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# A note on Gersten's conjecture for étale cohomology over two-dimensional henselian regular local rings

## *Une note sur la conjecture de Gersten pour la cohomologie étale sur des anneaux locaux réguliers henséliens à deux dimensions*

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**Abstract.** We prove Gersten's conjecture for étale cohomology over two dimensional henselian regular local rings without assuming equi-characteristic. As an application, we obtain the local-global principle for Galois cohomology over mixed characteristic two-dimensional henselian local rings.

**Résumé.** Nous montrons la conjecture de Gersten pour la cohomologie étale sur des anneaux locaux réguliers henséliens sans supposer de caractère équicaractéristique. En application, nous obtenons le principe local-global pour la cohomologie de Galois sur des anneaux locaux henséliens à deux dimensions de caractéristique mixte.

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## 1. Introduction

Let $R$ be an equi-characteristic regular local ring, $k(R)$ the field of fractions of $R$, $l$ a positive integer which is invertible in $R$ and $\mu_l$ the étale sheaf of $l$-th roots of unity. Then the sequence of étale cohomology groups

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^{n+1}_{\text{ét}}\left(R, \mu_l^{\otimes n}\right) &\longrightarrow \mathrm{H}^{n+1}_{\text{ét}}\left(k(R), \mu_l^{\otimes n}\right) \\ &\longrightarrow \bigoplus_{\substack{\mathfrak{p}\in \operatorname{Spec} R \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{H}^{n}_{\text{ét}}\left(\kappa(\mathfrak{p}), \mu_l^{\otimes(n-1)}\right) \\ &\longrightarrow \bigoplus_{\substack{\mathfrak{p}\in \operatorname{Spec} R \\ \mathrm{ht}(\mathfrak{p})=2}} \mathrm{H}^{n-1}_{\text{ét}}\left(\kappa(\mathfrak{p}), \mu_l^{\otimes(n-2)}\right) \longrightarrow \cdots \end{aligned} \tag{1}$$

is exact by Bloch–Ogus ([2]) and Panin ([10]). Here $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p} \in \operatorname{Spec} R$.

By using the exactness of the complex (1) at the first two terms, Harbater–Hartmann–Krashen ([7]) and Hu ([8]) proved the local-global principle as follows.

Let $K$ be a field of one of the following types:

(a) (semi-global case) The function field of a connected regular projective curve over the field of fractions of a henselian excellent discrete valuation ring $A$.

(b) (local case) The function field of a two-dimensional henselian excellent normal local domain $A$.

Then the following question was raised by Colliot-Thélène ([3]):

Let $n \geq 1$ be an integer and $l$ a positive integer which is invertible in $R$. Is the natural map

$$\mathrm{H}^{n+1}_{\text{ét}}\left(K, \mu_l^{\otimes n}\right) \longrightarrow \prod_{v\in\Omega_K} \mathrm{H}^{n+1}_{\text{ét}}\left(K_v, \mu_l^{\otimes n}\right) \tag{2}$$

injective ?

Here $\Omega_K$ is the set of normalized discrete valuations on $K$ and $K_v$ is the corresponding henselization of $K$ for each $v \in \Omega_K$.

Suppose that $A$ is equi-characteristic. Harbater–Hartmann–Krashen ([7, Theorem 3.3.6]) proved that the local-global map (2) is injective in the semi-global case. Later, Hu ([8, Theorem 2.5]) proved that the local-global map (2) is injective in both the semi-global case and the local case by an alternative method.

If the sequence (1) is exact (at the first two terms) in the case where $R$ is a mixed characteristic two-dimensional excellent henselian local ring, then the local-global map (2) is injective even without assuming equi-characteristic (cf. [7, Remark 3.3.7] and [8, Remark 2.6 (2)]).

In the case where $R$ is a local ring of a smooth algebra over a (mixed characteristic) discrete valuation ring, the sequence (1) is exact (cf. [6, Theorem 1.2 and Theorem 3.2 b)]).

