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Abstract

We present here the results concerning the influence of a thin obstacle on the behavior of incompressible flow. We extend the works made by Itimie, Lopes Filho, Nussenzveig Lopes and Kelliher where they consider that the obstacle shrinks to a point. We begin by working in two-dimension, and thanks to complex analysis we treat the case of ideal and viscous flows around a curve. Next, we consider three-dimensional viscous flow in the exterior of a surface/curve. We finish by giving uniqueness of the vortex-wave system with a single point vortex introduced by Marchioro and Pulvirenti, in the case where the initial vorticity is constant near the point vortex. This last result gives, in particular, the uniqueness of the limit system obtained in the case of a perfect fluid around a point. We choose here to give the main steps of this uniqueness result, obtained in collaboration with E. Miot.

1. Introduction

We study the influence of a thin obstacle on the behavior of incompressible flow, when the obstacle tends to a curve or a surface. The small obstacle limit is an instance of the general problem of PDE on singularly perturbed domains. There is a large literature on such problems, specially in the elliptic case. Asymptotic behavior of fluid flow on singularly perturbed domains is a natural subject for analytical investigation which is virtually unexplored.

The first works were made in 2003 and 2006 by Iftimie, Lopes Filho and Nussenzveig Lopes concerning two dimensional incompressible ideal flow (governed by the Euler equations) [5] and viscous flow (governed by the Navier-Stokes equations) [6] when the obstacle shrinks homothetically to a point. Iftimie and Kelliher worked in 2008 on the three dimensional incompressible viscous flow around an obstacle which shrinks to a point [4].

In the following section, we will expose the main results concerning incompressible ideal [9] and viscous [10] flows around a curve (in 2D) and viscous flow around a surface/curve (in 3D). In the last part, we present the uniqueness of the vortex-wave system with a single point vortex introduced by Marchioro and Pulvirenti, in the case where the initial vorticity is constant near the point vortex [11]. In fact, the limit system obtained in [5] (case of a perfect fluid around an obstacle which shrinks to a point) corresponds to the vortex-wave system with a point fixed in an eulerian formulation. Our uniqueness result holds also in this case. In [5], the authors just proved the existence of a subsequence $(u^\varepsilon, \omega^\varepsilon)$ which tends to a solution of their limit system, and a consequence of [11] is that all the sequence converges to the unique solution of the limit system.

The article [11] was made in collaboration with Evelyne Miot. We choose in this presentation to present its main steps, whereas the results concerning fluids around thin obstacles will be given without proof.

General problem

Let us consider a fluid occupying a domain Ω in $\mathbb{R}^2$. A classical macroscopic description of its state may be made in terms of the density ρ, the velocity $u = (u_1, u_2, u_3)$ and the pressure p. The motion of an incompressible viscous flow is governed by the Navier-Stokes equation:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u = -\nabla p + g,$$

with g the exterior force and ν the viscosity of the fluid. The justification of this equation is largely detailed in literature (see for example [13]). In all the following works, we do not considered the exterior force (i.e. $g = 0$). Moreover, for the main part of flows, it seems reasonable to add the incompressibility condition:

$$\operatorname{div} u = 0.$$

For the Navier-Stokes equations, the classical condition at the boundary is the no-slid boundary condition (or Dirichlet condition) :

$$u = 0 \quad \text{at the boundary.}$$

In the case of an unbounded domain, we assume here that the velocity tends to zero at infinity.

If the resistance of the fluid is not insignificant, sometimes ν becomes very small after scaling. For example, it was computed in a case of a tuna moving in water that $\nu \approx 10^{-7}$. Therefore, sometimes it is reasonable to let $\nu = 0$ and we obtain the Euler equations which govern the incompressible perfect flow:

$$\partial_t u + u \cdot \nabla u = -\nabla p.$$

In this case, the no-slip boundary condition should be replaced by:

$$u \cdot n = 0 \quad \text{at the boundary,}$$

where n denotes the unit normal on $\partial\Omega$.

The Navier-Stokes and Euler equations can be interpreted in any number of space dimensions. We exclude the case of $n = 1$ because the incompressibility condition makes this problem uninteresting. The physical interpretation applies in principle only to the three dimensional case. We observe, however, that any solutions $u = (u_1, u_2)$ of the two dimensional equations gives rise to a special solution of the 3D equations, of the form

$$u = u(x_1, x_2, x_3, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0).$$

Common examples are an infinite cylinder or an infinite air plane wing.

Moreover, we will have to fix an initial data to make the problem well-posed.

Let us define now an important quantity for the study of these equations. Let ω be the vorticity defined as follow:

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

if the space dimension is 2, and

$$\omega = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$$

if the space dimension is 3.

The works [9, 10], presented in the following section, treat a certain kind of singular limit for these equations. Precisely, we consider Navier-Stokes and Euler equations posed in the exterior of an obstacle which is more and more thin, tending to a curve. The goal is to determine the limit equation.

2. A singular limit

Let Ω_ε be an obstacles family (smooth, bounded, open, connected and simply connected domains) which shrink as $\varepsilon \to 0$ to a curve if the space dimension is two or to a surface in 3D. We consider the Navier-Stokes and Euler equations on the exterior domains $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \overline{\Omega_\varepsilon}$. We assume finally that the initial vorticity ω_0 is smooth and that its support does not intersect the obstacles. As the goal is to compare flows defined on different domains, we extend the functions on all the space as follow: for a function f defined on Π_ε, we denote by Ef the extension of f on $\mathbb{R}^n$, such that $Ef = f$ on Π_ε and zero otherwise.

The behavior of these singular limits depends on the space dimension and the equations chosen. We study three cases: Euler equations in two dimensional space and Navier-Stokes equations in two and three dimensional spaces.

This kind of limit was treated in the three same situations in [5, 6, 4] when the obstacles tend to a point, instead of a curve or surface. We will explain the main differences between their and our results.

