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Profile decompositions and applications to Navier-Stokes

Gabriel S. Koch

Abstract

In this expository note, we collect some recent results concerning the applications of methods from dispersive and hyperbolic equations to the study of regularity criteria for the Navier-Stokes equations. In particular, these methods have recently been used to give an alternative approach to the $L_{3,\infty}$ Navier-Stokes regularity criterion of Escauriaza, Seregin and Šverák. The key tools are profile decompositions for bounded sequences of functions in critical spaces.

1. Introduction

In the recent paper [12], the author and C. Kenig showed that one can apply a robust method from nonlinear dispersive (e.g. nonlinear Schrödinger) and hyperbolic (e.g. nonlinear wave) equations to obtain important regularity results for a parabolic equation, namely the Navier-Stokes system. The method, which we shall describe below, is based strongly on the use of “profile decompositions” of bounded sequences in certain function spaces where the underlying space is \mathbb{R}^d . Bounded sequences in such spaces fail in general to be pre-compact. The profile decompositions allow one to isolate the defects of compactness, thus allowing one to regain sufficient compactness to prove the existence of solutions enjoying important minimality properties. The results in [12] were made possible by the profile decompositions of P. Gérard [10] and I. Gallagher [6]. Recently in [17], the author adapted the method of S. Jaffard [11] to construct new profile decompositions for other spaces relevant to Navier-Stokes. Based on these, in collaboration with Gallagher and F. Planchon (see [9]) we give an alternative proof of the $L_{3,\infty}$ regularity criterion of Escauriaza-Seregin-Šverák [5], a special case of which was accomplished in [12].

The paper is organized as follows: in sections 2-4 we give the background, motivation and main tools for the results which follow; in sections 5-7 we outline the applications to Navier-Stokes as mentioned above and in section 8 we give some details of the proof. Finally in the last section we describe the method used by the author to prove the new profile decompositions which were used to complete the proofs of the main results.

2. Background and Motivations

Consider the standard Navier-Stokes equations for incompressible fluids:

$$(NSE) \quad \left\{ \begin{array}{l} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0 \quad (\nabla \cdot u_0 = 0) \end{array} \right\} \quad \text{in } \mathbb{R}^3 \times (0, T)$$

A velocity vector field $u(x, t)$ is said to be a solution to (NSE) if there exists an associated pressure function $p(x, t)$ such that (NSE) is satisfied.

For a "critical space" $X = X(\mathbb{R}^3)$, we are interested in establishing the following regularity criterion for Navier-Stokes:

$$(R)_X \quad \left\{ \begin{array}{l} \text{Suppose } u \text{ solves (NSE) and satisfies } \sup_{t \in (0, T)} \|u(t)\|_X < \infty \\ \text{for some } T > 0. \text{ Then } u \text{ is smooth on } \mathbb{R}^3 \times (0, T]. \end{array} \right.$$

To define the notion of critical space, we consider the natural scaling of (NSE). Due to the invariance of the equation under the transformation $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$, the initial data scale as $u_0(x) \mapsto \lambda u_0(\lambda x)$. We therefore say that a Banach space X of initial data is a critical space if $\|\lambda u_0(\lambda \cdot)\|_X = \|u_0\|_X$ for any $u_0 \in X$ and $\lambda > 0$. For example, the chain of continuous embeddings

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow \dot{B}_{p_1, q_1}^{-1 + \frac{3}{p_1}} \xrightarrow[\substack{p_1 < p_2 \\ q_1 < q_2}]{\hookrightarrow} \dot{B}_{p_2, q_2}^{-1 + \frac{3}{p_2}} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}_{\infty, \infty}^{-1} \quad (2.1)$$

consists of spaces which are all critical with respect to (NSE).

