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Guy Métivier

Lecture notes : Stability of Noncharacteristic Viscous Boundary Layers

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1. Introduction

The material included in these lectures is taken from a series of joint papers with O. Guès, M. Williams and K. Zumbrun. They concern the linear and nonlinear stability of viscous boundary layers which arise when one considers small viscosity parabolic perturbations of noncharacteristic multidimensional hyperbolic boundary value problems. The analysis of boundary layers is a major issue in many applications, for instance in fluid mechanics, and there is a huge literature in books of mechanics concerning all kinds of layers. Moreover, after suitable modifications, the study of layers applies to the analysis of shock waves: shock waves can be seen as (smooth) solutions of a *free boundary* value problem, or more accurately as solutions of a transmission problem across a free interface. Classical Lax's shocks are noncharacteristic, and constitute a major motivation for the analysis presented here. In this approach, the conservative character of the equations as well as the classical Lax's shock conditions are unessential, so that the analysis presented here also applies to nonclassical shocks (overcompressive or nonconservative) as long as they remain noncharacteristic.

Several references for the mathematical analysis of boundary layers for linear equations are [BBB], [BaRa], [Lio]. The semilinear symmetric dissipative case is completely solved in [Gue1]. This paper gives a rigorous complete asymptotic description of the layer, at all order of approximation. It also concerns characteristic and noncharacteristic problems, pointing out a fundamental difference between these two cases. For noncharacteristic boundaries, the layer has a characteristic width of order ε, the magnitude of the viscosity, and the layer profile is given by an ordinary differential equation; for characteristic boundaries, the characteristic width of the layer is of order $\sqrt{\varepsilon}$ and the layer is given by a partial differential equation.

The analysis of noncharacteristic boundary layers for quasilinear equations, is started in [GrGu] for small layers and symmetric systems (see also the results in one space dimension given in [Gis, GiSe, GrRo] and in [GoXi, Rou2] for shocks). The case of characteristic boundaries is much more delicate: in the example of Navier-Stokes equations, the PDE's governing the layer which are expected after formal computations, are Prandtl's equations, which are known to present strong unstabilities, in accordance with well known physical unstabilities. This is why, motivated by the analysis of shock waves and some particular examples of in or outflow boundary conditions for Navier-Stokes and MHD equations (see [GMWZ8] and the references therein), we will restrict our attention to noncharacteristic boundaries.

In [GrGu], the linear and nonlinear stability of the layer is proved under a sufficient *smallness condition*, whose analogue for shocks is never satisfied. The sharp conditions for stability can be analyzed using a spectral (Fourier-Laplace) analysis. The *spectral stability* conditions have a nice natural formulation in terms of Evans function. Evans functions have been introduced in the study of the stability of planar viscous shocks and boundary layers (see, e.g., [GaZu, ZuHo, ZuSe, Zum2, Ser, Rou1], and references therein). They play the role of the Lopatinski determinant for boundary value problems with constant coefficients. When they vanish in the open left half plane, the problem is strongly unstable and when they do not vanish in the closed half space, the problem is expected to be strongly stable. In space dimension one, this has been proved to be correct for boundary layers in [GrRo] and in [Rou2] for shocks. Next, this has been extended to any dimension in [MéZu1], for fully parabolic perturbations of hyperbolic systems with constant multipliciy (see also [GMWZ1, GMWZ2, GMWZ3] for shocks). The case of partial viscosities such as Navier-Stokes equations and hyperbolic systems with variable multiplicity such as magneto-hydodynamics is treated in [GMWZ4, GMWZ5, GMWZ6].

We now briefly describe the main lines of these lectures. We denote by t the time variable and by $(y, x) \in \mathbb{R}^{d-1} \times \mathbb{R}$ the space variables. We consider a boundary value problems, for simplicity in the half space $\{x > 0\}$,

$$L(u, \partial)u - \varepsilon B(u, \partial u, \partial^2 u) = 0, \quad \Upsilon(u, \varepsilon\partial u) = 0. \tag{1.1}$$

Here L is a first order hyperbolic problem, B a second order term, partially elliptic in the spatial derivatives and Υ denotes the boundary conditions. The precise assumptions are given in Section 2. The main goal is to study the small viscosity limit $\varepsilon \to 0$. The limiting hyperbolic equation is

$$L(u, \partial)u = 0 \tag{1.2}$$

for which the boundary conditions Υ are not (in general) adapted. In the interior, the solutions u_ε of (1.1) converge to a solution u_0 of (1.2). Part of the problem is to find the boundary conditions

$$\Gamma_0(u) = 0 \tag{1.3}$$

satisfied by the limit u_0. In general, $u_\varepsilon - u_0$ is small in the interior but is large at the boundary because of the mismatch of the boundary values. This means that $u_\varepsilon - u_0$ has a rapid variation near the boundary : this is the *boundary layer*.

Example 1.1. Consider the model case

$$\begin{cases} \partial_t u_\varepsilon - \partial_x u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = f, & x > 0, \\ u_\varepsilon(0) = 0. \end{cases} \tag{1.4}$$

The limiting problem is

$$\partial_t u - \partial_x u = f,$$

without any boundary condition (the field is propagating to the left). For $f = e^{-x}$, the stationary solution is

$$u_\varepsilon(x) = \frac{1}{1-\varepsilon} e^{-x} - \frac{1}{1-\varepsilon} e^{-x/\varepsilon}.$$

The first term converges to $u_0(x) = e^{-x}$ and the second term is the boundary layer which makes the connection with the boundary condition $u_\varepsilon(0) = 0$. It is exponentially decaying in x/ε; this will be a general feature of the layers considered in these notes.

The main idea is that the solutions u_ε of (1.1) look like

$$u_\varepsilon(t, y, x) = u_0(t, y, x) + U(t, y, \frac{x}{\varepsilon}) + O(\varepsilon) \tag{1.5}$$

where u_0 is a solution of (1.2) and U satisfies

$$\lim_{z \to +\infty} U(t, y, z) = 0.$$

Plugging (1.5) into the equation, reveals a singular term in ε^{-1}. Equating this term to 0 shows that for fixed (t, y, x), the function $z \mapsto w(z) := u(t, y, x) + U(t, y, z)$ satisfies a second order ordinary differential system

$$\mathcal{L}_0'(w, \partial_z w, \partial_z^2 w) = 0. \tag{1.6}$$

Moreover, for $x = 0$, w must satisfy the boundary condition

$$\Upsilon(w, \partial_z w)_{|z=0} = 0 \tag{1.7}$$

and the end point condition

$$\lim_{z \to +\infty} w(z) = u_0(t, y, 0). \tag{1.8}$$

This indicates what are the natural boundary conditions for u_0:

Definition 1.2. *Let $\mathcal{C}$ denote the set of end points $\underline{u}$ such that there is a solution of* (1.6) – (1.7) *which converges to $\underline{u}$ at infinity. Then the natural boundary conditions for the limiting hyperbolic system* (1.2) *are*

$$u_{0|x=0} \in \mathcal{C}. \tag{1.9}$$

In these lectures, we will discuss the following aspects of the problem:

- *Construction of layer profiles*; they are exact solutions

$$u_\varepsilon(x) = w\Big(\frac{x}{\varepsilon}\Big), \tag{1.10}$$

of (1.1). The profiles $w(z)$ are solutions of (1.6) - (1.7) They converge to a limit as z tends to infinity:

$$\lim_{z \to \infty} w(z) = \underline{w}. \tag{1.11}$$

The form of the equations $\mathcal{L}_0' = 0$ implies that constants are solutions of (1.6). The central-stable manifold theorem gives the solutions of the profile equation near infinity, with end state close to a given $\underline{w}$. The problem is to find solutions which extend to $z \in [0, \infty[$ and satisfy the boundary condition (1.7). Given such a solution, the regularity properties of the set $\mathcal{C}$ near the end point $\underline{w}$ depend on *transversality* properties of the central-stable manifold and the boundary conditions. This is detailed in Section 3.

- *Spectral stability*; it expresses the well posedness of the linearized equations near a layer profile. These linearized equations have the form :

$$L\Big(\frac{x}{\varepsilon}, \varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x\Big)\dot{u} = \dot{f}, \quad \Upsilon'(\dot{u}, \varepsilon\partial_y\dot{u}, \varepsilon\partial_x\dot{u})_{|x=0} = \dot{g}. \tag{1.12}$$

They have constant coefficients in (t, y), and following the usual theory of constant coefficients evolution equations, one performs a Laplace-Fourier transform in (t, y). After rescaling, one obtains the *spectral equation* :

$$L(z, \gamma + i\tau, i\eta, \partial_z)u = f, \quad \Upsilon'(u, i\eta u, \partial_z u)_{|z=0} = g. \tag{1.13}$$

The spectral stability conditions concern the well posedness of the equations (1.13) for $\gamma \geq 0$. They can be formulated using a suitable Evans function. A key point is to link the spectral stability of the viscous and inviscid problems: this was done first in [ZuSe] (see also [Rou1]). We will present here a new proof which reduces the analysis to a nonsingular perturbation problem and does not require any constant multiplicity assumption. This is explained in Sections 4 and 5.

- *Symbolic symmetrizers*; their construction is the main topic discussed in these lectures, developed in Sections 6 to 9. They are Fourier multipliers that are used to prove L^2 a-priori estimates for the solutions of (1.13). Indeed the one space dimension methods used in [GrRo, Rou2] do not extend to the multidimensional case and the L^p estimates of the Green's function (see [Zum2] and the references therein) are insufficient. This is a consequence of focusing and spreading in the underlying hyperbolic propagation. We want uniform estimates which give the hyperbolic estimates in the limit $\varepsilon \to 0$, which translates in the limit $\zeta := (\tau, \eta, \gamma) \to 0$ in (1.13) due to the rescaling. This leads to look for methods which are suitable for the analysis of both hyperbolic boundary-value problems and their parabolic regularizations. In multi-dimensions we are restricted by the hyperbolic limit to seeking $L^2 \to L^2$ bounds. To satisfy these requirements, we follow Kreiss' analysis of hyperbolic equations based on the construction of symmetrizers. The basic estimate concerns the L^2 stability of the linearized equations, and is proved using symmetrizers and a suitable extension of Kreiss' analysis to parabolic-hyperbolic problems.

- *The linear stability*; the question is to prove that the linearized equations near an approximate solution u_ε of the form (1.5) are well-posed. Keeping only the main terms, they read

$$L\Big(t, y, x, \frac{x}{\varepsilon}, \varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x\Big)\dot{u} = \dot{f}, \quad \Upsilon'(\dot{u}, \varepsilon\partial_y\dot{u}, \varepsilon\partial_x\dot{u})_{|x=0} = \dot{g}. \tag{1.14}$$

They are *slow perturbations* in (t, y, x) of (1.12). The main goal is to prove that the L^2 estimates obtained by using Fourier multipliers when u_ε is a profile (1.10) extend to the case where u_ε has the more general form (1.5). This is done using a suitable pseudo or para-differential calculus, which transforms the Fourier multiplier calculus into an operator calculus. In this analysis, the Fourier multipliers are used as the symbols of the operators. However, several calculi are needed to reflect both the semi-classical aspects and the different homogeneities. This is briefly discussed in Section 10.

- *The nonlinear stability*; the objective is to prove the existence, on an interval of time independent of ε, of exact solutions satisfying (1.5) of the parabolic-hyperbolic equations (1.1). In a first step, one constructs approximate solutions

$$u_\varepsilon^{app}(t,y,x) = \sum_{k=0}^{n} \varepsilon^k \Big(u_k(t,y,x) + U_k(t,y,\frac{x}{\varepsilon}) \Big) \tag{1.15}$$

which satisfy the equation up to source term of size $O(\varepsilon^n)$. Next one constructs exact solutions of the form

$$u_\varepsilon = u_\varepsilon^{app} + \varepsilon^n v_\varepsilon. \tag{1.16}$$

The equations for v_ε are solved by iterative methods which involve the resolution of linearized problems. Bounds for the iterates and convergence follow from the L^2 and Sobolev estimates which have been obtained for the linearized equations. Two methods are presented in Section 11: for fully elliptic viscosities, strong parabolic estimates are available for the linearized equations and the standard implicit function theorem yields the results with first order approximate solutions ($n = 1$). In the general case, one uses hyperbolic type iterates starting from an accurate approximate solution, that is with n large in (1.15).

2. Hyperbolic-parabolic boundary value problems

2.1. Structure of the equations

Consider a system of equations

$$\mathcal{L}_\varepsilon(u) := A_0(u)\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j \Big(B_{j,k}(u)\partial_k u \Big) = 0. \tag{2.1}$$

When $\varepsilon = 0$, $\mathcal{L}_0$ is first order and assumed to hyperbolic; ε plays the role of a non-dimensional viscosity and for $\varepsilon > 0$, the system is assumed to be parabolic or at least partially parabolic, see below. Systems of conservation laws

$$\mathcal{L}_\varepsilon(u) := \partial_t f_0(u) + \sum_{j=1}^{d} \partial_j f_j(u) - \varepsilon \sum_{j,k=1}^{d} \partial_j \Big(B_{j,k}(u)\partial_k u \Big) = 0 \tag{2.2}$$

are particular cases of systems (2.1) with $A_j(u) = \nabla_u f_j(u)$. Classical examples are Navier-Stokes equations of gas dynamics, or equations of magneto-hydrodynamics.

The form of the equations is preserved by changes of unknowns $u = \Phi(\tilde{u})$ and by multiplying on the left the equations by *constant* invertible matrices. To cover the case of partial viscosity and motivated by the examples of Navier-Stokes equations and MHD, we make the following assumption

Assumption 2.1. (H0) (Smooth fluxes and viscosity) *The matrices A_j and $B_{j,k}$ are C^∞ $N \times N$ real matrices of the variable $u \in \mathcal{U}^* \subset \mathbb{R}^N$. Moreover, for all $u \in \mathcal{U}^*$, the matrix $A_0(u)$ is invertible.*

(H1) (Block form) *Possibly after a change of unknowns u and multiplying the system on the left by an invertible constant coefficient matrix, there are coordinates $u = (u^1, u^2) \in \mathbb{R}^{N^1} \times \mathbb{R}^{N^2}$ and $f = (f^1, f^2) \in \mathbb{R}^{N^1} \times \mathbb{R}^{N^2}$, with $N^1 + N^2 = N$, such that the following block structure condition is satisfied :*

$$A_0(u) = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix}, \quad B_{jk}(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_{j,k}^{22} \end{pmatrix}. \tag{2.3}$$

We refer to [GMWZ4] for a geometric formulation of condition (H1), independent of coordinates $u \in \mathcal{U}^*$ and we also refer to [Zum3] for further comments and explanations. From now on we work with variables $u = (u^1, u^2) \in \mathcal{U}^*$ such that (2.3) holds. We set

$$A_j = f_j', \quad \overline{A}_j = A_0^{-1} A_j, \quad \overline{B}_{j,k} = A_0^{-1} B_{j,k}, \tag{2.4}$$

and systematically use the notation $M^{\alpha\beta}$ for the sub-blocks of a matrix M corresponding to the splitting $u = (u^1, u^2)$. Note that

$$\overline{B}_{j,k}(u) := A_0(u)^{-1} B_{j,k}(u) = \begin{pmatrix} 0 & 0 \\ 0 & \overline{B}^{22}_{j,k}(u) \end{pmatrix}. \tag{2.5}$$

The triangular form of the equations also reveals the importance of the $(1,1)$ block which plays a special role in the analysis :

$$L^{11}(u, \partial) = \sum_{j=0}^{d} A_j^{11}(u) \partial_j \,, \quad \text{or} \quad \overline{L}^{11}(u, \partial) = \Big(A_0^{11}(u) \Big)^{-1} L^{11}(u, \partial). \tag{2.6}$$

In this spirit, *the high frequency principal part* of the equation is

$$\begin{cases} \overline{L}^{11}(u, \partial) u^1 \\ \partial_t u^2 - \varepsilon \overline{B}^{22}(u, \partial) u^2 \end{cases} \tag{2.7}$$

with $\overline{B}^{22}(u, \xi) = \sum_{j,k=1}^{d} \xi_j \xi_k \overline{B}^{22}_{j,k}(u)$. The first natural hypothesis is that $L^{11}(u, \partial)$ is hyperbolic and $\partial_t - \overline{B}^{22}(u, \partial)$ is parabolic in the direction dt.

Assumption 2.2. (H2) (Partial parabolicity) *There is $c > 0$ such that for all $u \in \mathcal{U}^*$ and $\xi \in \mathbb{R}^d$, the eigenvalues of $\overline{B}^{22}(u, \xi)$ satisfy* $\operatorname{Re} \mu \geq c|\xi|^2$.

(H3) (Hyperbolicity of the 1-1 block) *For all $u \in \mathcal{U}^*$ and all $\xi \in \mathbb{R}^d \backslash \{0\}$, $\overline{A}^{11}(u, \xi) = \sum_{j=1}^{d} \xi_j \bar{A}_j^{11}(u)$ has only real eigenvalues.*

For the applications we have in mind such as Navier-Stokes and MHD, the operator $\overline{L}^{11}$ is a transport field and (H3) is trivially satisfied.

Next we assume that the inviscid equations are hyperbolic and that Kawashima's *genuine coupling* condition is satisfied for u, in some open subdomain $\mathcal{U} \subset \mathcal{U}^*$. Let

$$\overline{A}(u, \xi) = \sum_{j=1}^{d} \xi_j \overline{A}_j(u) \quad \text{and} \quad \overline{B}(u, \xi) = \sum_{j,k=1}^{d} \xi_j \xi_k \overline{B}_{j,k}(u). \tag{2.8}$$

Assumption 2.3. (H4) (Strict dissipativity near the end states) *There is $c > 0$ such that for $u \in \mathcal{U}$ and $\xi \in \mathbb{R}^d$, the eigenvalues of $i\overline{A}(u, \xi) + \overline{B}(u, \xi)$ satisfy*

$$\operatorname{Re} \mu \geq c \frac{|\xi|^2}{1 + |\xi|^2} \,. \tag{2.9}$$

Remark 2.4. (H4) implies *hyperbolicity* of the inviscid equation : for all $u \in \mathcal{U}$ and $\xi \in \mathbb{R}^d \backslash \{0\}$ the eigenvalues of $\overline{A}(u, \xi)$ are real. It is important for applications that $\mathcal{U}$, the domain of hyperbolicity which will contain *end states* of the layers, can be strictly smaller than $\mathcal{U}^*$.

Symmetric systems play an important role, and symmetry will be an important assumption in some of our results. In particular, the Assumption (H4) is satisfied when the following conditions are satisfied (see [KaS1, KaS2]):

Definition 2.5. *The system* (2.1) *is said to be symmetric dissipative if there exists a real matrix $S(u)$, which depends smoothly on $u \in \mathcal{U}$, such that for all $u \in \mathcal{U}$ and all $\xi \in \mathbb{R}^d \backslash \{0\}$, the matrix SA_0 is symmetric definite positive, $S(u)A(u, \xi)$ is symmetric and* $\operatorname{Re} S(u)B(u, \xi)$ *is non negative with kernel of dimension $N - N^2$.*

Proposition 2.6. *If the system is symmetric dissipative,* (2.9) *is equivalent to the genuine coupling condition of Kawashima: no eigenvector of $\bar{A}(u, \xi)$ lies in the kernel of $\bar{B}(u, \xi)$ for $\xi \in \mathbb{R}^d \setminus \{0\}$.*

Navier-Stokes equations satisfy this condition (see e.g. [KaS2]).

Remark 2.7. For systems of conservation laws (2.2), symmetry is implied by the existence of a strictly convex entropy, see [KaS2].

2.2. Boundary conditions

We consider a boundary value problem for (2.1) in the model case of a half space, which is given by $\{x > 0\}$, in some coordinates $(y_1, \dots, y_{d-1}, x)$ for the space variables. We assume that the boundary is not characteristic both for the viscous and the inviscid equations. The principal term of the viscous equation is block diagonal as indicated in (2.7). The B^{22} block is noncharacteristic by (H2). Restricting $\mathcal{U}^*$ to a component where the profiles will take their values, the condition for the $\overline{A}^{11}$ block reads

Assumption 2.8. (H5) *$\mathcal{U}^*$ is connected and for all $u \in \mathcal{U}^*$,* $\det A_d^{11}(u) \neq 0$.

For the inviscid equation, restricting $\mathcal{U}$ to the component where the hyperbolic solutions will take their value, the condition reads

Assumption 2.9. (H6) *$\mathcal{U}$ is connected and for all $u \in \mathcal{U}$,* $\det \Big(A_d(u) \Big) \neq 0$.

By Assumption (H3) and Remark 2.4, $\overline{A}_d^{11}(u)$ and $\overline{A}_d(u)$ have only real eigenvalues, which by (H5) and (H6) never vanish. This leads to two important indices :

Notations 2.10. With assumptions as above, N_+ denotes the number of positive eigenvalues of $\overline{A}_d(u)$ for $u \in \mathcal{U}$ and N_+^1 the number of positive eigenvalues of $\overline{A}_d^{11}(u)$ for $u \in \mathcal{U}^*$. We also set $N_b = N^2 + N_+^1$.

The block structure (2.7) suggests that N_b is the correct number of boundary conditions for the well posedness of (2.1), for solutions with values in $\mathcal{U}^*$. Indeed, the high frequency decoupling (2.7) suggests that N^2 boundary conditions for u^2 and N_+^1 boundary conditions for u^1 are required. On the other hand, N_+ is the correct number of boundary conditions for the inviscid equation for solutions with values in $\mathcal{U}$. Thus we supplement (2.1) with boundary conditions

$$\Upsilon(u, \varepsilon\partial_y u^2, \varepsilon\partial_x u^2)_{|x=0} = 0. \tag{2.10}$$

Without pretending to maximal generality, we assume that they decouple into zero-th order boundary conditions for u^1 and zero-th order and first order conditions for u^2:

$$\begin{cases} \Upsilon_1(u^1)_{|x=0} = 0, \\ \Upsilon_2(u^2)_{|x=0} = 0, \\ \Upsilon_3(u, \varepsilon\partial_y u^2, \varepsilon\partial_x u^2)_{|x=0} = 0. \end{cases} \tag{2.11}$$

with

$$\Upsilon_3(u, \partial_y u^2, \partial_x u^2) = K_d \partial_x u^2 + \sum_{j=1}^{d-1} K_j(u) \partial_j u^2.$$

Assumption 2.11. (H7) (Smooth boundary conditions) Υ_1, Υ_2 *and* Υ_3 *are smooth functions of their arguments with values in* $\mathbb{R}^{N^1_+}$, $\mathbb{R}^{N^2-N^3}$ *and* $\mathbb{R}^{N^3}$ *respectively, where* $N^3 \in \{0, 1, \dots, N^2\}$. *Moreover,* K_d *has maximal rank* N^3 *and for all* $u \in \mathcal{U}^*$ *the Jacobian matrix* $\Upsilon_1'(u^1)$ *and* $\Upsilon_2'(u^2)$ *have maximal rank* N^1_+ *and* $N^2 - N^3$ *respectively.*

3. Layers profiles

3.1. The profile equation

To match constant solutions $\underline{u}$ of the inviscid problem to solutions satisfying the boundary conditions, one looks for exact solutions of (2.1) (2.10) of the form:

$$u_\varepsilon(t, y, x) = w\Big(\frac{x}{\varepsilon}\Big), \tag{3.1}$$

such that

$$\lim_{z \to +\infty} w(z) = \underline{u}\,. \tag{3.2}$$

The equation and boundary conditions for w read

$$A_d(w)\partial_z w - \partial_z\Big(B_{d,d}(w)\partial_z w\Big) = 0, \quad z \geq 0 \tag{3.3}$$

$$\Upsilon(w, 0, \partial_z w^2)_{|z=0} = 0. \tag{3.4}$$

Solutions are called *layer profiles*. This equation can be written as a first order system for $U = (w, \partial_z w^2)$, which is nonsingular if and only if A^{11}_d is invertible (this indicates the strong link between Assumption 2.8 and the ansatz (3.1)):

$$\begin{cases} \partial_z w^1 = -(A^{11}_d)^{-1} A^{12}_d w^3, \\ \partial_z w^2 = w^3, \\ \partial_z\Big(B_{d,d} w^3\Big) = \Big(A^{22}_d - A^{21}_d (A^{11}_d)^{-1} A^{12}_d\Big) w^3, \end{cases} \tag{3.5}$$

and the matrices A_d and $B_{d,d}$ are evaluated at $w = (w^1, w^2)$.

For conservative systems, the equations read

$$\partial_z f_d(w) - \partial_z(B_{d,d}(w)\partial_z w) = 0. \tag{3.6}$$

They can be integrated once and, splitting the components w^1 and w^2 they are equivalent to

$$\begin{cases} f^1_d(w^1, w^2) - k^1 = 0 \\ B_{d,d}(w^1, w^2)\partial_z w^2 = f^2_d(w^1, w^2) - k^2, \end{cases} \tag{3.7}$$

with $k = (k^1, k^2)$ constant.

3.2. Existence of profiles

The constants are trivial solutions of the layer equation (3.3). The invariant manifold theorem implies that, near $\underline{u} \in \mathcal{U}$, there is a variety of dimension $N + N_b - N_+$ of solutions

$$\Phi(z, p, a), \tag{3.8}$$

depending on the parameters p near $\underline{u}$ and a in a neighborhood of 0 in $\mathbb{R}^{N_b - N_+}$, and such that

$$\lim_{z \to \infty} \Phi(z, p, a) = p \quad \text{and} \quad \Phi(z, p, 0) \equiv p,$$

(see [Mét4] for fully parabolic viscosity and [GMWZ5] for partial viscosity). Extend these solutions as maximal solutions. The layer profiles are then determined by solving the boundary conditions (3.4):

$$\mathrm{T}(p, a) := \Upsilon(\Phi, 0, \partial_z \Phi^2)_{|z=0} = 0. \tag{3.9}$$

The existence of small layers can be proved by perturbation arguments (see e.g. [GiSe] or [GrGu, Mét4]). For shocks the existence of small amplitude profiles is proved in [MaPe, Peg]. In [Gil] the existence of large profiles for gas dynamics is studied.

3.3. The inviscid boundary conditions

The natural limiting boundary conditions for the inviscid problem read

$$u_{|x=0} \in \mathcal{C}, \tag{3.10}$$

where $\mathcal{C}$ denotes the set of end points $\underline{u}$ such that there is a layer profile $w \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{U}^*)$ satisfying (3.2) (3.3). The properties of the set $\mathcal{C}$ depend on the well posedness of the equation (3.9), which is a system of N_b equations for $N + N_b - N_+$ unknows. This property is called *transversality* in [MéZu1, Mét4, GMWZ5, GMWZ6].

Definition 3.1. *The layer profile $w(z) = \Phi(z, \underline{p}, \underline{a})$, supposed to be defined on $[0, +\infty[$, is said to be transversal if the following two conditions holds*

$$\mathrm{rank} \nabla_a \mathrm{T}(\underline{p}, \underline{a}) \ = N_b - N_+ \,, \tag{3.11}$$

$$\mathrm{rank} \nabla_{a,p} \mathrm{T}(\underline{p}, \underline{a}) = N_b \,. \tag{3.12}$$

This definition immediately implies the following

Proposition 3.2. *If the profile $w(z) = \Phi(z, \underline{p}, \underline{a})$, supposed to be defined on $[0, +\infty[$ is transversal, then near $\underline{p}$, $\mathcal{C}$ is s smooth manifold of dimension $N - N_+$, defined by N_+ independent equations, $T(p) = 0$.*

Thus, the limiting inviscid boundary conditions (3.10), can be written $T(u) = 0$. Note that N_+ is the correct number of boundary conditions for the inviscid equation. For small layers, the transversality condition is satisfied (see e.g. [GrGu]). It is noticeable that for Lax shocks, the analogue limiting boundary condition always reduces to the usual Rankine-Hugoniot conditions. Note also that for extreme Lax shocks, the transversality condition is automatic. The discussion in [Gil] also includes transversality.

These geometric conditions can be rephrased in terms of properties of the linearized equations from (3.3) (3.4) near $w(z)$, since the derivatives $\partial_{p,a} \Phi(z, \underline{p}, \underline{a})$ are solutions of these linearized equations and form a basis of the space of bounded solutions. We abbreviate the linearized equation and boundary condition as

$$\begin{cases} \underline{L}(z, \partial_z)\dot{w}, & z \geq 0, \\ \underline{\Upsilon}'(\dot{w}, \partial_z \dot{w}^2)_{|z=0} \end{cases} \tag{3.13}$$

Proposition 3.3. *The layer profile w is transversal if and only if*

i) there is no nontrivial solution $\dot{w}$ of $\underline{L}\dot{w}$ which satisfies the boundary conditions $\underline{\Upsilon}'(\dot{w}, \partial_z \dot{w}^2)_{|z=0} = 0$,

ii) the mapping $\dot{w} \mapsto \underline{\Gamma}(\dot{w}, \partial_z \dot{w}^2)_{|z=0}$ from the space of solutions of $\underline{L}\dot{w} = 0$ to $\mathbb{C}^{N_b}$ has rank N_b.

3.4. Examples

1. *Burgers equation.* In space dimension one, consider for $x \geq 0$ the Burgers-Hopf equation:

$$\partial_t u + u\partial_x u - \varepsilon \partial_x^2 u = 0\,, \quad u(0) = 0. \tag{3.14}$$

In this case, the inner-layer o.d.e is

$$\partial_z^2 u = u \partial_z u\,, \quad u(0) = 0\,. \tag{3.15}$$

The equation can be integrated once yielding

$$\partial_z u = \frac{1}{2} u^2 + k\,, \quad u(0) = 0\,.$$

Depending on the sign of the constant k, there are two families of solutions:

$$\begin{aligned} &1) \quad u(z) = -\lambda \tanh\big(\lambda z/2\big) \\ &2) \quad u(z) = \mu \tan\big(\mu z/2\big)\,. \end{aligned}$$

Changing λ into $-\lambda$ or μ into $-\mu$ does not change the solution, so we can assume that the parameters are nonnegative. The two families intersect only on the constant solution $u = 0$. Solutions of the second family, have a finite time of existence: they do not provide solutions of (3.14) on the half line. Thus, we restrict attention to solutions of the first family, which are globally defined. In this case, we have

$$\lim_{z \to +\infty} u(z) = -\lambda \leq 0\,.$$

The end state $-\lambda$ is noncharacteristic (i.e. satisfies (H4)) if $\lambda \neq 0$. Thus we have shown:

for the Burgers equation (3.14), *the set of noncharacteristic end states p which can be connected to 0 by a solution of* (3.15) *is* $\widetilde{\mathcal{C}} =]-\infty, 0[$.

2. *The linear case.* Suppose that A_d and $B_{d,d}$ are constant (independent of u). The o.d.e. reads

$$\begin{cases} \partial_z u^1 = G_d^{12} \partial_z u^2\,, \\ \partial_z^2 u^2 = G_d^{22} \partial_z u^2\,, \end{cases} \tag{3.16}$$

with $G_d^{12} = -(A_d^{11})^{-1} A_d^{12}$ and $G_d^{22} = (B_{dd}^{22})^{-1}(A_d^{22} + A_d^{21} G_d^{12})$. The solutions of the o.d.e are

$$\begin{cases} u^1(z) = u^2(z) + p^1\,, \\ u^2(z) = p^2 + e^{zG_d^{22}} a\,, \end{cases} \tag{3.17}$$

with arbitrary constants p and a. Because the eigenvalues of G_d^{22} are real and different from zero, the explicit formula implies the following results.

1. *The solution is bounded if and only if* $a \in \mathbb{E}^-(G_d^{22})$, *the invariant space for* G_d^{22} *associated to eigenvalues in the left half space* $\{\operatorname{Im} \mu < 0\}$.

2. *Bounded solutions of* (3.16) *converge at an exponential rate at infinity.*

3. *The bounded solutions of* (3.16) *form a manifold of dimension* $N + N^2_-$, *where* $N^2_- = \dim \mathbb{E}^-(G_d^{22})$.

We now add boundary conditions for u^1 and Dirichlet boundary conditions for u^2

$$\Gamma^1 {u^1}_{|z=0} = 0, \quad {u^2}_{|z=0} = 0.$$

For the solutions of (3.17) this is equivalent to

$$\Gamma^1 p^1 = 0, \quad p^2 = -a.$$

Therefore:

4. *The set of end states* $p = (p^1, p^2)$ *which can be connected to a data satisfying the boundary condition is the linear space* $\ker \Gamma^1 \times \mathbb{E}_-(G_d^{22})$.

3. *Shock profiles for isentropic Navier-Stokes equations.* Consider in $\mathbb{R}^d$ the isentropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \varepsilon \Delta u, \end{cases} \tag{3.18}$$

with $p = P(\rho)$. The profile equations with speed velocity σ, that is relative to the front $x_d = \sigma t$, read

$$\begin{cases} \partial_z m = 0, \\ \partial_z(p + m u_d) = \partial_z^2 u_d, \\ \partial_z(m u_{\mathrm{tg}}) = \partial_z^2 u_{\mathrm{tg}} \end{cases} \tag{3.19}$$

with $m := \rho(u_d - \sigma)$. We look for solutions defined for $z \in \mathbb{R}$ which end points (ρ^-, u^-) and (ρ^+, u^+). Integrating once the equations and taking the limits at $+\infty$ and $-\infty$ yields the necessary Rankine-Hugoniot conditions:

$$\begin{cases} [m] = 0 \qquad \Leftrightarrow \qquad [u_d] = m[\tau] \\ [p] + m[u_d] = 0, \\ m[u_{\mathrm{tg}}] = 0. \end{cases} \tag{3.20}$$

with $\tau = 1/\rho$. Thus,

$$m^2 = -[p]/[\tau], \quad [u_d] = m[\tau], \quad \sigma = u_d^+ - m\tau^+ = u_d^- - m\tau^-. \tag{3.21}$$

The integrated system (3.19) reads:

$$\begin{cases} u_d(z) - \sigma = m\tau(z) \\ \partial_z(u_d(z) - \sigma) = m(u_d(z) - \sigma) + p(z) - k \\ u_{\mathrm{tg}}(z) = u_{\mathrm{tg}}^- = u_{\mathrm{tg}}^+ \end{cases} \tag{3.22}$$

with m and k constant. We end up with a scalar equation

$$\partial_z \tau = m\tau + \frac{1}{m}\psi(\tau) - b, \tag{3.23}$$

with $\psi(\tau) = P(1/\tau)$ and parameters which satisfy the constraints

$$m = [u_d]/[\tau], \quad b = m\tau^+ + \frac{1}{m}\psi(\tau^+) = m\tau^- + \frac{1}{m}\psi(\tau^-). \tag{3.24}$$

The constant k in (3.22) is $k = m(u_d^+ - \sigma) + p^+ = m(u_d^- - \sigma) + p^-$.

A 1-shock satisfies

$$u^+ - c^+ < \sigma < u^+, \quad u^- - c^- > \sigma, \tag{3.25}$$

where $c = c(\rho) = (\frac{1}{\rho}P'(\rho))^{\frac{1}{2}}$ is the sound speed for the state ρ. Thus

$$0 < m\tau^+ = u^+ - \sigma < c^+, \quad m\tau^- = u^- - \sigma > c^-. \tag{3.26}$$

This is equivalent to

$$m > 0, \quad (c^-/\tau^-)^2 < m^2 < (c^+/\tau^+)^2. \tag{3.27}$$

Since $c^2/\tau^2 = -\psi'(\tau)$, the Lax conditions for 1-shocks reduce to

$$\psi'(\tau^+) < \frac{\psi(\tau^+) - \psi(\tau^-)}{\tau^+ - \tau^-} = -m^2 < \psi'(\tau^-) < 0, \qquad m > 0 \tag{3.28}$$

The sign condition for $\psi'(\tau^-)$ ensures the hyperbolicity of the end states.

Similarly, for 3-shocks the conditions read

$$u^+ + c^+ < \sigma \quad u^- + c^- > \sigma > u^-, \tag{3.29}$$

$$m\tau^+ = u^+ - \sigma < -c^+, \quad 0 > m\tau^- = u^- - \sigma > -c^-, \tag{3.30}$$

$$m < 0, \quad (c^+/\tau^+)^2 < m^2 < (c^-/\tau^-)^2. \tag{3.31}$$

We end up with the conditions

$$\psi'(\tau^-) < \frac{\psi(\tau^+) - \psi(\tau^-)}{\tau^+ - \tau^-} = -m^2 < \psi'(\tau^+) < 0, \qquad m < 0 \tag{3.32}$$

As expected, we get the same condition as (3.28) with τ^+ and τ^- exchanged, since one passes from 1-shocks to 3-shocks changing x to $-x$, u, σ, m to $-u, -\sigma, -m$, keeping ρ and inverting the indices $+$ and $-$.

The equation (3.23) reads

$$\partial_z \tau = F(\tau). \tag{3.33}$$

The Rankine Hugoniot condition (3.24) implies that $F(\tau^-) = F(\tau^+) = 0$. Thus, for all initial data in the open interval limited by τ^+ and τ^-, the solution of (3.23) is globally defined and remains in this interval. The stability conditions at $\pm\infty$ read

$$F'(\tau^+) < 0, \qquad F'(\tau^-) > 0,$$

that is

$$\begin{aligned} &\frac{1}{m}\Big\{\psi'(\tau^+) - \frac{\psi(\tau^+) - \psi(\tau^-)}{\tau^+ - \tau^-}\Big\} < 0, \\ &\frac{1}{m}\Big\{\psi'(\tau^-) - \frac{\psi(\tau^+) - \psi(\tau^-)}{\tau^+ - \tau^-}\Big\} > 0. \end{aligned} \tag{3.34}$$

As expected, they are implied by the Lax shock conditions (3.28) (3.32).

Moreover, for a, and therefore all, solution to pass from τ^- to τ^+, it is necessary and sufficient that F has no rest points in the open interval I limited by τ^- and τ^+:

$$\forall \tau \in I: \quad F(\tau) = \frac{\tau - \tau^+}{m}\Big(\frac{\psi(\tau) - \psi(\tau^+)}{\tau - \tau^+} - \frac{\psi(\tau^-) - \psi(\tau^+)}{\tau^- - \tau^+}\Big) \neq 0. \tag{3.35}$$

If ψ is strictly decreasing and strictly convex, all these conditions are satisfied when

$$m^2 = -\frac{\psi(\tau^+) - \psi(\tau^-)}{\tau^+ - \tau^-}, \quad m(\tau^+ - \tau^-) < 0. \tag{3.36}$$

Lemma 3.4. *Assume that $\psi' < 0$ and $\psi'' > 0$ on an interval $I \subset]0, +\infty[$. Then for all $(\tau^-, \tau^+) \in I \times I$ and m satisfying the stability condition (3.36) and for all initial data in the open interval limited by τ^+ and τ^-, the solution of (3.23) is globally defined, remain in this interval and converges at exponential rate to $\tau^\pm$ at $\pm\infty$.*

When the profile for τ is known, we deduce the profile for u using (3.22).