2.1. Two dimensional ideal flow (corresponding to [9])

In two dimensional space, the vorticity is not sufficient to uniquely determine a divergence free vector field, tangent to the boundary $\Gamma_\varepsilon \equiv \partial\Omega_\varepsilon$ and vanishing at infinity. We need the circulation of the velocity around the obstacle. Then, we should fix the circulation of the initial velocity:

$$\gamma = \oint_{\Gamma_\varepsilon} u_0^\varepsilon . \mathbf{ds}$$

independently of ε. Then, we can show that, given the geometry of the obstacle Ω_ε, the initial velocity is uniquely determined in terms of ω_0 and γ. In fact we have an explicit formula of the Biot-Savart law: law which gives the velocity in terms of the vorticity and the circulation around the obstacle. We use complex analysis (Riemann theorem) to construct a biholomorphism T_ε between Π_ε and the exterior of the unit disk, which allows us to establish explicit formula:

$$u_0^\varepsilon = u_0^\varepsilon(x,t) = K^\varepsilon[\omega_0(\cdot,t)](x) + \alpha H^\varepsilon(x)$$

with K^ε, H^ε and α given in terms of ω_0, γ and T_ε. We see here the main advantage of the two dimensional space: we identify $\mathbb{R}^2$ to the complex plan $\mathbb{C}$ and we use complex analysis to obtain these explicit formulas.

The motion of an incompressible ideal flow in Π_ε is governed by the Euler equations:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon = -\nabla p^\varepsilon & \text{in } \Pi_\varepsilon \times (0,\infty) \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Pi_\varepsilon \times (0,\infty) \\ u^\varepsilon \cdot \hat{n} = 0 & \text{on } \Gamma_\varepsilon \times (0,\infty) \\ \lim\limits_{|x|\to\infty} |u^\varepsilon| = 0 & \text{for } t \in [0,\infty) \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{in } \Pi_\varepsilon \end{cases}$$

where $p^\varepsilon = p^\varepsilon(x,t)$ is the pressure. Kikuchi established in [7] that the previous system admits an unique global solution u^ε. A characteristic of this solution is the conservation of the velocity circulation on the boundary, and that $m \equiv \int \operatorname{curl} u^\varepsilon = \int \omega_0$. This implies in particular that α (in the Biot-Savart law) is constant and is equal to $\gamma + m$ (see [5]). In fact, to study the two dimensional ideal flows, we work on the vorticity equations $\omega^\varepsilon \equiv \operatorname{curl} u^\varepsilon$ which are equivalent to the previous system:

$$\begin{cases} \partial_t \omega^\varepsilon + u^\varepsilon \cdot \nabla \omega^\varepsilon = 0 & \text{in } \Pi_\varepsilon \times (0,\infty) \\ u^\varepsilon = K^\varepsilon[\omega^\varepsilon] + \alpha H^\varepsilon & \text{in } \Pi_\varepsilon \times (0,\infty) \\ \omega^\varepsilon(x,0) = \omega_0(x) & \text{in } \Pi_\varepsilon. \end{cases}$$

The interest of such a formulation is that we recognise a transport equation. The transport nature allows us to conclude that the $L^p(\Pi_\varepsilon)$ norms of the vorticity are conserved, for $p \in [1,\infty]$, which gives us directly an estimate and a weak convergence in L^p for the vorticity.

The goal is to determine the limit of $(u^\varepsilon, \omega^\varepsilon)$ when $\varepsilon \to 0$.

This kind of work was initiated by Iftimie, Lopes Filho and Nussenzveig Lopes in [5] when the obstacle shrink homothetically to a point. The authors set $\Omega_\varepsilon \equiv \varepsilon\Omega$, with Ω a fix obstacle (smooth, bounded, connected, simply connected, containing 0) and they show the following result:

Theorem 2.1. *There is a subsequence $\varepsilon = \varepsilon_k \to 0$ such that*

(a) $\Phi^\varepsilon u^\varepsilon \to u$ *strongly in* $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^2)$*;*

(b) $\Phi^\varepsilon \omega^\varepsilon \to \omega$ *weak* $*$ *in* $L^\infty(\mathbb{R}_+; L^4_{\text{loc}}(\mathbb{R}^2))$*;*

(c) *the limit pair (u,ω) verify in the sense of distributions:*

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ \operatorname{curl} u = \omega + \gamma\delta_0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ \omega(x,0) = \omega_0(x) & \text{in } \mathbb{R}^2 \end{cases}$$

with δ_0 the Dirac function at 0.

In this result, Φ^ε is a cut-off function of an ε-neighbourhood of Ω_ε. The limits (a) and (b) are independents of the choice of the cut-off functions, and we can give the same theorem with Eu^ε and $E\omega^\varepsilon$. Therefore, they obtain at the limit the

Euler equations in the full plane, where a Dirac mass at the origin appears. This additional term is a reminiscent of the circulation γ of the initial velocities around the obstacles, and we note that this term does not appear if $\gamma = 0$.

The work done in [9] describes the case where the obstacle shrinks to a smooth curve Γ. Just before to give the main result, we should precise the sense of this limit. Indeed, the goal in the two articles [5, 9] is to obtain some velocity estimates thanks to the Biot-Savart law, in order to pass to the limit. For this, we need some estimates of T_ε, the biholomorphism between Π_ε and the exterior of the unit disk. In [5], choosing a homothically convergence of the obstacles, the authors set $T_\varepsilon(x) = T(x/\varepsilon)$, with T the biholomorphism between Ω and the exterior of the unit disk. This simplifies the a priori estimates. In our case, the first step is to think about the existence of a conformal mapping T between the exterior of the curve $\Pi \equiv \mathbb{R}^2 \setminus \Gamma$ and the exterior of the unit disk. We use some important results in complex analysis (see [18, 19]) to establish this existence and we define the convergence of the obstacles Ω_ε to Γ by a convergence of T_ε to T. For details, see [9]. The convergence of T_ε to T, in a particular sense, is the key of the determination of the limit velocity and we will deduce properties on the limit velocity thanks to properties of T.

In [5], the authors do not have this convergence but their estimates are easier. Indeed, they decompose the velocity u^ε in two parts: v^ε uniformly bounded in ε and an harmonic part γH^ε. Moreover, when the obstacles shrink to a point, the cut-off functions verify:

$$mes(\text{supp } \nabla\Phi^\varepsilon) = O(\varepsilon^2) \text{ and } \|\nabla\Phi^\varepsilon\|_{L^p} \to 0, \forall 1 \leq p < 2.$$

Then they find a weak limit for the curl and the divergence:

$$\begin{aligned} \text{div}\,(\Phi^\varepsilon v^\varepsilon) &= 0 + \nabla\Phi^\varepsilon \cdot v^\varepsilon \to 0 \\ \text{curl}\,(\Phi^\varepsilon v^\varepsilon) &= \Phi^\varepsilon \omega_\varepsilon + \nabla^\perp\Phi^\varepsilon \cdot v^\varepsilon \to \omega, \end{aligned}$$

which allows us to extract a strong limit in L^2 of the sequence $\{\Phi^\varepsilon v^\varepsilon\}$, thanks to the div-curl lemma.