In this paper, we show the following result:

**Theorem 1 (Theorem 9).** *Let $R$ be a mixed characteristic two-dimensional excellent henselian local ring and $l$ a positive integer which is invertible in $R$. Then Gersten's conjecture for étale cohomology with $\mu_l^{\otimes n}$ coefficients holds over* $\operatorname{Spec} R$. *That is, the sequence* (1) *is exact.*

See Remark 8 (iii) for the reason why we assume $\dim(R) = 2$ in Theorem 1. We obtain the following result as an application of Theorem 1 :

**Theorem 2.** *With notations as above, assume that $A$ is mixed characteristic and $l$ is a positive integer which is invertible in $A$.*

*In both the semi-global case and the local case, the local-global principle for the Galois cohomology group $\mathrm{H}^{n+1}(K, \mu_l^{\otimes n})$ holds for $n \geq 1$. That is, the local-global map* (2) *is injective for $n \geq 1$.*

V. Suresh also proved Theorem 2 by an alternative method (cf. [8, Remark in Theorem 1.2]).

### 1.1. *Notations*

For a scheme $X$, $X^{(i)}$ is the set of points of codimension $i$, $k(X)$ is the ring of rational functions on $X$ and $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p} \in X$. If $X = \operatorname{Spec} R$, $k(\operatorname{Spec} R)$ is abbreviated as $k(R)$. The symbol $\mu_l$ denotes the étale sheaf of $l$-th roots of unity.

## 2. Proof of the main result (Theorem 1)

In this section, we use the following results (Theorem 3 and Theorem 4) repeatedly:

**Theorem 3 (cf.** [4, Theorem B.2.1 and Examples B.1.1.(2)]**).** *Let $A$ be a discrete valuation ring, $K$ the function field of $A$ and $l$ a positive integer which is invertible in $A$. Then the homomorphism*

$$\mathrm{H}^i_{\text{ét}}(A, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^i_{\text{ét}}(K, \mu_l^{\otimes n})$$

*is injective for any $i \geq 0$.*

**Theorem 4 (The absolute purity theorem** [5, p. 159, Theorem 2.1.1]**).** *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension $c$. Let $n$ be an integer which is invertible on $X$, and let $\Lambda = \mathbf{Z}/n$. Then the cycle class (cf.* [5, 1.1]*) give an isomorphism*

$$\Lambda_Y \xrightarrow{\sim} Ri^! \Lambda(c)[2c]$$

*in $D^+(Y_{\text{ét}}, \Lambda)$. Here $D^+(Y_{\text{ét}}, \Lambda)$ is the derived category of complexes bounded below of étale sheaves of $\Lambda$-modules on $Y$.*

In this section, we use Theorem 4 in the case where $\dim X \leq 2$. In this case, Theorem 4 was proved much earlier by Gabber in 1976. See also [11, §5, Remark 5.6] for a published proof.

**Proposition 5.** *Let $R$ be a henselian regular local ring, $\mathfrak{m}$ the maximal ideal of $R$ and $K$ the function field of $R$. Let $l$ be a positive integer such that $l \notin \mathfrak{m}$. Then the homomorphism*

$$\mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R, \mu_l^{\otimes n}\right) \longrightarrow \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} K, \mu_l^{\otimes n}\right) \tag{3}$$

*is injective for any $i \geq 0$.*

**Proof.** We prove the statement by induction on $\dim(R)$. Let $R$ be a discrete valuation ring (which does not need to be henselian). Then the homomorphism (3) is injective by Theorem 3.

Assume that the statement is true for a henselian regular local ring of dimension $d$.

Let $R$ be a henselian regular local ring of dimension $d+1$, $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathfrak{p} = (a)$. Then $R/\mathfrak{p}$ is a henselian regular local ring of dimension $d$ and

$$k(R/\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

where $k(R/\mathfrak{p})$ is the function field of $R/\mathfrak{p}$.