Remark 3.5. The profile is determined up to an arbitrary choice of $\tau(0)$ in the interval $]\tau^+, \tau^-[$. This means that there is a one parameter family of solutions of the profile equations (3.19). Since the profile τ is strictly monotone, (because F does not vanish) this is equivalent to the expected translation indeterminacy due to the translation invariance of the shock profiles equations.

4. Spectral stability

4.1. The linearized equations

For further use, it is convenient to enlarge the class of functions w: consider a function $C^\infty(\overline{\mathbb{R}}_+; \mathcal{U}^*)$ which converges at an exponential rate to and end state $\underline{u} \in \mathcal{U}$: there is $\delta > 0$ such that for all $k \in \mathbb{N}$

$$e^{\delta z} \Big| \partial_z^k (w(z) - \underline{u}) \Big| \in L^\infty(\overline{\mathbb{R}}_+). \tag{4.1}$$

We refer to such a function as a *profile*; it need not be a solution of (3.3), though it will be in applications. Note that solutions of (3.3) (3.2) satisfy the exponential convergence above.

Consider the linearized equations from (2.1) (2.10) around $u_\varepsilon = w(x/\varepsilon)$:

$$\mathcal{L}'_{u_\varepsilon} \dot{u} = \dot{f}, \quad \Upsilon'(\dot{u}, \varepsilon\partial_y \dot{u}, \varepsilon\partial_x \dot{u})_{|x=0} = \dot{g}. \tag{4.2}$$

Here Υ' is the differential of Υ at $(w(0), 0, \partial_z w(0))$. $\mathcal{L}'_{u_\varepsilon}$ is a differential operator with coefficients that are smooth functions of $z := x/\varepsilon$. Factoring out ε^{-1} it also appears as an operator in $\varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x$:

$$\mathcal{L}'_{u_\varepsilon} = \frac{1}{\varepsilon} L\Big(\frac{x}{\varepsilon}, \varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x\Big). \tag{4.3}$$

It has constant coefficients in (t, y), and following the usual theory of constant coefficient evolution equations, one performs a Laplace-Fourier transform in (t, y), with frequency variables denoted by $\tilde{\gamma} + i\tilde{\tau}$ and $\tilde{\eta}$ respectively, yielding the systems

$$\frac{1}{\varepsilon} L\Big(\frac{x}{\varepsilon}, \varepsilon(\tilde{\gamma} + i\tilde{\tau}), i\varepsilon\tilde{\eta}, \varepsilon\partial_x\Big).$$

Next, we introduce explicitely the fast variable $z = x/\varepsilon$, rescale the frequency variables as $\zeta = (\tau, \eta, \gamma) = \varepsilon(\tilde{\tau}, \tilde{\eta}, \tilde{\gamma})$ and multiply the equation by ε, revealing the equation

$$L(z, \gamma + i\tau, i\eta, \partial_z) u = f, \quad \Upsilon'(u, i\eta u, \partial_z u)_{|z=0} = g, \tag{4.4}$$

with

$$L = -\mathcal{B}(z)\partial_z^2 + \mathcal{A}(z, \zeta)\partial_z + \mathcal{M}(z, \zeta) \tag{4.5}$$

with in particular, $\mathcal{B}(z) = B_{d,d}(w(z))$ and $\mathcal{A}^{11}(z,\zeta) = A_d^{11}(w(z))$. We do not give here the explicit form of $\mathcal{A}$ and $\mathcal{M}$. Using (H2) and Assumption 2.2, the equation is written as a first order system

$$\partial_z U = \mathcal{G}(z,\zeta)U + F, \quad \Gamma(\zeta)U_{|z=0} = g. \tag{4.6}$$

for

$$U = \begin{pmatrix} u \\ \partial_z u^2 \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \\ \partial_z u^2 \end{pmatrix} \in \mathbb{C}^{N+N^2} \tag{4.7}$$

$$F = \begin{pmatrix} (\mathcal{A}^{11}(z))^{-1} f^1 \\ 0 \\ (\mathcal{B}^{22}(z))^{-1}(-f^2 + \mathcal{A}^{21}(z)(\mathcal{A}^{11}(z))^{-1} f^1) \end{pmatrix}. \tag{4.8}$$

Similarly, one considers the linearized equations from the inviscid hyperbolic problem $\mathcal{L}_0(u) = 0$ around the constant solution $\underline{u}$:

$$\mathcal{L}'_{0,\underline{u}}\dot{u} = \dot{f}. \tag{4.9}$$

After performing a Laplace-Fourier transform, this equation reads

$$L_0(\underline{u}, \gamma + i\tau, i\eta, \partial_x)u = f \tag{4.10}$$

or

$$\partial_x u = H_0(\underline{u},\zeta)u + A_d^{-1}(\underline{u})f, \tag{4.11}$$

with

$$H_0(u,\zeta) := -(A_d(u))^{-1}\Big((i\tau + \gamma)A_0(u) + \sum_{j=1}^{d-1} i\eta_j A_j(u)\Big). \tag{4.12}$$

4.2. Structure of the linearized equations

The analysis of (4.4) or (4.6) depends on the size of the frequencies ζ. When ζ is large, the parabolic character is prominent for the component u^2. For small or bounded frequencies ζ, we use the conjugation lemma of [MéZu1]. The condition (4.1) implies that there is $\delta > 0$ and an end state matrix $G(\underline{u},\zeta)$, depending on the endstate $\underline{u}$ of w, such that

$$\partial_z^k(\mathcal{G}(z,\zeta) - G(\underline{u},\zeta)) = O(e^{-\delta z}). \tag{4.13}$$

Lemma 4.1. *Given $\underline{\zeta} \in \mathbb{R}^{d+1}$, there is a smooth invertible matrix $\Phi(z,\zeta)$ for $z \in \overline{\mathbb{R}}_+$ and ζ in a neighborhood of $\underline{\zeta}$, such that* (4.6) *is equivalent to*

$$\partial_z \tilde{U} = G(\underline{u},\zeta)\tilde{U} + \tilde{F}, \quad \tilde{\Gamma}(\zeta)\tilde{U}_{|z=0} = g. \tag{4.14}$$

with $U = \Phi(z,\zeta)\tilde{U}$, $F = \Phi(z,\zeta)\tilde{F}$ and $\tilde{\Gamma}(\zeta) = \Gamma(\zeta)\Phi(0,\zeta)$. In addition, Φ and Φ^{-1} converge to the identity matrix at an exponential rate when $z \to \infty$.

Moreover, if the coefficients of the operator and w depend smoothly on extra parameters p (such as the end state $\underline{u}$), then Φ can also be chosen to depend smoothly on p, on a neighborhood of a given $\underline{p}$.

Remark 4.2. The linearized profile equations from (3.3) around w, are exactly (4.4) at the frequency $\zeta = 0$. In particular, Lemma 4.1 implies that these equations are conjugated to constant coefficient equations, via the conjugation by $\Phi(\cdot, 0)$.

Next we investigate the spectral properties of the matrix G. Below, $\mathbb{R}^{d+1}_+$ denotes the open half space $\{\zeta = (\tau, \eta, \gamma) : \gamma > 0\}$ and $\overline{\mathbb{R}}^{d+1}_+$ its closure $\{\gamma \geq 0\}$. In addition to H_0 defined in (4.12), we also introduce the matrix

$$P_0(u) \ := \ (B^{22}_{dd})^{-1}\Big(A^{22}_d - A^{21}_d (A^{11}_d)^{-1} A^{12}_d\Big), \tag{4.15}$$

Lemma 4.3. *i) For $u \in \mathcal{U}$, $P_0(u)$ has no eigenvalue on the imaginary axis. We denote by N^2_- the number of its eigenvalues in $\{\operatorname{Re}\mu < 0\}$.*

ii) For $u \in \mathcal{U}$ and $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash\{0\}$, $G(u, \zeta)$ has no eigenvalue on the imaginary axis. The number of its eigenvalues, counted with their multiplicity, in $\{\operatorname{Re}\mu < 0\}$ is equal to $N_+ + N^2_- = N_b := N^2 + N^1_+$.

iii) For a given $\underline{u} \in \mathcal{U}$, there are smooth matrices $V(u,\zeta)$ on a neighborhood of $(\underline{u}, 0)$ such that

$$V^{-1} G V = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} \tag{4.16}$$

with $H(u,\zeta)$ of dimension $N \times N$, $P(u,\zeta)$ of dimension $N^2 \times N^2$, and

a) the eigenvalues of P satisfy $|\operatorname{Re}\mu| \geq c$ for some $c > 0$,

b) there holds

$$H(u,\zeta) = H_0(u,\zeta) + O(|\zeta|^2) \tag{4.17}$$

c) at $\zeta = 0$, V has a triangular form

$$V(u,0) = \begin{pmatrix} \mathrm{Id} & \overline{V} \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{4.18}$$

Proof. i) Take $u \in \mathcal{U}$. If $v^2 \in \ker P_0(u)$, then ${}^t\Big(-(A^{11}_d)^{-1} A^{12}_d v^2, v^2\Big) \in \ker A_d$, implying that 0 is not an eigenvalue of P_0. Similarly, if $i\xi$ is an eigenvalue of P then 0 is an eigenvalue of $i\xi \overline{A}_d + \xi^2 \overline{B}_d$, which is impossible by (H4) if $\xi \neq 0$ is real.

ii) Direct computations show that $G(u,\zeta) = G_d(u,\zeta)^{-1} M(u,\zeta)$ with

$$G_d(u,\zeta) = \begin{pmatrix} -\tilde{A}_d & \tilde{B}_d \\ J & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \tilde{M} & 0_{N\times N^2} \\ 0_{N^2 \times N} & \mathrm{Id}_{N^2\times N^2} \end{pmatrix}$$

with, in the splitting $u = (u^1, u^2)$,

$$\tilde{B}_d(u) = \begin{pmatrix} 0_{N-N^2 \times N^2} \\ \overline{B}^{22}_{d,d}(u) \end{pmatrix}, \quad J = \begin{pmatrix} 0_{N^2 \times N - N^2} & \mathrm{Id}_{N^2 \times N^2} \end{pmatrix}.$$

and

$$\begin{cases} \tilde{A}(u,\zeta) = A_d(u) - \sum_{j=1}^{d-1} i\eta_j (B_{j,d}(u) + B_{d,j}(u)) \\ \tilde{M}(u,\zeta) = (i\tau + \gamma) A_0(u) + \sum_{j=1}^{d-1} i\eta_j A_j(u) + \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(u)\,. \end{cases}$$

In particular, $i\xi$ is an eigenvalue of $G(u,\zeta)$ if and only if $-(\gamma + i\tau)$ is an eigenvalue of $i\overline{A}(\eta,\xi) + \overline{B}(\eta,\xi)$, which, by (H4), implies either that $\gamma < 0$ if ξ is real and $(\eta,\xi) \neq 0$ or that $\zeta = 0$.

Thus $G(u,\zeta)$ has no eigenvalues on the imaginary axis and the number $\tilde{N}$ of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is constant for $u \in \mathcal{U}$ and $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash\{0\}$. That this number is equal to $N_b = N^1_+ + N^2$ is a consequence of the high frequency analysis in Lemma 9.3 below (see also Lemma 1.7 in [Zum3]).

iii) Because $\tilde{M}(u,0) = 0$ and $\tilde{A}(u,0) = A_d(u)$, there holds

$$G(u,0) = \begin{pmatrix} 0_{N\times N} & \begin{pmatrix} -(A_d^{11})^{-1}A_d^{12} \\ \mathrm{Id}_{N^2\times N^2} \end{pmatrix} \\ 0_{N^2\times N} & P_0(u) \end{pmatrix} \tag{4.19}$$

Since P_0 is invertible, G can be smoothly conjugated to a block diagonal matrix as in (4.16), with V satisfying (4.18) and $H(u,0) = 0$. More precisely, the matrix $\overline{V}$ is

$$\overline{V} = \begin{pmatrix} -(A_d^{11})^{-1}A_d^{12}P_0^{-}1 \\ P_0^{-1} \end{pmatrix} \tag{4.20}$$

The expansion (4.17) can be easily obtained by standard perturbation expansions.

For ζ small, the number of eigenvalues of P in $\{\mathrm{Re}\,\mu < 0\}$ is equal to N_-^2, and for $\gamma > 0$, the number of eigenvalues of $H_0(u,\zeta)$ in the negative half space is constant, by hyperbolicity, and equal to N_+. This implies that $\tilde{N} = N_+ + N_-^2$. $\square$

As mentioned in Remark 4.2, the linearized equations from (3.1) around w correspond exactly to the first order system (4.4) with $\zeta = 0$. Thus the homogeneous problem for (3.13) read

$$\begin{cases} L(z,0,\partial_z)\dot{w} = 0, & z \geq 0, \\ \Upsilon'(\dot{w},0,\partial_z\dot{w}^2)_{|z=0} = 0. \end{cases} \tag{4.21}$$

A corollary of Lemmas 4.1 and 4.3 is that the solutions of the homogeneous equation $L(z,0,\partial_z)\dot{w} = 0$ form a space of dimension $N + N^2$, parametrized by $(u_H, u_P) \in \mathbb{C}^N \times \mathbb{C}^{N^2}$:

$$\dot{w}(z) = \Phi_H(z,0)u_H + \Phi_P(z,0)e^{zP_0(\underline{u})}u_P \tag{4.22}$$

where the matrices $\Phi_H(z,0)$ and $\Phi_P(z,0)$ are smooth and bounded on $\mathbb{R}_+$ and $\Phi_H(z,0) \to \mathrm{Id}$ as $z \to \mathrm{Id}$. The solution is bounded if and only if u_P belongs to the negative space $\mathbb{E}^-(P_0(\underline{u}))$ of $P_0(\underline{u})$, that is the invariant space of $P_0(\underline{u})$ associated to the spectrum lying in $\{\mathrm{Re}\,\mu < 0\}$; thus the space $\mathcal{S}$ of bounded solutions has dimension $N + N_-^2$. The space of solutions that tend to zero at infinity, denoted by $\mathcal{S}_0$, has dimension N_-^2, corresponding to the conditions $u_H = 0$ and $u_P \in \mathbb{E}^-(P_0(\underline{u}))$.

The boundary conditions in (3.13) read

$$\underline{\Gamma}_H u_H + \underline{\Gamma}_P u_P := \underline{\Gamma}(\dot{w},\partial_z\dot{w}^2)_{|z=0} = 0. \tag{4.23}$$

Because of Proposition 3.3, the next definition extends to profiles the previous definition of *transversality* given in Definition 3.1 for layer profiles.

Definition 4.4. *The profile w is said to be transversal if*

i) there is no nontrivial solution $\dot{w} \in \mathcal{S}_0$ which satisfies the boundary conditions $\underline{\Gamma}(\dot{w},\partial_z\dot{w}^2)_{|z=0} = 0$,

ii) the mapping $\dot{w} \mapsto \underline{\Gamma}(\dot{w},\partial_z\dot{w}^2)_{|z=0}$ from $\mathcal{S}$ to $\mathbb{C}^{N_b}$ has rank N_b.

Equivalently, *i*) means that $\ker \underline{\Gamma}_P \cap \mathbb{E}^-(P_0(\underline{u})) = \{0\}$ and *ii*) that the rank of the matrix $(\underline{\Gamma}_H, \underline{\Gamma}_P)$ from $\mathbb{C}^N \times \mathbb{E}^-(P_0(\underline{u}))$ to $\mathbb{C}^{N_b}$ is N_b.

If the profile satisfies condition i), there is a decomposition

$$\mathbb{C}^{N_b} = \underline{\mathbb{F}}_H \oplus \underline{\mathbb{F}}_P, \quad \underline{\mathbb{F}}_P := \underline{\Gamma}_P \mathbb{E}^-(P_0(\underline{u})) \tag{4.24}$$

with $\dim \underline{\mathbb{F}}_H = N_+$ and $\dim \underline{\mathbb{F}}_P = N_-^2$. Denote by $\underline{\pi}_H$ and $\underline{\pi}_P$ the projections associated to this splitting.

For $\dot{w} \in \mathcal{S}$ given by (4.22), one can eliminate u_P from the boundary conditions (4.23) and write them

$$\underline{\Gamma}_{red} u_H = 0, \quad u_P = \underline{R}_P u_H, \tag{4.25}$$

with

$$\underline{\Gamma}_{red} := \pi_H \underline{\Gamma}_H, \quad \underline{R}_P := -(\underline{\Gamma}_P)^{-1} \pi_P \underline{\Gamma}_H \tag{4.26}$$

and $(\underline{\Gamma}_P)^{-1}$ is the inverse of the mapping $\underline{\Gamma}_P$ from $\mathbb{E}^-(P_0(\underline{u}))$ to $\underline{\mathbb{F}}_P$.

With these notations, ii) means that $\underline{\Gamma}_{red}$ has rank N_+. Its kernel $\ker \underline{\Gamma}_{red}$ is the space of $\dot{u} \in \mathbb{R}^d$ such that there is a solution of $\dot{w}$ of (3.13) with end point $\dot{u}$. It has dimension $N - N_+$.

Remark 4.5. When w is a layer profile, solution of (3.3), the transversality condition implies that near the end point $\underline{u}$, the set $\mathcal{C}$ in (3.10) which describes the limiting hyperbolic conditions is a smooth manifold of dimension $N_- = N - N_+$ and $\ker \underline{\Gamma}_{red}$ is the tangent space to $\mathcal{C}$ at $\underline{u}$. Therefore, the natural boundary conditions for the linearized hyperbolic equation, and in particular for (4.9), are

$$\underline{\Gamma}_{red} u = h. \tag{4.27}$$

4.3. Evans functions and Lopatinski determinant; weak stability

For a given $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash \{0\}$, we now investigate the well-posedness of equation (4.4) or equivalently (4.6) or (4.14). Introduce the space $\mathbb{E}^-(\zeta)$ of initial conditions $(u(0), \partial_z u^2(0))$ (or equivalently $U(0)$) such that the corresponding solution of $L(z, \zeta, \partial_z)u = 0$ (or $\partial_z U - \mathcal{G}(z,\zeta)U = 0$) is exponentially decaying at $+\infty$. Lemmas 4.1 and 4.3 show that

$$\mathbb{E}^-(\zeta) = \Phi(0,\zeta)\mathbb{E}^-(G(\underline{u},\zeta)) \tag{4.28}$$

where we use the following notations:

Notations 4.6. Given a square matrix M, $\mathbb{E}^-(M)$ [resp. $\mathbb{E}^+(M)$ denotes the invariant space of M associated to the spectrum of M contained in $\{\operatorname{Re}\mu < 0\}$ [resp. $\{\operatorname{Re}\mu > 0\}$].

In particular, by Lemma 4.3, $\mathbb{E}^-(\zeta)$ is a smooth vector bundle for $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash \{0\}$ and $\dim(\mathbb{E}^-(\zeta)) = N_b$.

The problems (4.4), (4.6) or (4.14) are well posed if and only if

$$\mathbb{E}^-(\zeta) \cap \ker \Gamma(\zeta) = \{0\} \quad \text{or} \quad \mathbb{E}^-(G(\underline{u},\zeta)) \cap \ker \tilde{\Gamma}(\zeta) = \{0\}. \tag{4.29}$$

Note that, because the rank of Γ is at most N_b and the dimension of $\mathbb{E}^-$ is N_b, this condition implies and is equivalent to

$$\mathbb{C}^{N+N^2} = \mathbb{E}^-(\zeta) \oplus \ker \Gamma(\zeta) \quad \text{or} \quad \mathbb{C}^{N+N^2} = \mathbb{E}^-(G(\underline{u},\zeta)) \oplus \ker \tilde{\Gamma}(\zeta). \tag{4.30}$$

This condition can be expressed using the notion of *Evans function*, which measures the "angle" between the two spaces (see e.g. [Zum2, Zum3] and the references therein). It is defined as

$$D(\zeta) = \Big| \det{}_{N+N^2}\Big(\mathbb{E}^-(\zeta), \ker\Gamma(\zeta)\Big)\Big| \tag{4.31}$$

where, for subspaces $\mathbb{E}$ and $\mathbb{F}$ of $\mathbb{C}^n$, $\det_n(\mathbb{E},\mathbb{F})$ is equal to 0 if $\dim\mathbb{E}+\dim\mathbb{F}\neq n$ and is the $n\times n$ determinant formed by orthonormal bases in $\mathbb{E}$ and $\mathbb{F}$ if $\dim\mathbb{E}+\dim\mathbb{F} = n$.

Remark 4.7. The definition of the determinant above depends on choices of bases. Note that changing orthonormal bases in $\mathbb{E}$ and $\mathbb{F}$ changes the determinant by a complex number of modulus one, thus leaves $|\det(\mathbb{E},\mathbb{F})|$ invariant. But it also depends on the choice of a scalar product on $\mathbb{C}^n$. Changing the scalar products (or arbitrary changes of bases in $\mathbb{C}^n$)) changes the function $\det(\mathbb{E},\mathbb{F})$ to a new function $\widetilde{\det}(\mathbb{E},\mathbb{F})$ such that $c|\det(\mathbb{E},\mathbb{F})| \leq |\widetilde{\det}(\mathbb{E},\mathbb{F})| \leq c^{-1}|\det(\mathbb{E},\mathbb{F})|$ where $c>0$ is independent of the spaces $\mathbb{E}$ and $\mathbb{F}$. We will denote by

$$\det\approx\widetilde{\det} \quad \text{or} \quad D\approx\widetilde{D} \tag{4.32}$$

this property. In particular all the stability conditions stated below are independent of orthonormal bases in $\mathbb{E}^-$ and $\ker\Gamma$ and independent of the choice of the scalar product.

Remark 4.8. If the coefficients of the operator and the profile depend smoothly on parameters p, then the Evans function is also a smooth function of the parameters.

These notations being settled, the *weak stability* condition is a necessary condition for the well posedness of (4.2) in Sobolev spaces. It reads:

Definition 4.9. *Given a profile w, the linearized equation* (4.4) *satisfies the weak spectral stability condition if $D(\zeta)\neq 0$ for all $\zeta\in\overline{\mathbb{R}}^{d+1}_+\backslash\{0\}$.*

The next lemma is useful and elementary.

Lemma 4.10. *Suppose that $\mathbb{E}\subset\mathbb{C}^n$ and $\Gamma:\mathbb{C}^n\mapsto\mathbb{C}^m$, with* $\operatorname{rank}\Gamma=\dim\mathbb{E}=m$. *If $|\det(\mathbb{E},\ker\Gamma)|\geq c>0$, then there is C, which depends only on c and $|\Gamma^*(\Gamma\Gamma^*)^{-1}|$ such that*

$$\forall U\in\mathbb{E},\ |U|\leq C|\Gamma U|.$$

Conversely, if this estimate is satisfied then $|\det(\mathbb{E},\ker\Gamma)|\geq c$ where $c>0$ depends only on C and $|\Gamma|$.

Proof. Let $\pi=\Gamma^*(\Gamma\Gamma^*)^{-1}\Gamma$ denote the orthogonal projector on $(\ker\Gamma)^\perp$. Diagonalizing the hermitian form $(\pi e,\pi e)$, yields orthonormal bases $\{e_j\}$ and $\{f_j\}$ in $\mathbb{E}$ and $(\ker\Gamma)^\perp$ respectively, such that $\pi e_j=\lambda_j f_j$ with $0<\lambda_j\leq 1$. Take any basis $\{g_k\}$ of $\ker\Gamma$. Expressing the e_j in the base $\{f_k,g_l\}$, implies that $|\det(\mathbb{E},\ker\Gamma)|=\prod\lambda_j$. Since $\lambda_j\leq 1$ for all j, if this determinant is larger than or equal to $c>0$, then $\min\lambda_j\geq c$ and for all $e\in\mathbb{E}$

$$c|e|\leq|\pi e|\leq|\Gamma^*(\Gamma\Gamma^*)^{-1}|\,|\Gamma e|.$$

Conversely, if the estimate is satisfied, then $|e|\leq C|\Gamma|\,|\pi e|$ since $\Gamma e=\Gamma\pi e$ for all $e\in\mathbb{E}$. Therefore $\lambda_jC|\Gamma|\geq 1$ and the determinant is at least equal to $(C|\Gamma|)^{-m}$. $\square$

Proposition 4.11. *For fixed* $\zeta \in \overline{\mathbb{R}}_+^{d+1} \backslash \{0\}$*, the following properties are equivalent:*

i) the weak stability condition $D(\zeta) \neq 0$ *is satisfied,*

ii) there is a constant C *such that for all* $f \in L^2(\mathbb{R}_+)$ *and* $g \in \mathbb{C}^{N_b}$*, the problem* (4.4) *has a unique solution* $u \in H^1(\mathbb{R}_+)$ *and*

$$\|u\|_{L^2} + \|\partial_z u^2\|_{L^2} + |u(0)| + |\partial_z u^2(0)| \leq C(\|f\|_{L^2} + |g|). \tag{4.33}$$

iii) there is a constant C *such that for all* $F \in L^2(\mathbb{R}_+)$ *and* $g \in \mathbb{C}^{N_b}$*, the problem* (4.6) *has a unique solution* $U \in L^2(\mathbb{R}_+)$ *and*

$$\|U\|_{L^2} + |U(0)| \leq C(\|F\|_{L^2} + |g|). \tag{4.34}$$

Proof. We show that *ii)* $\Rightarrow$ *i)* $\Rightarrow$ *iii)*.

a) Uniqueness in *ii)* implies that $\mathbb{E}^-(\zeta) \cap \ker \Gamma = \{0\}$, thus *i)*.

b) By Lemma 4.1, the linearized equation (4.6) is conjugated to the constant coefficient system (4.14). By Lemma 4.3 the kernel of $\partial_z - G(\underline{u}, \zeta)$ in L^2 has dimension equal to N_b, which is the number of boundary condition. Thus the operator $(\partial_z - G(\underline{u}, \zeta), \Gamma)$ has index 0 from H^1 to $L^2 \times \mathbb{C}^{N_b}$. Therefore, condition *i)* which means that it is injective, implies that it is surjective and *iii)* follows.

c) By reduction to first order (4.4) is equivalent to (4.6) for particular F. Thus *iii)* immediately implies *ii)*. □

There are analogous definitions for the linearized hyperbolic problem (4.9) with boundary conditions (4.27). For $\gamma > 0$, $H_0(\underline{u}, \zeta)$ has no eigenvalues on the imaginary axis, as a consequence of the hyperbolicity assumption (see Remark 2.4). The *Lopatinski determinant* is defined for $\zeta \in \mathbb{R}_+^{d+1} := \{\gamma > 0\}$ by

$$D_{Lop}(\zeta) = \Big| \det \Big(\mathbb{E}^-(H_0(\underline{u}, \zeta)), \ker \underline{\Gamma}_{red} \Big) \Big|. \tag{4.35}$$

By homogeneity of H_0, this determinant is homogeneous of degree zero in ζ and it is sufficient to consider the case where $\zeta \in S^d = \{|\zeta| = 1\}$.

Definition 4.12. *The linearized equation* (4.9) (4.27) *satisfies the weak spectral stability condition if* $D_{Lop}(\zeta) \neq 0$ *for all* $\zeta \in \mathbb{R}_+^{d+1}$*.*

Moreover, there is an analogue of Proposition 4.11, for $\gamma > 0$.

4.4. Maximal estimates and uniform spectral stability conditions

The next step in the study of the linearized equation is to perform an inverse Fourier-Laplace transform. This requires suitable estimates for the solutions of (4.4), with a precise description of the constants in the estimate (4.33) above.

By continuity in ζ, the weak stability condition implies that the estimate (4.33) is satisfied with a uniform constant C when ζ remains in a compact subset of $\overline{\mathbb{R}}_+^{d+1} \backslash \{0\}$. Thus the true question is to get a detailed behavior of the estimate when $\zeta \to 0$ and when $|\zeta| \to \infty$.

4.4.1. Low and medium frequencies

Consider first the *low frequency* case. Following [MéZu1], the expected *maximal estimates* for low and medium frequencies for the solutions of (4.4) read

$$\varphi\|u\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)} + |u(0)| + |\partial_z u^2(0)| \leq C\Big(\frac{1}{\varphi}\|f\|_{L^2(\mathbb{R}_+)} + |g|\Big) \tag{4.36}$$

where $\varphi = (\gamma + |\zeta|^2)^{\frac{1}{2}}$ with C independent of $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \backslash \{0\}$, $|\zeta| \leq \rho_0$. Note that for fixed $|\zeta| > 0$, this estimate is equivalent to (4.33).

Similarly, the maximal estimates for solutions of the first order system (4.6) read :

$$\varphi\|u\|_{L^2(\mathbb{R}_+)} + \|u^3\|_{L^2(\mathbb{R}_+)} + |U(0)| \leq C\Big(\frac{1}{\varphi}\|F\|_{L^2(\mathbb{R}_+)} + |g|\Big) \tag{4.37}$$

where $U = (u, u^3) \in \mathbb{C}^N \times \mathbb{C}^{N^2}$. For the constant coefficient system (4.14) the analogous expected estimates read :

$$\varphi\|\tilde{u}\|_{L^2(\mathbb{R}_+)} + \|\tilde{u}^3\|_{L^2(\mathbb{R}_+)} + |\widetilde{U}(0)| \leq C\Big(\frac{1}{\varphi}\|\widetilde{F}\|_{L^2(\mathbb{R}_+)} + |g|\Big). \tag{4.38}$$

Lemma 4.13. *The estimates* (4.38) *imply* (4.37) *which imply* (4.36).

Proof. (See [MéZu1]). Clearly, (4.36) is a particular case of (4.37) applied to source terms F of the special form (4.8). Moreover, using the conjugation Lemma 4.1, there holds $U = O(1)\widetilde{U}$ and $\widetilde{U} = O(1)U$ and similar estimates for F and $\widetilde{F}$. Moreover,

$$U^1 = O(1)\widetilde{U}, \quad U^2 = O(e^{-\theta z})\widetilde{U}^1 + O(1)\widetilde{U}^2$$

with $\theta > 0$. We use the inequality

$$\|e^{-\theta z}\widetilde{U}^1\|_{L^2} \lesssim |\widetilde{U}^1(0)| + \|\partial_z \widetilde{U}^1\|_{L^2}.$$

Moreover, the form of $G(\underline{u}, \zeta)$ at $\zeta = 0$ shows that

$$\partial_z \widetilde{U}^1 = O(|\zeta|)\widetilde{U}^1 + O(1)\widetilde{U}^2 + \widetilde{F}^1.$$

Therefore,

$$\|U^2\|_{L^2} \lesssim \|\widetilde{U}^2\|_{L^2} + |\widetilde{U}^1(0)| + |\zeta|\|\widetilde{U}^1\|_{L^2} + \|\widetilde{F}^1\|_{L^2}.$$

Since $|\zeta| \leq \varphi$, this shows that (4.38) implies (4.37). □

Taking $f = 0$, we point out the following necessary condition for the validity of the maximal estimates:

Proposition 4.14. *A necessary condition for* (4.36) *to be valid for* $0 < |\zeta| \leq \rho_0$, *is that there are* C *and* $\rho_0 > 0$ *such that*

$$\forall \zeta \in \overline{\mathbb{R}}^{d+1}_+, \quad 0 < |\zeta| \leq \rho_0, \quad \forall U \in \mathbb{E}^-(\zeta) : \quad |U| \leq C|\Gamma(\zeta)U|. \tag{4.39}$$

By Assumption 2.11, the rank of $\Gamma(\zeta)$ is always N_b, and the norms of $\Gamma(\zeta)$ and $(\Gamma\Gamma^*)^{-1}$ are uniformly bounded for ζ bounded. Thus, by Lemma 4.10, the condition (4.39) is equivalent to requiring that D is bounded from below by a positive constant for $0 < |\zeta| \leq \rho_0$. This leads to the following definition (see [MéZu1])

Definition 4.15. *Given a profile* w, *the uniform spectral stability condition for the linearized equation* (4.2) *is satisfied for low frequencies when there are* $c > 0$ *and* $\rho_0 > 0$ *such that* $D(\zeta) \geq c$ *for all* $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ *with* $0 < |\zeta| \leq \rho_0$.

We have seen that the low frequency uniform stability condition is necessary for the validity of the maximal estimates. *A major issue, which is discussed in Sections 6 to 8 is to prove a converse statement.*

4.4.2. High frequencies

For the *high frequency* analysis, we use the special form (2.11) of the boundary conditions. Their linearized version, $\Upsilon'(u, i\eta u^2, \partial_z u^2) = g$ reads

$$\begin{cases} \Gamma_1 u^1(0) := \Upsilon_1'(w^1(0)) \cdot u^1(0) = g^1, \\ \Gamma_2 u^2(0) := \Upsilon_2'(w^2(0)) \cdot u^2(0) = g^2, \\ \Gamma_3(\zeta)(u^2(0), \partial_z u^2(0)) := K_d \partial_z u^2(0) + K_{\text{tg}}(\eta) u^2(0) = g^3, \end{cases} \tag{4.40}$$

with

$$K_{\text{tg}}(\eta) = \sum_{j=1}^{d-1} i\eta_j K_j(w(0)). \tag{4.41}$$

The maximal extimates that are proven in [GMWZ4, GMWZ6] read

$$\begin{aligned} (1+\gamma)\|u^1\|_{L^2(\mathbb{R}_+)} &+ \Lambda\|u^2\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)} \\ +(1+\gamma)^{\frac12}|u^1(0)| &+ \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|\partial_z u^2(0)| \le \\ & C\Big(\|f^1\|_{L^2(\mathbb{R}_+)} + \Lambda^{-1}\|f^2\|_{L^2(\mathbb{R}_+)}\Big) \\ & + C\Big((1+\gamma)^{\frac12}|g^1| + \Lambda^{\frac12}|g^2| + \Lambda^{-\frac12}|g^3|\Big) \end{aligned} \tag{4.42}$$

with C independent of $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ large. Here, Λ is the natural parabolic weight

$$\Lambda(\zeta) = \Big(1 + \tau^2 + \gamma^2 + |\eta|^4\Big)^{1/4}. \tag{4.43}$$

Remark 4.16. The balance between the weights for u^1 and for u^2 is subtle: these components are decoupled in the high frequency principal system (2.7) and the choice of the weights depends on the actual coupling of u^1 and for u^2 through the nondiagonal lower order terms and the boundary conditions.

Taking $f = 0$, (4.42) implies the following necessary condition : there are C and $\rho_1 > 0$ such that

$$\begin{aligned} &\forall \zeta \in \overline{\mathbb{R}}^{d+1}_+, \quad |\zeta| \ge \rho_1, \quad \forall U = (u^1, u^2, u^3) \in \mathbb{E}^-(\zeta) : \\ &(1+\gamma)^{\frac12}|u^1| + \Lambda^{\frac12}|u^2| + \Lambda^{-\frac12}|u^3| \le \\ &\quad C\Big((1+\gamma)^{\frac12}|\Gamma_1 u^1| + \Lambda^{\frac12}|\Gamma_2 u^2| + \Lambda^{-\frac12}|\Gamma_3(\zeta)(u^2, u^3)|\Big). \end{aligned} \tag{4.44}$$

This can be reformulated in terms of a *rescaled Evans function* (see [MéZu1] : In $\mathbb{C}^{N+N^2}$ and $\mathbb{C}^{N_b}$ introduce the mappings

$$\begin{aligned} J_\zeta(u^1, u^2, u^3) &:= \Big((1+\gamma)^{\frac12} u^1, \Lambda^{\frac12} u^2, \Lambda^{-\frac12} u^3\Big) \\ J_\zeta(g^1, g^2, g^3) &:= \Big((1+\gamma)^{\frac12} g^1, \Lambda^{\frac12} g^2, \Lambda^{-\frac12} g^3\Big). \end{aligned} \tag{4.45}$$

Note that $J_\zeta \Gamma(\zeta) U = \Gamma^{sc}(\zeta) J_\zeta U$ with

$$\Gamma^{sc} U = \Big(\Gamma_1 u^1, \Gamma_2 u^2, K_d u^3 + \Lambda^{-1} K_{\text{tg}}(\eta) u^2\Big). \tag{4.46}$$

Thus (4.44) reads

$$\forall U \in J_\zeta \mathbb{E}^-(\zeta): \quad |U| \le C|J_\zeta \Gamma(\zeta) J_\zeta^{-1} U| \tag{4.47}$$

Introducing the *rescaled Evans function*

$$D^{sc}(\zeta) = \Big| \det \Big(J_\zeta \mathbb{E}^-(\zeta), J_\zeta \ker \Gamma(\zeta) \Big) \Big|. \tag{4.48}$$

we see that this stability condition is equivalent to the following definition:

Definition 4.17. *Given a profile w, the linearized equation* (4.2) *satisfies the uniform spectral stability condition for high frequencies when there are $c > 0$ and $\rho_1 > 0$ such that $D^{sc}(\zeta) \ge c$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $|\zeta| \ge \rho_1$.*

Note that for ζ in bounded sets, J_ζ and J_ζ^{-1} are uniformly bounded and $D(\zeta) \approx D^{sc}(\zeta)$, thus the condition $D^{sc}(\zeta) \ne 0$ is nothing but a reformulation of the weak stability condition.

By Lemma 4.10, the high frequency uniform stability is equivalent to (4.44). In section 9, we will recall from [GMWZ4] that *the uniform spectral stability implies the high frequency maximal estimates* (4.42)*, under structural assumptions on the system that are satisfied in many examples*, including Navier-Stokes and MHD.

4.4.3. The inviscid case

There are analogous definitions for the linearized hyperbolic problem (4.9) with boundary conditions (4.27). Recall that the Lopatinski determinant is defined at (4.35). Definition 4.12 of weak stability is strengthened as follows.

Definition 4.18. *The linearized equation* (4.9) (4.27) *satisfies the uniform spectral stability condition when there are $c > 0$ such that $D_{Lop}(\zeta) \ge c$ for all $\zeta \in S^d_+ := S^d \cap \{\gamma > 0\}$.*

This uniform stability condition is equivalent to a uniform estimate for all $\zeta \in S^d_+$:

$$\forall u \in \mathbb{E}^-(H_0(\underline{u}, \zeta)): \quad \Big|u\Big| \le C\Big|\underline{\Gamma}_{red} u\Big| \tag{4.49}$$

The expected maximal estimates for solutions of (4.9) (4.27) are

$$\gamma^{\frac12} \|u\|_{L^2} + |u(0)| \le C\Big(\gamma^{-\frac12} \|f\|_{L^2} + |h|\Big) \tag{4.50}$$

with C independent of $\zeta \in \mathbb{R}^{d+1}_+$.