For $X = L^3(\mathbb{R}^3)$, $(R)_{L^3(\mathbb{R}^3)}$ was verified in the famous paper of L. Escauriaza, G. Seregin and V. Šverák [5] in the setting of the standard Leray-Hopf weak solutions (roughly, solutions in $L^2(\mathbb{R}^3)$ satisfying an energy estimate) of (NSE). This is known as the $L_{3, \infty}$ (i.e., $L_t^\infty(L_x^3)$) Navier-Stokes regularity criterion. To put this in historical perspective, they establish the "endpoint" of a range of Navier-Stokes regularity criteria developed around the 1960s (see [18, 22, 23]). These "Ladyzhenskaya-Prodi-Serrin" conditions for regularity may be characterized (in the global setting) as follows:

Theorem 1 (LPS). *Let $p > 3$ and choose s so that $\frac{3}{p} + \frac{2}{s} = 1$. Let u be a Leray-Hopf weak solution to Navier-Stokes on $(0, T)$ such that $u \in L_{p, s} := L_t^s(L_x^p)$, i.e.,*

$$\left\| \|u(\cdot, t)\|_{L^p(\mathbb{R}^3)} \right\|_{L^s(0, T)} < \infty .$$

Then u is smooth and unique on $\mathbb{R}^3 \times (0, T]$.

The spaces $L_{p, s}$ are invariant under the Navier-Stokes scaling $u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t)$ and can therefore also be thought of as critical spaces for Navier-Stokes. Note also that due to the restriction that $p > 3$, and hence $s < \infty$, such spaces are "locally small." That is, for any finite subdomain $Q \subset\subset \mathbb{R}^3 \times (0, T)$, $u \in L_{p, s}$ implies that $\|u\|_{L_{p, s}(Q)} \rightarrow 0$ as $|Q| \rightarrow 0$. This fact is crucial in the proof of Theorem 1. At the endpoint of this range, i.e. $p = 3$ and $s = \infty$, it is not obvious whether there is a local smallness that one can take advantage of, which makes the proof significantly more difficult. In [5], Escauriaza, Seregin and Šverák treat this case,

but more precisely they show that a solution belonging to $L_{3,\infty}$ actually belongs to the Ladyzhenskaya-Prodi-Serin space $L_{5,5}$, which is quite surprising. Their proof, which is fairly involved, uses a “blow-up procedure” and is concluded using backward uniqueness and unique continuation results for parabolic inequalities, made possible in part by Carleman-type estimates.

In the papers [12, 9], we give an alternative proof of $(R)_{L^3(\mathbb{R}^3)}$ in the setting of “mild” solutions (locally smooth solutions satisfying an integral version of (NSE)). Our proof uses the powerful “critical element” method developed by Kenig and Merle for dispersive equations (where all solutions are “mild”), see e.g. [13, 14, 15]. The main tool of this method is a “profile decomposition” in a critical space, which we’ll describe below. Although we conclude in a similar way to the original proof of $(R)_{L^3(\mathbb{R}^3)}$ in [5], we hope that our proof may appear more transparent and potentially lead to further advances. Moreover, our result is important as it shows that it is possible to apply the critical element method in the context of a parabolic equation.

3. Profile Decompositions

The embeddings $X \hookrightarrow Y$ in (2.1) are not compact, as can be seen from the following simple example: for a fixed non-zero “profile” $f \in X$, we have the bounded sequence $f_n(x) := \frac{1}{\lambda_n} f\left(\frac{x-x_n}{\lambda_n}\right)$, where $\lambda_n > 0$ and $x_n \in \mathbb{R}^3$. If $\lim_{n \rightarrow \infty} \lambda_n \in \{0, \infty\}$ or $\lambda_n = 1$ and $|x_n| \rightarrow \infty$ and if $f_n \rightarrow \bar{f}$ in Y along some subsequence, then necessarily $\bar{f} = 0$. This however is impossible since $\|f_n\|_Y \equiv \|f\|_Y > 0$.