In our case, we do not obtain L^∞ estimates, but only in L^p for $p < 4$, because DT blows up at the end points of the curve like the inverse of the square root of the distance. Moreover $mes(\text{supp } \nabla\Phi^\varepsilon) = O(\varepsilon)$ and we remark that we can not apply a similar argument than [5]. However, noting that $\nabla\Phi^\varepsilon$ is normal to the boundary, whereas u^ε is tangent, a step consists to prove the limit $\nabla\Phi^\varepsilon \cdot u^\varepsilon \to 0$ holds true in our case. Next, the idea is to use the weak convergence of the vorticities and the convergence of T_ε to T in order to pass directly to the limit in the Biot-Savart law, thanks to the dominated convergence theorem. We obtain a strong convergence in L^2 for the velocity. By a weak-strong pair vorticity-velocity, we can pass to the limit in the Euler equations. The original thing here is that we obtain an explicit form of the limit velocity. We use it to deduce some properties as the behavior of the flow near the curve, the relation between limit velocity and limit vorticity, ... The result concerning the Euler equations in the case of a curve in 2D is the following theorem.

Theorem 2.2. *There is a subsequence $\varepsilon = \varepsilon_k \to 0$ such that*

(a) $\Phi^\varepsilon u^\varepsilon \to u$ *strong in* $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^2)$*;*

(b) $\Phi^\varepsilon \omega^\varepsilon \to \omega$ *weak* $*$ *in* $L^\infty(\mathbb{R}_+; L^4_{\text{loc}}(\mathbb{R}^2))$*;*

(c) u *is explicitly given in terms of* ω*;*

(d) *u and ω are weak solutions of $\partial_t\omega + u.\nabla\omega = 0$ in $\mathbb{R}^2 \times (0,\infty)$.*

The limit velocity u is explicitly given in terms of ω and γ and it corresponds to a divergence-free vector field, tangent to the curve Γ, vanishing at infinity, whose the curl is equal to ω on $\mathbb{R}^2 \setminus \Gamma$ and whose the circulation around Γ is γ. We also note that the velocity is continuous up to the curve (with different values on each side of the curve), except to the end points where it blows up like the inverse of the square root of the distance. However, the velocity remains bounded in L^p_{loc} for $p < 4$. If we compute the curl in the full plane, we obtain in fact that $\operatorname{curl} u = \omega + g_\omega(s)\delta_\Gamma$ where δ_Γ is the Dirac on the curve Γ, and where the density g_ω depends only on ω and γ. The function g_ω is continuous on Γ and blows up at the end-points. Moreover, we note that g_ω corresponds to the jump though the curve of the velocity tangent part. In fact, the presence of the additional term g_ω in the curl of the velocity, compared to the Euler equations in the full plane, is necessary to obtain a vector field tangent to the curve, with a circulation γ around the curve.

If we assume that $\gamma = 0$, we obtain a contrast between [5] and our case. In the case of a small obstacle, we do not observe any reminiscence of the obstacle, whereas in the case of a thin obstacle, $g_\omega \neq 0$. Moreover, in the case of a thin obstacle, this density depends on the time. In other words, if the fluid does not feel the effect of a point, it always feels a curve.

We also find a formulation on $\mathbb{R}^2 \setminus \Gamma$ which corresponds to the Euler equations in the exterior of a curve. Therefore, a consequence of this work is the global existence of a weak solution to the Euler equations in such a domain.

Remark 2.3. In the case of two dimensional ideal flow around small obstacle, we should mention that Lopes Filho worked in [12] about several obstacles where just one shrinks to a point. However, the author had to work in bounded domain. In his case, we do not have explicit form of the Biot-Savart law and the complex analysis techniques are replaced by variational methods and the maximum principle. His result is equivalent to Theorem 2.1.

Remark 2.4. We can also think about the uniqueness of solutions of the limit problems. The existence is established by passing to the limit for solutions defined on smooth domains. Moreover, in each theorem, we state that we extract a subsequence. If we show an uniqueness result, it would mean that $\Phi^\varepsilon u^\varepsilon \to u$ for all the sequence $\varepsilon_k \to 0$. Uniqueness in the case of the curve is too complicated, because the density g_ω depends on the time and ω. However, in the case where the obstacle shrinks to a point, we obtain in [11] the uniqueness of solutions of equation (c) in Theorem 2.1. This result is detailed in the last section.

2.2. Two dimensional viscous flow (corresponding to [10])

As we explain in the previous subsection, we should fix γ the circulation of the initial velocity around the obstacle. This quantity and the initial vorticity ω_0, chosen independently of ε, allow us to uniquely determine a divergence-free vector field u_0^ε, tangent to the boundary and vanishing at infinity.

The behaviour of an incompressible viscous flow in Π_ε is governed by the Navier-Stokes equations:

$$\begin{cases} \partial_t u^\varepsilon - \nu\Delta u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon = -\nabla p^\varepsilon & \text{in } \Pi_\varepsilon\times(0,\infty) \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Pi_\varepsilon\times(0,\infty) \\ u^\varepsilon = 0 & \text{on } \Gamma_\varepsilon\times(0,\infty) \\ \lim\limits_{|x|\to\infty} |u^\varepsilon| = 0 & \text{for } t\in[0,\infty) \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{in } \Pi_\varepsilon \end{cases}$$

where $p^\varepsilon = p^\varepsilon(x,t)$ is the pressure. Existence and uniqueness of global solution of such problem were given by the Kozono and Yamazaki's work [8]. Let us note that the vorticity equation does not give any control. In the viscous case, a dissipative term is added to the transport equation, which, in the full plane, allows us to conclude that the L^p norms of the vorticity are decreasing. However, it does not remain true if we work in a domain with boundaries (our case) because the vorticity does not verify well conditions at the boundary. In fact, estimates of one velocity derivative will be obtained directly from the Navier-Stokes equations, thanks to the second order term $-\nu\Delta u^\varepsilon$.