Remark 4.19. The uniform stability condition is satisfied for small amplitude layers (see [GrGu] for artificial viscosity and [Rou1] for real viscosity in 1D). In [GMWZ8] layers for Navier-Stokes equations and in or out-flow boundary conditions are studied. The analogous uniform stability condition for weak Lax shocks has been recently proved in [PlZu]. In the inviscid case, we also refer to [Maj1] for the verification of the uniform Lopatinski condition for Euler's equation and to [Mét1] for weak Lax shocks.

5. The Zumbrun-Serre-Rousset Theorem and the reduced low frequency problem

A famous result of [ZuSe] and [Rou1] links the low frequency uniform stability of the viscous regularizations and the uniform stability of the limiting inviscid problem. We give here the extension proved in [GMWZ6].

5.1. Transversality is necessary

Proposition 5.1. *Given a profile w, if the low frequency uniform spectral stability condition is satisfied, then w is transversal.*

Proof. Lemma 4.3 implies that for $\zeta \neq 0$ small enough, $\widetilde{U}$ is a solution of (4.14) if and only if ${}^t(u_H, u_P) = V^{-1}(\zeta)\widetilde{U}$ satisfies

$$\begin{aligned} \partial_z u_H &= H(\underline{u}, \zeta)u_H + f_H, && (5.1)\\ \partial_z u_P &= P(\underline{u}, \zeta)u_P + f_P, && (5.2)\\ \Gamma_H(\zeta)u_H(0) + \Gamma_P(\zeta)u_P(0) &:= \widetilde{\Gamma}(\zeta)\widetilde{U}(0) = g, && (5.3)\end{aligned}$$

where ${}^t(f_H, f_P) = V^{-1}(\zeta)\widetilde{F}$ and Γ_H [resp. Γ_P] denotes the restriction of $\widetilde{\Gamma}V$ to $\mathbb{C}^N \times \{0\}$ [resp. $\{0\} \times \mathbb{C}^{N^2}$]. In particular,

$$\mathbb{E}^-(G(\underline{u}, \zeta)) = V(\zeta)\Big(\mathbb{E}^-(H(\underline{u}, \zeta)) \oplus \mathbb{E}^-(P(\underline{u}, \zeta))\Big).$$

With (4.39), this shows that the low frequency uniform stability condition holds if and only if there are C and $\rho_0 > 0$ such that for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $0 < |\zeta| \leq \rho_0$

$$\begin{aligned} &\forall u_H \in \mathbb{E}^-(H(\underline{u}, \zeta)), \quad \forall u_P \in \mathbb{E}^-(P(\underline{u}, \zeta)) :\\ &\qquad \Big|u_H\Big| + \Big|u_P\Big| \leq C\Big|\Gamma_H(\zeta)u_H + \Gamma_P(\zeta)u_P\Big|. \end{aligned} \tag{5.4}$$

In particular,

$$\forall u_P \in \mathbb{E}^-(P(\underline{u}, \zeta)) : \quad \Big|u_P\Big| \leq C\Big|\Gamma_P(\zeta)u_P\Big|. \tag{5.5}$$

By Lemma 4.3, $\mathbb{E}^-(P(\underline{u}, \zeta))$ is a smooth bundle for ζ in a neighborhood of 0. Moreover, $\widetilde{\Gamma}(\zeta)$ and $\Gamma_P(\zeta)$ are smooth around the origin. This implies that $\Big|u_P\Big| \leq C\Big|\Gamma_P(0)u_P\Big|$ on $\mathbb{E}^-(P(\underline{u}, 0))$, implying that condition i) of Definition 4.4 is satisfied.

Since $\dim(\mathbb{E}^-(G(\zeta))) = \operatorname{rank} \widetilde{\Gamma}(\zeta) = N_b$, (5.4) implies that for all $h \in \mathbb{C}^{N_b}$ and all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $0 < |\zeta| \leq \rho_0$, there is $\tilde{U}(\zeta) = V(\zeta)\Big(u_H(\zeta), u_P(\zeta)\Big)$ in $\mathbb{E}^-(\zeta) \subset V(\zeta)\Big(\mathbb{C}^N \oplus \mathbb{E}^-(P(\zeta))\Big)$ such that $\widetilde{\Gamma}(\zeta)\tilde{U}(\zeta) = h$ and $|\tilde{U}(\zeta))| \leq c|h|$. By compactness and continuity, letting ζ tend to zero, implies that there is $\tilde{U} = V(0)\Big(u_H, u_P\Big)$ in $V(0)\Big(\mathbb{C}^N \oplus \mathbb{E}^-(P(0))\Big)$ such that $\widetilde{\Gamma}(0)\tilde{U} = h$, showing that condition ii) of Definition 4.4 is also satisfied. ☐

5.2. The reduced problem

Suppose that the profile w is transversal. Then, by i) of Definition 4.4 and Remark 4.2, $\Gamma_P(\zeta)$ is an isomorphism from $\mathbb{E}^-(P(\underline{u}, \zeta))$ to its image $\mathbb{F}_{0,P}$ when $\zeta = 0$;

by continuity this extends to a neighborhood of the origin and the decomposition (4.24) valid at $\zeta = 0$, extends smoothly on a neighborhood of the origin:

$$\mathbb{C}^{N_b} = \mathbb{F}_H \oplus \mathbb{F}_P(\zeta), \quad \mathbb{F}_P(\zeta) := \Gamma_P(\zeta)\mathbb{E}^-(P(\underline{u}, \zeta)). \tag{5.6}$$

Denote by $\pi_H(\zeta)$ and $\pi_P(\zeta)$ the projections associated to this splitting and define the *reduced boundary operator* as

$$\Gamma_{red}(\zeta) := \pi_H(\zeta)\Gamma_H(\zeta), \tag{5.7}$$

as well as the *reduced boundary value problem*

$$\partial_z u_H - H(\underline{u}, \zeta)u_H = f_H, \quad \Gamma_{red}(\zeta)u_H(0) = h. \tag{5.8}$$

The *reduced Evans function* is

$$D_{red}(\zeta) = \Big| \det \Big(\mathbb{E}^-(H(\underline{u}, \zeta)), \ker \Gamma_{red}(\zeta)\Big)\Big|. \tag{5.9}$$

Definition 5.2. *The reduced uniform stability condition is satisfied if $D_{red}(\zeta) \geq c > 0$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}\backslash\{0\}$ with $|\zeta|$ small enough.*

This is equivalent to the condition

$$\forall u \in \mathbb{E}^-(H(\underline{u}, \zeta)) : \quad |u| \leq C|\Gamma_{red}(\zeta)u|, \tag{5.10}$$

for $\zeta \in \overline{\mathbb{R}}^{d+1}\backslash\{0\}$ small.

Theorem 5.3. *Given a profile w, the linearized equation* (4.4) *satisfies the low frequency uniform spectral stability condition if and only if*

i) w is transversal,

ii) the reduced problem (5.8) *satisfies the reduced uniform stability condition.*

Proof. We have already shown that the low frequency uniform stability requires that w is transversal. Moreover, using the splitting (5.6), we see that the uniform stability conditions (4.39) or (5.4) are equivalent to

$$\big|u_H\big| + \big|u_P\big| \leq C\Big(\big|\Gamma_{red}u_H\big| + \big|\Gamma_P u_P + \pi_P\Gamma_H u_H\big|\Big) \tag{5.11}$$

for all $u_H \in \mathbb{E}^-(H)$ and $u_P \in \mathbb{E}^-(P)$ (to lighten notations we have omitted the ζ dependance). Since Γ_P is surjective from $\mathbb{E}^-(P)$ onto $\mathbb{F}_P$, for all $u_H \in \mathbb{E}^-(H)$ there is $u_P \in \mathbb{E}^-(P)$ such that $\Gamma_P u_P = -\pi_P\Gamma_H u_H$ and (5.11) implies (5.10).

Conversely, if the profile is transverse, the estimate (5.5) is valid at $\zeta = 0$ and extend by continuity to ζ in a neighborhood of 0. With (5.10), this clearly implies (5.11). □

5.3. The $\rho \to 0$ limit for Evans functions

It remains to link the reduced uniform stability condition to the uniform (Lopatinski) stability condition for the hyperbolic boundary value problem, that is for the problem (4.9) with boundary conditions (4.27). Note that these boundary conditions are given by $\underline{\Gamma}_{red} = \Gamma_{red}(0)$ (see Remark 4.5).

Because H vanishes at $\zeta = 0$, it is natural to use polar coordinates:

$$\zeta = \rho\check{\zeta}, \quad \rho = |\zeta|, \quad \check{\zeta} \in S^d. \tag{5.12}$$

In these coordinates

$$H(\underline{u}, \zeta) = \rho\check{H}(\underline{u}, \check{\zeta}, \rho), \quad \check{H}(\underline{u}, \check{\zeta}, \rho) = H_0(\underline{u}, \check{\zeta}) + O(\rho). \tag{5.13}$$

Changing z to $\check z = \rho z$, $u(z)$ to $\check u(\check z)$ and $f(z)$ to $\rho \check f(\check z)$ the reduced problem (5.8) is equivalent to

$$\partial_{\check z} \check u_H - H(\underline{u}, \check\zeta, \rho)\check u_H = \check f_H, \quad \Gamma_{red}(\zeta)\check u_H(0) = h, \tag{5.14}$$

which, for $\rho = 0$, is exactly the inviscid problem (4.11) (4.27). We are thus led to a *nonsingular* perturbation problem.

Clearly, for $\zeta \in \overline{S}^d_+ := S^d \cap \{\check\gamma \geq 0\}$, there holds $\mathbb{E}^-(H(\underline{u}, \zeta)) = \mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho))$ and $D_{red}(\zeta) = \check D(\check\zeta, \rho)$ with

$$\check D(\check\zeta, \rho) = \Big| \det \Big(\mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho)), \ker \Gamma_{red}(\rho\check\zeta) \Big) \Big| \tag{5.15}$$

Remark 5.4. For $\check\gamma > 0$, $H_0(\underline{u}, \check\zeta)$ has no eigenvalues on the imaginary axis, as a consequence of hyperbolicity (see Remark 2.4). By perturbation, this property holds true for $\check H(\underline{u}, \check\zeta, \rho)$ for ρ small enough (depending on $\check\gamma > 0$). This shows that the vector bundle $\mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho))$ which was defined on $\overline{S}^d_+ \times]0, \rho_0]$ has a smooth extension to $\in S^+ \times [0, \rho_0]$, as well as $\check D$. Comparing with the definition of the Lopatinski determinant (4.35), we see that

$$D_{Lop}(\check\zeta) = \check D(\check\zeta, 0), \quad \text{for } \check\gamma > 0. \tag{5.16}$$

The next theorem, combined with Theorem 5.3, extends Rousset's theorem [Rou1] (see also [ZuSe] for shocks).

Theorem 5.5. *Given a transverse profile w, if the reduced uniform spectral stability condition is satisfied, then the linearized hyperbolic problem* (4.9) (4.27) *satisfies the reduced uniform stability condition.*

Conversely, if the linearized hyperbolic problem is uniformly stable and the vector bundle $\mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho)$ has a continous extension to $\overline{S}^d_+ \times [0, \rho_0]$, then the reduced uniform spectral stability condition is satisfied and the linearized problem (4.2) *satisfies the uniform low frequency stability condition.*

Proof. The uniform estimate (5.10) implies that

$$|u| \leq C|\Gamma_{red}(\zeta)u|$$

for $u \in \mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho))$, $\check\zeta \in \overline{S}^d_+$ and $\rho > 0$ small. If $\check\gamma > 0$, every term is continuous up to $\rho = 0$ and the estimate above implies (4.49), that is

$$|u| \leq C|\Gamma_{red}(0)u|$$

for $u \in \mathbb{E}^-(H_0(\underline{u}, \check\zeta))$, $\check\zeta \in S^d_+$. This implies that the hyperbolic problem is uniformly stable.

If $\mathbb{E}^-(\check H(\underline{u}, \check\zeta, \rho))$ has a continous extension to $\overline{S}^d_+ \times [0, \rho_0]$, the reduced Evans function has a continuous extension to $\overline{S}^d_+ \times [0, \rho_0]$. The hyperbolic uniform stability and (5.16) imply that

$$\check D(\check\zeta, \rho) \geq c > 0$$

for $\check\zeta \in S^d_+$ and $\rho = 0$. By continuity, this extends first to $\check\zeta \in \overline{S}^d_+$ and next to $\rho \in [0, \rho_1]$ for some $\rho_1 > 0$. □

Remark 5.6. It is proved in [MéZu3] that when the eigenvalues of the hyperbolic symbol $\overline{A}(u, \xi)$ have constant multiplicity, and more generally when there is a smooth K family of symmetrizers (see the definition below), then the vector bundle

$\mathbb{E}^-(\check{H}(\underline{u},\check{\zeta},\rho))$ has a continuous extension to $\rho=0$. The main concern of this paper is to construct K-families for systems with variable multiplicity. This is possible under suitable assumptions, and therefore the two theorems above extend a result of F.Rousset [Rou1]. However, we will also show that the bundle $\mathbb{E}$ does not always admit a continuous extension, with the result that the hyperbolic problem can be uniformly stable while the viscous problem is strongly unstable in the low frequency regime. This seems to be a new phenomenon.

5.4. The $\rho \to 0$ limit for maximal estimates

Assuming transversality of w, Theorem 5.3 implies that the uniform spectral stability for low frequency is equivalent to the spectral stability for the reduced problem. There is an analogue for maximal estimates. The *maximal estimates for the reduced problem* (5.14) read

$$(\check{\gamma}+\rho)^{\frac{1}{2}}\|\check{u}_H\|_{L^2}+|\check{u}_H(0)| \le C\Big((\check{\gamma}+\rho)^{-\frac{1}{2}}\|\check{f}_H\|_{L^2}+|h|\Big) \tag{5.17}$$

with C independent of $\check{\zeta}\in\overline{S}^d_+$ and $\rho\in]0,\rho_0]$. Note that for $\rho=0$ and $\check{\gamma}>0$, this is the maximal estimate for the inviscid problem. Scaling back to the original variables, this estimate is equivalent to

$$(\gamma+|\zeta|^2)^{\frac{1}{2}}\|u_H\|_{L^2}+|u_H(0)| \le C\Big((\gamma+|\zeta|^2)^{-\frac{1}{2}}\|f_H\|_{L^2}+|h|\Big) \tag{5.18}$$

for the solutions of (5.8).

Theorem 5.7. *Suppose that the profile w is transversal. Then the maximal estimates* (4.38) *are valid for low frequencies if and only if the maximal estimates* (5.17) *for the reduced problem hold true.*

Proof. By Lemma 4.3 $P(\underline{u},\zeta)$ has no purely imaginary eigenvalues. Thus, using symmetrizers (see e.g. [MéZu1] and Section 6 below), there holds

$$\|u_P^+\|_{L^2}+|u_P^+(0)| \lesssim \|f_P^+\|_{L^2}, \tag{5.19}$$
$$\|u_P^-\|_{L^2} \lesssim \|f_P^-\|_{L^2}+|u_P^-(0)|, \tag{5.20}$$

where $\pm$ denotes the smooth projections on the spaces $\mathbb{E}^\pm(P(\underline{u},\zeta))$.

The splitting (5.6) implies that the boundary condition (5.3) reads

$$\pi_H g=\Gamma_{red}u_H(0)+\pi_H\Gamma_P u_P^+(0),$$
$$\pi_P g=\Gamma_P u_P^-(0)+\pi_P\Gamma_H u_H(0)+\pi_H\Gamma_P u_P^+(0).$$

Moreover Γ_P is invertible on $\mathbb{E}^-(P)$, hence $|\Gamma_P u_P^-(0)|\approx|u_P^-(0)|$ and

$$|\Gamma_{red}u_H(0)| \lesssim |\pi_H g|+|u_P^+(0)|,$$
$$|u_P^-(0)| \lesssim |\pi_P g|+|u_H(0)|+|u_P^+(0)|.$$

Suppose that the estimate (5.18) is satisfied. Then,

$$\varphi\|u_H\|_{L^2}+|u_H(0)| \lesssim \varphi^{-1}\|f_H\|_{L^2}+|\pi_H g|+|u_P^+(0)|.$$

With (5.19), this implies that

$$\begin{aligned}\varphi\|u_H\|_{L^2}+\|u_P^-\|_{L^2}+&|u_H(0)|+|u_P^-(0)|\\ &\lesssim \varphi^{-1}\|f_H\|_{L^2}+\|f_P^-\|_{L^2}+|g|+|u_P^+(0)|.\end{aligned}$$

Thus, with (5.19), we obtain that

$$\varphi\|u_H\|_{L^2} + \|u_P\|_{L^2} + |u_H(0)| + |u_P(0)| \lesssim \varphi^{-1}\|f_H\|_{L^2} + \|f_P\|_{L^2} + |g|.$$

Because $V(\underline{u}, 0)$ has the special form (4.18), $\widetilde{U} = V(u_H, u_P) = (\widetilde{U}^1, \widetilde{U}^2)$ satisfies

$$\widetilde{U}^1 = O(1)u_H + O(1)u_P, \quad \widetilde{U}^1 = O(|\zeta|)u_H + O(1)u_P$$

Therefore, the solutions of (4.14) satisfy

$$\varphi\|\widetilde{U}^1\|_{L^2} + \|\widetilde{U}^2\|_{L^2} + |\widetilde{U}(0)| \lesssim \varphi^{-1}\|\widetilde{F}\|_{L^2} + |g|.$$

that is the maximal estimate (4.38).

Conversely, assume that the maximal estimate (4.38) is satisfied. Suppose that u_H is a solution of (5.1). By transversality, Γ_P is surjective from $\mathbb{E}^-(P, \zeta)$ to its image $\mathbb{F}_P(\zeta)$ and there exists there is $u_P(0)$ in $\mathbb{E}^-(P, \zeta)$ such that

$$\Gamma_P u_P(0) = -\pi_P \Gamma_H u_H(0) \in \mathbb{F}_P(\zeta). \tag{5.21}$$

Consider $u_P = e^{zP}u_P(0)$ which is well defined and rapidly decaying at infinity since $u_P(0) \in \mathbb{E}^-(P, \zeta)$. It is a solution of (5.2) with $f_P = 0$. Then $\widetilde{U} = V(u_H, u_P)$ is a solution of (4.14) with $\widetilde{F} = V(f_H, 0)$. Thus $(u_H, u_P) = V^{-1}\widetilde{U}$ and there holds

$$\|u_H\|_{L^2} \lesssim \|\widetilde{U}\|_{L^2}, \quad |u_H(0)| \lesssim |\widetilde{U}(0)|, \quad \|\widetilde{F}\|_{L^2} \lesssim \|f_H\|_{L^2}$$

and, by (5.21), $\widetilde{\Gamma}\widetilde{U}(0) = \Gamma_H u_H(0) + \Gamma_P u_P(0) = \Gamma_{red} u_H(0)$. Thus the estimate (4.38) immediately implies (5.18). □

5.5. Viscous unstabilities

The analysis above indicates that when the negative space $\mathbb{E}^-$ is not continuous in $(\check{\zeta}, \rho)$, then the Evans function is likely not continuous and one can expect that the low frequency uniform stability condition for the viscous problem is strictly stronger than the similar condition for the inviscid problem. In particular, the inviscid problem can be strongly stable while the viscous one is strongly unstable. We illustrate here this phenomenon on an explicit example.

1. *An example.* Consider the system

$$\begin{cases} (\partial_t + \partial_y)u_1 + \partial_x u_2 = \varepsilon\mu\Delta u_1, \\ (\partial_t + \partial_y)u_2 + \partial_x u_1 = \varepsilon\nu\Delta u_2. \end{cases} \tag{5.22}$$

Taking linear combinations and changing ε, the system is equivalent to

$$(\partial_t + \partial_y)\mathrm{Id} + A\partial_x - \varepsilon B\Delta, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \tag{5.23}$$

with $a = |\nu - \mu|/(\nu + \mu) \in [0, 1[$. This system is symmetric and satisfy the assumptions (H1) and (H2).

The hyperbolic part is diagonal: the eigenvalues are

$$\lambda_1 = \eta + \xi, \quad \lambda_2 = \eta - \xi. \tag{5.24}$$

They cross on the line $\xi = 0$ and are trivially geometrically regular (see Definition 8.1 below) since the system is already in diagonal form. One of the eigenvalue is incoming, one is outgoing. The decoupling condition (8.9) is satisfied if and only if $a = 0$. In the sequel, we assume that $a > 0$.

2. *Boundary conditions.* Next, consider boundary conditions for (5.23):

$$u_{|x=0} + \varepsilon\Gamma\partial_x u_{|x=0} = 0. \tag{5.25}$$

We first compute the limiting inviscid boundary conditions, using boundary layers. The bounded solutions $u = w(x/\varepsilon)$ of (5.23) are

$$w(z) = u + e^{zB^{-1}A}h, \quad h \in \mathbb{E}^-_{B^{-1}A}, \; u \in \mathbb{C}^2. \tag{5.26}$$

where $\mathbb{E}^-_{B^{-1}A}$ is the negative space of $B^{-1}A$. Therefore, u is the endpoint of a profile which satisfies the boundary condition (5.25), if and only if

$$u \in \Big(\mathrm{Id} + \Gamma B^{-1}A\Big)\mathbb{E}^-_{B^{-1}A}. \tag{5.27}$$

Note that given any complex number $\underline{c}$, one can choose Γ such that this boundary condition reads

$$u_1 = \underline{c}u_2 \tag{5.28}$$

3. *Low frequency stability.* The first order system (4.6) reads

$$\partial_z U - G(\zeta)U, \quad G(\zeta) = \begin{pmatrix} 0 & \mathrm{Id} \\ \sigma B^{-1} + \eta^2\mathrm{Id} & B^{-1}A \end{pmatrix}, \tag{5.29}$$

with $\zeta = (\tau, \eta, \gamma)$ and $\sigma = \gamma + i(\tau + \eta)$. Perform the small frequency reduction (4.16), using the change of unknows

$$\begin{pmatrix} u \\ \partial_z u \end{pmatrix} = V(\zeta)\begin{pmatrix} u_H \\ u_P \end{pmatrix}.$$

Then, by Lemma 8.24, there holds

$$V^{-1}GV = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix}$$

with $P(0) = B^{-1}A$ and

$$H(\zeta) = -\sigma A + (\sigma^2 - \eta^2)AB + O(|\zeta|^3), \tag{5.30}$$

Since $V(0)$ has the triangular form (4.18), we see that the boundary condition reads

$$u_H + \tilde\Gamma(\zeta)u_P = 0, \quad \tilde\Gamma(0) = \Gamma + A^{-1}B. \tag{5.31}$$

The Evans condition is violated at ζ if there is $u_H \in \mathbb{E}^-_H(\zeta)$ and $u_P \in \mathbb{E}^-_P(\zeta)$ satisfying this boundary condition. The negative space of $P(\zeta)$, $\mathbb{E}^-_P(\zeta)$ is smooth in ζ and equal to $\mathbb{E}^-_{B^{-1}A}$ when $\zeta = 0$. Thus, the Evans condition is violated at ζ if

$$\mathbb{E}^-_H(\zeta) \cap \tilde\Gamma(\zeta)\mathbb{E}^-_P(\zeta) \neq \{0\}.$$

Since $A^{-1}B = (B^{-1}A)^{-1}$, there holds

$$\tilde\Gamma(0)\mathbb{E}^-_P(0) = (\mathrm{Id} + \Gamma B^{-1}A)\mathbb{E}^-_{B^{-1}A}.$$

Comparing with (5.27) and (5.28), we see that for ζ small, the space $\tilde\Gamma(\zeta)\mathbb{E}^-_P(\zeta)$ is generated by ${}^t(c(\zeta), 1)$ where $c(\zeta)$ is a smooth function such that $c(0) = \underline{c}$. Therefore, the Evans condition is violated at ζ if and only if

$$\begin{pmatrix} c(\zeta) \\ 1 \end{pmatrix} \in \mathbb{E}^-_H(\zeta). \tag{5.32}$$

Remark 5.8. The analysis above shows that the *reduced boundary condition* for the hyperbolic part $H(\zeta)$ reads

$$u_1 = c(\zeta)u_2. \tag{5.33}$$

Taking $\zeta = 0$ *in this equation, we recover that* (5.28) *is the natural limiting boundary condition for the hyperbolic operator* H_0*.*

Proposition 5.9. *There are choices of* a *and* Γ*, such that*

i) the inviscid problem (5.23) *for* $\varepsilon = 0$ *with the boundary condition* (5.28) *is maximal striclty dissipative thus uniformly stable,*

ii) the viscous problem with boundary conditions (5.25) *is strongly unstable for small frequencies, in the sense that there are arbitrarily small frequencies* ζ *with* $\gamma > 0$ *where the Evans functions vanishes.*

Proof. The matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad s > 0 \tag{5.34}$$

is a symmetrizer for the inviscid problem. If

$$|\underline{c}|^2 < s, \tag{5.35}$$

the boundary condition is strictly dissipative for S. This implies that the uniform Lopatinski condition is satisfied.

We consider frequencies $\zeta = \rho\check{\zeta}$ with $\check{\zeta}$ close to $(-1, 1, 0)$ where $H_0(\check{\zeta}) = 0$ has a double eigenvalue. More precisely we consider frequencies

$$\zeta = (-\rho + \rho^2\hat{\tau}, \rho, \rho^2\hat{\gamma}). \tag{5.36}$$

In this case, we see that G is a function of $\hat{\sigma} = \hat{\gamma} + i\hat{\tau}$ and ρ, holomorphic in $\hat{\sigma}$, as well as V, P, H and c. Moreover

$$H(\zeta) = -\rho^2(\hat{\sigma}A + A^{-1}B + O(\rho)) = \rho^2\hat{H}(\hat{\sigma}, \rho). \tag{5.37}$$

The model operator is

$$\hat{H}(\hat{\sigma}, 0) = -\hat{\sigma}A - A^{-1}B = \begin{pmatrix} -\hat{\sigma} - 1 & -a \\ a & \hat{\sigma} + 1 \end{pmatrix}$$

$\hat{H}(1, 0)$ has one eigenvalue with positive real part, with eigenvector ${}^t(\underline{b}, 1)$ with $\underline{b} = (2 + \sqrt{4 - a^2})/a$ (Note here the importance of the assumption $a \neq 0$). Therefore, for $\hat{\sigma}$ close to 1 and ρ small , the negative space of $\hat{H}(\hat{\sigma}, \rho)$ is generated by ${}^t(b(\hat{\sigma}, \rho), 1)$ where b is smooth and holomorphic in $\hat{\sigma}$ and $b(1, 0) = \underline{b}$. Moreover

$$\partial_{\hat{\sigma}} b(1, 0) = \frac{1}{a}\Big(1 + \frac{2}{\sqrt{4 - a^2}}\Big) \neq 0. \tag{5.38}$$

Comparing with (5.32), we see that the stability condition is violated at ζ given by (5.36), if and only if

$$b(\hat{\sigma}, \rho) = c(\zeta) = \hat{c}(\hat{\sigma}, \rho). \tag{5.39}$$

Given $a \in]0, 1[$, we choose $\underline{c} = \underline{b}$ and Γ such that the inviscid boundary condition reads (5.28). Note that $\hat{c}(\hat{\sigma}, 0) = \underline{c}$ for all $\hat{\sigma}$. Thus the equation (5.39) holds at $\hat{\sigma} = 1$ and $\rho = 0$. Moreover, with (5.38), the implicit function theorem shows that for $\rho > 0$ small, there is $\hat{\sigma}(\rho$ close to 1 solution of (5.39), providing frequencies $\zeta(\rho) = O(\rho)$ with $\gamma(\rho) \sim \rho^2 > 0$, where the stability condition is violated. ☐

4. *Smooth symmetrizers.* We briefly discuss here the existence of smooth symmetrizers for the hyperbolic operator $\check H$ (6.24). In the present case, we deduce from (5.30) that in polar coordinates $\zeta = \rho\check\zeta$, there holds

$$\check H(\check\zeta, \rho) = -\check\sigma A + \rho(\check\sigma^2 - \check\eta^2)AB + O(\rho^2), \quad \check\sigma = \check\gamma + i(\check\tau + \check\eta). \tag{5.40}$$

Fix $\underline{\check\zeta} = (1, -1, 0)$, which corresponds to a multiple root of the hyperbolic part. Then $\underline{\check\sigma} = 0$, and near $(\underline{\check\zeta}, 0)$

$$\check H(\check\zeta, \rho) = -A(\check\sigma \mathrm{Id} + \rho\beta(\check\zeta)B) + O(\rho^2) \tag{5.41}$$

with $\beta(\underline{\check\zeta}) = 1$. Dropping the $\check{}$, and changing ρb to ρ, the matrix $\check H$ is a perturbation for (σ, ρ) close to $(0, 0)$ of the following *canonical example*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \sigma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \quad \mathrm{Re}\,\sigma \geq 0, \ \rho \geq 0. \tag{5.42}$$

Note that (5.37) derives from (5.41) choosing $\check\sigma = \rho\hat\sigma$.

Denote by $\mathbb{E}^-$ the negative space of $\check H$ for $\mathrm{Re}\,\sigma + \rho > 0$. On can check directly on this example that the negative spaces have no limit as $(\sigma, \rho) \to (0, 0)$: the limits are different when $\rho = 0$ and $\sigma = 0$, since the positive spaces of A and AB are different when $a \neq 0$.

On the other hand, blowing up once more the local coordinates near $\underline{\check\zeta}$, that is taking polar coordinates $(\sigma, \rho) = r(\hat\sigma, \hat\rho)$, is is clear from (5.41) that $\mathbb{E}^-$ is a smooth function of $(\hat\sigma, \hat\rho)$.

If $\Sigma(\check\zeta, \rho)$ is a smooth symmetrizer for $\check H$, then (6.25) implies that $\underline{\Sigma} = \Sigma(\underline{\check\zeta}, 0)$ must be a symmetrizer for $-(\sigma A + \rho AB)$ for all σ and ρ, equivalently that $S = \underline{\Sigma} A$ is a symmetrizer for (5.42), that is

$$S = S^* \gg 0, \quad SA = AS, \quad \mathrm{Re}\,(SB) \gg 0. \tag{5.43}$$

The first two conditions are satisfied if and only if S is diagonal and positive. Multiplying it by a positive factor, it must be of the form (5.34).

The third condition holds if and only if

$$s > a^2(1 + s)^2/4.$$

Denoting by $s_{\min}(a) < 1 < s_{\max}(a) < \infty$ the roots of the equation $4s = a^2(1 + s^2)$, the condition reads

$$s_{\min}(a) < s < s_{\max}(a). \tag{5.44}$$

This shows that the choice of symmetrizers is much more limited in the viscous case compared to the inviscid one. In particular, when a is close to 1, (5.44) forces to choose s in a small interval around 1.

The boundary condition (5.33) is strictly dissipative for Σ, then (5.28) is strictly dissipative for $\underline{\Sigma}$. This holds if and only if $s > |\underline{c}|^2$. Therefore:

There is a smooth symmetrizer $\Sigma(\check\zeta, \rho)$ *for* $\check H$ *on a neighborhood of* $(\underline{\check\zeta}, 0)$*, adapated to the boundary conditions* (5.33) *only if*

$$|\underline{c}|^2 < s_{\max}(a). \tag{5.45}$$

6. LF and MF symmetrizers

6.1. The method of symmetrizers

This "method" applies to general boundary value problems

$$\partial_x u = G(x)u + f\,, \quad \Gamma u(0) = g\,. \tag{6.1}$$

Here, u and f are functions on $[0, \infty[$ with values in some Hilbert space $\mathcal{H}$, and $G(x)$ is a C^1 family of (possibly unbounded) operators defined on $\mathcal{D}$, dense subspace of $\mathcal{H}$. In this section we apply the method to finite dimensional spaces $\mathcal{H} = \mathbb{C}^{N+N^2}$. However, after inverse tangential Fourier transform, the space which is actually considered is rather $L^2(\mathbb{R}^d; \mathbb{C}^{N+N^2})$. Moreover, when passing to variable coefficients, one has to work directly in this infinite dimensional space.

A *symmetrizer* is a family of C^1 functions $x \mapsto S(x)$ with values in the space of operators in $\mathcal{H}$ such that there are C_0 and $c > 0$ such that

$$\forall x\,, \quad S(x) = S(x)^* \quad \text{and} \quad |S(x)| \le C_0\,, \tag{6.2}$$

$$\forall x\,, \quad \operatorname{Re} S(x)G(x) + \frac{1}{2}\,\partial_x S(x) \ge c\mathrm{Id}\,. \tag{6.3}$$

In (6.2), the norm of $S(x)$ is the norm in the space of bounded operators in $\mathcal{H}$. Similarly $S^*(x)$ is the adjoint operator of $S(x)$. The notation $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ is used in (6.3) for the real part of an operator T. When T is unbounded, the meaning of $\operatorname{Re} T \ge \lambda$, is that all $u \in \mathcal{D}$ belongs to the domain of T and satisfies

$$\operatorname{Re}\Big(Tu, u\Big)_{\mathcal{H}} \ge \lambda |u|^2\,, \tag{6.4}$$

where $(\cdot\,,\cdot)_{\mathcal{H}}$ is the scalar product in $\mathcal{H}$. The property (6.3) has to be understood in this sense.

Taking the scalar product of Su with the equation (6.1) and integrating over $[0, \infty[$, (6.2) and (6.3) imply

$$c\|u\|^2 + \Big(S(0)u(0), u(0)\Big)_{\mathcal{H}} \le \frac{C_0^2}{c}\|f\|^2, \tag{6.5}$$

where $f = \partial_x u - Gu$. Here, $\|\cdot\|$ is the norm in $L^2([0, \infty[; \mathcal{H})$.

The symmetrizer S is *adapted* to the boundary condition Γ if there are constants δ and C_1 such that:

$$S(0) \ge \delta Id - C_1 \Gamma^* \Gamma\,. \tag{6.6}$$

Hence,

Lemma 6.1. *If there is a symmetrizer S adapted to the boundary condition Γ, then for all $u \in C_0^1([0, \infty[; \mathcal{H}) \cap C^0([0, \infty[; \mathcal{D})$, one has*

$$\lambda\|u\|^2 + \delta|u(0)|^2 \le \frac{C_0^2}{\lambda}\|f\|^2 + C_1|\Gamma u(0)|^2\,, \tag{6.7}$$

where $f := \partial_x u - Gu$.

In the finite dimensional constant coefficients case, the usual construction of symmetrizers has two parts: first, one constructs families of symmetrizers S_κ satisfying (6.2) and (6.3). This only uses the structural hyperbolicity-parabolicity Assumptions and is independent of the boundary conditions. Second, one chooses κ such that the third condition (6.6) holds. There we use the stability condition for the boundary condition. In this spirit, we end this section with noticing a general recipe linking Evans-Lopatinski conditions to (6.6).

Proposition 6.2. *Suppose that G is a $n \times n$ matrix with no eigenvalues on the imaginary axis and n^- of them in the half space $\{\operatorname{Re}\mu > 0\}$. Denote by $\mathbb{E}^-$ the invariant space associated to these eigenvalues. Suppose that Γ is a $n^- \times n$ matrix and*

$$| \det(\mathbb{E}^-, \ker \Gamma)| \geq c > 0. \tag{6.8}$$

Suppose that S_κ is a symmetrizer for G such that

$$\Big(S_\kappa u, u\Big) \geq \kappa |\Pi^+ u|^2 - |\Pi^- u|^2, \tag{6.9}$$

where $\Pi^\pm$ denote the projectors associated to a decomposition

$$\mathbb{C}^n = \tilde{\mathbb{E}}^- \oplus \tilde{\mathbb{E}}^+, \qquad \dim \tilde{\mathbb{E}}^- = n^-. \tag{6.10}$$

Then, there are κ_0 and C_1 which depend only on c, $|\Gamma|$ and $|\Gamma^*(\Gamma\Gamma^*)^{-1}|$ such that for $\kappa \geq \kappa_0$ there holds

$$\Big(S_\kappa u, u\Big) \geq |u|^2 - C_1 |\Gamma u|^2. \tag{6.11}$$

Proof. By Lemma 4.10, there is C such that

$$\forall u \in \mathbb{E}^- : \quad |u| \leq C |\Gamma u|. \tag{6.12}$$

Next, we note that all element in $\mathbb{E}^-$ is an initial data for an exponentially decaying solution of $\partial_x u - Gu = 0$. Therefore, (6.5) implies that for all symmetrizer S of G there holds

$$\forall u \in \mathbb{E}^- : \quad (Su, u) \leq 0. \tag{6.13}$$

In particular, the assumption implies that

$$\forall u \in \mathbb{E}^- : \quad |\Pi^+ u| \leq \varepsilon |\Pi^- u|, \qquad \varepsilon = \kappa^{-\frac{1}{2}}. \tag{6.14}$$

Thus the mapping Π^- from $\mathbb{E}^-$ to $\tilde{\mathbb{E}}^-$ is injective. Because $\dim \mathbb{E}^- = \dim \tilde{\mathbb{E}}^-$ this implies that there is a linear mapping A from $\tilde{\mathbb{E}}^-$ to $\tilde{\mathbb{E}}^+$ with $|A| \leq \varepsilon$ such that

$$\mathbb{E}^- = \Big\{ u^- + A u^- ; \ u^- \in \tilde{\mathbb{E}}^- \Big\}. \tag{6.15}$$

Combining with (6.12), we see that if $\varepsilon(1 + C|\Gamma|) \leq \frac{1}{2}$, then

$$\forall u^- \in \tilde{\mathbb{E}}^- : \quad |u^-| \leq 2C |\Gamma u^-|. \tag{6.16}$$

This inequality implies that for $\kappa \geq 2 + 24 C^2 |\Gamma|^2$

$$\kappa |u^+|^2 - |u^-|^2 \geq |u^+ + u^-|^2 - C_1 |\Gamma u^+ + \Gamma u^-|^2$$

and the proposition is proved. □

Remark 6.3. In the limit $\kappa \to +\infty$, (6.14) implies that $\tilde{\mathbb{E}}^- \to \mathbb{E}^-$. This shows that in the splitting (6.10) there is little choice for $\tilde{\mathbb{E}}^-$. On the contrary, there is no need for $\tilde{\mathbb{E}}^+$ to be close to the positive invariant space $\mathbb{E}^+$. This is important for the construction of symmetrizers for $G(\zeta)$ when the frequency is close to “glancing frequencies”.