Therefore the diagram

$$\begin{array}{ccc} \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R, \mu_l^{\otimes n}\right) & \longrightarrow & \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R_{\mathfrak{p}}, \mu_l^{\otimes n}\right) \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R/\mathfrak{p}, \mu_l^{\otimes n}\right) & \longrightarrow & \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} k(R/\mathfrak{p}), \mu_l^{\otimes n}\right) \end{array} \tag{4}$$

is commutative. Then the left vertical map in the diagram (4) is an isomorphism by [1, p. 93, Theorem (4.9)] and the bottom horizontal map in the diagram (4) is injective by the induction hypothesis. Hence the homomorphism

$$\mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R, \mu_l^{\otimes n}\right) \longrightarrow \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R_{\mathfrak{p}}, \mu_l^{\otimes n}\right)$$

is injective. Moreover the homomorphism

$$\mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} R_{\mathfrak{p}}, \mu_l^{\otimes n}\right) \longrightarrow \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} K, \mu_l^{\otimes n}\right)$$

is injective by Theorem 3. Therefore the statement follows. □

**Proposition 6 (cf.** [12, Proposition 4.7]**).** *Let $R$ be a regular local ring and $l$ a positive integer which is invertible in $R$. Suppose that* $\dim(R) = 2$. *Then the sequence*

$$\mathrm{H}^i_{\text{ét}}(R, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^i_{\text{ét}}(k(R), \mu_l^{\otimes n}) \xrightarrow{(*)} \bigoplus_{\mathfrak{p} \in (\operatorname{Spec} R)^{(1)}} \mathrm{H}^{i-1}_{\text{ét}}(\kappa(\mathfrak{p}), \mu_l^{\otimes (n-1)})$$

*is exact for any $i \geq 0$.*

**Proof.** Let $A$ be a Dedekind ring, $\mathfrak{q}$ a maximal ideal of $A$. Then

$$\mathrm{H}^{i+1}_{\mathfrak{q}}((\operatorname{Spec} A)_{\text{ét}}, \mu_l^{\otimes n}) = \mathrm{H}^{i-1}_{\text{ét}}(\kappa(\mathfrak{q}), \mu_l^{\otimes (n-1)})$$

by Theorem 4. Hence the sequence

$$\mathrm{H}^i_{\text{ét}}(A, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^i_{\text{ét}}(U, \mu_l^{\otimes n}) \longrightarrow \bigoplus_{\mathfrak{q} \in Z^{(1)}} \mathrm{H}^{i-1}_{\text{ét}}(\kappa(\mathfrak{q}), \mu_l^{\otimes (n-1)})$$

is exact where $Z$ is a closed subscheme of $\operatorname{Spec} A$ and $U = \operatorname{Spec} R \setminus Z$. Since

$$\varinjlim_{U} \mathrm{H}^i_{\text{ét}}(U, \mu_l^{\otimes n}) = \mathrm{H}^i_{\text{ét}}(k(A), \mu_l^{\otimes n})$$

by [9, pp. 88–89, III, Lemma 1.16], the sequence

$$\mathrm{H}^i_{\text{ét}}(A, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^i_{\text{ét}}(k(A), \mu_l^{\otimes n}) \longrightarrow \bigoplus_{\mathfrak{q} \in (\operatorname{Spec} A)^{(1)}} \mathrm{H}^{i-1}_{\text{ét}}(\kappa(\mathfrak{q}), \mu_l^{\otimes (n-1)}) \tag{5}$$

is exact.

Let $\mathfrak{m}$ be the maximal ideal of $R$. Let $g \in \mathfrak{m} \setminus \mathfrak{m}^2$, $\mathfrak{p} = (g)$ and $Z = \operatorname{Spec} R/\mathfrak{p}$. Then $R/\mathfrak{p}$ is a regular local ring and we have

$$\mathrm{H}^{i+1}_{Z}((\operatorname{Spec} R)_{\text{ét}}, \mu_l^{\otimes n}) = \mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})$$

by Theorem 4.