More generally, suppose $X = X(\mathbb{R}^d)$ is a Banach space such that for some $\alpha > 0$, $\|\frac{1}{\lambda^\alpha} f(\frac{\cdot-x}{\lambda})\|_X = \|f\|_X$ for all $\lambda > 0$, $x \in \mathbb{R}^d$ and $f \in X$ (e.g. any space in (2.1) with $\alpha = 1$). One can hope to characterize the lack of compactness of an embedding $X \hookrightarrow Y$ by the following statement, which would show that the above example is in some sense the only obstacle:

Statement 2 (Profile Decomposition in X). *Let $\{\varphi_n\}_{n=1}^\infty$ be a bounded sequence in X . Then there exists a sequence of profiles $\{\phi_j\}_{j=1}^\infty \subset X$, and for each integer $j \geq 1$ a sequence $\{(\lambda_{j,n}, x_{j,n})\}_{n=1}^\infty \subset (0, \infty) \times \mathbb{R}^d$ of “scales/cores” such that after possibly passing to a subsequence in n one may write*

$$\varphi_n(x) = \sum_{j=1}^J \frac{1}{(\lambda_{j,n})^\alpha} \phi_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}\right) + \psi_n^J(x) \quad (3.1)$$

for any $n, J \in \mathbb{N}$, and the following properties hold:

1. The scales/cores are “orthogonal” in the sense that:

$$j \neq j' \implies \lim_{n \rightarrow +\infty} \frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} = +\infty ; \quad (3.2)$$

2. The remainder ψ_n^J is small in the sense that $\lim_{J \rightarrow \infty} \left(\overline{\lim}_{n \rightarrow \infty} \|\psi_n^J\|_Y \right) = 0$;
3. The profiles and remainders satisfy the following “stability” estimates (for some $\tau \geq 1$):

$$\sum_{j=0}^\infty (\|\phi_j\|_X)^\tau \leq \overline{\lim}_{n \rightarrow \infty} (\|\varphi_n\|_X)^\tau , \quad \sup_{n,J} \|\psi_n^J\|_X \lesssim \sup_n \|\varphi_n\|_X .$$

The orthogonality condition on the “scales” $\lambda_{j,n} > 0$ and “cores” $x_{j,n} \in \mathbb{R}^3$ ensures that interactions between profiles becomes negligible for large n . Such decompositions are very useful as the functions ϕ_j in the profiles are *fixed* for all n . Hence often a limit of the sequence φ_n can be replaced by ϕ_{j_0} for some j_0 .

Such a statement holds for example for the standard Sobolev embeddings $X \hookrightarrow Y$ of the form

$$\dot{H}^{s,p}(\mathbb{R}^d) \hookrightarrow L^{\frac{dp}{d-sp}}(\mathbb{R}^d) \quad (s < d/p)$$

with $\alpha = \frac{d}{p} - s$ and $\tau = p$. This was first established¹ for $p = 2$ by P. Gérard [10] using properties of Hilbert spaces, which in particular established a profile decomposition for the first embedding in (2.1) and is crucial in the proof of Theorem 6 below. This was then extended to the non-Hilbert setting $p \neq 2$ by S. Jaffard [11] using a different method involving wavelets.

In [17], we proved the following theorems (in the spirit of [11]) which in particular give profile decompositions for the second and third embeddings in (2.1). Here, we have limited the more general statements to critical spaces for (NSE):

Theorem 3 (Profile Decomposition in $L^d(\mathbb{R}^d)$). *Suppose $p, q \in (d, \infty]$, $d \geq 2$, and set $s_p := -1 + \frac{d}{p} < 0$. Then Statement 2 holds with $X = L^d(\mathbb{R}^d)$ and $Y = \dot{B}_{p,q}^{s_p}$ ($\alpha = 1, \tau = d$) equipped with norms which are equivalent to the standard ones.*

Theorem 4 (Profile Decomposition in $\dot{B}_{p,q}^{s_p}(\mathbb{R}^d)$). *Suppose $1 \leq a < p \leq +\infty$, $1 \leq b < b(p/a) \leq q \leq +\infty$, and set $s_r := -1 + \frac{d}{r}$ for $r \in \{a, p\}$, $d \in \mathbb{N}$. Then Statement 2 holds with $X = \dot{B}_{a,b}^{s_a}(\mathbb{R}^d)$ and $Y = \dot{B}_{p,q}^{s_p}(\mathbb{R}^d)$ ($\alpha = 1, \tau = \max\{a, b\}$) equipped with norms equivalent to the standard ones.*