Iftimie, Lopes Filho and Nussenzveig Lopes [6] treated the case of an obstacle which homothetically shrinks to a point. They prove that in the case of small circulation the limit equations are always the Navier-Stokes equations where the additional Dirac mass appears only on the initial data. This is due to the fact that the circulation of the initial velocity on the boundary of the obstacle does vanish for $t > 0$ when we consider the no-slip condition. They use again the change of variables $y = x/\varepsilon$ to work in a fixed domain.

Here, we assume that the obstacles shrink to a curve. The first step is to determine the limit of the initial data. We use here the Biot-Savart law, and we find an equivalent limit to the case of an ideal flow. Then, a priori estimates are simplified compared to [6], because our limit initial velocity u_0 belongs to L^p_{loc} for $p < 4$, whereas in the case of small obstacles, limit velocity does not belong to L^2_{loc}. Therefore, in [6], the authors have to work in a small time interval, in order to the velocity becomes L^2_{loc} and after they use classical estimates for Navier-Stokes equations. In the two cases u_0 is not square integrable at infinity. We should estimate the square integrable part of the velocity. This part, denoted by W^ε, corresponds to u^ε where we remove at infinity the harmonic part. For such an initial data, we define a solution of the Navier-Stokes equations in the exterior of the curve as a vector field verifying the equation in the sense of distributions and such that the difference between the solution and a fixed smooth vector field behaving like $\gamma\frac{x^\perp}{2\pi|x|^2}$ at infinity has the regularity expected from Leray solution. The main theorem of [10] is the following.

Theorem 2.5. *Let ω_0 and γ be independent of ε as defined above. Let u^ε be the solution of the Navier-Stokes equations on $\Pi_\varepsilon \equiv \mathbb{R}^2\setminus\overline{\Omega_\varepsilon}$ with initial velocity u_0^ε and denote by Eu^ε the extension of u^ε to $\mathbb{R}^2$ with values 0 on Ω_ε. Then $\{Eu^\varepsilon\}$ converges in $L^2_{\text{loc}}([0,\infty)\times(\mathbb{R}^2\setminus\Gamma))$ to a solution of the Navier-Stokes equations in $\mathbb{R}^2\setminus\Gamma$ with an initial vorticity $\omega_0 + g_{\omega_0}\delta_\Gamma$. Moreover, such a solution is unique.*

The initial velocity is given by the relation

$$u_0 = K[\omega_0] + \alpha H,$$

with K and H depend only on the Γ shape, and with $\alpha = \gamma + \int \omega_0$. Then, this initial velocity is explicitly given in terms of ω_0 and γ and can be viewed as the divergence free vector field which is tangent to Γ, vanishing at infinity, with curl in $\mathbb{R}^2 \setminus \Gamma$ equal to ω_0 and with circulation around the curve Γ equal to γ. This velocity is blowing up at the endpoints of the curve Γ as the inverse of the square root of the distance and has a jump across Γ.

The existence of solutions in the Navier-Stokes equations has been studied in general domains in [1] for the dimension two or three for square-integrable data, and in [17] for the dimension three and $H^{\frac{1}{2}}$ initial data. Kozono and Yamazaki [8] treated the case of $L^{2,\infty}$ data but for exterior domains which are smooth. A byproduct of the previous theorem is the existence and uniqueness of solutions of the Navier-Stokes equations on $\mathbb{R}^2 \setminus \Gamma$ in a case which is not covered in previous work. Indeed, the result of [1] does not apply because the initial data of our limit velocity is not square-integrable at infinity. Our extension from square-integrable velocities to velocities that decay like $1/|x|$ is physically meaningful: it allows nonvanishing initial circulation around the obstacle, something which can happen in impulsively started motions. On the other hand, our initial data u_0 satisfies the smallness condition of Kozono and Yamazaki [8], but the domain $\mathbb{R}^2 \setminus \Gamma$ is not smooth, as required in [8].

At opposite of [5, 6, 9], we do not find here a formulation in the full plane. Choosing to look for a formulation on $\mathbb{R}^2 \setminus \Gamma$, we work about convergences on compact sets K included in the exterior of the curve. There is ε_K such that $K \cap \Omega_\varepsilon = \emptyset$ for all $\varepsilon \leq \varepsilon_K$, then $\nabla \Phi^\varepsilon$ corresponding to the boundary effects does not appear in the estimates. The difficulty with Navier-Stokes equations around a curve comes with the presence of a second order operator Δ and the fact that the Lebesgue measure of the set where Φ^ε is not constant is $O(\varepsilon)$. In the case where the obstacle shrinks to a point, this measure is equal to $O(\varepsilon^2)$, so $\|\Delta \Phi^\varepsilon\|_{L^1}$ is $O(1)$, whereas in our case, this norm is equal to $O(1/\varepsilon)$. This blow up in the case of the viscous flow around the curve is mathematically difficult but physically meaningful. Indeed, we hope that the limit velocity verifies:

$$\partial_t u - \nu \Delta u + u \cdot \nabla u = \nu f m_G - \nabla p^\varepsilon \text{ in } \mathbb{R}^2 \times (0, \infty),$$

where m_G is a measure supported on the curve Γ (maybe the Dirac ?) and where f corresponds to the lift force applied by the viscous flow on the curve.

2.3. Three dimensional viscous flow

At the opposite of two dimensional flow, the initial vorticity ω_0 is sufficient to uniquely determine a divergence-free vector field, tangent to the boundary and vanishing at infinity. There is not any work about Euler equations because we can not obtain estimates independently of ε of one velocity derivative. Indeed, in three dimensional space, the vorticity equation is not any more a transport equation and we can not pretend that the L^p norms of the vorticity are conserved. The Navier-Stokes equations allow us to obtain this control, thanks to the second order term $\nu \Delta u$. Kelliher and Iftimie in [4] worked on the case of an obstacle shrinking to a point, and they prove that the limit velocity verifies the Navier-Stokes equations in the full space. The limit of the initial velocity is $u_0 = -\int_{\mathbb{R}^3} \frac{x-y}{4\pi|x-y|^3} \times \omega_0(y)\, dy$, and we do not observe any effect of the small obstacle, even on the initial data.