6.2. Elliptic points and MF symmetrizers

Lemma 6.4. *Suppose that G is a matrix with spectrum in $\{\mathrm{Re}\,\mu > 0\}$. Then, there is a symmetric definite positive matrix S such that $\mathrm{Re}\, SG \geq \mathrm{Id}$.*

Proof. The assumption implies that e^{-tG} and e^{-tG^*} are exponentially decaying. A symmetrizer is

$$S = 2 \int_0^\infty e^{-tG^*} e^{-tG} dt.$$

□

Note that this formula shows that one can choose S depending smoothly on G in the space of matrices with spectrum in the right open half space.

Proposition 6.5. *Suppose that $G(p)$ is a $n \times n$ matrix which depend smoothly on parameters p in a neighborhood of $\underline{p}$, with no eigenvalues on the imaginary axis. Then, for p in a neighborhood of $\underline{p}$, there is a family of symmetrizers $S_\kappa(p)$ for $G(p)$ which satisfy (6.9) in the decomposition of $\mathbb{C}^n$ in invariant spaces $\mathbb{E}^\pm(p)$ for $G(p)$ associated to the eigenvalues in $\{\pm\mathrm{Re}\,\mu > 0\}$.*

Proof. The spaces $\mathbb{E}^\pm(p)$ depend smoothly on p, in a neighborhood of p and there is a smooth matrix $V(p)$ such that

$$VGV^{-1} = \begin{pmatrix} G^+ & 0 \\ 0 & G^- \end{pmatrix}$$

with $G^\pm$ having their spectrum in $\{\pm\mathrm{Re}\,\mu > 0\}$. Then there are self adjoint matrices $S^\pm(p)$ such that $\pm S^\pm G^\pm \geq \mathrm{Id}$. Then,

$$S_\kappa = V^* \begin{pmatrix} \kappa S^+ & 0 \\ 0 & -S^- \end{pmatrix} V$$

symmetrizes G and satisfies (6.9) □

Thanks to Lemma 4.3, this proposition directly applies to the linearized equations (4.14) for $\zeta \neq 0$. For clarity, we drop the tildes and reserve the notations u, U... for the unknowns and call $p \in \mathcal{U}$ the parameter called $\underline{u}$ in this equation, which now reads

$$\partial_z U = G(p,\zeta)U + F, \quad \Gamma(p,\zeta)U(0) = g. \tag{6.17}$$

We assume that the assumptions of Section 2 are satisfied.

Proposition 6.6. *For all $\underline{\zeta} \in \overline{\mathbb{R}}^{d+1}\backslash\{0\}$, there is a neighborhood of $(\underline{p}, \underline{\zeta})$ in $\mathcal{U} \times \mathbb{R}^{d+1}$ such that for (p,ζ) in this neighborhood there is a smooth splitting*

$$\mathbb{C}^{N^2} = \mathbb{E}^-(p,\zeta) \oplus \mathbb{E}^+(p,\zeta). \tag{6.18}$$

where $\mathbb{E}^\pm(p,\zeta)$ denote the invariant space of $G(p,\zeta)$ associated to the spectrum in $\{\pm\mathrm{Re}\,\mu > 0\}$. Denoting by $\Pi^\pm(p,\zeta)$ the smooth spectral projectors associate to this splitting, there is a smooth family $\Sigma^\kappa(p,\zeta)$ of self adjoint matrices such that for all (p,ζ) in the given neighborhood and all $\kappa \geq 1$:

$$\begin{aligned} &i) \quad \mathrm{Re}\,\Sigma^\kappa G > 0, \\ &ii) \quad \mathrm{Re}\,\Sigma^\kappa \geq \kappa(\Pi^+)^*\Pi^+ - (\Pi^-)^*\Pi^-. \end{aligned} \tag{6.19}$$

This provides another proof of the estimates (4.34):

Corollary 6.7. *If the weak spectral stability condition is satisfied, then for all* $\underline{\zeta} \in \overline{\mathbb{R}}^{d+1}\backslash\{0\}$*, there are a constant* C *and a neighborhood of* $(\underline{p}, \underline{\zeta})$ *in* $\mathcal{U} \times \mathbb{R}^{d+1}$ *such that for* (p, ζ) *in this neighborhood the solutions of* (6.17) *satisfy*

$$\|U\|_{L^2} + |U(0)| \leq C\Big(\|F\|_{L^2} + |g|\Big). \tag{6.20}$$

6.3. LF symmetrizers

We now concentrate on low frequencies. By Lemma 4.3, the matrix $G(p, \zeta)$ is locally smoothly conjugated to a block diagonal matrix (4.16) with diagonal blocks with $H(p, \zeta)$ of dimension $N \times N$ and $P(p, \zeta)$ of dimension $N^2 \times N^2$. The system (6.17) is therefore equivalent to the equations (5.1) (5.2) coupled by the boundary conditions (5.3).

In the block diagonal reduction (4.16), we construct symmetrizers

$$\Sigma^\kappa = \begin{pmatrix} \Sigma_H^\kappa & 0 \\ 0 & \Sigma_P^\kappa \end{pmatrix} \tag{6.21}$$

such that the property (6.9) is satisfied for each block independently.

The construction for the elliptic block P is given by Proposition 6.5, since $P(\underline{p}, 0)$ has no eigenvalues on the imaginary axis. Denote by $\mathbb{E}_P^\pm(p, \zeta)$ the subspaces of $\mathbb{C}^{N^2}$, invariant for $P(p, \zeta)$, associated to the spectrum in $\{\pm \mathrm{Re}\, \mu > 0\}$. Thus, for (p, ζ) in a neighborhood of $(\underline{p}, 0)$, there is a smooth splitting

$$\mathbb{C}^{N^2} = \mathbb{E}_P^-(p, \zeta) \oplus \mathbb{E}_P^+(p, \zeta). \tag{6.22}$$

Denote by $\Pi_P^\pm(p, \zeta)$ the smooth spectral projectors associate to this splitting.

Proposition 6.8. *There is a smooth family of self adjoint matrices* Σ_P^κ *on a neighborhood of* $(\underline{p}, 0)$ *such that*

$$\begin{aligned} &i) \quad \mathrm{Re}\, \Sigma_P^\kappa P > 0, \\ &ii) \quad \mathrm{Re}\, \Sigma_P^\kappa \geq \kappa (\Pi_P^+)^* \Pi_P^+ - (\Pi_P^-)^* \Pi_P^- \end{aligned} \tag{6.23}$$

This implies the estimates (5.19) (5.20) which where used in the previous section.

To analyze H, we use polar coordinates for $\zeta = \rho\check{\zeta}$ as in (5.12) so that

$$H(p, \zeta) = \rho \check{H}(p, \check{\zeta}, \rho), \quad \check{H}(p, \check{\zeta}, \rho) = H_0(p, \check{\zeta}) + O(\rho). \tag{6.24}$$

By Lemma 4.3, for $\zeta \in \mathbb{R}_+^{d+1}\backslash\{0\}$, $\check{H}$ has no eigenvalue on the imaginary axis, hence the number N^- of eigenvalues of $\check{H}$ in $\{\mathrm{Re}\, \mu < 0\}$ is constant.

We fix a point $\underline{\check{\zeta}} \in \overline{S}_+^d$, that is $\underline{\check{\zeta}} = (\underline{\check{\tau}}, \underline{\check{\eta}}, \underline{\check{\gamma}})$ in the unit sphere with $\underline{\check{\gamma}} \geq 0$. The goal is to construct smooth symmetrizers for $\check{H}$, for $(p, \check{\zeta}, \rho)$ close to $(\underline{p}, \underline{\check{\zeta}}, 0)$. For convenience we introduce the following terminology.

Definition 6.9. *A smooth symmetrizer for* $\check{H}$ *on a neighborhood* ω *of* $(\underline{p}, \underline{\check{\zeta}}, 0)$ *is a smooth self adjoint matrix* $\check{\Sigma}^H(p, \check{\zeta}, \rho)$ *such that*

$$\mathrm{Re}\, \check{\Sigma}^H \check{H} = \sum V_k^* \Sigma_k V_k, \tag{6.25}$$

where the V_k *and* Σ_k *are smooth matrices on* ω *of appropriate dimension so that the products make sense, satisfying*

i) $\sum V_k^* V_k$ *is definite positive,* *ii) either Σ_k is definite positive or $\Sigma_k = \gamma\Sigma_{k,1} + \rho\Sigma_{k,2}$ with $\Sigma_{k,1}$ and $\Sigma_{k,2}$ definite positive.*

Definition 6.10. *A family of smooth symmetrizers Σ^κ on neigborhoods ω^κ of $(\underline{p}, \underline{\check\zeta}, 0)$ is called a K-family of symmetrizers for $\check H$ if there are a decomposition*

$$\mathbb{C}^N = \underline{\mathbb{E}}_H^- \oplus \underline{\mathbb{E}}_H^+ \tag{6.26}$$

with $\dim \underline{\mathbb{E}}^- = N_-$ and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$ such that for all κ

$$\Sigma^\kappa(\underline{p}, \underline{\check\zeta}, 0) \geq m(\kappa)\Pi_+^*\Pi_+ - \Pi_-^*\Pi_-. \tag{6.27}$$

where $\Pi_\pm$ are the projectors associated to the splitting (6.26).

Using (6.14), one proves the following result (see [MéZu3]):

Theorem 6.11. *Suppose that there exists a K-family of symmetrizers near $(\underline{p}, \underline{\check\zeta}, 0)$. Then $\underline{\mathbb{E}}_H^-$ is the limit of the negative spaces $\mathbb{E}_H^-(p, \check\zeta, \rho)$ as $(p, \check\zeta, \rho)$ tends to $(\underline{p}, \underline{\check\zeta}, 0)$ with $\rho > 0$.*

Remark 6.12. This theorem shows that $\underline{\mathbb{E}}_H^-$ is uniquely determined. On the other hand, $\underline{\mathbb{E}}_H^+$ is arbitrary, provided that the the splitting (6.26) holds: if (6.27) holds for some choice of $\underline{\mathbb{E}}^+$, then it also holds for another choice for a multiple of Σ^κ with some other function $m(\kappa)$.

Remark 6.13. The advantage of the notion of K-families is that it is independent of the boundary conditions. Therefore, their construction depends only on an analysis of $\check H$. In particular, we can use a spectral block decompositions of $\check H$.

Fix $\underline{\check\zeta} \in \overline{S}_+^d$. Consider the *distinct* eigenvalues $\underline{\mu}_k$ of $H_0(\underline{p}, \underline{\check\zeta})$. For $(p, \check\zeta, \rho)$ in a neighborhood of $(\underline{p}, \underline{\check\zeta}, 0)$, there is a smooth block reduction

$$V^{-1}\check H V = \mathrm{diag}(\check H_k) \tag{6.28}$$

where the H_k have their spectrum in small discs centered at $\underline{\mu}_k$ that are pairwise disjoints. Equivalently, there is a smooth decomposition

$$\mathbb{C}^N = \bigoplus_k \mathbb{E}_k(p, \check\zeta, \rho) \tag{6.29}$$

in invariant spaces for $\check H(p, \check\zeta, \rho)$ and $\check H_k$ is the restriction of $\check H$ to $\mathbb{E}_k$. We denote by N_k the dimension of $\mathbb{E}_k$, that is the size of $\check H_k$.

The K-families of symmetrizers are constructed for each block $\check H_k$ separately. If Σ_k^κ is a K-family for $\check H_k$, it is clear that $\Sigma^\kappa = V^*\mathrm{diag}(\Sigma_k^\kappa)V$ has the form (6.25) and is a K-family for $\check H$.

When the mode is *elliptic*, that is when $\mathrm{Re}\,\underline{\mu}_k \neq 0$, the construction of symmetrizers is given by Proposition 6.5

Proposition 6.14. *Suppose that $\underline{\mu}_k$ is an eigenvalue of $H_0(\underline{p}, \underline{\check\zeta})$ with $\mathrm{Re}\,\underline{\mu}_k \neq 0$. Then is a smooth family of self adjoint matrices Σ_k^κ on a neighborhood of $(\underline{p}, \underline{\check\zeta}, 0)$ such that*

$$\begin{aligned} &i) \quad \mathrm{Re}\,(\Sigma_k^\kappa \check H_k) > 0,\\ &ii) \quad \mathrm{Re}\,\Sigma_k^\kappa \geq \kappa\mathrm{Id} \quad \textit{if} \;\; \mathrm{Re}\,\underline{\mu}_k > 0,\\ &\qquad\; \mathrm{Re}\,\Sigma_k^\kappa \geq -\mathrm{Id} \quad \textit{if} \;\; \mathrm{Re}\,\underline{\mu}_k < 0. \end{aligned} \tag{6.30}$$

Therefore we now restrict our attention to a nonelliptic mode:

$$\underline{\mu}_k = i\underline{\check{\xi}}_d, \quad \underline{\check{\xi}}_d \in \mathbb{R}. \tag{6.31}$$

By definition of H_0, this implies that $-\underline{\check{\tau}} + i\underline{\check{\gamma}}$ is an eigenvalue $\underline{\lambda}$ of $A(\underline{p}, \underline{\check{\xi}})$ with $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$. In particular, by hyperbolicity, this can only happen when $\underline{\check{\gamma}} = 0$. By Lemma 4.3, $\check{H}_k$ has no eigenvalue on the imaginary axis when $\rho > 0$, thus the number of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is constant. We call it N_k^-. The next definition reformulates Definitions 6.9 and 6.10 for nonelliptic blocks $\check{H}_k$.

Definition 6.15. *A smooth symmetrizer for a nonelliptic block $\check{H}_k$ on a neighborhood ω of $(\underline{p}, \underline{\check{\zeta}}, 0)$ is a smooth self adjoint matrix $\Sigma(p, \check{\zeta}, \rho)$ such that, for some C, $c > 0$, there holds for all $(p, \check{\zeta}, \rho) \in \omega$,*

$$\operatorname{Re} \Sigma \check{H}_k = \check{\gamma}\Sigma_1 + \rho\Sigma_2, \tag{6.32}$$

with $\Sigma_1(\underline{p}, \underline{\check{\zeta}}, 0)$ and $\Sigma_2(\underline{p}, \underline{\check{\zeta}}, 0)$ definite positive.

A family of smooth symmetrizers Σ_k^κ on neigborhoods ω^κ of $(\underline{p}, \underline{\check{\zeta}}, 0)$ is called a K-family of symmetrizers for $\check{H}_k$ if there are a decomposition

$$\mathbb{E}_k(\underline{p}, \underline{\check{\zeta}}, 0) = \underline{\mathbb{E}}_k^- \oplus \underline{\mathbb{E}}_k^+ \tag{6.33}$$

with $\dim \underline{\mathbb{E}}_k^-$ equal to N_k^- and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$ such that for all κ

$$\Sigma_k^\kappa(\underline{p}, \underline{\check{\zeta}}, 0) \geq m(\kappa)(\Pi_k^+)^*\Pi_k^+ - (\Pi_k^-)^*\Pi_k^-, \tag{6.34}$$

where $\Pi_k^\pm$ are the projectors associated to the splitting (6.33).

7. Symmetrizers for nonelliptic blocks; Examples

In this section and the next one, we consider a block $\check{H}_k$ associated to a purely imaginary eigenvalue $\underline{\mu}_k = i\underline{\check{\xi}}_d$ of $H_0(\underline{p}, \underline{\check{\zeta}})$ with $\underline{\check{\zeta}} = (\underline{\check{\tau}}, \underline{\check{\eta}}, 0)$. Equivalently, $-\underline{\check{\tau}}$ is an eigenvalue of $A(\underline{p}, \underline{\check{\xi}})$, with $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$. To build symmetrizers, some knowledge of properties of $\check{H}_k$ is necessary. Part of the analysis, is to relate them to properties of the eigenvalue $-\underline{\tau}$. In this section, we give examples which help to understand the general analysis.

7.1. Simple hyperbolic points

Suppose that $-\underline{\tau}$ is a *simple* eigenvalue of $A(\underline{p}, \underline{\check{\xi}})$. Thus, in the vicinity of $(\underline{p}, \underline{\check{\xi}}, 0)$, $iA(p, \check{\xi}) + \rho B(p, \check{\xi})$ has a simple eigenvalue

$$i\lambda(p, \check{\xi}, \rho), \quad \text{with} \quad \underline{\check{\tau}} + \lambda(\underline{p}, \underline{\check{\xi}}, 0) = 0. \tag{7.1}$$

Moreover, by Assumption (H4),

$$\operatorname{Im} \lambda(p, \xi, 0) = 0, \quad \partial_\rho \operatorname{Im} \lambda(\underline{p}, \underline{\check{\xi}}, 0) < 0. \tag{7.2}$$

In a neighborhood of $(\underline{p}, \underline{\check{\zeta}}, 0)$, the eigenvalue $H(p, \check{\zeta}, \rho)$ close to $\underline{\mu}$ are $\mu = i\check{\xi}$ where $\check{\xi}$ solves

$$\check{\tau} - i\check{\gamma} + \lambda(p, \check{\eta}, \check{\xi}, \rho) = 0. \tag{7.3}$$

The easiest case occurs when $\check{\underline{\xi}}$ is a *simple root* of this equation at the base point $(\underline{p}, \check{\underline{\zeta}}, 0)$, that is when

$$\beta := \partial_{\xi_d}\lambda(\underline{p}, \check{\underline{\xi}}, 0) \neq 0. \tag{7.4}$$

Note that $\beta \in \mathbb{R}$. In this case, $(\check{\underline{\tau}}, \check{\underline{\xi}})$ is called a *simple hyperbolic characteristic point.*

The condition (7.4) implies that there is a simple eigenvalue $\mu(p, \check{\zeta}, \rho)$ of $H(p, \check{\zeta}, \rho)$ such that

$$\begin{aligned} &\mu(\underline{p}, \check{\underline{\zeta}}, 0) = \underline{\mu}, && (7.5)\\ &\operatorname{sign} \partial_\rho \mathrm{Re}\, \mu(\underline{p}, \check{\underline{\zeta}}, 0) = \operatorname{sign} \partial_{\check{\gamma}} \mathrm{Re}\, \mu(\underline{p}, \check{\underline{\zeta}}, 0) = -\operatorname{sign} \beta. && (7.6)\end{aligned}$$

From that, we see that the invariant space $\mathbb{E}_k(p, \check{\zeta}, \rho)$ has dimension equal to one, and that

$$\begin{aligned} &\mathbb{E}_k^- = \mathbb{E}_k, \quad \mathbb{E}_k^+ = \{0\}, \qquad \text{when} \quad \beta > 0,\\ &\mathbb{E}_k^- = \{0\}, \quad \mathbb{E}_k^+ = \mathbb{E}_k, \qquad \text{when} \quad \beta < 0. \end{aligned} \tag{7.7}$$

Moreover, $\check{H}_k$ is the multiplication by μ, and therefore, K-families of symmetrizers for $\check{H}_k$ are multiplications by

$$\begin{aligned} &\Sigma_k^\kappa = -1 \qquad \text{when} \quad \beta > 0,\\ &\Sigma_k^\kappa = \kappa \qquad \text{when} \quad \beta < 0. \end{aligned} \tag{7.8}$$

7.2. Simple glancing modes

We still assume that $-\underline{\tau}$ is a *simple* eigenvalue but that (7.4) is not satisfied. In geometric optics which applies to the analysis of propagation of singularities or oscillations, $\nabla_\xi \lambda(p, \xi, 0)$ is the group velocity at frequency ξ, and the lines $x + t\nabla_\xi \lambda(p, \xi, 0)$ are the rays of propagation (the equation has constant coefficients). The condition $\partial_{\xi_d}\lambda = 0$ means that the corresponding ray is parallel to the boundary, it is called a *glancing* and we say that $(\check{\underline{\tau}}, \check{\underline{\xi}})$ is a *glancing mode.*

The simplest case occurs when

$$\partial_{\xi_d}\lambda(\underline{p}, \check{\underline{\xi}}, 0) = 0, \quad \beta := \partial_{\xi_d}^2\lambda(\underline{p}, \check{\underline{\xi}}, 0) \neq 0. \tag{7.9}$$

In this case, one can show that $\dim \mathbb{E}_k = 2$ and that in a smooth basis $\check{H}_k$ has the form

$$\check{H}_k = i\underline{\xi}_d \mathrm{Id} + i \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \tag{7.10}$$

with

$$\begin{aligned} &a(\underline{p}, \check{\underline{\zeta}}, 0) = b(\underline{p}, \check{\underline{\zeta}}, 0) = 0, && (7.11)\\ &\mathrm{Im}\, a = \mathrm{Im}\, b = 0 \qquad \text{when} \quad \check{\gamma} = \rho = 0, && (7.12)\\ &\partial_{\check{\gamma}} \mathrm{Im}\, b \neq 0, \qquad \partial_\rho \mathrm{Im}\, b \neq 0, && (7.13)\\ &\operatorname{sign} \partial_{\check{\gamma}} \mathrm{Im}\, b = \operatorname{sign} \partial_\rho \mathrm{Im}\, b = \operatorname{sign} \beta. && (7.14)\end{aligned}$$

The prototype for $\check{H}_k$ at $p = \underline{p}$, $\check{\zeta} = (\check{\underline{\tau}}, \check{\underline{\eta}}, \check{\gamma})$ and $\rho = 0$ is

$$\check{H} = \begin{pmatrix} 0 & i \\ -\beta\check{\gamma} & 0 \end{pmatrix}. \tag{7.15}$$

For $\check\gamma > 0$ the invariant spaces are

$$\mathbb{E}^- = \mathbb{C}\begin{pmatrix}1\\ \mu\end{pmatrix}, \quad \mathbb{E}^+ = \mathbb{C}\begin{pmatrix}1\\ -\mu\end{pmatrix} \tag{7.16}$$

where μ is the square root of $-i\beta\check\gamma$ such that $\operatorname{Re}\mu < 0$. Note that these spaces have the same limit as $\check\gamma \to 0$. Hence, the spectral projections *are not uniformly bounded* so that they *cannot* be used simultaneously in the construction of symmetrizers. However, Theorem 6.11 clearly imposes the choice

$$\underline{\mathbb{E}}^- = \mathbb{C}\begin{pmatrix}1\\ 0\end{pmatrix} \tag{7.17}$$

which we supplement with

$$\underline{\mathbb{E}}^+ = \mathbb{C}\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{7.18}$$

Using Taylor expansions at $\check\gamma = 0$ and $\rho = 0$

$$\check H_k = G_0 + \check\gamma G_1 + \rho G_2.$$

The symmetrizers are constructed as

$$\Sigma^\kappa = \Sigma_0^\kappa + \check\gamma\Sigma_1^\kappa + \rho\Sigma_2^\kappa.$$

The first term is searched as

$$\Sigma_0^\kappa = \begin{pmatrix}\sigma^\kappa & \epsilon\\ \epsilon & \kappa\end{pmatrix}$$

with $\sigma^\kappa(p, \check\tau, \check\eta)$ real, vanishing at $(\underline{p}, \underline{\check\tau}, \underline{\check\eta})$, and $\epsilon \in \{-1, +1\}$ to be chosen later. In particular, with $\underline{\mathbb{E}}^\pm$ as above, there holds at $(\underline{p}, \underline{\check\zeta})$:

$$(\Sigma_0^\kappa u, u) \geq (\kappa - 1)|u_2|^2 - |u_1|^2.$$

Thus, the condition (6.34) will be satisfied with $m(\kappa) = \kappa - 2$ on a small neighborhood of $(\underline{p}, \underline{\zeta})$. The function σ^κ is determined by requiring that

$$\operatorname{Re}\Sigma_0^\kappa G_0 = 0.$$

Denoting by a_0 and b_0 the restrictions of a and b respectively at $\check\gamma = \rho = 0$, which are real by (7.12), the condition above reads

$$\sigma^\kappa = \epsilon a_0 + \kappa b_0.$$

Next, Σ_1^κ and Σ_2^κ are searched under the form

$$\Sigma_1^\kappa = i\begin{pmatrix}0 & \sigma_1^\kappa\\ -\sigma_1^\kappa & 0\end{pmatrix}, \quad \Sigma_2^\kappa = i\begin{pmatrix}0 & \sigma_2^\kappa\\ -\sigma_2^\kappa & 0\end{pmatrix}$$

with σ_1^κ and σ_2^κ real. Since

$$\operatorname{Re}\Sigma^\kappa\check H_k = \check\gamma\operatorname{Re}(\Sigma_0^\kappa G_1 + \Sigma_1^\kappa G_0) + \rho\operatorname{Re}(\Sigma_0^\kappa G_2 + \Sigma_2^\kappa G_0) + O(\check\gamma^2 + \rho^2),$$

for the condition (6.32) to be satisfied on a neigborhood of $(\underline{p}, \underline{\check\zeta})$, it is sufficient that

$$\operatorname{Re}(\Sigma_0^\kappa G_1 + \Sigma_1^\kappa G_0) > 0 \quad \text{and} \quad \operatorname{Re}(\Sigma_0^\kappa G_2 + \Sigma_2^\kappa G_0) > 0$$

at the base point. There,

$$G_1 = i\begin{pmatrix}a_1 & 0\\ b_1 & 0\end{pmatrix}, \quad G_2 = i\begin{pmatrix}a_2 & 0\\ b_2 & 0\end{pmatrix}$$

with $\operatorname{Im} b_1 \neq 0$ and $\operatorname{Im} b_2 \neq 0$, having the same sign as β. At the base point,

$$\operatorname{Re}(\Sigma_0^\kappa G_1 + \Sigma_1^\kappa G_0) = \operatorname{Re}\begin{pmatrix} i\epsilon b_1 & 0 \\ i\epsilon a_1 + i\kappa b_1 & \sigma_1^\kappa \end{pmatrix}.$$

Choosing $\varepsilon = -\operatorname{sign}\beta$ and σ_1^κ large enough this matrix is definite positive. Proceeding similarly for σ_2^κ, this finishes the construction of a K-family of symmetrizers. Note that the condition (7.14) is essential for the simultaneous construction of Σ_1^κ and Σ_2^κ.

7.3. Hyperbolic modes with constant multiplicity

We suppose here that for $(p, \check\xi)$ near $(\underline{p}, \underline{\check\xi})$, $A(p, \check\sigma)$ has a smooth real and semi-simple eigenvalue $\lambda(p, \check\xi)$ of constant multiplicity m, such that

$$\underline{\check\tau} + \lambda(\underline{p}, \underline{\check\xi}) = 0. \tag{7.19}$$

This eigenvalue has a complex analytic extension in $\check\xi$, near $\underline{\check\xi}$. We further assume that this mode in nonglancing, that is

$$\beta := \partial_{\check\xi_d}\lambda(\underline{p}, \underline{\check\xi}) \neq 0. \tag{7.20}$$

In this case, near $(\underline{p}, \underline{\check\xi}, 0)$, $iA(p, \check\xi) + \rho B(p, \check\xi)$ has an invariant space $\mathbb{E}$ of dimension m and its restriction to this space has the form

$$i\lambda(p, \check\xi)\mathrm{Id} + \rho B^\sharp(p, \check\xi, \rho). \tag{7.21}$$

Moreover, Assumption (H4) implies that the spectrum of $B^\sharp(\underline{p}, \check\xi, 0)$ is contained in the right half plane $\{\operatorname{Re}\mu > 0\}$.

For $(p, \check\zeta)$ near $(\underline{p}, \underline{\check\zeta})$, the equation in $\check\xi_d$

$$\check\tau - i\check\gamma + \lambda(p, \check\eta, \check\xi_d) = 0 \tag{7.22}$$

has a unique solution near $\underline{\check\xi}$ and $\mu(p, \check\zeta) = i\check\xi_d$ is a semi-simple eigenvalue of $\check H_0(p, \check\zeta)$ with $\mu(\underline{p}, \underline{\check\zeta}) = \underline{\mu}_k$. Thus

$$\check H_k(p, \check\zeta, \rho) = \mu(p, \check\zeta)\mathrm{Id} + \rho B^\flat(p, \check\zeta, \rho). \tag{7.23}$$

Moreover, denoting by ϵ the sign of β, there holds

$$\operatorname{Re}\mu(p, \check\zeta) = 0 \quad \text{when} \quad \check\gamma = 0, \tag{7.24}$$

$$\operatorname{sign}\partial_{\check\gamma}\operatorname{Re}\mu(\underline{p}, \underline{\check\zeta}) = -\epsilon, \tag{7.25}$$

$$\text{spectrum } B^\flat(\underline{p}, \underline{\check\zeta}) \subset \{-\epsilon\operatorname{Re}\mu > 0\}. \tag{7.26}$$

This implies that

$$\begin{aligned} &\mathbb{E}_k^- = \mathbb{E}_k, \quad \mathbb{E}_k^+ = \{0\}, \qquad \text{when} \quad \epsilon = +1, \\ &\mathbb{E}_k^- = \{0\}, \quad \mathbb{E}_k^+ = \mathbb{E}_k, \qquad \text{when} \quad \epsilon = -1. \end{aligned} \tag{7.27}$$

By Lemma 6.4, there is a smooth symmetric definite positive matrix $S(p, \check\zeta, \rho)$ on $\mathbb{E}_k$ such that $-\epsilon SB^\flat$ is definite positive. Therefore, a smooth K-family of symmetrizers is

$$\begin{aligned} &\Sigma_k^\kappa = -S \qquad \text{when} \quad \beta > 0, \\ &\Sigma_k^\kappa = \kappa S \qquad \text{when} \quad \beta < 0. \end{aligned} \tag{7.28}$$

7.4. Smoothly diagonalizable hyperbolic modes

Consider here the more general situation where $-\underline{\check\tau}$ is a semi-simple eigenvalue of $A(\underline{p},\underline{\check\xi})$ of mutliplicity m and $A(p,\check\xi)$ has, near $(\underline{p},\underline{\check\xi})$, m smooth eigenvalues $\lambda_j(p,\check\xi)$, such that

$$\underline{\check\tau} + \lambda_j(\underline{p},\underline{\check\xi}) = 0. \tag{7.29}$$

We further assume that there exist m smooth eigenvectors, $e_j(p,\check\xi)$, linearly independent:

$$A(p,\check\xi)e_j(p,\check\xi) = \lambda_j(p,\check\xi)e_j(p,\check\xi). \tag{7.30}$$

We assume that the λ_j's and e_j's have complex analytic extension in $\check\xi$, near $\underline{\check\xi}$. We denote by ℓ_j a dual basis of left eigenvectors:

$$\ell_j(p,\check\xi)A(p,\check\xi) = \lambda_j(p,\check\xi)\ell_j(p,\check\xi), \qquad \ell_j\cdot e_{j'} = \delta_{j,j'}. \tag{7.31}$$

Thus, near $(\underline{p},\underline{\check\xi},0)$, $iA(p,\check\xi)+\rho B(p,\check\xi)$ has an invariant space $\mathbb{E}$ of dimension m and its restriction to this space has the form

$$i\,\mathrm{diag}\{\lambda_j(p,\check\xi)\} + \rho B^\sharp(p,\check\xi,\rho). \tag{7.32}$$

Moreover, Assumption (H4) implies that the spectrum of $B^\sharp(\underline{p},\check\xi,0)$ is contained in the right half plane $\{\mathrm{Re}\,\mu>0\}$.

Assume that none of the mode in glancing, that is

$$\forall j, \quad \beta_j := \partial_{\check\xi_d}\lambda_j(\underline{p},\underline{\check\xi}) \neq 0. \tag{7.33}$$

7.4.1. The inviscid case

In the inviscid case, which corresponds exactly to the case $\rho=0$, the analysis is parallel to the constant multiplicity case. For $(p,\check\zeta)$ near $(\underline{p},\underline{\check\zeta})$, the equations in $\check\xi_d$

$$\check\tau - i\check\gamma + \lambda_j(p,\check\eta,\check\xi_d) = 0$$

have unique solutions $\xi_{d,j}(p,\check\zeta)$ near $\underline{\check\xi}$ which define eigenvalues $\mu_j(p,\check\zeta)$ of $\check H_0(p,\check\zeta)$ with $\mu(\underline{p},\underline{\check\zeta}) = \underline{\mu}_k$. Moreover the

$$e_j^\sharp(p,\zeta) = e_j(p,\check\eta,\check\xi_{d,j}(p,\check\zeta))$$

are eigenvectors of $H_0(p,\check\zeta)$ associated to the eigenvalues μ_j, and there are linearly independent near $(\underline{p},\underline{\check\zeta})$. Thus

$$\check H_k(p,\check\zeta,0) = \mathrm{diag}\{\mu_j(p,\check\zeta)\}. \tag{7.34}$$

As above, denoting by ϵ_j the sign of β_j, there holds

$$\mathrm{Re}\,\mu_j(p,\check\zeta) = 0 \quad \text{when} \quad \check\gamma = 0,$$
$$\mathrm{sign}\,\partial_{\check\gamma}\mathrm{Re}\,\mu_j(\underline{p},\underline{\check\zeta}) = -\epsilon_j.$$

This implies that

$$\begin{aligned}\mathbb{E}_k^- &= \mathrm{span}\{e_j^\flat :\ \epsilon_j = +1\},\\ \mathbb{E}_k^+ &= \mathrm{span}\{e_j^\flat :\ \epsilon_j = -1\}.\end{aligned} \tag{7.35}$$

One obtains a smooth K-family of symmetrizers setting

$$\Sigma_k^\kappa = \mathrm{diag}\{\sigma_j^\kappa\} \tag{7.36}$$

with

$$\sigma_j^\kappa = \begin{cases} -1 & \text{when} \quad \epsilon_j = +1, \\ \kappa & \text{when} \quad \epsilon_j = -1. \end{cases} \tag{7.37}$$

7.4.2. The viscous case

The viscous case is much more delicate. In the basis $e_j^\flat$ there holds

$$\check H_k(p, \check\zeta, \rho) = \mathrm{diag}\{\mu_j(p, \check\zeta)\} + \rho B^\flat(p, \check\zeta, \rho). \tag{7.38}$$

In general, $B^\sharp$ is a full matrix and, if the β_j's have different signs, the real part of eigenvalues of $B^\flat$ have different signs. As seen in (7.36), the symmetrizers Σ^κ do not give the same weight to indices with positive and negative ϵ_j. This leads to impose a natural *decoupling condition*, which means that all the entries $B^\flat_{j,j'}$ of $B^\flat$ with $\epsilon_j \neq \epsilon_{j'}$, vanish. But this condition is not yet sufficient: for instance, if all the ϵ_j's are equal, the diagonal form (7.36) is necessary to symmetrize $\mathrm{diag}\{\mu_j\}$. On the other hand, the spectral condition on $B^\sharp$ is too weak to ensure the existence of a symmetrizer of $B^\flat$ having this diagonal form. This leads to a second condition, which requires that there exists a *basis* $\{e_j\}$ *adapted to* $B^\sharp$ (see Definition 8.8 below). These questions will be discussed in Section 8.2 below.

7.5. Totally nonglancing modes and symmetrizable systems

Proposition 7.1 ([GMWZ6]). *Suppose that the system is symmetric in the sense of Definition 2.5. Then, there are K-families of symmetrizers for $\check H_k$ if either $\mathbb{E}_k^- = \mathbb{E}_k$ or $\mathbb{E}_k^- = \{0\}$.*

Proof. By symmetry, $-\check{\underline\tau}$ is a semi-simple eigenvalue of $A(\underline p, \check{\underline\xi})$, say of multiplicity m. In [MéZu2], it is proved that the assumption on $\mathbb{E}_k^-$ implies that the multiplicity of $\underline\mu_k$ as an eigenvalue of $H_0(\underline p, \check{\underline\zeta}) = \check H(\underline p, \check{\underline\zeta}, 0)$ is equal to m. Denote by V_k a $N\times m$ matrix the columns of which form a basis of $\mathbb{E}_k$, so that $\mathbb{E}_k(p, \check\zeta, \rho) = V_k(p, \check\zeta, \rho)\mathbb{C}^m$. Thus

$$V_k \check H_k = \check H V_k. \tag{7.39}$$

By assumption, there is a definite positive matrix $S(p)$ such that the SA_j and $SB_{j,k}$ are symmetric.

Lemma 7.2. *The symmetric matrix*

$$\Sigma_{k,0}(p, \zeta) = -V_k^*(p, \zeta, 0) S(p) A_d(p) V_k(p, \zeta, 0)\,. \tag{7.40}$$

is a symmetrizer for $\check H_k$ on a neighborhood of $(\underline p, \check{\underline\zeta}, 0)$. More precisely, there holds

$$\mathrm{Re}\, \Sigma_k \check H_k = \gamma R_1 + \rho R_2 \tag{7.41}$$

with $\Sigma_1(\underline p, \check{\underline\zeta}, 0)$ and $\Sigma_2(\underline p, \check{\underline\zeta}, 0)$ definite positive.

In addition, $\Sigma_k(\underline p, \underline\zeta, 0)$ is definite positive [resp. negative] when $\mathbb{E}_k^- = \mathbb{E}_k$ [resp. $\mathbb{E}_k^- = \{0\}$].

Proof. According to (6.24), there holds

$$\check H(p, \check\zeta, \rho) = H_0(p, \check\zeta) + \rho H'(p, \check\zeta, \rho).$$

Using (7.39) and the definition (4.12) of H_0, one obtains the identity (7.41) with

$$R_1 = V_k^* S V_k, \tag{7.42}$$

$$R_2 = V_k^* (\mathrm{Re}\, S A_d H') V_k. \tag{7.43}$$

Because S is definite positive, R_1 also has this property. Next, Lemma 8.24 implies that $H'(p, \check\zeta, 0) = -H_1(p, \check\zeta)$ with H_1 given by (8.52). Since $H_0(\underline{p}, \underline{\check\zeta}) = \mu_k \mathrm{Id} = -i\xi_k \mathrm{Id}$ on $\mathbb{E}_k(\underline{p}, \underline{\check\zeta}, 0)$, there holds

$$H'(\underline{p}, \underline{\check\zeta}, 0) V_k(\underline{p}, \underline{\check\zeta}, 0) = -A_d^{-1}(\underline{p}) B(\underline{p}, \underline{\check\xi}).$$

Therefore, at the base point $(\underline{p}, \underline{c}z, 0)$, there holds

$$R_2(\underline{p}, \underline{c}z, 0) = V_k^* (\mathrm{Re}\, SB) V_k.$$

The symmetry assumption implies that SB is definite positive on the space $\mathbb{E}_k(\underline{p}, \underline{\check\zeta}, 0) = \ker(A(\underline{p}, \underline{\check\xi}) + \underline{\check\tau}\mathrm{Id})$, implying that R_2 is definite positive at $(\underline{p}, \underline{c}z, 0)$, hence on a neighborhood of that point.