We consider the commutative diagram

$$\begin{array}{cccccccc} & & \mathrm{H}^i_{\text{ét}}(R, \mu_l^{\otimes n}) & \longrightarrow & \mathrm{H}^i_{\text{ét}}(R_g, \mu_l^{\otimes n}) & \longrightarrow & \mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})' & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathrm{Ker}(*) & \longrightarrow & \mathrm{H}^i_{\text{ét}}(R_g, \mu_l^{\otimes n})' & \longrightarrow & \mathrm{H}^{i-1}_{\text{ét}}(k(R/\mathfrak{p}), \mu_l^{\otimes (n-1)}) & \end{array} \tag{6}$$

where

$$\mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})' = \mathrm{Im}\left(\mathrm{H}^i_{\text{ét}}(R_g, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})\right)$$

and

$$\mathrm{H}^i_{\text{ét}}(R_g, \mu_l^{\otimes n})' = \mathrm{Ker}\left(\mathrm{H}^i_{\text{ét}}(k(R_g), \mu_l^{\otimes n}) \longrightarrow \bigoplus_{\mathfrak{q} \in (\operatorname{Spec} R_g)^{(1)}} \mathrm{H}^{i-1}_{\text{ét}}(\kappa(\mathfrak{q})), \mu_l^{\otimes (n-1)})\right).$$

Then the rows in the diagram (6) are exact by Theorem 4. Since $R_g$ is a Dedekind domain, the middle map in the diagram (6) is surjective by (5). Moreover, since

$$\mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})' \subset \mathrm{H}^{i-1}_{\text{ét}}(R/\mathfrak{p}, \mu_l^{\otimes (n-1)})$$

and $R/\mathfrak{p}$ is a discrete valuation ring, the right map in the diagram (6) is injective by Theorem 3. Therefore the statement follows from the snake lemma. □

**Corollary 7.** *Let $R$ be the henselization of a regular local ring which is essentially of finite type over a mixed characteristic discrete valuation ring. Suppose that* $\dim(R) = 2$. *Then*

$$H^{n+1}_{\mathrm{Zar}}(R, \mathbb{Z}/l(n)) = 0$$

*for a positive integer $l$ which is invertible in $R$. Here $\mathbb{Z}(n)$ is Bloch's cycle complex and $\mathbb{Z}/l(n) = \mathbb{Z}(n) \otimes \mathbf{Z}/l$ (cf.* [6, p. 779]*).*

**Proof.** Let $\mathfrak{m}$ be the maximal ideal of $R$. Let $g \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathfrak{p} = (g)$. Then the homomorphism

$$H^{n+1}_{\acute{e}t}(R, \mu_l^{\otimes n}) \longrightarrow H^{n+1}_{\acute{e}t}(R_g, \mu_l^{\otimes n})$$

is injective by Proposition 5. Hence the homomorphism

$$H^{n}_{\acute{e}t}(R_g, \mu_l^{\otimes n}) \longrightarrow H^{n-1}_{\acute{e}t}(R/\mathfrak{p}, \mu_l^{\otimes n-1})$$

is surjective by Theorem 4. Therefore the homomorphism

$$H^{n}_{\mathrm{Zar}}(R_g, \mathbb{Z}/l(n)) \longrightarrow H^{n-1}_{\mathrm{Zar}}(R/\mathfrak{p}, \mathbb{Z}/l(n-1))$$

is surjective by [6, p. 774, Theorem 1.2] and [14]. Moreover the homomorphism

$$H^{n+1}_{\mathrm{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\mathrm{Zar}}(R_g, \mathbb{Z}/l(n))$$

is injective by the localization theorem [6, p. 779, Theorem 3.2]. We consider the commutative diagram

$$\begin{array}{ccc} H^{n+1}_{\mathrm{Zar}}(R_g, \mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}_{\acute{e}t}(R_g, \mathbb{Z}/l(n)) \\ \downarrow & & \downarrow \\ H^{n+1}_{\mathrm{Zar}}(k(R_g), \mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}_{\acute{e}t}(k(R_g), \mathbb{Z}/l(n)). \end{array} \tag{7}$$

