha}}.$$

The expression $\frac{\sqrt{2} \log(1+\alpha)}{\sqrt{\alpha}}$ takes its maximal value, if α is a root of the equation

$$2\alpha = (\alpha + 1) \log(\alpha + 1).$$

Calculation shows that $\alpha \approx 3.92155\dots$. For this value of α we get

$$\mathcal{E}_f(r) \sim \frac{2\sqrt{2\alpha}}{(\alpha + 1)\sqrt{1-r}} \approx \frac{1.13808\dots}{\sqrt{1-r}}, \text{ as } r \rightarrow 1.$$

From Theorem 2 we know that the upper estimate for the constant is $\sqrt{2 \log 2} = 1.17741\dots$. Therefore, there exists a small difference between the common estimate and the value of the constant for this particular example. Hence, it would be nice to know the sharp asymptotic behaviour for $\sup_{f \in \mathcal{B}} \mathcal{E}_f(r)$ as $r \rightarrow 1$.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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