Theorems 3 and 4 are proved using the fact that for many spaces one has a universal unconditional basis of “wavelets” of the form

$$(2^j)^\alpha \psi^{(i)}(2^j x - k), \quad (3.3)$$

where $\psi^{(i)} \in C_0^\infty$ and $(i, j, k) \in (1, \dots, 2^d - 1) \times \mathbb{Z} \times \mathbb{Z}^d$. Since (3.3) has a form similar to the profiles in (3.1), one can expand a bounded sequence in such a basis, rearrange the components into groups with comparable scales/cores and pass to a subsequence (made considerably easier by the fact that we work on lattices) to achieve the desired result. The norms we use are defined in terms of the wavelet basis. (See the last section for more details.)

4. The “Critical Element” Method

In a series of recent works [13, 14, 15], C. Kenig and F. Merle developed the method of “critical elements” which uses profile decompositions to approach the questions of global existence and “scattering” for nonlinear hyperbolic and dispersive equations in critical settings. For example, in [15] they prove the following theorem for the 3D defocusing cubic non-linear Schrödinger equation:

$$(NLS) \quad \begin{cases} iu_t + \Delta u - |u|^2 u = 0 & \text{in } \mathbb{R}^3 \times (0, T) \\ u|_{t=0} = u_0 \end{cases}$$

¹See also [24] where certain cases were treated with slightly weaker results.

Theorem 5 (Kenig-Merle [15]). *Suppose $u_0 \in \dot{H}^{\frac{1}{2}}$ and u solves (NLS) on $\mathbb{R}^3 \times [0, T^*(u_0))$, where $T^*(u_0)$ is the maximal time of existence of the solution. Suppose moreover that $\|u\| < \infty$, where $\|u\| := \sup_{0 \leq t < T^*(u_0)} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}$. Then $T^*(u_0) = +\infty$ and u scatters, i.e. $\|u(t) - L(t)u_0^+\|_{\dot{H}^{\frac{1}{2}}} \rightarrow 0$ as $t \rightarrow +\infty$ for some $u_0^+ \in \dot{H}^{\frac{1}{2}}$, where $L(t)$ is the associated linear solution operator.*

Note that (NLS) has the same scaling as (NSE), and Theorem 5 holds in the (NSE) setting by [5] with $u_0^+ = 0$ (i.e. $\|u(t)\|_{\dot{H}^{\frac{1}{2}}} \rightarrow 0$ as $t \rightarrow \infty$, see [7]). We will momentarily describe how to apply the following method of proof to (NSE).

The proof of Theorem 5 (and its analogues in [13, 14] as well as Theorems 6 and 7 below) can be described as “concentration-compactness” (i.e., profile decompositions) + “rigidity theorem”. It is a proof by contradiction, so we first assume the theorem is false. The method has three steps as follows:

1. *Existence of a “Critical Element”:* Let $A_c < \infty$ be the infimum of values $\|u\|$ of solutions for which the statement of the theorem fails. Then there exists a solution u_c with $\|u_c\| = A_c$, for which the statement fails – that is, the infimum is attained.
2. *Compactness:* Up to norm-invariant transformations in space, the set $\{u_c(t)\}$ satisfies a compactness property (e.g. pre-compactness) in a critical space.
3. *Rigidity:* The “compactness” of u_c together with known results implies that $u_c = 0$, which is impossible as zero is a global, scattering solution.

Steps 1 and 2 are proved using profile decompositions and are fairly independent of the specific structure of the equation, while Step 3 typically requires a result specific to the equation. For example, the Morawetz identity for (NLS) was used in [15] to complete this step.