Then the upper map in the commutative diagram (7) is injective by the Beilinson–Lichenbaum conjecture ([6, p. 774, Theorem 1.2], [14]) and the right map in the commutative diagram (7) is injective by the commutative diagram (6) in the proof of Proposition 6. Hence the homomorphism

$$H^{n+1}_{\mathrm{Zar}}(R_g, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\mathrm{Zar}}(k(R_g), \mathbb{Z}/l(n))$$

is injective and the homomorphism

$$H^{n+1}_{\mathrm{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\mathrm{Zar}}(k(R_g), \mathbb{Z}/l(n))$$

is also injective. Since

$$H^{n+1}_{\mathrm{Zar}}(k(R_g), \mathbb{Z}/l(n)) = 0,$$

we have

$$H^{n+1}_{\mathrm{Zar}}(R, \mathbb{Z}/l(n)) = 0.$$

This completes the proof. □

**Remark 8.**

(i) If $R$ is a local ring of a smooth algebra over a discrete valuation ring, then

$$H^{i}_{\mathrm{Zar}}(R, \mathbb{Z}/m(n)) = 0$$

for $i > n$ and any positive integer $m$ (cf. [6, p. 786, Corollary 4.4]).

(ii) If we have

$$H^{n+1}_{\mathrm{Zar}}(R, \mathbb{Z}/l(n)) = 0$$

for any regular local ring $R$ which is finite type over a discrete valuation ring and a positive integer $l$ which is invertible in $R$, we can show that the homomorphism

$$H^{n+1}_{\acute{e}t}(R, \mu_l^{\otimes n}) \longrightarrow H^{n+1}_{\acute{e}t}(k(R), \mu_l^{\otimes n})$$

is injective by a similar argument as in the proof of [13, Theorem 4.2].

(iii) The reason why we assume $\dim(R) = 2$ in Propositin 6 and Theorem 9 is that we have to show that the middle map in the diagram (6), i.e., the homomorphism

$$\mathrm{H}^{n+1}_{\text{ét}}(R_g, \mu_l^{\otimes n}) \longrightarrow \mathrm{H}^{n+1}_{\text{ét}}(R_g, \mu_l^{\otimes n})'$$

is surjective for an element $g$ of $\mathfrak{m} \setminus \mathfrak{m}^2$. Here $\mathfrak{m}$ is the maximal ideal of $R$ and

$$\mathrm{H}^{n+1}_{\text{ét}}(R_g, \mu_l^{\otimes n})' = \operatorname{Ker}\left(\mathrm{H}^{n+1}_{\text{ét}}(k(R), \mu_l^{\otimes n}) \longrightarrow \bigoplus_{\mathfrak{q} \in (\operatorname{Spec} R_g)^{(1)}} \mathrm{H}^{n}_{\text{ét}}(\kappa(\mathfrak{q}), \mu_l^{\otimes (n-1)})\right).$$

If we have

$$\mathrm{H}^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) = \mathrm{H}^{n+2}_{\text{Zar}}(R, \mathbb{Z}/l(n)) = 0$$

for any regular local ring $R$ which is finite type over a discrete valuation ring and a positive integer $l$ which is invertible in $R$, then

$$\mathrm{H}^{n+1}_{\text{Zar}}(R_g, \mathbb{Z}/l(n)) = \mathrm{H}^{n+2}_{\text{Zar}}(R_g, \mathbb{Z}/l(n)) = 0$$

by the localization theorem ([6, p. 779, Theorem 3.2]) and we can show that

$$\mathrm{H}^{n+1}_{\text{ét}}(R_g, \mu_l^{\otimes n}) = \Gamma(\operatorname{Spec} R_g, R^{n+1}\epsilon_*(\mu_l^{\otimes n})) = \mathrm{H}^{n+1}_{\text{ét}}(R_g, \mu_l^{\otimes n})'$$

and Proposition 6 holds. Here $\epsilon : (\operatorname{Spec} R_g)_{\text{ét}} \to (\operatorname{Spec} R_g)_{\text{Zar}}$ is the change of site maps.