That $\Sigma_k(\underline{p}, \underline{\zeta}, 0)$ is definite positive [resp. negative] when the mode is totally incoming [resp. outgoing] is proved in [MéZu2]. □

This implies that

$$\Sigma_k^\kappa = \begin{cases} \Sigma_k & \text{in the incoming case,} \\ \kappa\Sigma_k & \text{in the outgoing case.} \end{cases} \tag{7.44}$$

are K-familes of symmetrizers for $\check H_k$.

□

8. Main results from [MéZu2] and [GMWZ6]

The reader is refereed to [MéZu2] and [GMWZ6] for complete proofs of the results quoted in this section.

8.1. Hyperbolic multiple roots

We first recall several notations and definitions concerning the characteristic roots of the hyperbolic part L. For simplicity, we suppose, as we may, that the coefficient of ∂_t is $A_0 = \mathrm{Id}$, so that, with notations (2.4), $L = \overline{L}$. The characteristic determinant is denoted by

$$\Delta(p, \tau, \xi) := \det(\tau \mathrm{Id} + A(p, \xi)). \tag{8.1}$$

Definition 8.1. *Consider a root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of $\Delta(\underline{p}, \underline{\tau}, \underline{\xi})) = 0$, of algebraic multiplicity m in τ.*

i) $(\underline{p}, \underline{\tau}, \underline{\xi})$ is algebraically regular, if on a neighborhood ω of $(\underline{p}, \underline{\xi})$ there are m smooth real functions $\lambda_j(p, \xi)$, analytic in ξ, such that $\lambda_j(\underline{p}, \underline{\xi}) = -\underline{\tau}$ and for $(p, \xi) \in \omega$:

$$\Delta(p, \tau, \xi) = e(p, \tau, \xi) \prod_{j=1}^{m} \Big(\tau + \lambda_j(p, \xi)\Big) \tag{8.2}$$

where e is a polynomial in τ with smooth coefficients such that $e(\underline{p}, \underline{\tau}, \underline{\xi}) \neq 0$.

ii) $(\underline{p}, \underline{\tau}, \underline{\xi})$ *is geometrically regular if in addition there are* m *smooth functions* $e_j(p,\xi)$ *on* ω *with values in* $\mathbb{C}^N$*, analytic in* ξ*, such that*

$$A(p,\xi)e_j(p,\xi) = \lambda_j(p,\xi)e_j(p,\xi), \tag{8.3}$$

and the $e_1, \ldots, e_m$ *are linearly independent.*

iii) $(\underline{p}, \underline{\tau}, \underline{\xi})$ *is semi-simple with constant multiplicity if all the* λ_j*'s are equal.*

Case *iii)* occurs when $\lambda(p,\xi)$ is a continuous semi-simple eigenvalue of $A(p,\xi)$ with constant multiplicity near $(\underline{p}, \underline{\xi})$, such $\underline{\tau} + \lambda(\underline{p}, \underline{\xi}) = 0$. This implies that λ is smooth and analytic in ξ as well as the eigenspace $\ker(A - \lambda)$. In this case, one can choose for $\{e_j\}$ any smooth basis of of this eigenspace.

If all the roots at $(\underline{p}, \underline{\xi})$ are geometrically regular, then, locally near $(\underline{p}, \underline{\xi})$, $A(p,\xi)$ is smoothly diagonalizable, meaning that it has a smooth basis of eigenvectors.

Example 8.2. For the inviscid MHD, the multiple eigenvalues are algebraically regular, but some are not geometrically regular (see [MéZu2]).

The second notion which plays an important role in the analysis of hyperbolic boundary value problems is the notion of *glancing modes* . Recall from [MéZu2] the following definition. If $\underline{\tau}$ is a root of multiplicity m of the polynomial $\Delta(\underline{p}, \cdot, \underline{\xi})$, then by hyperbolicity, the Taylor expansion of Δ at $(\underline{p}, \underline{\tau}, \underline{\xi})$ at the order $m-1$ vanishes so that

$$\Delta(\underline{p}, \underline{\tau} + \tau, \underline{\xi} + \xi) = \underline{\Delta}_m(\tau, \xi) + O(|\tau, \xi|^{m+1}) \tag{8.4}$$

and $\underline{\Delta}_m$ is homogeneous of degree m. Moreover, $\underline{\Delta}_m$ is hyperbolic in the time direction. Indeed, any direction of hyperbolicity for $\Delta(\underline{p}, \cdot)$ is a direction of hyperbolicity for $\underline{\Delta}_m$. Denote by $\underline{\Gamma}_+$ the open convex cone of hyperbolic directions fot $\underline{\Delta}_m$ which contains dt.

Definition 8.3. *The root* $(\underline{p}, \underline{\tau}, \underline{\xi})$ *of* Δ*, of multiplicity* m*, is said nonglancing when the boundary is noncharacteristic for* $\underline{\Delta}$*.*

It is totally incoming [resp. outgoing] when the inward [resp. outward] conormal to the boundary belongs to $\underline{\Gamma}_+$*.*

It is totally nonglancing if is either totally incoming or totally outgoing.

Example 8.4. This definition agrees with the usual one for simple roots, given by $\tau + \lambda(p,\xi) = 0$. In this case $\partial_t + \nabla_\xi \lambda \cdot \partial_x$ is is the Hamiltonian transport field for the propagation of singularities or oscillations and the glancing condition $\partial_{\xi_d}\lambda = 0$ precisely means that the field is tangent to the boundary. More generally, if the root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of Δ is algebraically regular, then, with notations as in (8.2)

$$\underline{\Delta}_m(\tau, \xi) = e(\underline{p}, \underline{\tau}, \underline{\xi}) \prod_{j=1}^{m} \Big(\tau + \xi \cdot \nabla_\xi \lambda_j(\underline{p}, \underline{\xi})\Big). \tag{8.5}$$

The mode is nonglancing if none of the tangential speed $\partial_{\xi_d}\lambda_j(\underline{p}, \underline{\xi})$ vanish. It is totally incoming [resp. outgoing] if they all are positive [resp. negative]. In particular, in the constant multiplicity case, all the λ_j are equal and they are all glancing, incoming or outgoing at the same time.

In the study of boundary value problems, the dichotomy incoming vs outgoing plays a crucial role: for instance, for transport equations one boundary condition is needed in the first case and none in the second. The symmetrizers are constructed in opposite ways. The general Kreiss construction also reflects this dichotomy. Introduce the following definition:

Definition 8.5. *Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is an algebraically regular root of Δ. With notations as in (8.2), denote by ν_j the order of $\underline{\xi}_d$ as a root of order of the equation $\underline{\tau} + \lambda_j(\underline{p}, \underline{\xi}_1, \ldots, \underline{\xi}_{d-1}, \cdot) = 0$, that is the positive integer such that*

$$\partial^a_{\xi_d}\lambda_j(\underline{p}, \underline{\xi}) = 0 \quad \text{for } a < \nu_j \quad \text{and} \quad \beta_j := \frac{1}{\nu_j!}\partial^{\nu_j}_{\xi_d}\lambda_j(\underline{p}, \underline{\xi}) \neq 0. \tag{8.6}$$

We say that λ_j is of type I when either ν_j is even or ν_j is odd and $\beta_j > 0$. It is of type O when ν_j is odd and $\beta_j < 0$.

We denote by J_O [resp. J_I] the set of indices j of the corresponding type.

Remark 8.6. When $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ is non glancing, then the all the ν_j are equal to 1, and being of type I [resp. type O] means to be incoming [resp. outgoing]. They are all of the same type exactly when the mode is totally nonglancing.

Remark 8.7. The details of the construction of Kreiss' symmetrizers depend strongly on being of type I or O, see [Kre, ChPi, Mét4]. There are no reason other than technical why even roots are of type I rather than O.

8.2. The decoupling condition

The spectral properties of $A(\xi)$ are modified by the perturbation B. In particular, since the construction of symmetrizers depends deeply on the property of being incoming/outgoing, it is very important that the perturbation respects the decoupling between the different type of modes.

Definition 8.8. *Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is a geometrically regular root of Δ of order m. Consider a basis $\{e_j\}$ as in (8.3) and dual left eigenvectors $\underline{\ell}_j$ such that*

$$\underline{\ell}_j\Big(\underline{\tau}\mathrm{Id} + A(\underline{p}, \underline{\xi})\Big) = 0, \quad \underline{\ell}_j \cdot e_{j'}(\underline{p}, \underline{\xi}) = \delta_{j,j'}. \tag{8.7}$$

Consider the $m \times m$ matrix with entries

$$B^\sharp_{j,j'} = \underline{\ell}_j B(\underline{p}, \underline{\xi}) e_{j'}(\underline{p}, \underline{\xi}). \tag{8.8}$$

i) We say that the decoupling condition is satisfied if

$$B^\sharp_{j,j'} = 0 \quad \textit{when} \quad (j, j') \in (J_O \times J_I) \cup (J_I \times J_O) \tag{8.9}$$

where J_0 and J_I are introduced in Definition 8.5.

ii) We say that the basis $\{e_j\}$ is adapted to B if

$$\mathrm{Re}\, B^\sharp > 0. \tag{8.10}$$

Definition 8.9. *We say that the root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of Δ satisfies the condition (BS) if it is a geometrically regular root, it satisfies the decoupling condition (8.9) and there is an eigenbasis $\{e_j\}$ adapted to B.*

We give now several examples and counterexamples.

Theorem 8.10 (Constant multiplicity)**.** *Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is a semi-simple characteristic root with constant multiplicity of Δ. Then the condition (BS) is satisfied.*

Proof. For semi-simple characteristic root λ with constant multiplicity either J_O or J_I is empty so that the decoupling condition (8.9) is trivially satisisfied. Moreover, it is proved in [MéZu1] that (H1) implies that the spectrum of $B^\sharp$ is located in $\{\operatorname{Re} z > 0\}$. Thus there is a basis $\{\underline{e}_j\}$ in $\ker(A(\underline{p},\underline{\xi}) + \underline{\tau}\mathrm{Id})$ such that $\operatorname{Re} B^\sharp$ is definite positive. Next, since any smooth basis $\{e_j\}$ in $\ker(A-\lambda)$ satisfies (8.3), one can choose it such that $e_j(\underline{p},\underline{\xi}) = \underline{e}_j$. □

Proposition 8.11 (Artificial viscosity). *Suppose that $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular for $iA+B$ in the sense that there are m smooth functions $\lambda_j(p,\xi,\rho)$ and m linearly independent smooth vectors $e_j(p,\xi,\rho)$ on a neighborhood of $(\underline{p},\underline{\xi},\rho)$, analytic in ξ, such that $\lambda_j(\underline{p},\underline{\xi},0) = -\underline{\tau}$ for all j and*

$$\Big(iA(p,\xi) + \rho B(p,\xi)\Big)e_j(p,\xi,\rho) = i\lambda_j(p,\xi,\rho)e_j(p,\xi,\rho). \tag{8.11}$$

Then, the decoupling condition is satisfied and the basis $\{e_{j|\rho=0}\}$ is adapted to B.

Proof. Alternately, differentiating (8.3) with respect to ρ and multiplying on the left by $\underline{\ell}_{j'}$, implies that $B^\sharp_{j',j} = 0$ when $j \neq j'$. Moreover, (H1) implies that $B^\sharp_{j,j} > 0$. □

For example, if $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular for A in the sense of Definition 8.1 and if $B = \Delta_x\mathrm{Id}$ is an artificial viscosity, then $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular for $iA+B$. However, this condition is too restrictive for applications, in particular when A and B do not commute.

Example 8.12. If the root is totally nonglancing, then the decoupling condition is trivially satisfied since either J_I or J_O is empty. This applies to fast shocks in MHD.

Counter example 8.13. Slow shocks in MHD do not satisfy the decoupling condition, see [GMWZ6].

The decoupling condition is crucial in the construction of symmetrizers. The second condition (8.10) is more technical. One could expect that with the positivity Assumption (H1), one could always find an adapted basis. This is not clear, except for mutliplicity 2 or symmetric systems.

Proposition 8.14. *Suppose that $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular of multiplicity m. Assume that either $m=2$ or that the symmetry assumption (H1') is satisfied. There is a basis $\{e_j\}$ adapted to B.*

If in addition all the eigenvalues λ_j are of the same type O or I, then the condition (BS) is satisfied.

Finally, we recall from [GMWZ6] that the decoupling condition is necessary for the existence of K-family of symmetrizers and even more, for the continuity of the negative space $\mathbb{E}^-_k$.

Theorem 8.15. *Suppose that $(\underline{p},\underline{\check\tau},\underline{\check\xi})$ is geometrically regular and nonglancing and suppose that there exist $j \in J_I$ and $j' \in J_O$ such that*

$$B^\sharp_{j',j} \neq 0. \tag{8.12}$$

Then the negative space $\mathbb{E}^-_k(\underline{p},\check\zeta,\rho)$ has no limit as $(\check\zeta,\rho) \to (\underline{\check\zeta},0)$.

In particular, there are no smooth K-families of symmetrizers for $\check H_k$ near $(\underline{p},\underline{\check\zeta})$.

8.3. The hyperbolic block structure condition

We turn back to the construction of symmetrizers for nonelliptic blocks $\check{H}_k$ in the splitting (6.28). The construction of K-families is performed in [MéZu1] provided that $\check{H}_k$ can be put in a suitable normal form. This is the so called *block structure condition.* We first review this condition in the hyperbolic case, and next extend it to the hyperbolic-parabolic case.

Consider $\underline{p}$ and a frequency $\underline{\check{\zeta}} = (\underline{\check{\tau}}, \underline{\check{\eta}}, 0) \neq 0$ and a purely imaginary eigenvalue (6.31) $\underline{\mu}_k = i\underline{\check{\xi}}_d$ of $H_0(\underline{p}, \underline{\check{\zeta}})$. Let $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$. Then $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is a root of Δ. We consider the block $\check{H}_k$ associated to $\underline{\mu}_k$ and denote by $\mathbb{E}_k$ the corresponding invariant space of $\check{H}$. We use the notations $\check{H}_{k,0}(p, \check{\zeta}) = \check{H}_k(p, \check{\zeta}, 0)$ and $\mathbb{E}_{k,0}(p, \check{\zeta}) = \mathbb{E}_k(p, \check{\zeta}, 0)$.

Definition 8.16. *$\check{H}_{k,0}$ has the block structure property near $(\underline{p}, \underline{\check{\zeta}})$ if there exists a smooth invertible matrix $V_{k,0}$ on a neighborhood of that point such that $V_{k,0}^{-1}\check{H}_{k,0}V_{k,0}$ is block diagonal,*

$$V_{k,0}^{-1}\check{H}_{k,0}V_{k,0} = \begin{bmatrix} Q_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & Q_{m'} \end{bmatrix}, \tag{8.13}$$

with diagonal blocks Q_j of size $\nu_j \times \nu_j$ such that :

$Q_j(p, \check{\zeta})$ has purely imaginary coefficients when $\check{\gamma} = 0$,

$$Q_j(\underline{p}, \underline{\check{\zeta}}) = \underline{\mu}_k \mathrm{Id} + i \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & \cdots & 0 \end{bmatrix}, \tag{8.14}$$

and the real part of the lower left hand corner of $\partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\zeta})$, denoted by $q_j^\flat$, does not vanish.

When $\nu_j = 1$, $Q_j(p, \check{\zeta})$ is a scalar. In this case, (8.14) has to be understood as $Q_j(\underline{p}, \check{\zeta}) = \underline{\mu}_k$, with no Jordan's block. The lower left hand corner of the matrix is Q_j itself and the condition reads $q_j^\flat := \partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\check{\zeta}}) \neq 0$.

Proposition 8.17 ([MéZu2]). *If the root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is geometrically regular in the sense of Definition* 8.1*, the corresponding block $\check{H}_{k,0}$ satisfies the block structure condition.*

Conversely, if $\check{H}_{k,0}$ satisfies the block structure condition with matrices V that are real analytic in $\check{\zeta}$, then the root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is geometrically regular.

Remark 8.18. There is a slight discrepancy here between the necessary and the sufficient condition, due to analyticity conditions. Definition 8.1 requires analyticity in $\check{\xi}$. This is used in the proof of sufficiency. In addition, it implies that the block structure condition holds with matrices V that are real analytic in $\check{\zeta}$. Thus, there is an "if and only if" theorem. However, for the construction of symmetrizers, analyticity of V_k is not needed, this is why we do not insist on it in the definition above. In addition, note that *for fixed* p, the existence of C^∞ eigenvalues and eigenvectors for A, implies that these eigenvalues are real analytic in ξ and that one can choose analytic eigenvectors (see e.g [Shi, Mal]). The question is to control the domain of analyticity as p varies. In applications, for this problem, proving analyticity is not harder than proving the C^∞ smoothness.

To prepare the hyperbolic-parabolic analysis, we have to review the proof of Proposition 8.17. In particular, we reformulate the conditions of Definition 8.16 in a more intrinsic way. The choice of a smooth matrix $V_{k,0}$ is equivalent to the choice of a smooth basis of $\mathbb{E}_{k,0}$, denoted by $\{\varphi_{j,a}(p,\check\zeta)\}_{1\le j\le m',1\le a\le \nu_j}$. The property (8.14) reads

$$(H_0(\underline{p},\underline{\check\zeta}) - \underline{\mu}_k)\varphi_{j,1}(\underline{p},\underline{\check\zeta}) = 0, \tag{8.15}$$

$$(H_0(\underline{p},\underline{\check\zeta}) - \underline{\mu}_k)\varphi_{j,a}(\underline{p},\underline{\check\zeta}) = i\varphi_{j,a-1}(\underline{p},,\underline{\check\zeta}), \quad 2\le a\le \nu_j. \tag{8.16}$$

With (6.29), there is a unique smooth dual basis $\psi_{j,a}(p,\check\zeta)$ such that

$$\begin{aligned} &\psi_{j,a}\cdot \mathbb{E}'_{k,0} = 0,\\ &\psi_{j,a}\cdot\varphi_{j',a'} = \delta_{j,j'}\delta_{a,a'}. \end{aligned} \tag{8.17}$$

Here, $\mathbb{E}'_{k,0}$ denotes the invariant space of $H_0(p,\check\zeta)$ such that $\mathbb{C}^N = \mathbb{E}_{k,0}\oplus\mathbb{E}'_{k,0}$. It is the sum of invariant subspaces associated to eigenvalues $\underline{\mu}_{k'}\ne\underline{\mu}_k$.

In the basis $\varphi_{j,a}$, the entries of the matrix $V_{k,0}^{-1}\check H_{k,0}V_{k,0}$ are $\psi_{j,a}H_0\varphi_{j',a'}$. The diagonal block structure means that

$$\psi_{j,a}H_0\varphi_{j',a'} = 0 \quad\text{when}\quad j\ne j'. \tag{8.18}$$

The other conditions read:

$$\operatorname{Re}(\psi_{j,a}H_0\varphi_{j,a'}) = 0 \quad\text{when}\quad \check\gamma = 0, \tag{8.19}$$

$$\operatorname{Re}\partial_{\check\gamma}(\psi_{j,\nu_j}H_0\varphi_{j,1})(\underline{p},\underline{\check\zeta}) \ne 0. \tag{8.20}$$

We first show how to compute this quantity in terms of A only.

Lemma 8.19. *Suppose that $\check H_{k,0}$ has a block diagonal decomposition* (8.13) *in a smooth basis $\varphi_{j,a}$ of $\mathbb{E}_k(p,\zeta,0)$ which satisfies* (8.15) (8.16)*. Let $\psi_{j,a}$ denote a dual basis satisfying* (8.17)*. The lower left hand corner entry of $\partial_{\check\gamma}Q_j(\underline{p},\underline{\check\zeta})$ is equal to the lower left hand corner entry of $-i\partial_{\check\tau}Q_j(\underline{p},\underline{\check\zeta})$ and equal to*

$$\underline{q}_j = -\psi_{j,\nu_j}(\underline{p},\underline{\check\zeta})\,A_d^{-1}(\underline{p})\,\varphi_{j,1}(\underline{p},\underline{\check\zeta}). \tag{8.21}$$

Proof. Let $\underline{H}_0 = H_0(\underline{p},\underline{\check\zeta})$. Then $\underline{H}_0 - \underline{\mu}_k$ is invertible on $\mathbb{E}'_{k,0}(\underline{p},\underline{\check\zeta})$. With (8.15) (8.16), this implies that

$$\operatorname{range}\Big(\underline{H}_0 - \underline{\mu}_k\mathrm{Id}\Big) = \{\psi_{1,\nu_1}(\underline{p},\underline{\check\zeta}),\dots,\psi_{m',\nu_{m'}}(\underline{p},\underline{\check\zeta})\}^\perp, \tag{8.22}$$

$$\ker\Big(\underline{H}_0 - \underline{\mu}_k\mathrm{Id}\Big) = \{\varphi_{1,1}(\underline{p},\underline{\check\zeta}),\dots,\varphi_{m',1}(\underline{p},\underline{\check\zeta})\}. \tag{8.23}$$

In particular,

$$\Big(\underline{H}_0 - \underline{\mu}_k\mathrm{Id}\Big)\varphi_{j,1} = 0 \quad\text{and}\quad \psi_{j,\nu_j}\Big(\underline{H}_0 - \underline{\mu}_k\mathrm{Id}\Big) = 0. \tag{8.24}$$

The entry in consideration is

$$q_j(p,\check\zeta) = \psi_{j,\nu_j}H_0\varphi_{j,1} = \psi_{j,\nu_j}\Big(H_0 - \underline{\mu}_k\mathrm{Id}\Big)\varphi_{j,1} + \underline{\mu}_k\delta_{\nu_j,1}.$$

Therefore, differentiating in $\check\gamma$ and $\check\tau$ and using (4.12), implies that

$$\partial_{\check\gamma}q_j(\underline{p},\underline{\check\zeta}) = -i\partial_{\check\tau}q_j(\underline{p},\underline{\check\zeta}) = \underline{q}_j \tag{8.25}$$

is given by (8.21). $\square$

We now discuss how much flexibility there is in the choice of the basis $\varphi_{j,a}$. Recall that we are considering a purely imaginary eigenvalue $\underline{\mu}_k = i\underline{\xi}_d$ of $H_0(\underline{p}, \underline{\zeta})$, so that $-\underline{\check\tau}$ is an eigenvalue $\underline{\lambda}$ of $A(\underline{p}, \underline{\check\xi})$ with $\underline{\check\xi} = (\underline{\check\eta}, \underline{\check\xi}_d)$.

Lemma 8.20. *Suppose that $\check H_{k,0}$ has the block structure property near $(\underline{p}, \underline{\check\zeta})$ in a smooth basis $\varphi_{j,a}$ and denote by $\psi_{j,a}$ the dual basis* (8.17)*. Then,*

i) $\underline{\lambda}$ is a semi-simple eigenvalue of $A(\underline{p}, \underline{\check\xi})$ with multiplicity m equal to the number m' of blocks Q_j,

ii) on a neighborhood of $(\underline{p}, \underline{\check\xi})$, there are m smooth eigenvalues $\lambda_j(p, \check\xi)$ of $A(p, \check\xi)$ and m smooth linearly independent eigenvectors $e_j(p, \check\xi)$, such that

$$\lambda_j(\underline{p}, \underline{\check\xi}) = \underline{\lambda}, \tag{8.26}$$

$$A(p, \check\xi) e_j(p, \check\xi) = \lambda_j(p, \check\xi) e_j(p, \check\xi), \tag{8.27}$$

$$e_j(\underline{p}, \underline{\check\xi}) = \varphi_{j,1}(\underline{p}, \underline{\check\zeta}), \tag{8.28}$$

iii) the order of $\underline{\check\xi}_d$ as a root of $\underline{\check\tau} + \lambda_j(\underline{p}, \underline{\check\eta}, \cdot) = 0$ is equal to ν_j,

iv) denoting by $\{\underline{\ell}_j\}$ the left eigenvector dual basis of $\{e_j\}$ as in (8.7)*, there holds*

$$\underline{\ell}_j A_d(\underline{p}) = \beta_j \psi_{j,\nu_j}(\underline{p}, \underline{\check\zeta}). \tag{8.29}$$

with $\beta_j := \frac{1}{\nu_j!} \partial_{\xi_d}^{\nu_j} \lambda_j(\underline{p}, \underline{\check\xi})$ as in (8.6)*,*

v) the lower left hand corner entry of $\partial_{\check\gamma} Q_j(\underline{p}, \underline{\check\zeta})$ is

$$\underline{q}_j = -1/\beta_j \in \mathbb{R}. \tag{8.30}$$

Proof. **a)** Define $\tilde\varphi_{j,\nu_j} = \varphi_{j,\nu_j}$ and for $a < \nu_j$

$$\tilde\varphi_{j,a}(p, \zeta) = -i\Big(H_0(p, \zeta) - \underline{\mu}_k\Big)\varphi_{j,\nu_j}. \tag{8.31}$$

By (8.13)(8.14), there holds

$$\tilde\varphi_{j,a}(\underline{p}, \underline{\check\zeta}) = \varphi_{j,a}(\underline{p}, \underline{\check\zeta}). \tag{8.32}$$

Moreover, in the new basis $\tilde\varphi_{j,a}$, the matrix of Q_j has the form

$$Q_j = i\underline{\check\xi}_d \mathrm{Id} + i \begin{pmatrix} * & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ * & 0 & \dots & 1 \\ * & 0 & \dots & 0 \end{pmatrix}. \tag{8.33}$$

Thanks to (8.32), the dual basis $\{\tilde\psi_{j,a}\}$ associated to $\{\tilde\varphi_{j,a}\}$ also satisfies $\tilde\psi_{j,a}(\underline{p}, \underline{\check\zeta}) = \psi_{j,a}(\underline{p}, \underline{\check\zeta})$. This implies that the lower left hand corner of $\partial_{\check\gamma} Q_j(\underline{p}, \underline{\check\zeta})$ is unchanged in the new basis.

b) Consider the determinant

$$\Delta_j(p, \check\zeta, \check\xi_d) = \det\Big(\xi_d \mathrm{Id} + iQ_j(p, \check\zeta)\Big).$$

It is independent of the basis $\{\psi_{j,a}\}$ or $\{\tilde\psi_{j,a}\}$. Thus, it is real when $\check\gamma = 0$ and vanishes at $(\underline{p}, \underline{\check\zeta}, \underline{\check\xi}_d)$. Moreover, (8.14) implies that

$$\partial_{\check\tau} \Delta_j(\underline{p}, \underline{\check\zeta}, \underline{\xi}_d) = -\underline{q}_j.$$

As a byproduct, using also (8.25) this shows that

$$\underline{q}_j \in \mathbb{R} \quad \text{thus} \quad \underline{q}_j = \operatorname{Re} \underline{q}_j = q_j^\flat \neq 0. \tag{8.34}$$

In particular, the implicit function theorem implies that there is a smooth function $\lambda_j(p, \check\xi)$, in a real neighborhood of $(\underline{p}, \underline{\check\xi})$, such that $\lambda_j(\underline{p}, \underline{\check\xi}) = -\underline{\check\tau}$ and for $\check\zeta = (\check\tau, \check\eta, 0)$:

$$\Delta_j(p, \check\zeta, \check\xi_d) = \alpha_j(p, \check\zeta, \check\xi_d)\Big(\check\tau + \lambda_j(p, \check\xi)\Big) \tag{8.35}$$

with $\alpha_j(\underline{p}, \underline{\check\zeta}, \underline{\check\xi}_d) \neq 0$.

c) Consider next the eigenvector equation

$$\Big(\check\xi_d \mathrm{Id} + iQ_j(p, \check\zeta)\Big) e_j = 0. \tag{8.36}$$

By (8.33), in the basis $\{\tilde\psi_{j,a}\}$, the $\nu_j - 1$ first equations determine the last $\nu_j - 1$ components of e_j

$$(e_j)_a = (\check\xi_d - \underline{\check\xi}_d)^{a-1}(e_j)_1, \quad a \geq 2. \tag{8.37}$$

Substituting these values, the last equation is a scalar equation equivalent to $\Delta_j = 0$. Introduce

$$\zeta_j(p, \check\eta, \check\xi) = \Big(-\lambda_j(p, \check\xi), \check\eta, 0\Big),$$

and

$$e_j(p, \check\xi) = \tilde\varphi_{j,1}(p, \check\zeta) + \sum_{a=2}^{\nu_j} (\check\xi_d - \underline{\check\xi}_d)^{j-1} \tilde\varphi_{j,a}(p, \check\zeta). \tag{8.38}$$

This vector is smooth and satisfies (8.36), thus

$$\Big(A(p, \check\xi) - \lambda_j(p, \check\xi)\mathrm{Id}\Big) e_j(p, \check\xi) = A_d(p)\Big(iH_0(p, \check\zeta_j) + \check\xi_d \mathrm{Id}\Big) e_j(p, \check\xi) = 0.$$

Moreover, the $e_j(\underline{p}, \underline{\check\xi}) = \varphi_{j,1}(\underline{p}, \underline{\check\zeta})$ are linearly independent.

d) By (8.35), for $\check\zeta = (\check\tau, \check\eta, 0)$, there holds

$$\begin{aligned} \det\Big(\check\tau \mathrm{Id} + A(p, \check\xi)\Big) &= \det(A_d) \det\Big(iH_0(p, \check\zeta) + \check\xi_d \mathrm{Id}\Big) \\ &= \alpha(p, \check\tau, \check\xi) \prod_{j=1}^{m'} \Big(\check\tau + \lambda_j(p, \check\xi)\Big) \end{aligned}$$

where $\alpha(\underline{p}, \underline{\check\tau}, \underline{\check\xi}) \neq 0$ and m' is the number of blocks Q_j. This shows that $-\underline{\check\tau}$ is an eigenvalue of algebraic order m' of $A(\underline{p}, \underline{\check\xi})$. By step c), the geometric multiplicity is at least m', implying that $-\underline{\check\tau}$ is semi-simple of order m'.

Moreover, by (8.15), there holds

$$\Delta_j(\underline{p}, \underline{\check\zeta}, \check\xi_d) = (\check\xi_d - \underline{\check\xi}_d)^{\nu_j},$$

showing that $\underline{\check\xi}_d$ is a root of multiplicity ν_j of Δ_j, thus of $\underline{\check\tau} + \lambda_j(\underline{p}, \underline{\check\eta}, \check\xi) = 0$.

e) Let $\underline{\ell}_j$ satisfy (8.7). Thus

$$\begin{aligned} \mathrm{Range}\Big(\check H_0(\underline{p}, \underline{\check\zeta}) - \underline{\mu}_k \mathrm{Id}\Big) &= A_d^{-1}(\underline{p}) \mathrm{Range}\Big(\underline{\check\tau}\mathrm{Id} + A(\underline{p}, \underline{\check\xi})\Big) \\ &= A_d^{-1}(\underline{p})\{\underline{\ell}_1, \ldots, \underline{\ell}_m\}^\perp. \end{aligned}.$$

Comparing with (8.22), this implies that

$$\mathrm{span}\Big\{\psi_{j,\nu_j}(\underline{p}, \underline{\check\zeta}), 1 \leq j \leq m\Big\} = \mathrm{span}\Big\{\underline{\ell}_j, 1 \leq j \leq m\Big\}. \tag{8.39}$$

For $a \in \{1, \dots, \nu_j\}$, introduce

$$\underline{e}_{j,a} = \frac{1}{(a-1)!} \partial_{\xi_d}^{a-1} e_j(\underline{p}, \underline{\check{\xi}}). \tag{8.40}$$

Because $\underline{\check{\xi}}_d$ is a root of order ν_j of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\eta}}, \check{\xi}) = 0$, the definition (8.38) implies that

$$\underline{e}_{j,a} = \tilde{\varphi}_{j,a}(\underline{p}, \underline{\check{\zeta}}) = \varphi_{j,a}(\underline{p}, \underline{\check{\zeta}}) \quad \text{for } 1 \leq a \leq \nu_j.$$

In particular, (8.17) implies that

$$\psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}}) \cdot \underline{e}_{j,\nu_j} = \psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}}) \cdot \varphi_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}) = \delta_{j,j'}. \tag{8.41}$$

Differentiating the equation

$$\Big(A(\underline{p}, \check{\xi}) - \lambda_j(\underline{p}, \check{\xi}) \Big) e_j(\underline{p}, \check{\xi}) = 0 \tag{8.42}$$

with respect to $\check{\xi}_d$ and at order ν_j yields

$$\Big(\underline{\check{\tau}} \mathrm{Id} + A(\underline{p}, \underline{\check{\xi}}) \Big) \partial_{\xi_j}^{\nu_j} e_j(\underline{p}, \underline{\check{\xi}}) = -\nu_j A_d(\underline{p}) \partial_{\xi_d}^{\nu_j - 1} e_j(\underline{p}, \underline{\check{\xi}}) + \partial_{\xi_j}^{\nu_j} \lambda_j(\underline{p}, \underline{\check{\xi}}) e_j(\underline{p}, \underline{\check{\xi}}).$$

Multiplying on the left by $\underline{\ell}_{j'}$ annihilates the left hand side, implying

$$\underline{\ell}_{j'} A_d(\underline{p}) e_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}) = \beta_j \underline{\ell}_{j'} \cdot e_j(\underline{p}, \underline{\check{\xi}}) = \beta_j \delta_{j',j}.$$

By (8.39), the $\underline{\ell}_j A_d$ and ψ_{j,ν_j} span the same space. , Therefore, comparing with (8.41) implies that $\underline{\ell}_{j'} A_d(\underline{p}) = \beta_j \psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}})$.

f) By (8.21) and (8.29), we have

$$-\beta_j \underline{q}_j = \underline{\ell}_j \varphi_{j,1}(\underline{p}, \underline{\check{\zeta}}) = \underline{\ell}_j e_j(\underline{p}, \underline{\check{\xi}}) = 1.$$

The proof of the lemma is complete. □

Remark 8.21. This lemma is a variation on the necessary part in Proposition 8.17 (see [MéZu2]), with useful additional remarks. It shows that the block structure condition is closely related to a smooth diagonalisation of A. Conversely, if one starts from a smooth basis e_j and a root of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\xi}})$ with (8.6), one constructs a basis $\varphi_{j,a}$ such that $\varphi_{j,a}(\underline{p}, \underline{\check{\zeta}})$ is given by (8.40), using an holomorphic extension of e_j to complex values of $\check{\xi}_d$ (see [MéZu2]). Lemma 8.20 implies that the change of bases which preserve the block structure form are linked to change of bases which preserve the smooth diagonalization of A.

The construction of K-families of symmetrizers for the blocks Q_j is performed in [Kre, Maj1, Mét4]. The sign of β_j and the parity of ν_j play an important role. Hyperbolicity implies that H_0 and thus the $\check{H}_k$ and Q_j have no purely imaginary eigenvalues when $\check{\gamma} > 0$. Denote by $\mathbb{E}^-_{Q_j}$ the invariant space of Q_j associated to the spectrum in $\{\mathrm{Re}\, \mu < 0\}$ since the definition of the limiting space $\underline{\mathbb{E}}^-_{Q_j}$. Recall that the limit space at $(\underline{p}, \underline{\check{\zeta}})$ is

$$\underline{\mathbb{E}}^-_{Q_j} = \mathbb{C}^{\nu'_j} \times \{0\}^{\nu_j - \nu'_j} \tag{8.43}$$

with

$$\nu'_j = \begin{cases} \nu_j/2 & \text{when } \nu_j \text{ is even,} \\ (\nu_j+1)/2 & \text{when } \nu_j \text{ is odd and } \beta_j > 0, \\ (\nu_j-1)/2 & \text{when } \nu_j \text{ is odd and } \beta_j < 0. \end{cases} \tag{8.44}$$

Remark 8.22. As a corollary, we have the following characterization of the sets J_0 and J_I:

$$\begin{cases} j \in J_I & \textit{if } \nu_j \textit{ is even or } \nu_j \textit{ is odd and } q_j^\flat < 0, \\ j \in J_0 & \textit{if } \nu_j \textit{ is odd and } q_j^\flat > 0. \end{cases} \tag{8.45}$$

8.4. The hyperbolic-parabolic case

We still consider a block $\check{H}_k$ associated to a purely imaginary eigenvalue (6.31). In the next section, we show that the following technical conditions are the natural ones for the construction of Kreiss symmetrizers.

Definition 8.23. *$\check{H}_k$ has the generalized block structure property near $(\underline{p}, \underline{\check{\zeta}}, 0)$ if there exists a smooth invertible matrix V_k on a neighborhood of that point such that*

$$V_k^{-1} \check{H}_k V_k = \begin{pmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_m \end{pmatrix} + \rho \begin{pmatrix} \tilde{B}_{1,1} & \cdots & \tilde{B}_{1,m} \\ \vdots & \ddots & \vdots \\ \tilde{B}_{m,1} & \cdots & \tilde{B}_{m,m} \end{pmatrix} \tag{8.46}$$

where the $Q_j(p, \check{\zeta})$ satisfy the properties of Definition 8.16*. Moreover, the $m \times m$ matrix $B^\flat$ with entries $B^\flat_{j,j'}$ equal to the lower left hand corner of $\tilde{B}_{j,j'}(\underline{p}, \underline{\check{\zeta}}, 0)$ satisfies*

$$B^\flat_{j,j'} = 0 \quad \textit{when} \quad (j, j') \in (J_O \times J_I) \cup (J_I \times J_O) \tag{8.47}$$

where J_O and J_I are defined by (8.45) *and there is a real diagonal matrix $D^\flat$, with entries $d_j^\flat$ such that*

$$d_j^\flat q_j^\flat > 0, \quad \operatorname{Re} D^\flat B^\flat > 0. \tag{8.48}$$

We show that these conditions are related to the condition (BS) of Definition 8.9 formulated on the original system. We need first a more detailed form of the block reduction H in (4.16). Introduce the following notations:

$$B_{*,*}(p, \zeta) \ := \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(p), \tag{8.49}$$

$$B_{*,d}(p, \zeta) \ := \sum_{j=1}^{d-1} \eta_j (B_{j,d}(p) + B_{d,j}(p)). \tag{8.50}$$

Lemma 8.24. *One can choose the matrix V in* (4.16) *such that there holds*

$$H(p, \zeta) = H_0(p, \zeta) - H_1(p, \zeta) + O(|\zeta|^3) \tag{8.51}$$

where

$$H_1 = A_d^{-1} \Big(B_{*,*} - i B_{*,d} H_0 - B_{d,d} H_0^2 \Big). \tag{8.52}$$

Proof. Direct computations show that the kernel of $G(p, 0)$ is $\mathbb{C}^N \times \{0\}$ and, using that A_d is invertible, that $\ker G(p, 0) \cap \operatorname{range} G(p, 0) = \{0\}$. This shows that 0 is a semi-simple eigenvalue of $G(p, 0)$.