We set S, a smooth surface, bounded, with a boundary Γ and we assume that $\Pi_\varepsilon \equiv \mathbb{R}^3 \setminus \Omega_\varepsilon$ is an exterior domain simply connected with a smooth boundary, such that Ω_ε tends to S when $\varepsilon \to 0$ in the following sense: there is $M > 0$ such that $\Omega_\varepsilon \subset S + B(0, M\varepsilon)$ for all $\varepsilon > 0$. Then, we do not use here strong properties on the convergence assumption, comparing to the two dimensional case. Luckily because we do not have an equivalent to complex analysis. Then, we do not have any more explicit formula for the Biot-Savart law. This explicit form was necessary in 2D Euler, because we deduce velocity convergence from weak convergence of vorticities. In 2D viscous flow, we need explicit formula to estimate harmonic part v^ε at infinity, which allows us to work with square integrable vector fields $W^\varepsilon = u^\varepsilon - v^\varepsilon$. There is not this problem in three dimensional space because Π_ε is simply connected, then there are not circulation and harmonic part. However, it is harder to find some properties of the initial velocity, as the blow up near the boundary. We can show the strong convergence to a vector field verifying the Navier-Stokes in the exterior of the surface, with an divergence-free initial data, tangent to the surface. Except the explicit formula of the initial data, we find an equivalent result to the curve in two dimensional space. For the same reasons, we do not try to look for a formulation in the full space.

In the case where the obstacle shrinks to a curve, we prove the strong convergence in $L^2(\mathbb{R}^3)$ for the initial velocity to the initial velocity without obstacle u_0. Actually, we can not yet show that the limit velocity is a weak solution of the Navier-Stokes in the full space $\mathbb{R}^3$.

3. Uniqueness for the vortex-wave system

3.1. Introduction

In this section, we study a system occurring in two dimensional fluid dynamics. The motion of an ideal incompressible fluid in $\mathbb{R}^2$ with divergence-free velocity field $v = (v_1, v_2) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ and vorticity $\omega = \operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1 : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is given by the Euler equations

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ \omega = \operatorname{curl} v, \ \operatorname{div} v = 0, \end{cases} \tag{3.1}$$

where $\operatorname{div} v = \partial_1 v_1 + \partial_2 v_2$. For this system, Yudovich's Theorem [21] states global existence and uniqueness in $L^\infty(\mathbb{R}^+, L^1 \cap L^\infty(\mathbb{R}^2))$ for an initial vorticity $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$. Equation (3.1) is a transport equation with field v, therefore one may solve it with the method of characteristics. When v is smooth, it gives rise to a flow defined by

$$\begin{cases} \frac{d}{dt}\phi_t(x) = v\big(t, \phi_t(x)\big) \\ \phi_0(x) = x \in \mathbb{R}^2. \end{cases} \tag{3.2}$$

In view of (3.1), we then have

$$\frac{d}{dt}\omega\big(t, \phi_t(x)\big) \equiv 0, \tag{3.3}$$

which means that ω is constant along the characteristics. In the general case of a vorticity $\omega \in L^\infty(\mathbb{R}^+, L^1 \cap L^\infty(\mathbb{R}^2))$, these computations may be rigorously justified, so that the Eulerian formulation (3.1) and the Lagrangian one (3.2), (3.3) turn out to be equivalent.

Since equation (3.1) governs the evolution of the vorticity ω, it is natural to express the velocity v in terms of ω. This can be done by taking the orthogonal gradient in both terms in the relation $\omega = \operatorname{curl} v$ and using that v is divergence free. This yields $\nabla^\perp \omega = \Delta v$, so that under the additional constraint that v vanishes at infinity, we have

$$v = K * \omega. \tag{3.4}$$

Here $*$ denotes the convolution product and $K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ stands for the Biot-Savart Kernel defined by

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \qquad x \neq 0, \tag{3.5}$$

where $(x_1, x_2)^\perp = (-x_2, x_1)$. When the vorticity tends to be concentrated at points, one may modify equation (3.1) according to formulas (3.4) and (3.5) into a system of ordinary differential equations, called point vortex system, which governs the motion of these points. A rigorous justification for this system has been carried out in [16]. It is proved there that if the initial vorticity ω_0 is close to the weighted sum of Dirac masses $\sum d_i \delta_{z_i}$ in a certain sense, then $\omega(t)$ remains close to $\sum d_i \delta_{z_i(t)}$ for all time, where the vortices $z_i(t)$ evolve according to the point vortex system.

In the early 90s, Marchioro and Pulvirenti [14, 15] investigated the mixed problem in which the vorticity is composed of an L^∞ part and a sum of Dirac masses. They obtained the so-called vortex-wave system, which couples the usual point vortex system and the classical Lagrangian formulation for the two-dimensional fluid dynamics. In the case of a single point vortex (which will be the case studied here), these authors obtained the global existence of solutions of the vortex-wave system in Lagrangian formulation.

Definition 3.1 (Lagrangian solutions). *Let $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ and $z_0 \in \mathbb{R}^2$. We say that the triple (ω, z, ϕ) is a global Lagrangian solution to the vortex-wave system with initial condition (ω_0, z_0) if $\omega \in L^\infty(\mathbb{R}^+, L^1 \cap L^\infty(\mathbb{R}^2))$, $v = K*\omega \in C(\mathbb{R}^+ \times \mathbb{R}^2)$ and*

$$z : \mathbb{R}^+ \to \mathbb{R}^2, \qquad \phi : \mathbb{R}^+ \times \mathbb{R}^2 \setminus \{z_0\} \to \mathbb{R}^2$$

are such that $z \in C^1(\mathbb{R}^+, \mathbb{R}^2)$, $\phi(\cdot, x) \in C^1(\mathbb{R}^+, \mathbb{R}^2)$ for all $x \neq z_0$ and satisfy

$$\begin{cases} v(\cdot, t) = (K * \omega)(\cdot, t), \\ \dot{z}(t) = v(t, z(t)), \\ z(0) = z_0, \\ \dot{\phi}_t(x) = v(t, \phi_t(x)) + K\Big(\phi_t(x) - z(t)\Big), \\ \phi_0(x) = x, \ x \neq z_0, \\ \omega(\phi_t(x), t) = \omega_0(x), \end{cases} \tag{LF}$$

where $\phi_t = \phi(t, \cdot)$. In addition, for all t, ϕ_t is an homeomorphism from $\mathbb{R}^2 \setminus \{z_0\}$ into $\mathbb{R}^2 \setminus \{z(t)\}$ that preserves Lebesgue's measure.