**Theorem 9.** *Let $R$ be a henselian regular local ring with* $\dim(R) = 2$ *and $l$ a positive integer which is invertible in $R$. Then the sequence*

$$0 \longrightarrow \mathrm{H}^{i}_{\text{ét}}\left(R, \mu_l^{\otimes n}\right) \longrightarrow \mathrm{H}^{i}_{\text{ét}}\left(k(R), \mu_l^{\otimes n}\right) \longrightarrow \bigoplus_{\mathfrak{p} \in (\operatorname{Spec} R)^{(1)}} \mathrm{H}^{i-1}_{\text{ét}}\left(\kappa(\mathfrak{p}), \mu_l^{\otimes (n-1)}\right)$$
$$\longrightarrow \bigoplus_{\mathfrak{p} \in (\operatorname{Spec} R)^{(2)}} \mathrm{H}^{i-2}_{\text{ét}}\left(\kappa(\mathfrak{p}), \mu_l^{\otimes (n-2)}\right) \longrightarrow 0 \quad (8)$$

*is exact for any $i \geq 0$.*

**Proof.** The exactness of the complex (8) at the first two terms follows from Proposition 5 and Proposition 6.

We consider the coniveau spectral sequence

$$\mathrm{H}_1^{p,q} = \coprod_{x \in (\operatorname{Spec} R)^{(p)}} \mathrm{H}^{p+q}_{x}\left(\operatorname{Spec} R, \mu_l^{\otimes n}\right) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}\left(R, \mu_l^{\otimes n}\right) = \mathrm{H}^{p+q}$$

(cf. [4, §1]). Then we have a filtration

$$0 \subset \mathrm{H}^{p+q}_{p+q} \subset \cdots \subset \mathrm{H}^{p+q}_{1} \subset \mathrm{H}^{p+q}_{0} = \mathrm{H}^{p+q},$$

such that

$$\mathrm{H}^{p+q}_{p} / \mathrm{H}^{p+q}_{p+1} \simeq \mathrm{H}^{p,q}_{\infty}.$$

By Theorem 4, it suffices to show that

$$\mathrm{H}^{1,i-1}_{2} = \mathrm{H}^{2,i-2}_{2} = 0.$$

By Proposition 5, the morphism

$$\mathrm{H}^{i} \longrightarrow \mathrm{H}^{0,i}_{\infty}$$

is injective and

$$\mathrm{H}^{i}_{1} = \mathrm{H}^{i}_{2} = 0.$$

Hence we have

$$\mathrm{H}^{1,i-1}_{\infty} = \mathrm{H}^{2,i-2}_{\infty} = 0.$$

Since

$$\mathrm{H}^{p,i-p+1}_{r} = 0$$

for $p \geq 3$ and
$$\mathrm{H}_r^{1-r,i+r-2} = 0$$
for $r \geq 2$, we have
$$\mathrm{H}_2^{1,i-1} = \mathrm{H}_\infty^{1,i-1} = 0.$$
By the exactness of the complex (8) at the second term, we have
$$\mathrm{H}_2^{0,i-1} = \mathrm{H}_\infty^{0,i-1} = \mathrm{H}^{i-1}$$
and
$$\mathrm{Im}\left(\mathrm{H}_2^{0,i-1} \xrightarrow{\mathrm{d}_2^{0,i-1}} \mathrm{H}_2^{2,i-2}\right) = 0.$$
Hence we have
$$\mathrm{H}_2^{2,i-2} = \mathrm{H}_3^{2,i-2}.$$
Moreover, since
$$\mathrm{H}_r^{2-r,i+r-3} = 0$$
for $r \geq 3$, we have
$$\mathrm{H}_{r+1}^{2,i-2} = \frac{\mathrm{Ker}(\mathrm{d}_r^{2,i-2})}{\mathrm{Im}(\mathrm{d}_r^{2-r,i+r-3})} = \mathrm{H}_r^{2,i-2}$$
for $r \geq 3$. Therefore
$$\mathrm{H}_2^{2,i-2} = \mathrm{H}_3^{2,i-2} = \mathrm{H}_\infty^{2,i-2} = 0.$$
This completes the proof. □

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