If μ is a purely imaginary eigenvalue of $G(p, 0)$, then 0 is an eigenvalue of $iA(p, \xi) + B(p, \xi)$ with $\xi = (0, -i\mu)$. By Assumption (H1) this requires that $\xi = 0$, thus $\mu = 0$. This shows that the nonvanishing eigenvalues of $G(p, 0)$ are not on the imaginary axis.

This implies that there is a smooth matrix $V(p,\zeta)$ on a neighborhood of $(\underline{p},0)$ such that (4.16) holds with $H(p,0)=0$ and $P(p,0)$ invertible with no eigenvalue on the imaginary axis.

The image of the first N columns of V is the invariant space of G, and H is the restriction of G to that space. At $\zeta=0$ this space is $\ker G$, and performing a smooth change of basis in $\mathbb{C}^N$, we can always assume that the first N columns of V are of the form

$$V_I(p,\zeta)=\begin{pmatrix}\mathrm{Id}_{N\times N}\\ W(p,\zeta)\end{pmatrix} \tag{8.53}$$

with W of size $N^2\times N$ vanishing at $\zeta=0$. This implies (4.18).

By (4.16) $GV_I=V_IH$, hence $MV_I=G_dV_IH$ and

$$\mathcal{M}=-\mathcal{A}H+\overline{B}_dWH,\quad W=JH.$$

Therefore,

$$\mathcal{M}=-\mathcal{A}H+\overline{B}_dJH^2=-\mathcal{A}H+B_{d,d}H^2. \tag{8.54}$$

Taking the first order term at $\zeta=0$ shows that the first order term in H_0 in H satisfies

$$(i\tau+\gamma)\mathrm{Id}+\sum_{j=1}^{d-1}i\eta_jA_j=-A_d(p)H_0$$

and hence is given by (4.12). The second order term H_1 in H satsifies

$$B_{*,*}=-A_dH_1+iB_{*,d}H_0+B_{d,d}JH_0^2$$

implying (8.51) and (8.52). □

Parallel to Lemma 8.19, we can now state:

Lemma 8.25. *Suppose that the matrix of $\check{H}_k$ is given by the right hand side of* (8.46) *in a smooth basis $\varphi_{j,a}$ of $\mathbb{E}_k(p,\check\zeta,\rho)$ which satisfies* (8.15) *and* (8.16) *for $\rho=0$. Let $\{\underline{\ell}_j\}$ denote the dual basis of $\{e_j=\varphi_{j,1}\}$ satisfying* (8.7)*. The entries of $B^\flat$ are*

$$B^\flat_{j,j'}=-\frac{1}{\beta_j}\underline{\ell}_j\,B(\underline{p},\underline{\check\xi})\,\varphi_{j',1}(\underline{p},\underline{\check\zeta},0). \tag{8.55}$$

Proof. In the block reduction (8.46), the lower left hand corner entry of the (j,j')-block is

$$h_{j,j'}=\psi_{j,\nu_j}\check{H}\varphi_{j',1}=\psi_{j,\nu_j}\big(\check{H}-\underline{\mu}_k\big)\varphi_{j',1}+\underline{\mu}_k\delta_{j,j'}.$$

Differentiating in ρ and using the relations (8.24) yields

$$-B^\flat_{j,j'}=\partial_\rho h_{j,j'}(\underline{p},\underline{\check\zeta},0)=-\underline{\psi}_{j,\nu_j}\,\tilde{B}(\underline{p},\underline{\check\zeta})\,\underline{\varphi}_{j,1},$$

where $\underline{\psi}_{j,\nu_j}$ and $\underline{\varphi}_{j,1}$ stand for the evaluation at $(\underline{p},\underline{\check\zeta},0)$ of the corresponding function. Using the explicit form of $\tilde{B}$ and the relations

$$\underline{H}_0\underline{\varphi}_{j,1}=i\underline{\check\xi}_d\underline{\varphi}_{j,1},\quad \underline{\psi}_{j,\nu_j}\underline{H}_0=i\underline{\check\xi}_d\underline{\psi}_{j,\nu_j}$$

we obtain

$$\begin{aligned}\underline{\psi}_{j,\nu_j}\tilde{B}(\underline{p},\underline{\check\zeta})\underline{\varphi}_{j,1}&=\underline{\psi}_{j,\nu_j}A_d^{-1}\Big(B_{*,*}(\underline{p},\underline{\check\eta})+\underline{\check\xi}_dB_{*;d}(\underline{p},\underline{\check\eta})+\underline{\check\xi}_d^2B_{d,d}(\underline{p})\Big)\underline{\varphi}_{j,1}\\&=\underline{\psi}_{j,\nu_j}B(\underline{p},\underline{\check\xi})\underline{\varphi}_{j,1}\end{aligned}$$

With (8.29), this implies (8.55). □

Theorem 8.26. *If* $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ *is a geometrically regular characteristic root of* Δ *which satisfies the condition (BS) of Definition* 8.9*. Then the associated block* $\check{H}_k$ *satisfies the generalized block structure condition.*

Proof. Since $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ is geometrically regular, the hyperbolic part $\check{H}_{k,0}$ satisfies the block structure condition. Moreover, if e_j is a basis analytic in ξ, there is a basis $\varphi_{j,a}$ such that $\varphi_{j,a}(\underline{p}, \check{\underline{\zeta}}) = e_j(\underline{p}, \check{\underline{\xi}})$ (see Remark 8.21 or [MéZu2]). By Lemma 8.25, (8.9) is equivalent to (8.47).

If once can choose the basis $\{e_j\}$ such that (8.10) holds, then choose $d_j^\flat = -\beta_j$ and by (8.30) and (8.55) there holds $d_j^\flat q_j^\flat = 1$ so that $DB^\flat = B^\sharp$ satisfies (8.48). □

Remark 8.27. Conversely, if the generalized block structure condition holds with matrices V_k which are real analytic in ζ, then, by Proposition 8.17 $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ is geometrically regular. By (8.55), (8.47) is equivalent to the decoupling condition (8.9). Moreover, (8.48) implies that there is a diagonal matrix with positive entries $d_j^\sharp = d_j^\flat / q_j^\flat$ such that $\mathrm{Re}\, D^\sharp B^\sharp > 0$. Consider the diagonal matrix $C = (D^\sharp)^{-1/2} = \mathrm{diag}(c_j)$ and the new basis $\tilde{e}_j = c_j e_j$. The new dual basis is $\tilde{\underline{\ell}}_j = c_j^{-1} \underline{c}_j$ and the new matrix $\tilde{B}^\sharp$ is $C^{-1} B^\sharp C = C D^\sharp B^\sharp C$ and therefore $\mathrm{Re}\, \tilde{B}^\sharp = C \mathrm{Re}\, (D^\sharp B^\sharp) C$ is definite positive.

8.5. Existence of K-families of symmetrizers

We can now state the main results of [MéZu2] and [GMWZ6].

Theorem 8.28. *Suppose that the Assumptions of Section* 2 *are satisfied. Assume further that one of the following two condition is satisfied:*

i) all the real characteristic roots $(\underline{p}, \tau, \xi)$ *with* $|\xi| = 1$ *satisfy the condition (BS) of Definition* 8.9*.*

ii) the system is symmetric dissipative in the sense of Definition 2.5 *and the real characteristic roots* $(\underline{p}, \tau, \xi)$ *with* $|\xi| = 1$ *are either totally nonglancing in the sense of Definition* 8.3 *or satisfy the condition (BS) of Definition* 8.9*.*

Then, for all $\check{\underline{\zeta}} \in \overline{S}^d_+$*, there exist K-families of smooth symmetrizers for* $\check{H}(p, \zeta, \rho)$ *near* $(\underline{p}, \check{\underline{\zeta}}, 0)$*.*

Recall that Theorem 8.10 gives sufficient conditions for the condition (BS) to be satisfied. In particular, there holds

Corollary 8.29. *Suppose that the full system* (2.1) *is symmetric dispersive in the sense of Definition* 2.5*. Suppose in addition that the eigenvalues of the inviscid system are either semi-simple with constant multiplicity or totally nonglancing in the sense of Definition* 8.3*. Then, there are K-families of symmetrizers for the associated reduced system* $\check{H}$*.*

Finally, we recall that the existence of a K-family of symmetrizers implies that the maximal estimates are satisfied when the uniform spectral stability condition is verified.

Theorem 8.30. *Suppose that there exists a K-family of symmetrizers for* $\check{H}$ *near* $(\underline{p}, \check{\underline{\zeta}}, 0)$ *and suppose that the boundary conditions are such that the uniform spectral stability condition is satisfied for low frequencies. Then the uniform stability estimates* (4.38) *are satisfied.*

Similarly, if the reduced boundary conditions satisfy the reduced uniform stability condition then the uniform estimates (5.17) *and* (5.18) *hold true.*

9. The high frequency analysis

9.1. The main high frequency estimate

This section is devoted to an analysis of uniform maximal estimates for high frequencies. We still assume that the Assumptions of Section 2 are satisfied and we prove that the anticipated estimates (4.42) are satisfied when the uniform spectral stability conditions are satisfied, under the following additional structural assumptions which strengthens (H3):

Assumption 9.1. (H8) *For all $u \in \mathcal{U}^*$, $L^{11}(u, \partial)$ is hyperbolic with constant multiplicities in the direction dt.*
(H9) *$L^{11}(u, \partial)$ is also hyperbolic with respect to the normal direction dx_d.*

For Navier-Stokes and MHD equations and in many examples L^{11} is a transport field

$$L^{11} = \partial_t + \sum_{j=1}^{d} a_j(u)\partial_j \tag{9.1}$$

and the condition reduces to $a_d(u) \neq 0$ for $u \in \mathcal{U}^*$, that is to Assumption 2.8, which means inflow or outflow boundary conditions. The hyperbolicity condition (H9) in the normal direction is important as shown on an example below. On the other hand the constant multiplicity condition (H8) is more technical, and could be replaced by symmetry conditions: this is briefly discussed in Remark 9.12.

We consider the linearized equation (4.6):

$$\partial_z u = \mathcal{G}(z, \zeta)u + f, \quad \Gamma(\zeta)u(0) = g \tag{9.2}$$

with $u = {}^t(u^1, u^2, u^3)$, $f = {}^t(f^1, f^2, f^3)$, Γ as in (4.40) and $g = {}^t(g^1, g^2, g^3)$.

Theorem 9.2. *With assumptions as indicated above, assume that the uniform spectral stability condition is satisfied for high frequencies. Then there are $\rho_1 > 0$ and C such that for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $|\zeta| \geq \rho_1$, the solutions of* (9.2) *satisfy*

$$\begin{aligned}
&(1+\gamma)\|u^1\|_{L^2} + \Lambda\|u^2\|_{L^2} + \|u^3\|_{L^2} \\
&\quad +(1+\gamma)^{\frac12}|u^1(0)| + \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|u^3(0)| \\
&\qquad \leq C\Big(\|f^1\|_{L^2} + \|f^2\|_{L^2} + \Lambda^{-1}\|f^3\|_{L^2}\Big) \\
&\qquad\quad + C\Big((1+\gamma)^{\frac12}|g^1| + \Lambda^{\frac12}|g^2| + \Lambda^{-\frac12}|g^3|\Big).
\end{aligned} \tag{9.3}$$

High frequencies require a particular analysis for two reasons. First, the splitting hyperbolic vs parabolic is quite different in this regime and second the conjugation operator Φ of Lemma 4.1 is not uniform for large ζ. The analysis is made in [MéZu1] for full viscosities and Dirichlet boundary conditions. For partial viscosities and shocks, that is for transmission condition, the problem is solved in [GMWZ4]. The presentation below is more systematic and allows for more general boundary conditions of the form (2.11).

We now explain the general strategy of the proof. We use the notations

$$
\begin{aligned}
&\|u\|_{sc} = (1+\gamma)\|u^1\|_{L^2} + \Lambda\|u^2\|_{L^2} + \|u^3\|_{L^2}, \\
&\|f\|'_{sc} = \|f^1\|_{L^2} + \|f^2\|_{L^2} + \Lambda^{-1}\|f^3\|_{L^2}, \\
&|u(0)|_{sc} = (1+\gamma)^{\frac12}|u^1(0)| + \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|u^3(0)|, \\
&|g|_{sc} = (1+\gamma)^{\frac12}|g^1| + \Lambda^{\frac12}|g^2| + \Lambda^{-\frac12}|g^3|.
\end{aligned}
\tag{9.4}
$$

1) The main step in the proof of the theorem is to separate off the incoming and outgoing components of u. This is done using a change of variables $\hat{u} = \mathcal{V}^{-1}(z,\zeta)u$ which transforms the equation (9.2) to

$$
\partial_z \hat{u} = \hat{\mathcal{G}}(z,\zeta)\hat{u} + \hat{f}, \quad \hat{\Gamma}(\zeta)\hat{u}(0) = g. \tag{9.5}
$$

There are norms similar to (9.4) for $\hat{u}$ and $\hat{f}$ as well; with little risk of confusion, we use here the same notations. An important property is that:

$$
\begin{aligned}
&\|u\|_{sc} \le C\|\hat{u}\|_{sc}, \qquad \|\hat{f}\|'_{sc} \le C\|f\|'_{sc}, \\
&|u(0)|_{sc} \le C|\hat{u}(0)|_{sc}, \quad |\hat{u}(0)|_{sc} \le C|u(0)|_{sc},
\end{aligned}
\tag{9.6}
$$

with C independent of ζ. Moreover, $\hat{\Gamma}(\zeta) = \Gamma(\zeta)\mathcal{V}(0,\zeta)$ satisfies

$$
|\hat{\Gamma}(\zeta)\hat{u}(0)|_{sc} \le C|\hat{u}(0)|_{sc}. \tag{9.7}
$$

The new matrix $\hat{\mathcal{G}}$ has the important property that

$$
\hat{\mathcal{G}} = \begin{pmatrix} \hat{\mathcal{G}}^+ & 0 \\ 0 & \hat{\mathcal{G}}^- \end{pmatrix} + \hat{\mathcal{G}}' \tag{9.8}
$$

with

$$
\|\hat{\mathcal{G}}'\hat{u}\|'_{sc} \le \varepsilon(\zeta)\|\hat{u}\|_{sc} \tag{9.9}
$$

where $\varepsilon(\zeta)$ tends to 0 as $|\zeta|$ tends to infinity. The block structure corresponds to a splitting $\hat{u} = (\hat{u}^+, \hat{u}^-)$ with $\hat{u}^- \in \mathbb{C}^{N_b}$ and $\hat{u}^+ \in \mathbb{C}^{N+N^2-N_b}$ denoting the incoming and outgoing components respectively.

2) One proves separate estimates for the incoming and outgoing components:

$$
\|\hat{u}^+\|_{sc} + |\hat{u}^+(0)| \le C\|(\partial_z - \hat{\mathcal{G}}^+)\hat{u}^+\|_{sc}, \tag{9.10}
$$

$$
\|\hat{u}^-\|_{sc} \le C\|(\partial_z - \hat{\mathcal{G}}^-)\hat{u}^-\|_{sc} + C|\hat{u}^-(0)|, \tag{9.11}
$$

with C independent of ζ. (The norms are defined, identifying $\hat{u}^- \in \mathbb{C}^{N_b}$ to $(0, \hat{u}^-) \in \mathbb{C}^N$ etc). As a result, with (9.9), this implies that if $\hat{u}$ is a solution of (9.5), then

$$
\|\hat{u}^+\|_{sc} + |\hat{u}^+(0)| \le C\|\hat{f}\|_{sc} + \varepsilon(\zeta)\|\hat{u}\|_{sc}, \tag{9.12}
$$

$$
\|\hat{u}^-\|_{sc} \le C\|\hat{f}\|_{sc} + \varepsilon(\zeta)\|\hat{u}\|_{sc} + C|\hat{u}^-(0)|, \tag{9.13}
$$

3) We show that the estimates above imply that, if the uniform spectral stability condition is satisfied, then the solutions of (9.5) satisfy for $|\zeta|$ large enough

$$
\|\hat{u}\|_{sc} + |\hat{u}(0)|_{sc} \le C\Big(\|\hat{f}\|_{sc} + |g|_{sc}\Big) \tag{9.14}
$$

implying that the solutions of (9.2) satisfy

$$
\|u\|_{sc} + |u(0)|_{sc} \le C\Big(\|f\|_{sc} + |g|_{sc}\Big) \tag{9.15}
$$

that is (9.3).

• Indeed, by definition, $h \in \mathbb{E}^-(\zeta)$ if and only if there is u solution of $\partial_z u = \mathcal{G}u$ with $u(0) = h$. The corresponding $\hat{u} = \mathcal{V}^{-1}u$ satisfies by (9.13)

$$\|\hat{u}^-\|_{sc} \leq C|u^-(0)| + \varepsilon(\zeta)\|\hat{u}^+\|_{sc}$$

if ζ is large enough. Therefore, (9.12) implies that for ζ large and all $h \in \mathbb{E}^-(\zeta)$, $\hat{h} = \mathcal{V}^{-1}(0,\zeta)h = (\hat{h}^+, \hat{h}^-)$ satisfies

$$|\hat{h}^+|_{sc} \leq \varepsilon(\zeta)|\hat{h}^-|_{sc}. \tag{9.16}$$

• In addition $\hat{\mathbb{E}}^-(\zeta) := \mathcal{V}^{-1}(0,\zeta)\mathbb{E}^-(\zeta)$ has dimension equal to N_b, as the space of the $\hat{h}^-$. Therefore, (9.16) shows that for ζ large, the projection $h \mapsto h^-$ is bijective from $\hat{\mathbb{E}}^-(\zeta)$ to $\mathbb{C}^{N_b}$, with inverse uniformly bounded in the norm $|\cdot|_{sc}$.

The uniform spectral stability condition reads

$$\forall h \in \mathbb{E}^-(\zeta), \qquad |h|_{sc} \leq C|\Gamma(\zeta)h|_{sc} \tag{9.17}$$

(see (4.44)). Using (9.6), this implies

$$\forall \hat{h} \in \hat{\mathbb{E}}^-(\zeta), \qquad |\hat{h}|_{sc} \leq C|\hat{\Gamma}(\zeta)\hat{h}|_{sc}. \tag{9.18}$$

Using the isomorphism between $\hat{\mathbb{E}}^-(\zeta)$ and $\mathbb{C}^{N_b}$, we see that for ζ large enough and $\hat{h}^- \in \mathbb{C}^{N_b}$, there is $\hat{h}^+$ such that $(\hat{h}^+, \hat{h}^-) \in \hat{\mathbb{E}}^-(\zeta)$. Together with (9.16) and (9.7), there holds

$$|\hat{h}^-|_{sc} \leq |\hat{h}|_{sc} \leq C|\hat{\Gamma}(\zeta)\hat{h}|_{sc} \leq C|\hat{\Gamma}(\zeta)(0, \hat{h}^-)|_{sc} + \varepsilon(\zeta)|\hat{h}^-|_{sc}.$$

For ζ large, the last term can be dropped, increasing C. Finally, we conclude that for all $\hat{h} \in \mathbb{C}^N$

$$|\hat{h}|_{sc} \leq C|\hat{\Gamma}(\zeta)\hat{h}|_{sc} + C|\hat{h}^+|_{sc}. \tag{9.19}$$

Applying this estimate to $\hat{u}(0)$, combining with (9.10) and (9.11) and absorbing the error term $\hat{\mathcal{G}}'\hat{u}$ for ζ large, we immediately obtain (9.14).

The third part of the proof will not be repeated. We will focus on the reduction (9.5) and on the proof of the estimates for $\hat{u}^\pm$.

9.2. Spectral analysis of the symbol

Consider the linearized operator (4.5)

$$-\mathcal{B}\partial_z^2 + \mathcal{A}\partial_z + \mathcal{M}.$$

The coefficients satisfy

$$\begin{aligned}
\mathcal{B}(z) &= B_{d,d}(w(z)) \\
\mathcal{A}(z,\zeta) &= A_d(w(z)) - \sum_{j=1}^{d-1} i\eta_j\Big(B_{j,d} + B_{d,j}\Big)(w(z)) + E_d(z) \\
\mathcal{M}(z,\zeta) &= (i\tau + \gamma)A_0(w(z)) + \sum_{j=1}^{d-1} i\eta_j\Big(A_j(w(z)) + E_j(z)\Big) \\
&\qquad + \sum_{j,k=1}^{d-1} \eta_j\eta_k B_{j,k}(w(z)) + E_0(z)
\end{aligned} \tag{9.20}$$

where the E_k are functions, independent of ζ, which involve derivatives of w and thus converge to 0 at an exponential rate when z tends to infinity. Moreover, we note that

$$E_k^{11} = 0, \quad E_k^{12} = 0 \quad \text{for } k > 0. \tag{9.21}$$

With (2.3), we also remark that $\mathcal{M}^{12}$ does not depend on τ and γ.

We start with a spectral analysis of the matrix $\mathcal{G}$ in (4.6). It is convenient to use here the notations $u = (u^1, u^2, u^3) \in \mathbb{C}^{N-N^2} \times \mathbb{C}^{N^2} \times \mathbb{C}^{N^2}$. In the corresponding block decomposition of matrices and using the notations above, there holds

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & \mathcal{G}^{13} \\ 0 & 0 & \mathrm{Id} \\ \mathcal{G}^{31} & \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix} \tag{9.22}$$

where

$$\begin{aligned} \mathcal{G}^{11} &= -\,(\mathcal{A}^{11})^{-1}\mathcal{M}^{11}, & \mathcal{G}^{31} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{11} + \mathcal{M}^{21}), \\ \mathcal{G}^{12} &= -\,(\mathcal{A}^{11})^{-1}\mathcal{M}^{12}, & \mathcal{G}^{32} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{12} + \mathcal{M}^{22}), \\ \mathcal{G}^{13} &= -\,(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}, & \mathcal{G}^{33} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{13} + \mathcal{A}^{22}). \end{aligned}$$

Note that $\mathcal{G}^{11}$, $\mathcal{G}^{12}$, $\mathcal{G}^{31}$ and $\mathcal{G}^{33}$ are first order (linear or affine in ζ), that $\mathcal{G}^{32}$ is second order (at most quadratic in ζ) and that $\mathcal{G}^{13}$ is of order zero (independent of ζ). We denote by $\mathcal{G}_{\mathrm{p}}^{ab}$ their principal part (leading order part as polynomials). We note that

$$\mathcal{G}_{\mathrm{p}}^{ab}(z,\zeta) = G_{\mathrm{p}}^{ab}(w(z),\zeta) \quad \text{when } (a,b) \neq (3,1), \tag{9.23}$$

with

$$\begin{aligned} G_{\mathrm{p}}^{11}(u,\zeta) &= -(A_d^{11}(u))^{-1}\Big((\gamma + i\tau)A_0^{11}(u) + \textstyle\sum_{j=1}^{d-1} i\eta_j A_j^{11}(u)\Big), \\ G_{\mathrm{p}}^{12}(u,\zeta) &= -(A_d^{12}(u))^{-1} \textstyle\sum_{j=1}^{d-1} i\eta_j A_j^{12}(u) \\ G_{\mathrm{p}}^{13}(u) &= -(A_d^{11}(u))^{-1} A_d^{12}(u) \\ G_{\mathrm{p}}^{32}(u,\zeta) &= (B^{22}(u))^{-1} \textstyle\sum_{j,k=1}^{d-1} \eta_j\eta_k B_{j,k}^{22}(u)\Big), \\ G_{\mathrm{p}}^{33}(u,\zeta) &= -(B^{22}(u))^{-1} \textstyle\sum_{j=1}^{d-1} i\eta_j \Big(B_{j,d}^{22}(u) + B_{d,j}^{22}(u)\Big). \end{aligned}$$

The principal term of $\mathcal{G}^{31}$ involves derivatives of the profile w. Denoting by $p = \lim_{z\to+\infty} w(z) = w(\infty)$ the end state of the profile w, we note that the end state of $\mathcal{G}_{\mathrm{p}}^{31}$ is

$$\mathcal{G}_{\mathrm{p}}^{31}(\infty,\zeta) = (B^{22}(p))^{-1}\Big((\gamma + i\tau)A_0^{21}(p) + \sum_{j=1}^{d-1} i\eta_j A_j^{21}(p) + A_d^{21}(p)G_{\mathrm{p}}^{11}(p,\zeta)\Big).$$

There are similar formulas using the matrices $\overline{A}_j$ and $\overline{B}_{j,k}$ of (2.4).

The spectral analysis is easier when all the terms are reduced to first order. If $u = (u^1, u^2, u^3)$ is replaced by $\tilde{u} = \mathrm{h}_{|\zeta|} u := (u^1, u^2, |\zeta|^{-1}u^3)$, $\mathcal{G}$ is replaced by

$$\tilde{\mathcal{G}} = \mathrm{h}_{|\zeta|}\mathcal{G}\mathrm{h}_{|\zeta|}^{-1} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & |\zeta|\mathcal{G}^{13} \\ 0 & 0 & |\zeta|\mathrm{Id} \\ |\zeta|^{-1}\mathcal{G}^{31} & |\zeta|^{-1}\mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix} := \begin{pmatrix} \mathcal{G}^{11} & \mathcal{P}^{12} \\ \mathcal{P}^{21} & \mathcal{P}^{22} \end{pmatrix} \tag{9.24}$$

with obvious definitions of $\mathcal{P}^{ab}$. Note that $\tilde{\mathcal{G}}$ is or order one, while $\mathcal{P}^{21}$ is of order zero. Thus

$$\tilde{\mathcal{G}}(z,\zeta) = \tilde{\mathcal{G}}_{\mathrm{p}}(z,\zeta) + O(1), \quad \tilde{\mathcal{G}}_{\mathrm{p}} = \begin{pmatrix} \tilde{\mathcal{G}}_{\mathrm{p}}^{11} & \mathcal{P}_{\mathrm{p}}^{12} \\ 0 & \mathcal{P}_{\mathrm{p}}^{22} \end{pmatrix} = O(|\zeta|). \tag{9.25}$$

Moreover, since the coefficients in $\mathcal{G}$ converge exponentially at infinity, the remainder in (9.25) is uniform in $z \in \mathbb{R}_+$ and $|\zeta| \geq 1$. Moreover, the principal part of $\tilde{\mathcal{P}}^{22}$ is if the form $\tilde{\mathcal{P}}^{22}_{\rm p}(z,\zeta) = P^{22}_{\rm p}(w(z),\zeta)$.

Lemma 9.3. *i) For all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $\gamma > 0$ and $\eta \neq 0$ and for all and $z \geq 0$, $\tilde{\mathcal{G}}_{\rm p}(z,\zeta)$ has no eigenvalues on the imaginary axis; moreover, the number of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is $N_b = N^1_+ + N^2$.*

ii) for all compact subset of $\mathcal{U}^*$, there are $c > 0$ and $\delta > 0$ such that for all u in the given compact and all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ such that either $\gamma \leq \delta|\zeta|$ or $|\eta| \leq \delta|\zeta|$, the distance between the spectrum of $G^{11}_{\rm p}(u,\zeta)$ and the spectrum of $P^{22}_{\rm p}(u,\zeta)$ is larger than $c|\zeta|$.

Proof. The spectrum of $\tilde{\mathcal{G}}_{\rm p}$ is the union of the spectra of $G^{11}_{\rm p}$ and $P^{22}_{\rm p}$. By homogeneity, it suffices to consider $\zeta \in \overline{S}^d_+$.

a) $G^{11}_{\rm p}$ is related to L^{11} since $A^{11}_d(i\xi + G^{11}_{\rm p}(u,\zeta)) = L^{11}(u, \gamma + i\tau, i\eta, i\xi)$. By Assumption (H3), L^{11} is hyperbolic in the time direction, hence $G^{11}_{\rm p}$ has no eigenvalues on the imaginary axis when $\gamma > 0$; moreover, the boundary is noncharacteristic for L^{11} by Assumption 2.8, implying that the number of eigenvalues of $G^{11}_{\rm p}$ in $\{\operatorname{Re}\mu < 0\}$ is equal to the number of positive eigenvalues of A^{11}_d, that is is N^1_+.

Next, note that

$$P^{22}_{\rm p} = \begin{pmatrix} 0 & |\zeta|\mathrm{Id} \\ |\zeta|^{-1}G^{32}_{\rm p} & G^{33}_{\rm p} \end{pmatrix}.$$

Thus, $i\xi$ is an eigenvalue of $P^{22}_{\rm p}$ if and only if 0 is an eigenvalue of $B^{22}(\eta,\xi)$, which is impossible by (H2) if $\eta \neq 0$. Thus, the eigenvalues of $P^{22}_{\rm p}$ are not purely imaginary when $\eta \neq 0$. Moreover, the number of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is N^2 (see [MéZu1]). This finishes the proof of i).

b) If $\eta = 0$, $G^{32}_{\rm p}$ and $G^{33}_{\rm p}$ vanish, hence the spectrum of $P^{22}_{\rm p}$ is $\{0\}$. On the other hand 0 is not an eigenvalue of $G^{11}_{\rm p} = -(\gamma + i\tau)(A^{11}_d)^{-1}A^{11}_0$ since A^{11}_d and A^{11}_0 are invertible and $|\gamma + i\tau| = |\zeta| = 1$.

If $\gamma = 0$ and $\eta \neq 0$, the eigenvalues of $P^{22}_{\rm p}$ are not in $i\mathbb{R}$. On the other hand, by Assumption (H9) the eigenvalues of $G^{11}_{\rm p}$ are purely imaginary, thus $P^{22}_{\rm p}$ and $G^{11}_{\rm p}$ have no common eigenvalue. This finishes the proof of ii). □

The analysis in a purely "elliptic" zone $\{\gamma \geq \delta|\zeta|$ and $|\eta| \geq \delta|\zeta|\}$ with $\delta > 0$, is easy, see below. The most difficult and important part is to understand the "hyperbolic-parabolic" decoupling in an arbitrarily small cone

$$\mathrm{C}_\delta = \{0 \leq \gamma \leq \delta|\zeta|\} \cup \{|\eta| \leq \delta|\zeta|\} \tag{9.26}$$

with δ such that property ii) of Lemma 9.3 holds for u in a simply connected neighborhood $\mathcal{U}^*_0$ of a compact set which contains the curve $\{w(z),\ z \in [0,+\infty[\}$. There, the usual homogeneity and the parabolic homogeneity are in competition, leading to different classes of symbols. We use the following terminology: let $\zeta = (\tau,\gamma,\eta)$ and for a multi-index $\alpha = (\alpha_\tau, \alpha_\eta, \alpha_\gamma) \in \mathbb{N} \times \mathbb{N}^{d-1} \times \mathbb{N}$, set

$$|\alpha| = \alpha_\tau + |\alpha_\eta| \quad \text{and} \quad \langle\alpha\rangle = 2(\alpha_\tau + \alpha_\gamma) + |\alpha_\eta|.$$

Recall that the parabolic weight is $\Lambda = (1 + \tau^2 + \gamma^2 + |\eta|^4)^{\frac{1}{4}}$.

Definition 9.4. *i) $\Gamma^m(\Omega)$ denotes the space of* homogeneous symbols *of order m, that is of functions $h(z,\zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$ such that there is $\theta > 0$ such that for all $\alpha \in \mathbb{N}^{d+1}$ and all $k \in \mathbb{N}$, there are constants $C_{\alpha,k}$ such that for $|\zeta| \geq 1$:*

$$|\partial_\zeta^\alpha h| \leq C_{\alpha,0}|\zeta|^{m-|\alpha|}, \qquad \text{if } k = 0, \tag{9.27}$$

$$|\partial_z^k \partial_\zeta^\alpha h| \leq C_{\alpha,k} e^{-\theta z}|\zeta|^{m-|\alpha|}, \qquad \text{if } k > 0, \tag{9.28}$$

ii) $\mathrm{P}\Gamma^m(\Omega)$ denotes the space of parabolic symbols *of order m, that is of functions $h(z,\zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$ satisfying similar estimates with $|\zeta|^{m-|\alpha|}$ replaced by $\Lambda^{m-\langle\alpha\rangle}$.*

We use the same notation for spaces of homogeneous or parabolic matrix symbols of any fixed dimension.

Lemma 9.5. *For all $\hat\zeta \in S^d \cap \mathrm{C}_\delta$, there is a a conical neighborhood Ω of $\hat\zeta$ and there are matrices $\mathcal{W}_\mathrm{p}^{12} \in \Gamma^0(\Omega)$ and $\mathcal{W}_\mathrm{p}^{21}$, homogenous of degree 0 in ζ for $u \in \mathcal{U}_0^*$ such that*

$$\mathcal{W}_\mathrm{p}^{21}\mathcal{G}_\mathrm{p}^{11} - \mathcal{P}_\mathrm{p}^{22}\mathcal{W}_\mathrm{p}^{21} = |\zeta|\mathcal{P}_\mathrm{p}^{21}, \tag{9.29}$$

$$\mathcal{G}_\mathrm{p}^{11}\mathcal{W}_\mathrm{p}^{12} - \mathcal{W}_\mathrm{p}^{12}\mathcal{P}_\mathrm{p}^{22} = -\mathcal{P}_\mathrm{p}^{12}. \tag{9.30}$$

Proof. By homogeneity, it is sufficient to construct $\mathcal{W}_\mathrm{p}^{21}$ for $|\zeta| = 1$. By Lemma 9.3, for $\zeta \in S^{d+1} \cap \mathrm{C}_\delta$ and $u \in \mathcal{U}_0^*$, the spectra of $G_\mathrm{p}^{11}(u,\zeta)$ and $P_\mathrm{p}^{22}(u,\zeta)$ do not intersect, so that the linear system of equation

$$XG_\mathrm{p}^{11}(u,\zeta) - P_\mathrm{p}^{22}(u,\zeta)X = Y$$

has a unique solution $X = \mathcal{X}(u,\zeta)Y$. Therefore $\mathcal{W}_\mathrm{p}^{21}(z,\zeta) = |\zeta|\mathcal{X}(w(z),\zeta)\mathcal{P}_\mathrm{p}^{21}(z,\zeta)$ satisfies (9.29) (Note that $\mathcal{P}^{21}$ is of degree 0).

The construction of $\mathcal{W}_\mathrm{p}^{12}$ is similar, noticing that $\mathcal{P}_\mathrm{p}^{12}$ is of degree 1. ∎

In the block structure of $\mathcal{G}$, there holds

$$\mathcal{W}_\mathrm{p}^{21} = \begin{pmatrix} \mathcal{V}_\mathrm{p}^{21} \\ \mathcal{V}_\mathrm{p}^{31} \end{pmatrix}, \qquad \mathcal{W}_\mathrm{p}^{12} = \begin{pmatrix} \mathcal{V}_\mathrm{p}^{12} & \mathcal{V}_\mathrm{p}^{13} \end{pmatrix} \tag{9.31}$$

and (9.29) reads

$$\mathcal{V}_\mathrm{p}^{21}\mathcal{G}_\mathrm{p}^{11} - |\zeta|\mathcal{V}_\mathrm{p}^{31} = 0, \tag{9.32}$$

$$\mathcal{V}_\mathrm{p}^{31}\mathcal{G}_\mathrm{p}^{11} - |\zeta|^{-1}\mathcal{G}_\mathrm{p}^{32}\mathcal{V}_\mathrm{p}^{21} - \mathcal{G}_\mathrm{p}^{33}\mathcal{V}_\mathrm{p}^{31} = \mathcal{G}_\mathrm{p}^{31}. \tag{9.33}$$

Similarly,

$$\mathcal{G}_\mathrm{p}^{11}\mathcal{V}_\mathrm{p}^{12} - |\zeta|^{-1}\mathcal{V}_\mathrm{p}^{13}\mathcal{G}_\mathrm{p}^{32} = -\mathcal{G}_\mathrm{p}^{12} \tag{9.34}$$

$$\mathcal{G}_\mathrm{p}^{11}\mathcal{V}_\mathrm{p}^{13} - |\zeta|\mathcal{V}_\mathrm{p}^{12} - \mathcal{V}_\mathrm{p}^{13}\mathcal{G}_\mathrm{p}^{33} = |\zeta|\mathcal{G}_\mathrm{p}^{13}. \tag{9.35}$$

For further use, we make the following remark : by (9.23), we see that $\mathcal{G}_\mathrm{p}^{12}$ and $\mathcal{G}_\mathrm{p}^{32}$ vanish when $\eta = 0$. Therefore, (9.34) implies that $\mathcal{V}^{12}$ also vanishes when $\eta = 0$ and hence

$$\mathcal{V}^{12}(z,\zeta) = O(|\eta|/|\zeta|). \tag{9.36}$$

With these notations, let

$$\mathcal{V}_I(z,\zeta) = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ |\zeta|^{-1}\mathcal{V}_\mathrm{p}^{21} & \mathrm{Id} & 0 \\ \mathcal{V}_\mathrm{p}^{31} & 0 & \mathrm{Id} \end{pmatrix}, \quad \mathcal{V}_{II}(z,\zeta) = \begin{pmatrix} \mathrm{Id} & \mathcal{V}_\mathrm{p}^{12} & |\zeta|^{-1}\mathcal{V}_\mathrm{p}^{13} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$$

and $\mathcal{V} = \mathcal{V}_I\mathcal{V}_{II}$. Using the conjugation $u = \mathcal{V}\hat{u}$, $f = \mathcal{V}\hat{f}$, for ζ in the in the cone C_δ, the equation (9.2) is transformed to

$$\partial_z \hat{u} = \hat{\mathcal{G}}\hat{u} + \hat{f}, \quad \hat{\Gamma}\hat{u}(0) = g \tag{9.37}$$

with $\hat{\mathcal{G}} = \mathcal{V}^{-1}\mathcal{G}\mathcal{V} - \mathcal{V}^{-1}\partial_z\mathcal{V}$ and $\hat{\Gamma}(\zeta) = \Gamma(\zeta)\mathcal{V}(0,\zeta)$.