This system involves two kinds of trajectories. The point vortex $z(t)$ moves under the velocity field v produced by the regular part ω of the vorticity. This regular part and the vortex point give rise to a smooth flow ϕ along which ω is constant. The main difference with the classical Euler dynamics is the presence of the field $K(x - z(t))$, which is singular at the point vortex but smooth elsewhere. Marchioro and Pulvirenti [14] proved global existence for (LF). The proof mainly relies on estimates involving the distance between $\phi_t(x)$ and $z(t)$ and uses almost-Lipschitz regularity for $v = K * \omega$ and the explicit form of K. It is shown in particular that a characteristic starting far apart from the point vortex cannot collide with $z(t)$ in finite time. Consequently, the singular term $K(\phi_t(x) - z(t))$ in (LF) remains well-defined for all time.

The notion of Lagrangian solutions is rather strong. One can define a weaker notion of solutions: solutions in the sense of distributions of the PDE (without involving the flow ϕ). We call these Eulerian solutions and we define them here below.

Definition 3.2 (Eulerian solutions)**.** *Let* $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ *and* $z_0 \in \mathbb{R}^2$. *We say that* (ω, z) *is a global Eulerian solution of the vortex-wave equation with initial condition* (ω_0, z_0) *if*

$$\omega \in L^\infty\left(\mathbb{R}^+, L^1 \cap L^\infty(\mathbb{R}^2)\right), \qquad z \in C(\mathbb{R}^+, \mathbb{R}^2)$$

and if we have in the sense of distributions

$$\begin{cases} \partial_t \omega + \operatorname{div}\left((v + H)\omega\right) = 0, \\ \omega(0) = \omega_0, \\ \dot{z}(t) = v\Big(t, z(t)\Big), \qquad z(0) = z_0, \end{cases} \tag{EF}$$

where v *and* H *are given by*

$$v(t, \cdot) = K *_x \omega(t), \qquad H(t, \cdot) = K(\cdot - z(t)).$$

In other words, we have [1] *for any test function* $\psi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$

$$-\int_{\mathbb{R}^2} \omega_0(x)\psi(0, x)\, dx = \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \omega(\partial_t \psi + (v + H) \cdot \nabla\psi)\, ds\, dx,$$

and [2]

$$z(t) = z_0 + \int_0^t v(s, z(s))\, ds$$

for all $t \in \mathbb{R}^+$.

This kind of Eulerian solutions appears for example in [5]. In that paper, a solution of the Euler equation with a fixed point vortex is obtained as the limit of the Euler equations in the exterior of an obstacle that shrinks to a point. The regularity of the limit solution obtained in [5] is not better than the one given in Definition 3.2.

In [11], we are concerned with the problems of uniqueness of Eulerian and Lagrangian solutions and with the related question of equivalence of Definitions 3.1 and 3.2.

[1] We can prove that the field defined by $v = K * \omega$ belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$. On the other hand, H belongs to $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, so that this definition makes sense.

[2] We can prove that $v(t)$ is defined for all time and is continuous in the space variable.

3.2. Lagrangian implies Eulerian.

We first prove the following Theorem, clarifying that a Lagrangian solution is an Eulerian solution.

Theorem 3.3. *Let* $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ *and* $z_0 \in \mathbb{R}^2$. *Let* (ω, z, ϕ) *be a global Lagrangian solution of the vortex-wave system with initial condition* (ω_0, z_0). *Then* (ω, z) *is a global Eulerian solution.*

Idea of the proof. Given (ω, z, ϕ) a solution of (LF), it actually suffices to show that

$$\partial_t \omega + \operatorname{div}((v+H)\omega) = 0 \tag{3.6}$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^2$.
We first give a formal proof of (3.6). Let us take a C^1 function $\psi(t,x)$ and define

$$f(t) = \int_{\mathbb{R}^2} \omega(t,y)\psi(t,y)\, dy.$$

We set $y = \phi_t(x)$. Since ϕ_t preserves Lebesgue's measure for all time and since ω is constant along the trajectories, we get

$$f(t) = \int_{\mathbb{R}^2} \omega_0(x)\psi(t,\phi_t(x))\, dx.$$

Differentiating with respect to time and using the ODE solved by $\phi_t(x)$, this leads to

$$f'(t) = \int_{\mathbb{R}^2} \omega_0(x)\Big(\partial_t\psi + u\cdot\nabla\psi\Big)(t,\phi_t(x))\, dx.$$

Using the change of variables $y = \phi_t(x)$ once more yields

$$f'(t) = \int_{\mathbb{R}^2} \omega(t,y)\Big(\partial_t\psi + u\cdot\nabla\psi\Big)(t,y)\, dy,$$

which is (3.6) in the sense of distributions. In order to justify the previous computation, we need to be able to differentiate inside the integral, and we proceed as follows.

Let $\psi = \psi(t,x)$ be any test function. For $0 < \delta < 1$, we set

$$\psi_\delta(t,x) \equiv \chi_\delta\left(x - z(t)\right)\, \psi(t,x),$$

where χ_δ is the map defined as follow: $\chi_\delta(z) = \chi_0\left(\frac{z}{\delta}\right)$ with $\chi_0 : \mathbb{R}^2 \to \mathbb{R}$ a smooth, *radial* cut-off map such that

$$\chi_0 \equiv 0 \ \text{ on } \ B(0,\frac{1}{2}), \quad \chi_0 \equiv 1 \ \text{ on } \ B(0,1)^c, \quad 0 \le \chi_0 \le 1.$$

Since we have $\psi_\delta(t) \equiv 0$ on the ball $B\left(z(t), \frac{\delta}{2}\right)$, we may apply the previous computation to ψ_δ, which yields for all t

$$\int_{\mathbb{R}^2} \omega(t,x)\psi_\delta(t,x)\, dx - \int_{\mathbb{R}^2} \omega_0(x)\psi_\delta(0,x)\, dx \\ = \int_0^t \int_{\mathbb{R}^2} \omega\left(\partial_t\psi_\delta + u\cdot\nabla\psi_\delta\right) dx\, ds.$$