Lemma 9.6. *The entries of $\hat{\mathcal{G}}$ satisfy:*

$$\begin{aligned}
&\hat{\mathcal{G}}^{11} - \Big(\mathcal{G}^{11} + |\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}\Big) \in \Gamma^{-1},\\
&\hat{\mathcal{G}}^{12} \in \Gamma^0, \qquad \hat{\mathcal{G}}^{13} \in \Gamma^{-1}, \qquad \hat{\mathcal{G}}^{21} \in \Gamma^{-1}, \qquad \hat{\mathcal{G}}^{31} \in \Gamma^0,\\
&\hat{\mathcal{G}}^{22} \in \Gamma^0, \qquad \hat{\mathcal{G}}^{23} - \mathrm{Id} \in \Gamma^{-1},\\
&\hat{\mathcal{G}}^{32} - (\mathcal{G}^{32} - V^{31}\mathcal{G}^{12}) \in \Gamma^0, \qquad \hat{\mathcal{G}}^{33} - \mathcal{G}^{33} \in \Gamma^0.
\end{aligned}$$

Proof. We first compute the entries of $\mathcal{G}_I = \mathcal{V}_I^{-1}\mathcal{G}\mathcal{V}_I$. Direct computations show that

$$\begin{aligned}
&\mathcal{G}_I^{11} = \mathcal{G}^{11} + |\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}, \qquad \mathcal{G}_I^{12} = \mathcal{G}^{12}, \qquad \mathcal{G}_I^{13} = \mathcal{G}^{13}\\
&\mathcal{G}_I^{32} = \mathcal{G}^{32} - V^{31}\mathcal{G}^{12}, \qquad \mathcal{G}_I^{33} = \mathcal{G}^{33} - V^{31}\mathcal{G}^{13}.
\end{aligned}$$

Moreover,

$$\mathcal{G}_I^{21} = -|\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{21}\mathcal{G}^{11} + \mathcal{V}^{31} - |\zeta|^{-1}\mathcal{V}^{21}\Big(|\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}\Big).$$

The first two terms are of degree zero, and by (9.32), the sum of their principal terms vanishes; the third term is of degree -1 thus $\mathcal{G}_I^{21} \in \Gamma^{-1}$. Similarly, $\mathcal{G}_I^{31}$ is of degree 1 and its principal part vanishes by (9.33). Thus,

$$\mathcal{G}_I^{21} \in \Gamma^{-1}, \qquad \mathcal{G}_I^{31} \in \Gamma^0.$$

Next

$$\mathcal{G}_I^{22} = -|\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{21}\mathcal{G}^{12} \in \Gamma^0, \qquad \mathcal{G}_I^{22} - \mathrm{Id} = -|\zeta|^{-1}\mathcal{V}^{21}\mathcal{G}^{13} \in \Gamma^{-1}.$$

The computations for $\mathcal{G}_{II} = \mathcal{V}_{II}^{-1}\mathcal{G}_I\mathcal{V}_{II}$ are quite similar. This new conjugation annihilates the principal parts of $\mathcal{G}_I^{12}$ and $\mathcal{G}_I^{13}$ and contributes to remainder terms in the other entries.

Finally, direct computations show that $\mathcal{V}^{-1}\partial_z\mathcal{V}$ only contributes to remainder. $\square$

The main idea is to consider (9.37) as a perturbation of the decoupled system

$$\partial_z \hat{u}^1 = \hat{\mathcal{G}}^{11}\hat{u}^1 + \hat{f}_1, \tag{9.38}$$

$$\partial_z \begin{pmatrix}\hat{u}^2\\ \hat{u}^3\end{pmatrix} = \begin{pmatrix}0 & \mathrm{Id}\\ \mathcal{G}^{32} & \mathcal{G}^{33}\end{pmatrix}\begin{pmatrix}\hat{u}^2\\ \hat{u}^3\end{pmatrix} + \begin{pmatrix}\hat{f}^2\\ \hat{f}^3\end{pmatrix}. \tag{9.39}$$

Introduce then

$$\mathcal{G}' = \hat{\mathcal{G}} - \begin{pmatrix}\hat{\mathcal{G}}^{11} & 0 & 0\\ 0 & 0 & \mathrm{Id}\\ 0 & \mathcal{G}^{32} & \mathcal{G}^{33}\end{pmatrix}. \tag{9.40}$$

The next lemma how the estimates are transported by the change of variables $u = \mathcal{V}\hat{u}$. We use the notations (9.4) for the scaled norms.

Lemma 9.7. *There are constant C and ρ_1 such that for all ζ in the cone C_δ with $|\zeta| \geq \rho_1$, there holds*

$$\begin{aligned} &\|\mathcal{V}^{-1}\hat{u}\|_{sc} \leq C\|\hat{u}\|_{sc}, \qquad &&\|\mathcal{V}f\|'_{sc} \leq C\|f\|'_{sc}, \\ &|\mathcal{V}^{-1}\hat{u}(0)|_{sc} \leq C|\hat{u}(0)|_{sc}, \quad &&|\mathcal{V}u(0)|_{sc} \leq C|u(0)|_{sc}, \end{aligned} \tag{9.41}$$

and

$$|\hat{\Gamma}(\zeta)\hat{u}(0)|_{sc} \leq C|\hat{u}(0)|_{sc}. \tag{9.42}$$

Moreover,

$$\|\mathcal{G}'\hat{u}\|_{sc} \leq C\Lambda^{-1}\|\hat{u}\|_{sc}. \tag{9.43}$$

Proof. Direct computations, using (9.36), show that $u = \mathcal{V}\hat{u}$ satisfies

$$\begin{aligned} u^1 &= O(1)\hat{u}^1 + O(|\eta|\,|\zeta|^{-1})\hat{u}^2 + O(|\zeta|^{-1})\hat{u}^3, \\ u^2 &= O(|\zeta|^{-1})\hat{u}^1 + O(1)\hat{u}^2 + O(|\zeta|^{-1})\hat{u}^3, \\ u^3 &= O(1)\hat{u}^1 + O(1)\hat{u}^2 + O(1)\hat{u}^3. \end{aligned}$$

This implies the first estimate in (9.41), using the inequalities

$$(1+\gamma)|\eta|/|\zeta| \lesssim \Lambda, \quad (1+\gamma)/|\zeta| \lesssim 1, \quad \Lambda/|\zeta| \lesssim 1.$$

The proof of the other estimates of (9.41) is similar, using in particular for the traces the inequality $(1+\gamma)^{\frac{1}{2}}|\eta|/|\zeta| \lesssim \Lambda^{\frac{1}{2}}$.

The inequality (9.42) follows from the second line of (9.41) and the estimate $|\Gamma u(0)|_{sc} \leq |u(0)|_{sc}$ which is a direct consequence of the form (4.40) of the boundary conditions.

Finally, Lemma 9.6 implies that $\hat{f} = \mathcal{G}'\hat{u}$ satisfies

$$\begin{aligned} \hat{f}^1 &= O(1)\hat{u}^2 + O(|\zeta|^{-1})\hat{u}^3, \\ \hat{f}^2 &= O(|\zeta|^{-1})\hat{u}^1 + O(1)\hat{u}^2 + O(|\zeta|^{-1})\hat{u}^3, \\ f^3 &= O(1)\hat{u}^1 + O(1)\hat{u}^2 + O(1)\hat{u}^3, \end{aligned}$$

and (9.43) follows. □

The parabolic bloc (9.39) is studied in [MéZu1]. We now focus on the hyperbolic block (9.38), recalling and extending the analysis of [GMWZ4].

9.3. Analysis of the hyperbolic block.

9.3.1. The genuine coupling condition

For $u \in \mathcal{U}^*$, denote by $\lambda_j(u,\xi)$ the distinct eigenvalues of $\overline{A}^{11}(u,\xi)$, which are real and have constant multiplicity ν_j by Assumption (H8). Assumption (H9) implies the following:

Lemma 9.8. *For all $u \in \mathcal{U}^*$, all $\xi \in \mathbb{R}^d$ and all j, there holds $\partial_{\xi_d}\lambda_j(u,\xi) \neq 0$, and all these derivatives have the same sign.*

Proof. If $\partial_{\xi_d}\lambda_j(u,\underline{\eta},\underline{\xi}_d) = 0$, then the equation $\tau + \lambda(\underline{\eta},\xi_d) = 0$ would have complex roots in ξ_d for some τ close to $\underline{\tau} = -\lambda_j(u,\underline{\eta},\underline{\xi}_d)$ (recall that λ_j is real analytic). Thus hyperbolicity in the normal direction prevents glancing. Moroever, by continuity the sign of $\partial_{\xi_d}\lambda_j(u,\eta,\xi_d)$ is constant for all $\xi_d \in \mathbb{R}$ when $\eta \neq 0$. Thus the functions $\xi_d \mapsto \lambda_j(u,\eta,\xi_d)$ are monotone and tend to infinity as ξ_d tends to $\pm\infty$. Since $\lambda_j \neq \lambda_k$

when $j \neq k$, they must be all increasing or all decreasing. This remains true for $\eta = 0$ by continuity. □

According to the terminology of Section 4, we will say that the hyperbolic block L^{11} is *incoming* [resp. *outgoing*] when the derivatives $\partial_{\xi_d}\lambda_j(u,\xi)$ are positive [resp. negative].

Corollary 9.9. *i) The matrix $G_{\rm p}^{11}(u,\zeta)$ has no purely imaginary eigenvalues when $\gamma > 0$. They are all lying in $\{\mathrm{Re}\,\mu > 0\}$ if the 11-block is outgoing and in $\{\mathrm{Re}\,\mu < 0\}$ if it is incoming.*

ii) Near points $\underline{\zeta}$ with $\underline{\gamma} = 0$, $G_{\rm p}^{11}(u,\zeta)$ has semi-simple eigenvalues $\mu_j(u,\zeta)$ of constant multiplicity ν_j, which are purely imaginary when $\gamma = 0$. Moreover, $\partial_\gamma \mathrm{Re}\,\mu_j > 0$ when the 11-block is outgoing and $\partial_\gamma \mathrm{Re}\,\mu_j < 0$ when the 11-block is incoming.

Proof. Note that μ is an eigenvalue of $G_{\rm p}^{11}(u,\zeta)$ if and only if $-\tau + i\gamma$ is an eigenvalue of $\overline{A}^{11}(u,\eta,\xi)$ with $\xi = -i\mu$.

Consider the equations in $\xi_d : \tau + \lambda_j(u,\eta,\xi_d) = 0$. Since λ_j is strictly monotone and tends to infinity at both infinity, it always have a unique solution, $\psi_j(u,\eta,\tau)$ and $\partial_\tau\psi_j$ has the same sign as $-\partial_{\xi_d}\lambda_j$. This solution extends analytically for $\mathrm{Im}\,\tau$ small. This yields distinct eigenvalues $\mu_j(u,\zeta) = i\psi_j(u,\eta,\tau - i\gamma)$ of $G_{\rm p}^{11}$ for ζ close to the real domain. In particular $\partial_\gamma\mu_j = \partial_\tau\psi_j$ and the eigenvalues all lie in $\{\mathrm{Re}\,\mu > 0\}$ if the 11-block is outgoing and in $\{\mathrm{Re}\,\mu > 0\}$ if it is incoming.

The kernel of $G_{\rm p}^{11} - \mu_j$ is the kernel of $\overline{A}^{11} - \lambda_j$, thus has dimension equal to the multiplicity of λ_j. Since these dimensions add up to N^1, this shows that $G_{\rm p}^{11}$ has only semi-simple eigenvalues of constant multiplicity, which all lie in a given half space when $\gamma > 0$.

Hyperbolicity of L^{11} implies that $G_{\rm p}^{11}(u,\zeta)$ has no purely imaginary eigenvalues when $\gamma \neq 0$ and by continuity they all lie in the same half space. □

Next we need more information on the zero-th order correction of $\hat{\mathcal{G}}^{11}$. From (9.20) (9.21) and (9.22) we see that

$$\hat{\mathcal{G}}^{11}(z,\zeta) - (\mathcal{V}^{-1}\partial_z\mathcal{V})^{11} = G_{\rm p}^{11}(w(z),\zeta) + \mathcal{E}(z,\zeta), \tag{9.44}$$

where $\mathcal{E} \in \Gamma^0$. Denote its principal part by $\mathcal{E}_{\rm p}$. Its limit at $z = \infty$ is

$$E_{\rm p}(p,\zeta) = |\zeta|^{-1} G_{\rm p}^{12}(p,\zeta) V_{\rm p}^{21}(p,\zeta) G_{\rm p}^{13}(p,\zeta) V_{\rm p}^{31}(p,\zeta) \tag{9.45}$$

where $p = \lim_{z\to+\infty} w(z)$ and $V_{\rm p}^{21}(p,\zeta)$, $V_{\rm p}^{31}(p,\zeta)$ denote the end points of $\mathcal{V}_{\rm p}^{21}$ and $\mathcal{V}_{\rm p}^{31}$, that is the solutions of the intertwining relations (9.32) (9.33) with matrices $\mathcal{G}_{\rm p}^{ab}$ replaced bay their endpoint values $G_{\rm p}^{ab}(p,\zeta)$. The next result is crucial and follows from the genuine coupling condition (H4).

Proposition 9.10. *Fix $\underline{\zeta}$ with $|\underline{\zeta}| = 1$ and $\underline{\gamma} = 0$. For ζ in a neighborhood of $\underline{\zeta}$, consider a basis where $G^{11}(u,\zeta)$ has the block diagonal form $\mathrm{diag}(\mu_j \mathrm{Id}_{\nu_j})$. Denote by $E_{j,k}(u,\zeta)$ the corresponding blocks of E is this basis. Then, for $u \in \mathcal{U}$ the eingenvalues of the diagonal blocks $\mathrm{Re}\,E_{j,j}$ have a positive [resp. negative] real part if the 11-block is outgoing [resp. incoming].*

Proof. It is sufficient to prove the positivity at $\underline{\zeta}$. Suppose that $\gamma = 0$, denote by $\varphi_{j,p}$ with $p \in \{1,\dots,\nu_j\}$ a basis of eigenvectors of $G^{11}(u,\zeta)$. Fix j and set

$\xi_d = -i\mu_j(u,\zeta) \in \mathbb{R}$, $\xi = (\eta, \xi_d)$. Then the $\varphi_{j,p}$ are right eigenvectors of $\overline{A}^{11}(u,\xi)$ associated to the eigenvalue $-\tau = \lambda_j(u,\xi)$.

Consider left eigenvectors $\ell_{j,p}$ of $\overline{A}^{11}(u,\xi)$, dual to the $\varphi_{j,p}$. Then, the left eigenvectors of $G^{11}_{\rm p}(u,\zeta)$ associated to μ_j are $\frac{1}{\beta_j}\ell_j\overline{A}^{11}_d$ with $\beta_j = \partial_{\xi_d}\lambda_j(u,\eta,\xi)$, see Lemma 8.20. The entries of the block $E_{j,j}$ are

$$\frac{1}{\beta_j}\ell_{j,p}\overline{A}^{11}_d E_{\rm p}(u,\zeta)\varphi_{j,p'}. \tag{9.46}$$

Computing the eigenvalues of order ε of $\overline{B}(u,\xi) + i\varepsilon\overline{A}(u,\xi)$, leads to consider the matrix

$$i\overline{A}^{11} + \varepsilon\overline{A}^{12}(\overline{B}^{22})^{-1}\overline{A}^{21}. \tag{9.47}$$

The genuine coupling condition (H4) implies that for $u \in \mathcal{U}$, its spectrum lies in $\operatorname{Re}\mu > c\varepsilon$ for ε small, and this implies that the matrix $F_{j,j}$ with entries

$$\ell_{j,p}\overline{A}^{12}(\overline{B}^{22})^{-1}\overline{A}^{21}\varphi_{j,p'} \tag{9.48}$$

has its eigenvalues in the right half plane $\{\operatorname{Re}\mu > 0\}$.

Because $G^{11}_{\rm p}\varphi_{j,p'} = i\xi_d\varphi_{j,p'}$, the relation (9.32) implies

$$V^{31}_{\rm p}\varphi_{j,p'} = |\zeta|^{-1}V^{21}_{\rm p}G^{11}_{\rm p}\varphi_{j,p'} = i\xi_d|\zeta|^{-1}V^{21}_{\rm p}\varphi_{j,p'},$$

and, using the expressions of the matrices $G^{a,b}$ yields

$$(|\zeta|^{-1}G^{12}_{\rm p}V^{21}_{\rm p} + G^{13}_{\rm p}V^{31}_{\rm p})\varphi_{j,p'} = -i|\zeta|^{-1}(\overline{A}^{11}_d)^{-1}\overline{A}^{12}(\eta,\xi)V^{21}_{\rm p}\varphi_{j,p'}$$

and

$$(|\zeta|^{-1}G^{32}_{\rm p}V^{21}_{\rm p} + G^{33}_{\rm p}V^{31}_{\rm p} - V^{31}_{\rm p}G^{11}_{\rm p})\varphi_{j,p'} = |\zeta|^{-1}(\overline{B}^{22}_{dd})^{-1}\overline{B}_{22}(\eta,\xi)V^{21}_{\rm p}\varphi_{j,p'}$$

By (9.33) this is equal to

$$-G^{31}_{\rm p}\varphi_{j,p'} = -i(\overline{B}^{22}_{dd})^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}.$$

Thus

$$|\zeta|^{-1}V^{21}_{\rm p}\varphi_{j,p'} = -i\Big(\overline{B}_{22}(\eta,\xi)\Big)^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}.$$

and

$$E_{\rm p}\varphi_{j,p'} = -(\overline{A}^{11}_d)^{-1}\overline{A}^{12}(\eta,\xi)\Big(\overline{B}_{22}(\eta,\xi)\Big)^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}.$$

Multiplying on the left by $\ell_j\overline{A}^{11}_d$, this shows that the coefficients in (9.46) and (9.48) only differ by the factor $-1/\beta_j$, and the proposition follows. ☐

9.3.2. Estimates

We are now in position to prove maximal estimates for the solutions of the equation (9.38).

Proposition 9.11. *There are constants C and $\rho_1 \ge 1$ such that for all ζ in the cone C_δ with $|\zeta| \ge \rho_1$ and all $\hat u^1$ and $\hat f^1$ in $L^2(\mathbb{R}_+)$ satisfying* (9.38)*, there holds*

$$\begin{aligned}(1+\gamma)\|\hat u^1\|_{L^2} + (1+\gamma)^{\frac12}|\hat u^{1+}(0)| \\ \le C\Big(\|\hat f^1\|_{L^2} + (1+\gamma)^{\frac12}|\hat u^{1-}(0)|\Big)\end{aligned} \tag{9.49}$$

where $\hat u^{1+} = \hat u^1$ and $\hat u^{1-} = 0$ if the 11-block is outgoing and $\hat u^{1+} = 0$ and $\hat u^{1-} = \hat u^1$ if it is incoming.

Proof. **a)** Fix $\underline{\zeta} \in \overline{S}^{d+1}_+$. We prove the estimate for ζ in a conical neighborhood of $\underline{\zeta}$. Suppose first that $\underline{\gamma} = 0$ (the most difficult case). By Corollary 9.9 there is a matrix $\mathcal{V}^{11}(z, \zeta)$ homogeneous of degree 0 such that $(\mathcal{V}^{11})^{-1}\mathcal{G}^{11}_{\rm p}\mathcal{V}^{11} = \mathrm{diag}(\mu_j(w(z), \zeta)\mathrm{Id}_{\nu_j})$. Setting $\hat{u}^1 = \mathcal{V}^{11}u^1$ transforms the equation to

$$\partial_z u^1 = (\mathrm{diag}(\mu_j(w(z), \zeta)\mathrm{Id}_{\nu_j}) + \widetilde{\mathcal{E}})u^1 + f^1 \tag{9.50}$$

with $\widetilde{\mathcal{E}} = \mathcal{E} - (\mathcal{V}^{11})^{-1}\partial_z\mathcal{V}^{11} \in \Gamma^0$, whose principal part $\widetilde{\mathcal{E}}_{\rm p}$ has the same end point $E_{\rm p}(p, \zeta)$ as $\mathcal{E}_{\rm p}$.

As usual, since the μ_j are pairwise distinct, there is a new change $u^1 = (\mathrm{Id}+\mathcal{V}_{-1})\tilde{u}^1$ with $\mathcal{V}^{11}_{-1} \in \Gamma^{-1}$, such that the resulting system has the same form with the additional property that the zero-th order part is also block diagonal, so that $\widetilde{\mathcal{E}}_{\rm p} = \mathrm{diag}(\mathcal{E}_{j,j})$ and the end points of the blocks $\mathcal{E}_{j,j}$ are $E_{j,j}$ introduced in Proposition 9.10.

The term $(\widetilde{\mathcal{E}} - \widetilde{\mathcal{E}}_{\rm p})u$ is $O(|\zeta|^{-1}|u|)$, is incorporated to f^1 and finally absorbed from the right to the left of the inequality by choosing $|\zeta|$ large enough. This reduces the proof to the case where the equation reads

$$\partial_z \hat{u}^1 = \mu_j(w(z), \zeta)\hat{u}^1 + E_{j,j}(\zeta)\hat{u}^1 + F_{j,j}(z, \zeta)\hat{u}^1 + \hat{f}^1 \tag{9.51}$$

with $|F_{j,j}| \leq C_0 e^{-\theta z}$.

Consider the outgoing case. Then, Corollary 9.9 implies that there is a constant $c > 0$ such that $\mathrm{Re}\,\mu_j(u, \zeta) \geq c\gamma$. Moreover, Proposition 9.10 implies that the eigenvalues of $E_{j,j}$ have a positive real part. Thus, there is a positive definite (constant) matrix $S(\zeta) \geq \mathrm{Id}$ such that $\mathrm{Re}\, SE_{j,j}$ is definite positive, say $\mathrm{Re}\, SE_{j,j} \geq \mathrm{Id}$. Introduce $a = C_0|S| \int_0^z e^{-\theta s} ds$ such that $\partial_z a \geq |SF_{j,j}|$ and a is bounded in L^∞ uniformly with respect to ζ. Therefore, multiplying the equation by $e^{2a(z)}S$ and taking the L^2 scalar product with $\hat{u}^1$ implies that

$$(1 + c\gamma)\|e^a\hat{u}^1\|^2_{L^2} + |\hat{u}^1(0)|^2 \leq C\|e^a\hat{u}^1\|_{L^2}\|e^a\hat{f}^1\|_{L^2}$$

which implies (9.49). The proof in the incoming case is similar.

b) Suppose next that $\underline{\gamma} = 0$. Consider again the outgoing case. Then, the eigenvalues of $G^{11}_{\rm p}$ satisfy $\mathrm{Re}\,\mu \geq c|\zeta|$ in a conical neighborhood of $\underline{\zeta}$. This is the classical "elliptic" case. There is a symmetric definite positive matrix $S(u, \zeta) \in \Gamma^0$ such that $\mathrm{Re}\, SG^{11} \geq c|\zeta|\mathrm{Id}$ and usual integrations by parts imply that

$$c|\zeta|\|\hat{u}^1\|^2_{L^2} + |\hat{u}^1(0)|^2 \leq C\|\hat{u}^1\|_{L^2}\|\hat{f}^1\|_{L^2} + C_1\|\hat{u}^1\|^2_{L^2}$$

where C_1 involve estimates of the zero-th order terms, which include $\partial_z S(w(z), \zeta)$. This term is eliminated choosing $|\zeta|$ large enough. The proof in the incoming case is similar. □

Remark 9.12. The proof above contains two ingredients. First, the 11-block is totally incoming or totally outgoing, in analogy with the terminology of Section 4. Thus the decoupling incoming/outgoing is trivial. More generally, this could be replaced by a decoupling condition in the spirit of Section 4. For instance, for shocks, such a decoupling is immediate in [GMWZ4] corresponding to equations on each side of the front. Next, we construct symmetrizers for the incoming and outgoing components. There we use the genuine coupling condition. If the eigenvalues are not of constant multiplicity one can introduce adapted bases or use symmetry also in the spirit of Section 4.

9.3.3. About Assumption (H9)

We show on an example that hyperbolicity in the normal direction is crucial in the proof of estimates of the form (9.49). Suppose that the L^{11}- block reads

$$\begin{cases} \partial_t u - \partial_y u + \partial_x v, \\ \partial_t v + \partial_y v + \partial_x u. \end{cases} \tag{9.52}$$

Then, on the Fourier side, the 11 equation will be of the form

$$\begin{cases} (i(\tau - \eta) + \gamma)u + \partial_z v + a(z)u = f, \\ (i(\tau + \eta) + \gamma)v + \partial_z u + a(z)v = g, \end{cases} \tag{9.53}$$

and the only information we have from the genuine coupling condition is that a is positive at $z = +\infty$. Suppose that $a(z_0) < 0$ for some $z_0 > 0$. Then glancing waves for (9.52) will propagate parallel to the boundary and thus may remain in a region where a is negative and thus may never be damped. This is illustrated by choosing $\tau = \eta$, large, $\gamma = -a(z_0) >$ and

$$u_\tau(z) = \chi(\tau^{\frac{1}{3}}(z - z_0)), \quad v_\tau(z) = \frac{-\partial_z u_\tau}{2i\tau + \gamma + a},$$

with $\chi \in C_0^\infty(\mathbb{R})$. Then (9.53) is satisfied with $f = (a(z) - a(z_0))u_\tau + \partial_z v_\tau$ and $g = 0$. Moreover, $\|f\|_{L^2} = O(\tau^{-\frac{1}{3}})\|u\|_{L^2}$ and $u(0) = v(0) = 0$, showing that no estimate of the form (9.49) can be valid.

9.4. Proof of Theorem 9.2

9.4.1. In the cone C_δ

We consider now the equation (9.39) and briefly recall the results from [MéZu1]. It is natural to rescale the problem using the parabolic weights: with $v^2 = \hat{u}^2$ and $v^3 = \Lambda^{-1}\hat{u}^3$ and $g^2 = \hat{f}^2$ and $g^3 = \Lambda^{-1}\hat{f}^3$the system reads

$$\partial_z \begin{pmatrix} v^2 \\ v^3 \end{pmatrix} = \mathcal{G}_P \begin{pmatrix} v^2 \\ v^3 \end{pmatrix} + \begin{pmatrix} g^2 \\ g^3 \end{pmatrix}, \tag{9.54}$$

with

$$\mathcal{G}_P = \begin{pmatrix} 0 & \Lambda \mathrm{Id} \\ \Lambda^{-1}\mathcal{G}^{32} & \mathcal{G}^{31} \end{pmatrix} \in \mathrm{P}\Gamma^1$$

of quasi-homogenenous degree one and principal part $G_P(w(z), \zeta)$ with

$$G_P(u,\zeta) = \begin{pmatrix} 0 & \Lambda \mathrm{Id} \\ \Lambda^{-1}\big((i\tau + \gamma)(\overline{B}^{22})^{-1} + G_{\mathrm{p}}^{32}(u,\eta)\big) & G_{\mathrm{p}}^{31}(u,\eta) \end{pmatrix}. \tag{9.55}$$

Lemma 9.13 ([MéZu1]). *There is $c > 0$ such that the spectrum of G_P lies in $\{|\mathrm{Re}\,\mu| \geq c\Lambda\}$, with N^2 eigenvalues, counted with their multiplicity, of positive real part. There is a smooth change of variables $\mathcal{W} \in \mathrm{P}\Gamma^0$ such that*

$$\mathcal{W}^{-1}\mathcal{G}_P\mathcal{W} = \begin{pmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{pmatrix},$$

with $\mathcal{P}_\pm \in \mathrm{P}\Gamma^1$ having their eigenvalues satisfying $\pm\mathrm{Re}\,\mu \geq c\Lambda$.

Introduce

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} v^2 \\ v^3 \end{pmatrix}.$$

Corollary 9.14 ([MéZu1]). *There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geq \rho_1$, there holds*

$$\begin{aligned} &\Lambda \|v^+\|_{L^2} + \Lambda^{\frac12} |v^+(0)| \leq C \|(\partial_z - \mathcal{P}^+) v^+\|_{L^2}, \\ &\Lambda \|v^-\|_{L^2} \leq C \|(\partial_z - \mathcal{P}^-) v^-\|_{L^2} + C\Lambda^{\frac12} |v^-(0)|. \end{aligned}$$

Scaling back, introduce

$$\begin{pmatrix} \hat u^{2,+} \\ \hat u^{3,+} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \Lambda \end{pmatrix} \mathcal{W} \begin{pmatrix} v^+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \hat u^{2,-} \\ \hat u^{3,-} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \Lambda \end{pmatrix} \mathcal{W} \begin{pmatrix} 0 \\ v^- \end{pmatrix}. \tag{9.56}$$

Because, $\mathcal{W}^{-1}\partial_z \mathcal{W}$ is uniformly bounded, the Corollary implies the following estimate:

Proposition 9.15. *There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geq \rho_1$, there holds*

$$\begin{aligned} \Lambda \|u^{2,+}\|_{L^2} + &\|u^{3,+}\|_{L^2} + \Lambda^{\frac12} |u^{2,+}(0)| + \Lambda^{-\frac12} |u^{3,+}(0)| \\ &\leq C \|\hat f^2\|_{L^2} + C\Lambda^{-1} \|\hat f^3\|_{L^2} + \|\hat u^2\|_{L^2} + C\Lambda^{-1} \|\hat u^3\|_{L^2}, \end{aligned}$$

$$\begin{aligned} \Lambda \|u^{2,-}\|_{L^2} + &\|u^{3,-}\|_{L^2} \leq C\Lambda^{\frac12} |u^{2,-}(0)| + C\Lambda^{-\frac12} |u^{3,-}(0)| \\ &+ C \|\hat f^2\|_{L^2} + C\Lambda^{-1} \|\hat f^3\|_{L^2} + \|\hat u^2\|_{L^2} + C\Lambda^{-1} \|\hat u^3\|_{L^2}. \end{aligned}$$

Finally, with $\hat u^{1,\pm}$ as in Proposition 9.11, introduce

$$\hat u^{\pm} = {}^t(\hat u^{1,\pm}, \hat u^{2,\pm}, \hat u^{2,\pm}). \tag{9.57}$$

Adding up the various estimates and using (9.43), one obtains the following estimates.

Proposition 9.16. *There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geq \rho_1$ and all $\hat u \in H^1(\overline{\mathbb{R}}_+)$:*

$$\|\hat u^+\|_{sc} + |\hat u^+(0)| \leq C \|(\partial_z - \mathcal{G})\hat u\|_{sc} + \Lambda^{-1} \|\hat u\|_{sc}, \tag{9.58}$$

$$\|\hat u^-\|_{sc} \leq C \|(\partial_z - \mathcal{G})\hat u\|_{sc} + \Lambda^{-1} \|\hat u\|_{sc} + C|\hat u^-(0)|. \tag{9.59}$$

As indicated at the end of Section 7.1, these estimates imply the maximal estimates of Theorem 9.2 provided that the boundary conditions are uniformly spectral stable.

9.4.2. Analysis in the central zone

We now consider the remaining cone where

$$\zeta \in \mathbb{R}^{d+1}, \quad \gamma \geq \delta|\zeta| \quad \text{and } |\eta| \geq \delta|\zeta|. \tag{9.60}$$

We consider the rescaled $\tilde{\mathcal{G}}$ matrix (9.25), for the rescaled unknows $\tilde u = \mathrm{h}_{|\zeta|} u := (u^1, u^2, |\zeta|^{-1} u^3)$, $\tilde f = \mathrm{h}_{|\zeta|} f := (f^1, f^2, |\zeta|^{-1} f^3)$. We note that in the region under consideration we now have $(1+\gamma) \approx \Lambda \approx |\zeta|$, so that the rescaled norms (9.4) are equivalent to

$$\begin{aligned} &\|u\|_{sc} \approx |\zeta| \|\tilde{u}\|_{L^2}, \\ &|u(0)|_{sc} \approx |\zeta|^{\frac{1}{2}} |\tilde{u}(0)|, \\ &\|f\|'_{sc} \approx \|\tilde{f}\|_{L^2}. \end{aligned} \tag{9.61}$$

By Lemma 9.3, there is a smooth matrix $\mathcal{V} \in \Gamma^0$ such that

$$\mathcal{V}^{-1}(z\zeta)\mathcal{G}_{\rm p}(z,\zeta)\mathcal{V}(z,\zeta) = \begin{pmatrix} \mathcal{G}_{\rm p}^{+} & 0 \\ 0 & \mathcal{G}_{\rm p}^{-} \end{pmatrix} := \mathcal{G}_{\rm p}^{\rm diag}$$

where the spectrum of $\mathcal{G}_{\rm p}^{\pm} \in \Gamma^1$ is contained in $\{\pm{\rm Re}\,\mu \geq c|\zeta|\}$. We use the notations

$$\hat{u} := \mathcal{V}\tilde{u} = \begin{pmatrix} \hat{u}^{+} \\ \hat{u}^{-} \end{pmatrix}. \tag{9.62}$$

$\tilde{u}^{+}$ has dimension $N + N^2 - N_b$ and u^{-} has dimension N_b. The equation for $\hat{u}$ reads

$$\partial_z \hat{u} = \hat{\mathcal{G}}\hat{u} + \hat{f}, \tag{9.63}$$

with $\hat{\mathcal{G}} = \mathcal{G}^{\rm diag} + O(1)$. The ellipticity of $\mathcal{G}^{\rm diag}$ immediately implies the following estimates.

Proposition 9.17. *There are constants C and ρ_1 such that for all ζ satisfying* (9.60) *and $|\zeta| \geq \rho_1$ and all $\tilde{u} \in H^1(\mathbb{R}_+)$ satisfying* (9.63)*, there holds*

$$|\zeta|\, \|\hat{u}^{+}\|_{L^2} + |\zeta|^{\frac{1}{2}} |u^{+}(0)| \leq C\|\hat{f}\|_{L^2} + C\|\hat{u}\|_{L^2}, \tag{9.64}$$

$$|\zeta|\, \|\hat{u}^{-}\|_{L^2} \leq C\|\hat{f}\|_{L^2} + C\|\hat{u}\|_{L^2} + C|\zeta|^{\frac{1}{2}} |\hat{u}^{-}(0)|^2. \tag{9.65}$$

Thanks to (9.61), this is the exact analogue of Proposition 9.16 and these estimates imply the maximal estimates of Theorem 9.2 provided that the boundary conditions are uniformly spectral stable, as explained in Section 7.1.

10. Linear stability

In the previous sections, we have studied the validity of maximal estimates (see (4.36) and (4.42) in Section 4) for the spectral equation. Scaling back to the original variables, and using Plancherel's formula for inverting the Laplace-Fourier transform, they imply weighted L^2 estimates for the linearized equations (4.3) near a function $u_\varepsilon(t,y,x) = w(x/\varepsilon)$ where w satisfies (4.1). The main goal of this section is to extend these estimates (see (10.14) below) for the linearized equation near slow perturbations of $w(x/\varepsilon)$, considering the Fourier-Laplace calculus developed in the preceding sections as a symbolic calculus for suitable pseudo-differential symmetrizers.

10.1. Linearized equations, spectral stability conditions

A possible formalism is the following. Instead of considering a single profile as in Section 4, we consider now a *family of profiles* $W(p,z)$, which are smooth functions defined for p in a domain $\mathcal{P}$ and $z \in \overline{\mathbb{R}}_+$, with values in $\mathcal{U}^*$ and such that their derivatives converge at an exponential rate as $z \to +\infty$: there are $\delta > 0$ and a smooth function $\underline{W}$ on $\mathcal{P}$ with values in $\mathcal{U}$, such that

$$\sup_{p,z} e^{\delta z}\Big|\partial^\alpha_{p,z}(W(p,z) - \underline{W}(p))\Big| < +\infty \tag{10.1}$$

We further assume that we are given a family of functions $p_\varepsilon(t,x,y)$ on $\mathbb{R}^{1+d}$, with values in a compact subset of $\mathcal{P}$, with at least

$$\sup_\varepsilon \|p_\varepsilon\|_{W^{1,\infty}(\mathbb{R}^{1+d})} < +\infty. \tag{10.2}$$

The functions p_ε stand for the coordinates themselves or some additional function depending on the iteration process in the resolution of the nonlinear problem.

For $\varepsilon \in]0,1]$, we consider the linearized equations from (2.1) (2.10) around

$$\tilde u_\varepsilon(t,x,y) = W\Big(p_\varepsilon(t,x,y), \frac{x}{\varepsilon}\Big). \tag{10.3}$$

With abbreviated notations, they read

$$\mathcal{L}'_{\tilde u_\varepsilon}\dot u = \dot f, \quad \Upsilon'_{\tilde u_\varepsilon}\dot u_{|x=0} = \dot g. \tag{10.4}$$

$\mathcal{L}'_{\tilde u_\varepsilon}$ is a differential operator with coefficients that are smooth functions of (t,y,x), $z := x/\varepsilon$ and $\varepsilon \in [0,1]$. Factoring out ε^{-1} it also appears as an operator in $\varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x$:

$$\mathcal{L}'_{\tilde u_\varepsilon} = \frac{1}{\varepsilon}\widetilde L_\varepsilon\Big(t,y,x,\frac{x}{\varepsilon},\varepsilon\partial_t,\varepsilon\partial_y,\varepsilon\partial_x\Big). \tag{10.5}$$

The analysis of Section 4 applies for all fixed $p \in \mathcal{P}$ to the linearized equations

$$\mathcal{L}'_{u_{p,\varepsilon}}\dot u = \dot f, \quad \Upsilon'_{u_{p,\varepsilon}}\dot u_{|x=0} = \dot g. \tag{10.6}$$

near

$$u_{p,\varepsilon}(x) = W\Big(p, \frac{x}{\varepsilon}\Big). \tag{10.7}$$

With notations parallel to (4.3), we have

$$\begin{cases} \mathcal{L}'_{u_{p,\varepsilon}}\dot u = \dfrac{1}{\varepsilon}L\Big(p,\dfrac{x}{\varepsilon},\varepsilon\partial_t,\varepsilon\partial_y,\varepsilon\partial_x\Big)\dot u, \\ \Upsilon'_{u_{p,\varepsilon}}\dot u = \Upsilon'(p,\dot u,\varepsilon\partial_y\dot u,\varepsilon\partial_x\dot u). \end{cases} \tag{10.8}$$

The main idea is that, as far as local stability properties are studied, the operator $\mathcal{L}_{\tilde u_\varepsilon}$ is sort of a perturbation of the family of operators $\mathcal{L}'_{u_\varepsilon(p)}$, meaning that

- the terms which involve derivatives of $\tilde u_\varepsilon$ with respect to the slow variables (t,y,x) in (10.3) (i.e. derivatives of p_ε) contribute only to admissible errors which do not change the form of the estimates;

- the stability conditions are expressed by freezing the slow coefficients of $\widetilde L_\varepsilon$ at each point $(\underline{t},\underline{y},\underline{x})$.

Accordingly, we set

Assumption 10.1. (H10) *For all $p \in \mathcal{P}$, the linearized equations* (10.6) *satisfy the uniform spectral stability conditions of Definitions* 4.15 *and* 4.17.