We first observe that thanks to the pointwise convergence of $\psi_\delta(t,\cdot)$ to $\psi(t,\cdot)$ as $\delta\to 0$, we have

$$\int_{\mathbb{R}^2}\omega(t,x)\psi_\delta(t,x)\,dx\;-\int_{\mathbb{R}^2}\omega_0(x)\psi_\delta(0,x)\,dx$$
$$\to\int_{\mathbb{R}^2}\omega(t,x)\psi(t,x)\,dx-\int_{\mathbb{R}^2}\omega_0(x)\psi(0,x)\,dx$$

by Lebesgue's dominated convergence Theorem. Then, we compute

$$\partial_t\psi_\delta+u\cdot\nabla\psi_\delta=\chi_\delta(x-z)(\partial_t\psi+u\cdot\nabla\psi)$$
$$+\,\psi(-\dot{z}+v+H)\cdot\nabla\chi_\delta(x-z).$$

Using that $H\cdot\nabla\chi_\delta(x-z)=0$ and that v is uniformly bounded, we obtain

$$\Big|\int_{\mathbb{R}^2}\omega[\partial_t\psi_\delta{+}u\cdot\nabla\psi_\delta-\chi_\delta(x-z)\Big(\partial_t\psi+u\cdot\nabla\psi\Big)]\,dx\Big|$$
$$\le C\|\psi\|_{L^\infty}\|v\|_{L^\infty}\|\omega\|_{L^\infty}\int_{\mathbb{R}^2}|\nabla\chi_\delta|(x)\,dx.$$

We now let δ tend to zero. Since H is locally integrable, we observe that

$$\int_0^t\int_{\mathbb{R}^2}\omega\chi_\delta\Big(\partial_t\psi{+}u\cdot\nabla\psi\Big)\,dx\,ds$$
$$\to\int_0^t\int_{\mathbb{R}^2}\omega(\partial_t\psi+u\cdot\nabla\psi)\,dx\,ds,$$

so that the conclusion finally follows.

□

The key of this proof is the remark: $H\cdot\nabla\chi_\delta(x-z)=0$ because χ_δ is radial. Although H is not bounded, we use its explicit form and the previous remark to establish equivalent theorem to the case of a vorticity belonging to $L^\infty(L^1\cap L^\infty)$. This previous method was the main tool in [14] and it allows us to extend renormalized solutions for velocity $v+H$.

We turn next to our main purpose and investigate uniqueness for Eulerian solutions, which will imply uniqueness for Lagrangian soltutions.

3.3. Conservation of the vorticity near the point vortex.

Uniqueness for Lagrangian solutions can be easily achieved when the support of ω_0 does not meet z_0; in that case, the support of $\omega(t)$ never meets $z(t)$ and the field $x\mapsto K(\phi_t(x)-z(t))$ is Lipschitz on supp ω_0.

Another situation that has been studied is the case where the vorticity is initially constant near the point vortex. Marchioro and Pulvirenti [14] suggested with some indications that uniqueness for Lagrangian solutions should hold in that situation. This was proved by Starovoitov [20] under the supplementary assumption that ω_0 is Lipschitz. In [11], we treat the general case where the initial vorticity is constant near the point vortex z_0 and belongs to $L^1\cap L^\infty(\mathbb{R}^2)$.

We first show that if (ω,z) is an Eulerian solution, then ω is a renormalized solution in the sense of DiPerna-Lions [3] of its transport equation. We consider equation (EF) as a linear transport equation with given velocity field $u=v+H$ and trajectory z. Our purpose is to show that if ω solves this linear equation, then so does $\beta(\omega)$ for a suitable smooth function β. When there is no point vortex, this directly follows from the theory developed in [3] (see also [2] for more details).

The results stated in [3] hold for velocity fields having enough Sobolev regularity; a typical relevant space is $L^1_{\text{loc}}\left(\mathbb{R}^+, W^{1,1}_{\text{loc}}(\mathbb{R}^2)\right)$. These results can actually be extended to our present situation, thanks to the regularity of H away from the point vortex and to its special form.

Lemma 3.4. *Let* (ω, z) *be a solution of* (EF). *Let* $\beta : \mathbb{R} \to \mathbb{R}$ *be a smooth function such that*

$$|\beta'(t)| \leq C(1 + |t|^p), \qquad \forall t \in \mathbb{R},$$

for some $p \geq 0$. *Then for all test function* $\psi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$, *we have*

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi\beta(\omega)\, dx = \int_{\mathbb{R}^2} \beta(\omega)(\partial_t\psi + u\cdot\nabla\psi)\, dx \;\; \text{in} \;\; L^1_{\text{loc}}(\mathbb{R}^+).$$

Remark 3.5. We let $1 \leq p < +\infty$. Approximating $\beta(t) = |t|^p$ by smooth functions and choosing $\psi \equiv 1$ in Lemma 3.4, we deduce that for an Eulerian solution ω to (EF), the maps $t \mapsto \|\omega(t)\|_{L^p(\mathbb{R}^2)}$ are continuous and constant. In particular, we have

$$\|\omega(t)\|_{L^1(\mathbb{R}^2)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \equiv \|\omega_0\|_{L^1(\mathbb{R}^2)} + \|\omega_0\|_{L^\infty(\mathbb{R}^2)},$$

and we denote by $\|\omega_0\|$ this last quantity.

Specifying our choice for β in Lemma 3.4, we are led to the following.

Proposition 3.6. *Let* (ω, z) *be an Eulerian solution of* (EF) *such that*

$$\omega_0 \equiv \alpha \qquad \text{on} \;\; B(z_0, R_0)$$

for some positive R_0. *Then there exists a continuous and positive function* $t \mapsto R(t)$ *depending only on* t, R_0 *and* $||\omega_0||$ *such that* $R(0) = R_0$ *and*

$$\forall t \in \mathbb{R}^+, \qquad \omega(t) \equiv \alpha \qquad \text{on} \;\; B\left(z(t), R(t)\right).$$

Idea of the proof. We set $\beta(t) = (t-\alpha)^2$ and use Lemma 3.4 with this choice. Let $\Phi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$. We claim that for all T

$$\int_{\mathbb{R}^2} \Phi(T,x)(\omega-\alpha)^2(T,x)\, dx - \int_{\mathbb{R}^2} \Phi(0,x)(\omega-\alpha)^2(0,x)\, dx \\ = \int_0^T \int_{\mathbb{R}^2} (\omega-\alpha)^2(\partial_t\Phi + u\cdot\nabla\Phi)\, dx\, dt.$$

Now, we choose a test function Φ centered at $z(t)$. More precisely, we let Φ_0 be a non-increasing function on $\mathbb{R}$, which is equal to 1 for $s \leq 1/2$ and vanishes for $s \geq 1$ and we set $\Phi(t,x) = \Phi_0(|x - z(t)|/R(t))$, with $R(t)$ a smooth, positive and decreasing function to be determined later on, such that $R(0) = R_0$. For this choice of Φ, we have $(\omega_0(x) - \alpha)^2\Phi(0,x) \equiv 0$.

We compute then

$$\nabla\Phi = \frac{x-z}{|x-z|}\frac{\Phi_0'}{R(t)}$$

and

$$\partial_t\Phi = -\frac{R'(t)}{R^2(t)}|x-z|\Phi_0' + \frac{\dot{z}\cdot(z-x)}{|x-z|}\frac{\Phi_0'}{R(t)}.$$

Since $u \cdot \nabla\Phi = (v+H)\cdot\nabla\Phi = v\cdot\nabla\Phi$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} &\Phi(T,x)(\omega-\alpha)^2(T,x)\,dx \\ &= \int_0^T\int_{\mathbb{R}^2}(\omega-\alpha)^2\frac{\Phi_0'(\frac{|x-z|}{R})}{R}\Big((v(x)-\dot z)\cdot\frac{(x-z)}{|x-z|}-\frac{R'}{R}|x-z|\Big)\,dx\,dt.\end{aligned} \tag{3.7}$$

Without loss of generality, we may assume that $R_0 \le 1$, so that $R \le 1$. As $\Phi_0'(\frac{|x-z|}{R}) \le 0$ for $R/2 \le |x-z| \le R$ and vanishes elsewhere and $R' < 0$, we can estimate the right-hand side term of (3.7) by:

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}&(\omega-\alpha)^2\frac{\Phi_0'(\frac{|x-z|}{R})}{R}\Big((v(x)-\dot z)\cdot\frac{(x-z)}{|x-z|}-\frac{R'}{R}|x-z|\Big)\,dx\,dt \\ &\le \int_0^T\int_{\mathbb{R}^2}(\omega-\alpha)^2\frac{|\Phi_0'|(\frac{|x-z|}{R})}{R}\left(|v(x)-v(z)|+\frac{R'}{2}\right)\,dx\,dt.\end{aligned}$$

Using regularity properties of v, we deduce from (3.7)

$$\begin{aligned}\int_{\mathbb{R}^2}&\Phi(T,x)(\omega-\alpha)^2(T,x)\,dx \\ &\le \int_0^T\int_{\mathbb{R}^2}(\omega-\alpha)^2\frac{|\Phi_0'|(\frac{|x-z|}{R})}{R}\left(CR(1-\ln R)+\frac{R'}{2}\right)\,dx\,dt,\end{aligned}$$

where C only depends on $\|\omega_0\|$. Taking $R(t) = \exp(1-(1-\ln R_0)e^{2Ct})$, we arrive at

$$\int_{\mathbb{R}^2}\Phi(T,x)(\omega-\alpha)^2(T,x)\,dx \le 0,$$

which ends the proof.

□

Remark 3.7. We assume that ω_0 has compact support. Considering $\beta(t) = t^2$ in Lemma 3.4 and adapting the proof of Proposition 3.6, we obtain that $\omega(t)$ remains compactly supported and its support grows at most linearly. Indeed, if we choose $\Phi(t,x) = 1-\Phi_0(|x-z(t)|/R(t))$, with $R(t)$ a smooth, positive and increasing function such that $R(0) = R_1$, where $\operatorname{supp}\omega_0 \subset B(z_0,R_1)$, then (3.7) becomes

$$\begin{aligned}\int_{\mathbb{R}^2}&\Phi(T,x)\omega^2(T,x)\,dx \\ &= \int_0^T\int_{\mathbb{R}^2}\omega^2\frac{-\Phi_0'(\frac{|x-z|}{R})}{R}\Big((v(x)-\dot z)\cdot\frac{(x-z)}{|x-z|}-\frac{R'}{R}|x-z|\Big)\,dx\,dt \\ &\le \int_0^T\int_{\mathbb{R}^2}\omega^2\frac{|\Phi_0'|(\frac{|x-z|}{R})}{R}\Big(2C-\frac{R'}{2}\Big)\,dx\,dt,\end{aligned}$$

where C depends only on $\|\omega_0\|$. The right-hand side is identically zero for $R(t) = R_1 + 4Ct$, and we conclude that $\operatorname{supp}(\omega(t)) \in B(0,R(t))$.

We then take advantage of the weak formulation (EF) to derive a partial differential equation satisfied by the velocity $v = K*\omega$. In order to compare two solutions, one not only has to compare the two regular parts, but also possibly the diverging trajectories of the two vortices. Given two Eulerian solutions (ω_1,z_1) and (ω_2,z_2), we therefore introduce the quantity

$$r(t) = |\tilde z(t)|^2 + \|\tilde v(t)\|_{L^2(\mathbb{R}^2)}$$

where $\tilde{z} = z_1 - z_2$, $\tilde{\omega} = \omega_1 - \omega_2$ and $\tilde{v} = v_1 - v_2 = K * \tilde{\omega}$. Since $\tilde{\omega}$ vanishes in a neighborhood of the point vortex, the velocity $\tilde{v}$ has to be harmonic in this neighborhood. This provides in particular a control of its L^∞ norm (as well as the L^∞ norm for the gradient) by its L^2 norm, whereas we take the norm L^1 for H. Far of the point vortex, H is smooth and we can use Yudovich's argument. Separating integral in this two part, ones ultimately yields a Gronwall-type estimate for $r(t)$ and allows to prove that it vanishes.

More precisely, we prove the following

Theorem 3.8. *Let $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ and $z_0 \in \mathbb{R}^2$ such that there exists $R_0 > 0$ and $\alpha \in \mathbb{R}$ such that*

$$\omega_0 \equiv \alpha \ \ on \ \ B(z_0, R_0).$$

Suppose in addition that ω_0 has compact support. Then there exists a unique Eulerian solution of the vortex-wave system with this initial data.

Finally, although we have chosen to restrict our attention to Eulerian solutions, we remark in the last section of [11] that the renormalization property established for the linear transport equation can be used to show the converse of Theorem 3.3. This implies that Definitions 3.1 and 3.2 are equivalent for any $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$, even if the vorticity is not initially constant in a neighborhood of the point vortex.

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