10.2. Maximal stability estimates

In order to use the symmetrizers of Sections 8 and 9, we supplement the structural assumptions of Section 2 with the following "technical" conditions:

Assumption 10.2. (H11) (Existence of low frequency symmetrizers) *Near all $p \in \mathcal{P}$ and $\check\zeta \in \overline{S}^d_+$, there exist a K-family of symmetrizers for the matrix $\check H(p, \check\zeta, \rho)$ associated to the profile $W(p, \cdot)$.*

We refer to Section 8 for geometric sufficient conditions which imply (H11). For the existence of high frequency symmetrizers, we will further assume that the Assumptions (H8) and (H9) of Section 9 are satisfied.

The stability estimates for (10.5) are expressed using weighted estimates. Define the weights

$$\Lambda_\epsilon(\zeta) = \Lambda(\epsilon\zeta) = (1 + (\epsilon\tau)^2 + (\epsilon\gamma)^2 + |\epsilon\eta|^4)^{\frac14}, \tag{10.9}$$

$$\lambda_\epsilon(\zeta) = \begin{cases} (\gamma + \epsilon|\zeta|^2)^{\frac12}, & \text{when } |\epsilon\zeta| \le 1,\\ \frac{1}{\sqrt{\epsilon}}, & \text{when } \epsilon|\zeta| \ge 1. \end{cases} \tag{10.10}$$

Observe that the expressions defining λ_ϵ in the two frequency regimes are of the same order when $|\epsilon\zeta| \approx 1$. Moreover, on any set of frequencies such that $0 \le |\epsilon\zeta| \le R$, we have $1 \le \Lambda_\epsilon \le C_R$.

Given a weight function $\phi(\zeta)$ we use the notation

$$|u|_\phi = \Big(\int_{\mathbb{R}^d} \phi(\tau,\gamma,\eta)^2 |\hat u(\tau,\eta)|^2 d\tau d\eta \Big)^{\frac12}. \tag{10.11}$$

where $\hat u$ denotes the Fourier transform of u. When u also depends on x, we set

$$\|u\|_\phi = \Big(\int_0^\infty |u(\,\cdot\,,x)|^2_\phi dx \Big)^{\frac12}. \tag{10.12}$$

Theorem 10.3 (L^2 estimate)**.** *Suppose that W and p_ε satisfy* (10.1) *and* (10.2) *respectively. Assume* (H1) *to* (H11)*. Then, there exist positive constants C, $\gamma_0 > 0$, and $\epsilon_0 > 0$ such that for γ and ϵ satisfying*

$$\gamma \ge \gamma_0,\ \epsilon \in (0, \epsilon_0] \tag{10.13}$$

and all solution $\dot u$ of the linearized boundary value problem (10.4)*, with $\dot u$, and $\dot f$ C^∞ with compact support on $\overline{\mathbb{R}}^{d+1}_+$ and $\dot g$ C^∞ with compact support on $\mathbb{R}^d$, there holds*

$$\begin{aligned} &\|e^{-\gamma t}u^1\|_{\lambda_\epsilon^2} + \|e^{-\gamma t}u^2\|_{\lambda_\epsilon^2\Lambda_\epsilon} + \sqrt{\epsilon}\|\partial_x e^{-\gamma t}u^2\|_{\lambda_\epsilon} \\ &\quad + |e^{-\gamma t}u^1{}_{|x=0}|_{\lambda_\epsilon} + |e^{-\gamma t}u^2{}_{|x=0}|_{\lambda_\epsilon\Lambda_\epsilon^{\frac12}} + \epsilon|e^{-\gamma t}\partial_x u_{2|x=0}|_{\lambda_\epsilon\Lambda_\epsilon^{-\frac12}} \\ &\le \\ &C\Big(\|e^{-\gamma t}f^1\| + \|e^{-\gamma t}f^2\|_{\Lambda_\epsilon^{-1}}\Big) \\ &\quad + C\Big(|e^{-\gamma t}g^1|_{\lambda_\epsilon} + |e^{-\gamma t}g^2|_{\lambda_\epsilon\Lambda_\epsilon^{\frac12}} + |e^{-\gamma t}g^3|_{\lambda_\epsilon\Lambda_\epsilon^{-\frac12}}\Big). \end{aligned} \tag{10.14}$$

For instance, this implies the following uniform estimates:

Corollary 10.4. *With assumptions as in Theorem* 10.3*, if $g = 0$, then*

$$\begin{aligned} \gamma\|e^{-\gamma t}u\|_{L^2(\mathbb{R}^{1+d}_+)} + \sqrt{\gamma}\|e^{-\gamma t}u_{|x=0}\|_{L^2(\mathbb{R}^d)} \le \\ C\|e^{-\gamma t}f\|_{L^2(\mathbb{R}^{1+d}_+)}. \end{aligned} \tag{10.15}$$

10.3. Hints for the proof

Consider the linearized equations (10.4). Introduce $u_\gamma = e^{-\gamma t}\dot u$, $f_\gamma = e^{-\gamma t}\dot f$ and $g_\gamma = e^{-\gamma t}\dot g$. Thus

$$\frac{1}{\varepsilon}\widetilde{L}_\varepsilon\Big(t, y, x, \frac{x}{\varepsilon}, \varepsilon(\partial_t + \gamma), \varepsilon\partial_y, \varepsilon\partial_x\Big)u_\gamma = f_\gamma, \quad \Upsilon'_{\tilde u_\varepsilon} u_{\gamma|x=0} = g_\gamma. \tag{10.16}$$

We prove estimates for the u_γ similar to (10.14), with additional "error" terms in the right hand side, which are arbitrarily small compared to the left hand side, uniformly in ϵ, when γ is large.

a) We note that the terms in

$$\frac{1}{\varepsilon}\Big(\widetilde{L}_\varepsilon\Big(t, y, x, \frac{x}{\varepsilon}, \varepsilon(\partial_t + \gamma), \varepsilon\partial_y, \varepsilon\partial_x\Big) - L\Big(p_\varepsilon(t, y, x), \frac{x}{\varepsilon}, \varepsilon(\partial_t + \gamma), \varepsilon\partial_y, \varepsilon\partial_x\Big)\Big)u_\gamma$$

and

$$\Upsilon'_{\tilde u_\varepsilon} u_\gamma - \Upsilon(p_\varepsilon(t, y, 0), u_\gamma, \varepsilon\partial_y u_\gamma, \varepsilon\partial_x u_\gamma)$$

only contribute to error terms and thus can be neglected. Therefore, it is sufficient to study the equations

$$\begin{cases} \dfrac{1}{\varepsilon}L\Big(p_\varepsilon, \dfrac{x}{\varepsilon}, \varepsilon(\partial_t + \gamma), \varepsilon\partial_y, \varepsilon\partial_x\Big)u = f, \\ \Upsilon'(p_\varepsilon, u, \varepsilon\partial_y u, \varepsilon\partial_x u)_{|x=0} = g. \end{cases} \tag{10.17}$$

and prove estimates

$$\begin{aligned} &\|u^1\|_{\lambda_\epsilon^2} + \|u^2\|_{\lambda_\epsilon^2\Lambda_\epsilon} + \sqrt{\epsilon}\|\partial_x u^2\|_{\lambda_\epsilon} \\ &\qquad + |u^1{}_{|x=0}|_{\lambda_\epsilon} + |u^2{}_{|x=0}|_{\lambda_\epsilon\Lambda_\epsilon^{\frac12}} + \epsilon|\partial_x u_{2|x=0}|_{\lambda_\epsilon\Lambda_\epsilon^{-\frac12}} \\ &\le C\Big(\|f^1\| + \|f^2\|_{\Lambda_\epsilon^{-1}} + |g^1|_{\lambda_\epsilon} + |g^2|_{\lambda_\epsilon\Lambda_\epsilon^{\frac12}} + |g^3|_{\lambda_\epsilon\Lambda_\epsilon^{-\frac12}}\Big) + \text{errors}, \end{aligned} \tag{10.18}$$

where the errors are arbitrarily small compared to the left hand side when γ is large.

Next, we transform (10.17) to a first order system in x, introducing $U = {}^t(u, \varepsilon\partial_x u^2)$:

$$\begin{cases} \partial_x U = \dfrac{1}{\varepsilon}\mathcal{G}\Big(p_\varepsilon, \dfrac{x}{\varepsilon}, \varepsilon(\partial_t + \gamma), \varepsilon\partial_y, \varepsilon\gamma\Big)U + F, \\ \Gamma(p_\varepsilon)U_{|x=0} = g. \end{cases} \tag{10.19}$$

b) In Sections 4 to 9, we have studied these equations, when p_ε is a constant, using localizations and symmetrizers. We follow the same analysis. The LF / MF / HF localizations are performed with *semi-classical* operators

$$U_I = \chi(\varepsilon D_t, \varepsilon D_y, \varepsilon\gamma)U,$$

with $\chi(\zeta)$ supported in a neighborhood of the origin / in compact sets in $\mathbb{R}^{d+1}\backslash\{0\}$ / in $|\zeta|$ large respectively. Here $D_t = \frac{1}{i}\partial_t$, $D_y = \frac{1}{i}\partial_y$ and $\chi(\varepsilon D_t, \varepsilon D_y, \varepsilon\gamma)$ is defined as a Fourier multiplier. We use the notation

$$D = (D_t, D_y, \gamma). \tag{10.20}$$

Rule 1 : *Commutators*

$$[\frac{1}{\varepsilon}\mathcal{G}(p_\varepsilon, \frac{x}{\varepsilon}, \varepsilon D), \chi(\varepsilon D)]$$

contribute to error terms.

Indeed, the semiclassical calculus does not touch the fast variable $\frac{x}{\varepsilon}$ and the commutators win one factor ε, erasing the singular factor $\frac{1}{\varepsilon}$ in front of $\mathcal{G}$, and one semiclassical derivative.

c) In the LF and MF regime, we use conjugation operators

$$U_{II} = \Phi(p_\varepsilon, \frac{x}{\varepsilon}, \varepsilon D) U_I$$

with symbols $\Phi(p, z, \zeta)$ given by Lemma 4.1 for all fixed p. Using the same commutator argument as in Rule 1, we see that

$$\begin{cases} \partial_x U_{II} = \frac{1}{\varepsilon} G\Big(p_\varepsilon(t,y,x), \varepsilon D\Big) U_{II} + F_{II}, \\ \widetilde{\Gamma}(p_\varepsilon) U_{II|x=0} = g_{II}, \end{cases} \tag{10.21}$$

where f_{II} and g_{II} are controlled by the right hand side of (10.18).

d) The symbols $S(p, \zeta)$ of the MF symmetrizers are given by Proposition 6.6. We consider the operators $S_\varepsilon := S(p_\varepsilon(t,x,y), \varepsilon D)$. The semiclassical calculus implies the following :

Rule 2 (Semi-classical elliptic Gårding's inequality) *Because* $\operatorname{Re} S(p,\zeta) G(p,\zeta) \geq c\mathrm{Id}$*, the following inequality is satisfied in the sense of symmetric operators:*

$$\operatorname{Re} S(p_\varepsilon, \varepsilon D) \frac{1}{\varepsilon} G(p_\varepsilon, \varepsilon D) + \frac{1}{2} \partial_x S(p_\varepsilon, \varepsilon D) \geq \frac{c}{\varepsilon} - O(1).$$

Together with similar estimates for the boundary terms (using now *ii*) of (6.19)), this implies L^2 estimates for $\chi_{MF}(\varepsilon D) U$ and its traces.

e) In the LF regime, we use now the block diagonalization

$$\begin{pmatrix} u_H \\ u_P \end{pmatrix} = V(p_\varepsilon, \varepsilon D) U_{II} \tag{10.22}$$

with symbol $V(p, \zeta)$ given by Lemma 4.3. This leads to equations

$$\partial_x u_H = \frac{1}{\varepsilon} H(p_\varepsilon, \varepsilon D) u_H + f_H, \tag{10.23}$$

$$\partial_x u_P = \frac{1}{\varepsilon} P(p_\varepsilon, \varepsilon D) u_P + f_P. \tag{10.24}$$

For the elliptic block u_P, we use the symmetrizers given by Proposition 6.8 and use Rule 2 again.

For the hyperbolic block u_H, we use the polar coordinates

$$H(p,\zeta) = |\zeta| \check{H}(p, \check{\zeta}, |\zeta|), \qquad \check{\zeta} = \frac{\zeta}{|\zeta|},$$

and we note that

Lemma 10.5.

$$\frac{1}{\varepsilon}H(p_\varepsilon,\varepsilon D)=|D|\check H(p_\varepsilon,\check D,\varepsilon|D|),\qquad \check D=\frac{D}{|D|}$$

is a classical pseudo-differential operator of degree 1 in (t,y) with parameter γ, depending smoothly on x.

As a consequence, for the component u_H, we use the *classical pseudo-differential* (tangential) calculus with parameter γ (see [ChPi]). In particular, the LF symmetrizers are given by symbols $S(p,\check\zeta,\rho)$ (see Definition 6.9) and we quantify them as classical tangential pseudo's of degree 0: $S_\varepsilon=S(p_\varepsilon,\check D,\varepsilon|D|)$.

Rule 3 (Classical elliptic Gårding's inequality) *The conditions of Definition* 6.9 *imply that*

$$\begin{aligned}\operatorname{Re}\Big(S(p_\varepsilon,\check D,\varepsilon|D|)|D|\check H(p_\varepsilon,\check D,\varepsilon|D|)\Big)&\geq c(\check\gamma+|\varepsilon D|)|D|-O(1)\\&\geq c(\gamma+\varepsilon|D|^2)-O(1).\end{aligned}$$

Note that the conditions of Definition 6.9 imply but are stronger than

$$\operatorname{Re}S(p,\check\zeta,\rho)\check H(p,\check\zeta,\rho)\geq c(\check\gamma+\rho).$$

Knowing only this weaker estimate would force to use the sharp Gårding's inequality. With the stronger conditions of Definition 6.9, one can use the usual elliptic estimates.

Combining the estimates for u_P and u_H, one obtains the desired interior estimates for U_{LF}. To deal with the traces, we note that semiclassical operators of degree 0 supported in a compact frequency set, are classical pseudo-differential operators of degree 0. Thus, we can again convert the ellipticity of the operators acting on the traces into estimates.

f) In the HF regime, part of the analysis is made with semi-classical operators $S(p_\varepsilon,\frac{x}{\varepsilon},\varepsilon D)$: this concerns the block reduction and the use of symmetrizers in the central zone and also in the cone $\{\gamma\leq\delta|\zeta|\}\cup\{|\eta|\leq\delta|\zeta|\}$ for the hyperbolic 1-1 block. In this cone, for the parabolic block, as already noticed in Section 9, the correct homogeneity is the parabolic homogeneity given by the weight Λ. There we use a *semi-classical quasi-homogeneous calculus*

$$S(p_\varepsilon,\frac{x}{\varepsilon},\varepsilon D),\qquad S(p,z,\zeta)\in\mathrm{P}\Gamma.$$

where $\mathrm{P}\Gamma$ denotes here classes of symbols analogous to those introduced in Definition 9.4, depending smoothly on the parameter p.

g) To deal with Lipschitzean coefficients, we use *para-differential* calculi in place of pseudo-differential calculi. To summarize, we need three different calculi:

- the semiclassical homogeneous calculus,
- the semiclassical quasi-homogeneous calculus,
- the classical homogeneous calculus,

all depending on the parameter γ. We also need to compare them and to combine them in certain zones.

For details on these calculi, we refer to the Appendix of [MéZu1] and for the complete proof of the estimates, to the papers [MéZu1, GMWZ3, GMWZ4, GMWZ5].

10.4. Further steps

Theorem 10.3 gives global in time weighted L^2 a-priori estimates. From here, the path to the nonlinear stability analysis follows more or less classical steps. We briefly overview four of them, without precise statements.

a) Sobolev estimates. The classical analysis for noncharacteristic problems is to obtain first tangential H^s estimates and then get the normal derivatives from the equation. However, in the equation (10.19),

$$\frac{1}{\varepsilon}\mathcal{G}(p_\varepsilon, \frac{x}{\varepsilon}, \varepsilon D)$$

contains 0-th order terms of amplitude $O(\frac{1}{\varepsilon})$. They depend on (t, y) and commuting the equation (10.19) with $\partial_{t,y}$ yields terms in $O(\frac{1}{\varepsilon})$ which are not admissible errors in the LF regime.

To obtain tangential H^s low frequency estimates, one can follow the scheme of the proof of the L^2 estimates. After the reduction to equations (10.23)-(10.24), the commutator

$$[\partial_{t,y}, \frac{1}{\varepsilon}P(p_\varepsilon, \varepsilon D)]u_P = O(\frac{1}{\varepsilon})u_P$$

is admissible because of $\varepsilon^{-1}u_P$ is controlled. Moreover, since $H(p, \zeta)$ vanishes at $\zeta = 0$, the commutator

$$[\partial_{t,y}, \frac{1}{\varepsilon}H(p_\varepsilon, \frac{x}{\varepsilon}, \varepsilon D)]u_H = O(1)u_H$$

is non singular and yields admissible errors.

Following these ideas, one can actually prove *tangential H^s estimates*, see [MéZu1] an also the next section. But, for purely technical reasons, the proof given in this paper requires $W^{s,\infty}$ smoothness for the coefficients in place of the expected H^s smoothness (for s large enough). Clearing up this difficulty is an open question.

On the other hand *semiclassical H^s tangential* estimates are easy to prove, since the commutators of $\varepsilon D_{t,y}$ with the equation (10.19) contains no singular terms.

One final word about the normal derivatives. Because the equation (10.19) is singular, the easy thing is to get estimates for $\varepsilon\partial_x$ derivatives. Moreover, the presence of the boundary layer implies large variations in x at the scale ε. Thus the $\varepsilon\partial_x$ derivatives are natural in the problem. However, note that the maximal estimate of Theorem 10.3 implies that $\sqrt{\varepsilon}\partial_x u^2 \in L^2$.

To make better the transition with the interior estimates, one can also introduce the $x\partial_x$ derivatives. This shows that *conormal Sobolev regularity* plays a natural role in this problem. This also indicates some similarity with characteristic problems in the proof of Sobolev estimates.

b) Existence of smooth solutions to the linear boundary value problem. For fully parabolic problems, at any fixed $\varepsilon > 0$, this follows from the general theory of parabolic equations. For hyperbolic-parabolic problems, the existence of smooth solutions is less classical. It follows from the usual semi-group approach at least for dissipative equations and boundary conditions. One can also follow the general scheme of hyperbolic equations (see [ChPi]): study the backward dual problem and prove that it has the same form as the original system with time reversed; prove that it satisfies the uniform stability conditions (only the HF part is necessary for fixed

ε); deduce uniform estimates for the backward dual problem which imply existence of weak solutions for the forward system; prove a weak=strong theorem, showing that the weak solutions are unique and satisfy the a-priori estimates; and finally prove smoothness of the solutions along the lines of a), for fixed ε.

c) The causality principle and local in time estimates. Roughly speaking, letting $\gamma \to \infty$ in the estimates, one proves local uniqueness : if the data vanish for $t < 0$, then $u = 0$ for $t < 0$. One can replace 0 by any time T, and this shows that the solution up to time t depends only on the data for smaller time. This implies that the estimates (10.14) or (10.15) can be localized on $]-\infty, T] \times \mathbb{R}^d_+$. We refer to [ChPi] or [Mét4] for details concerning this general consequence of the estimates.

d) The mixed initial boundary value problem and semi-group estimates. Using steps b) and c) one solves the initial boundary value (IBV) problem with vanishing initial data, provided that the data can be extended by 0 in the past. For general initial data and source terms, compatibility conditions at the edge $\{t = x = 0\}$ are needed, their number depending on the desired smoothness of the solution (see e.g. [ChPi] or [Mét4]). For general data, one can solve first the equation in the sense of Taylor expansions at $\{t = 0\}$ (that is one determines the traces $\partial_t^j u_{|t=0}$ and lift up the traces). Using the compatibility conditions, this reduces the question to finding smooth solutions of the boundary value problem which vanish in the past. For fully parabolic problem, this method is perfectly efficient (see e.g; [Mét4]). For hyperbolic problem this method yields a loss of 1/2 derivatives from the initial data to the solution. This difficulty persists for hyperbolic-parabolic problem. The important question hidden behind this difficulty is the proof of semi-group estimates (pointwise in time with values in Sobolev spaces). The general proof for hyperbolic problems is not easy (see [Rau1]), unless the system is symmetric, because then usual integrations by parts yield pointwise estimates in time, knowing L^2 in space-time estimates of the traces, which are given by the main Kreiss estimate (see e.g. [Mét3]). This extends to hyperbolic-parabolic systems.

11. Nonlinear stability

11.1. Statement of the problem and main result

We consider the viscous boundary value problem (2.1) (2.10), assuming that the Assumptions (H1)–(H10) are satisfied. For simplicity, we restrict here our attention to problems in a half space, but $\mathbb{R}^d_+$ could be replaced by any smooth open set $\Omega \subset \mathbb{R}^d$. To start the discussion, we assume that a family of solutions of the profile equation is chosen, connecting 0 to a set of end states $\mathcal{C} \subset \mathcal{U}$:

Assumption 11.1. (H12) *We are given a smooth manifold $\mathcal{C} \subset \mathcal{U}$ of dimension $N - N_+$ and a smooth function W from $\mathcal{C} \times [0, \infty[$ to $\mathcal{U}^*$, such that for all $u \in \mathcal{C}$,*

i) $W(u, \cdot)$ is a solution of (3.3)(3.4)*;*

ii) $W(u, z)$ converges to u when z tends to $+\infty$, at an exponential rate, which can be chosen uniform on compact subsets of $\mathcal{C}$;

iii) the layer profile $W(u, \cdot)$ is transversal in the sense of Definition 4.4*;*

iv) the uniform spectral stability condition is satisfied for the linearized equation associated to the profile $W(u, \cdot)$.

In particular, by Theorem 5.5, the inviscid problem

$$\mathcal{L}_0(u_0) = 0, \quad u_{0|x=0} \in \mathcal{C}. \tag{11.1}$$

satisfies the uniform stability condition. Following A. Majda [Maj2], this implies that the mixed boundary value-Cauchy problem

$$\mathcal{L}_0(u_0) = 0, \quad u_{0|x=0} \in \mathcal{C}, \quad u_{|t=0} = h_0, \tag{11.2}$$

is locally well posed, provided that h_0 satisfies sufficiently many compatibility conditions at the edge. In order to avoid technical discussions on compatibility conditions at the edge $\{t = x = 0\}$, we consider here the simple case where the Cauchy data is compactly supported away from the boundary and takes its values in $\mathcal{U}$ and

Assumption 11.2. (H13) $0 \in \mathcal{C}$ *and* $W(0, z) = 0$. *In particular,* $\Upsilon(0) = 0$.

Of course, the state 0 has no real significance and can be replaced by any fixed constant $\underline{u} \in \mathcal{U}$, changing u into $u - \underline{u}$.

For $\varepsilon > 0$, consider next the viscous problems:

$$\mathcal{L}_\varepsilon(u) = 0, \quad \Upsilon(u_\varepsilon)_{|x=0} = 0, \quad u_{\varepsilon|t=0} = h_\varepsilon, \tag{11.3}$$

where the family $\{h_\varepsilon\}_{\varepsilon \geq 0}$ is uniformly bounded in $H^s(\mathbb{R}^d_+)$, supported in a fixed compact subset of $\{x > 0\}$, and such that $h_\varepsilon \to h_0$ as $\varepsilon \to 0$. The goal is to prove that the solution exists on a fixed interval of time $[0, T]$ independent of ε and that $u_\varepsilon \to u_0$ as $\varepsilon \to 0$. However, the essence of the problem is that this convergence cannot be uniform, due to the existence of boundary layers. Instead, a first approximation of the expected solution is obtained by adding to u_0 a layer corrector, so that to satisfy the viscous boundary conditions:

$$\tilde{u}_{0,\varepsilon}(t, x, y) = W(u_0(t, y, 0), x/\varepsilon) - u_0(t, y, 0) + u_0(t, y, x). \tag{11.4}$$

Note that $\tilde{u}_{0,\varepsilon}$ converges to u_0 when $x > 0$.

Theorem 11.3 (Non linear stability). *Suppose that the Assumptions (H1) to (H10) and (H12), (H13) are satisfied. Then if the regularity index s is large enough, there are $T > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, the problem* (11.3) *has a unique solution $u_\varepsilon \in W^{2,\infty}([0, T] \times \mathbb{R}^d_+)$ and $u_\varepsilon - \tilde{u}_{0,\varepsilon}$ tends to 0 in $L^\infty([0, T] \times \mathbb{R}^d_+)$.*

Remark 11.4. In general, (H13) must be replaced by suitable compatibility conditions. Another possible statement is of the form : given a very smooth inviscid solution u_0 of (11.1), there are u_ε solutions of the viscous equations such that $u_\varepsilon - u_{0,\varepsilon} \to 0$. We refer to [GMWZ2, GMWZ3, GMWZ4] for such statements in the case of shock waves.

As usual, solving initial-boundary value problems involves technicalities, in part because the linearized estimates of Section 10 are not semi-group estimates. It is more convenient to reduce first the problem to solving a continuation theorem, that is constructing an extension of a solution known in the past $\{t < 0\}$, to the price of introducing source terms. One can proceed as follows.

First, using the equation, one computes the Taylor expansion at $t = 0$ of the solutions

$$\partial_t^j u_{\varepsilon|t=0} = h_{j,\varepsilon}.$$

They are uniformly bounded in $H^{s-2j-\frac{1}{2}}$, if s is large, and supported in a fixed compact subset of $\mathbb{R}^d_+$. Next we lift the traces to find $u_{Tay,\varepsilon} \in H^{s-J}(\mathbb{R}^{1+d}$ such that

$$\partial_t^j u_{Tay,\varepsilon|t=0} = h_{j,\varepsilon} \quad \text{for } j \leq J.$$

and $u_{Tay,\varepsilon} = 0$ near $x = 0$. $f_\varepsilon = \mathcal{L}_\varepsilon(u_{Tay,\varepsilon})$ satisfies

$$\partial_t^j f_{\varepsilon|t=0} = 0 \quad \text{for } j \leq J-1.$$

With $\tilde{f}_\varepsilon = f_\varepsilon$ in the past and $\tilde{f}_\varepsilon = 0$ for $t > 0$, we are thus reduced to solve

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \tilde{f}_\varepsilon, \quad \Upsilon(u_\varepsilon)_{|x=0} = 0, \quad u_\varepsilon = u_{Tay,\varepsilon} \ \text{ for } t < 0. \tag{11.5}$$

11.2. High order approximate solutions

By suitable extension of the methods of [GrGu, GuWi, GMWZ3, Mét4], one proves that provided (i) the inviscid solution u_0 satisfies the spectral stability condition imposed by Majda on his constructed solutions and (ii) each tangent profile equation has a transversal planar viscous profile, then we may construct a hierarchy of approximate solutions

$$u_a^{\varepsilon,M} = \sum_{0 \leq n \leq M} \varepsilon^n U_n(t, y, x, x/\varepsilon)\,, \tag{11.6}$$

of (2.1). In this expansion one can look for U_n as

$$U_n(t, y, x, z) = u_n(t, y, x) + V_n(t, y, z) \tag{11.7}$$

where the $V_n(t, y, z)$ decays exponentially to zero as $z \to +\infty$; they describe the successive terms of the viscous boundary layer. The first term u_0 is a solution of the limiting inviscid hyperbolic boundary value problem and $W_0(t, y, z) := U^0(t, y, 0, z)$ satisfies the viscous profile equation (3.3), (3.4). The approximate solution $u_a^{\varepsilon,M}$ solves the equation with an error $O(\varepsilon^M)$.

The terms U_n are obtained inductively:

- u_n satisfies the linearized hyperbolic boundary value problem at u_0, with source term depending on $(U_0, \ldots, U_{n-1})$;

- V_n satisfies the linearized profile equation and initial condition at W_0, with source term depending on $(U_0, \ldots, U_{n-1})$;

- the boundary conditions of u_n are precisely those needed for the existence of $W_n(t, x, y) = u_n(t, y, 0) + V_n(t, y, z)$ satisfying the linearized profile equation and initial condition, converging to $u_n(t, y, 0)$ when z tends to $+\infty$.

Knowing the approximate solution $u_a^{\varepsilon,M}$, we look for exact solutions as

$$u_\varepsilon = \tilde{u}_a^{\varepsilon,M} + v_\varepsilon.$$

11.3. Parabolic methods

When the perturbation is fully parabolic one can use the standard local inversion theorem to solve the equation for the remainder v_ε. We recall here the principle of the method developed in [MéZu1].

We assume here that *the perturbation is fully parabolic* that is $N^2 = N$. For simplicity we consider Dirichlet boundary conditions and the following problem similar to (11.5)

$$\mathcal{L}_\varepsilon(u_\varepsilon) = f, \quad u_{\varepsilon|x=0} = 0, \quad u_\varepsilon = 0 \text{ for } t < 0. \tag{11.8}$$

assuming that f_ε is smooth and vanishes for $t < 0$ (see the more general form in [MéZu1]).

Consider a smooth solution u_0 on $[-T, T] \times \mathbb{R}^d_+$, of the hyperbolic boundary value problem. Let $u_{0,\varepsilon}$ be given by (11.4) and consider a first order corrector $u_{1,\varepsilon}$ such that

$$u_{a,\varepsilon} = u_{0,\varepsilon} + \varepsilon u_{1,\varepsilon} \tag{11.9}$$

satisfies

$$\mathcal{L}_\varepsilon(u_{a,\varepsilon}) = f + O(\varepsilon), \quad u_{a,\varepsilon|x=0} = 0, \quad u_{a,\varepsilon} = 0 \text{ for } t < 0. \tag{11.10}$$

The solution of (11.8) is constructed as

$$u_\varepsilon = u_{a,\varepsilon} + \varepsilon v_\varepsilon. \tag{11.11}$$

The equation for v_ε is written as

$$\mathcal{L}'_{u_{a,\varepsilon}} v_\varepsilon + \varepsilon \mathcal{Q}_\varepsilon(v_\varepsilon) = e_\varepsilon, \tag{11.12}$$

where $\mathcal{Q}_\varepsilon$ is at least quadratic in v and $e_\varepsilon = 0(1)$ is a given source term which vanishes in the past.

As mentioned in the previous section, *conormal* Sobolev spaces play an important role. Such spaces have already been widely used in the study of boundary value problems, see e.g. [Rau2], [Gue2]. Let $\{Z_k\}_{0 \le k \le d}$ denote a finite set of generators of vector fields tangent to $\{x = 0\}$:

$$Z_0 = \partial_t, \quad Z_j = \partial_{y_j} \text{ for } 1 \le j \le d-1, \quad Z_d = \frac{x}{1+x}\partial_x.$$

For $U \subset \mathbb{R} \times \Omega$ and $m \in \mathbb{N}$, define the space

$$\begin{aligned} \mathcal{H}^m(U) := \Big\{ u \in L^2(U) \ : Z_{k_1} \dots Z_{k_p} u \in L^2(U)\,, \\ \forall p \le m\,, \forall (k_1, \dots, k_p) \in \{0, \dots d\}^p \Big\}. \end{aligned} \tag{11.13}$$

This space is equipped with the obvious norm, denoted by $\| \cdot \|_{\mathcal{H}^m(U)}$.

In order to solve nonlinear problems, we need work in Banach algebras which means here that we have to supplement the $\mathcal{H}^m$ estimates with L^∞ estimates. Introduce the following norms

$$\|u\|_{\mathcal{W}^\mu(U)} = \|u\|_{L^\infty} + \sum_{p=1}^{\mu} \sum_{1 \le k_1, \dots, k_p \le \underline{k}} \|Z_{k_1} \dots Z_{k_p} u\|_{L^\infty}\,. \tag{11.14}$$

We first give estimates for the linearized equation at $u_{a,\varepsilon}$:

$$\mathcal{L}'_{u_{a,\varepsilon}} u = f, \quad u_{|x=0} = 0, \quad u_{t<0} = 0. \tag{11.15}$$

Assume that on $\Omega_{T_0} := [-T_0, T_0] \times \Omega$,

$$\begin{cases} u_0 \in W^{m+2,\infty}(\Omega_{T_0})\,, \\ \sup\limits_{\varepsilon \in]0,1]} \|u_{1,\varepsilon}\|_{\mathcal{W}^m} + \varepsilon \|\nabla_{t,x} u_{1,\varepsilon}\|_{\mathcal{W}^m} + \varepsilon^2 \|\nabla_x^2 u_{1,\varepsilon}\|_{\mathcal{W}^m} < \infty\,. \end{cases} \tag{11.16}$$

Theorem 11.5. *There are $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $f \in \mathcal{H}^m([-T_0, T_0] \times \Omega)$ vanishing for $t < 0$, the solution of equation* (11.15) *satisfies*

$$\|u\|_{\mathcal{H}^m} + \sqrt{\varepsilon}\|\partial_x u\|_{\mathcal{H}^m} + \varepsilon^{3/2}\|\partial_x^2 u\|_{\mathcal{H}^m} \leq C\|f\|_{\mathcal{H}^m} \tag{11.17}$$

If in addition $m \geq 2 + \frac{d+1}{2}$ and $f \in L^\infty([-T_0, T_0] \times \Omega)$, then the solution u also satisfies

$$\|u\|_{\mathcal{W}^2} + \varepsilon\|\partial_x u\|_{\mathcal{W}^1} + \varepsilon^2\|\partial_x^2 u\|_{L^\infty} \leq C\Big(\|f\|_{\mathcal{H}^m} + \varepsilon\|f\|_{L^\infty}\Big)\,. \tag{11.18}$$

Denote by $\|\cdot\|_{\mathcal{X}_\varepsilon^m}$ [resp. $\|\cdot\|_{\mathcal{Y}_\varepsilon^m}$] the norm given by adding the left [resp. right] hand sides of (11.17) and (11.18). Then the theorem implies the estimates

$$\|u\|_{\mathcal{X}_\varepsilon^m} \leq C\|\mathcal{L}'_{u_{a,\varepsilon}} u\|_{\mathcal{Y}_\varepsilon^m} \tag{11.19}$$

with C independent of ε.

Suppose that (11.16) holds with indices m such that

$$m > \frac{d+1}{2}\cdot \tag{11.20}$$

Proposition 11.6. *The quadratic term in* (11.8) *satisfies the following estimates:*

$$\begin{aligned} &\|\varepsilon\mathcal{Q}_\varepsilon(v^\varepsilon)\|_{\mathcal{Y}_\varepsilon^m} \leq \varepsilon^{1/4}\,C(M)\,, \\ &\|\varepsilon(\mathcal{Q}_\varepsilon(v_1^\varepsilon) - \mathcal{Q}_\varepsilon(v_2^\varepsilon))\|_{\mathcal{Y}_\varepsilon^m} \leq \varepsilon^{1/4}C(M)\,\|v_1 - v_2\|_{\mathcal{X}_\varepsilon^m}\,, \end{aligned} \tag{11.21}$$

provided that

$$\begin{aligned} &\varepsilon\|v_1\|_{L^\infty} \leq 1\,, \qquad &&\varepsilon\|v_1\|_{L^\infty} \leq 1, \\ &\varepsilon\|v_1\|_{\mathcal{X}_\varepsilon^m} \leq M\,, &&\varepsilon\|v_1\|_{\mathcal{X}_\varepsilon^m} \leq M\,, \end{aligned} \tag{11.22}$$

where $C(M)$ is independent of $\varepsilon \in]0, 1]$.

The nonlinear equations (11.12) and thus (11.8) can be solved using (11.19)-(11.21) and the standard local inversion theorem in $\mathcal{X}_\varepsilon^m$ (see [MéZu1] for details).

Theorem 11.7. *There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the problem* (11.8) *has a unique solution $u^\varepsilon = u_{a,\varepsilon} + \varepsilon v_\varepsilon$ with v_ε bounded in $\mathcal{X}_\varepsilon^m$. In particular,*

$$\|u^\varepsilon - u_0^\varepsilon\|_{\mathcal{H}^m} + \|u - u_0^\varepsilon\|_{L^\infty} = O(\varepsilon)\,. \tag{11.23}$$

11.4. Hyperbolic-like methods

The method sketched in the preceding section relies on inverting the linearized operator $\mathcal{L}'_{u_{a,\varepsilon}}$ *at the approximate solution.* The key argument uses maximal parabolic type estimates (with precised dependence on ε) for *all the components of* u. For partial viscosities this property fails and, one has to switch to iterative schemes which use the linearized operators $\mathcal{L}'_{u_n}$ for a sequence of u_n, following the lines of iterative schemes for hyperbolic equations.

We mention here a method of proof for Theorem 11.3 which has been used for example in [GrGu, Gue1, GMWZ4]. The idea is to start form a high order approximate solution

$$u_{a,\varepsilon}^M = \sum_{k=0}^{M} \varepsilon^k U_k(t, y, x, \frac{x}{\varepsilon})$$

which solves the equation up to an error $O(\varepsilon^M)$ (see Section 11.2). Next one looks for the solution as

$$u_\varepsilon = u^M_{a,\varepsilon} + \varepsilon^M v_\varepsilon,$$

where v_ε solves an equation of the form

$$\mathcal{L}'_{u^M_{a,\varepsilon}+\varepsilon^M v_\varepsilon} v_\varepsilon = f_\varepsilon = O(1).$$

To solve this equation, one uses Picard's iterates

$$\mathcal{L}'_{u^M_{a,\varepsilon}+\varepsilon^M v^n_\varepsilon} v^{n+1}_\varepsilon = f_\varepsilon$$

in semiclassical Sobolev spaces H^s_ε. The core of the analysis is to prove maximal estimate in H^s_ε-type spaces for v^{n+1}_ε using only the same control for v^n_ε. They are obtained by commutator arguments, as mentioned in Section 10.4, using Moser-Gagliardo-Nirenberg inequalities in H^s_ε. Just to give one typical ingredient, a minimal requirement is a Lipschitzean control of the coefficients $a(u^M_{a,\varepsilon} + \varepsilon^M vn_\varepsilon)$ of the linearized equation which follows from the estimate

$$\|\varepsilon^M \nabla v\|_{L^\infty} \leq C\|v\|_{H^s_\varepsilon}$$

if $s > 1 + d/2$ and $M \geq 1 + d/2$. Here we see that it is crucial for this method that M can be chosen large enough. This means here that we do start from a high order approximate solution. We refer to the cited references for details and precise statements.

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IMB Université de Bordeaux I, 33405 Talence Cedex
metivier@math.u-bordeaux.fr