5. Critical Elements for Navier-Stokes

In collaboration with C. Kenig [12], we give a critical element proof of (R) $_{\dot{H}^{\frac{1}{2}}}$ in the form of the following theorem which is a consequence of [5]:

Theorem 6. *Suppose $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and u is the associated mild solution to (NSE) on $\mathbb{R}^3 \times [0, T^*(u_0))$, where $T^*(u_0)$ is the maximal time of existence of the solution. Suppose moreover that $\sup_{0 \leq t < T^*(u_0)} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} < \infty$. Then $T^*(u_0) = +\infty$.*

As a technical point, we note that the solutions considered in [5] were Leray-Hopf weak solutions while we consider mild solutions. The reason is that mild solutions have an integral representation formula and other properties similar to the notion of solutions of dispersive equations and hence are well-adapted to the critical element method. In particular, for fixed $u_0 \in \dot{H}^{\frac{1}{2}}$ a unique smooth mild solution to (NSE) exists in the class $\mathcal{C}([0, T]; \dot{H}^{\frac{1}{2}}) \cap L^2((0, T); \dot{H}^{3/2})$ ($\hookrightarrow \mathcal{C}([0, T]; L^3) \cap L_{5,5}(\mathbb{R}^3 \times (0, T))$), the class of solutions considered in Theorem 7 below, and hence smooth by Theorem 1) for all $T < T^*(u_0)$, so that in this setting the above is equivalent to (R) $_{\dot{H}^{\frac{1}{2}}}$. In what follows, we’ll say that u is “singular” if $T^*(u_0) < \infty$.

The main tool we use is the result of I. Gallagher [6], who showed that one could evolve the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ profile decomposition of Gérard (note that at the time of writing [12], Theorem 3 was not yet available) by the Navier-Stokes flow when the sequence φ_n are divergence-free vector fields. Specifically, letting u_n and U_j be the unique mild solutions to (NSE) with initial data φ_n and ϕ_j respectively and letting w_n^J be the heat flow of the error ψ_n^J , one has

$$u_n(x, t) = \sum_{j=1}^J \frac{1}{\lambda_{j,n}} U_j \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t}{\lambda_{j,n}^2} \right) + w_n^J(x, t) + r_n^J(x, t), \quad (5.1)$$

for any t such that the sum of profiles is defined (note that profiles are global for large j due to the stability estimates for (3.1)), where again r_n^J is a small error for large J and n . (This is the counterpart of similar results which had been established in the hyperbolic/dispersive settings, see e.g. [2, 13, 16].) Using this result, we proceed as above as follows:

Step 1 (Existence of a Critical Element) As in Theorem 5, we consider the profile decomposition for a sequence of initial data $u_{0,n}$ with associated singular solutions u_n satisfying $\sup_t \|u_n(t)\|_{\dot{H}^{\frac{1}{2}}} \rightarrow A_c < \infty$. Due to (5.1), there must be at least one singular profile U_{j_0} . Moreover, using (5.1) and the decay result in [7] we may establish for some such j_0 and any $t \in T^*(U_{j_0}(0))$, an identity of the form

$$\|u_n(\lambda_{j_0,n}^2 t)\|_{\dot{H}^{\frac{1}{2}}}^2 = \sum_{j=0}^J \|U_j(\lambda_{j_0,n}^2 t / \lambda_{j,n}^2)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|w_n^{l,J}(\lambda_{j_0,n}^2 t)\|_{\dot{H}^{\frac{1}{2}}}^2 + o(1) \quad (5.2)$$

as $n, J \rightarrow \infty$ along certain subsequences. This implies that $\sup_t \|U_{j_0}(t)\|_{\dot{H}^{\frac{1}{2}}} \leq A_c$. However, since U_{j_0} is singular, we must have equality and hence $u_c := U_{j_0}$ is a critical element.

Step 2 (Compactness) Using profile decompositions of initial data of the form $u_{0,n} := u_c(t_n)$, one uses (5.2) to show that there exists a sequence $s_n \nearrow T^*(u_c(0))$ such that, up to norm-invariant transformations, $\{u_c(s_n)\}$ is pre-compact in L^3 . Moreover, this compactness implies that the solution tends to zero in L^2 on any bounded subset of \mathbb{R}^3 as $s_n \rightarrow T^*(u_{0,c})$.

Step 3 (Rigidity) Using known ‘‘partial regularity results’’ for NSE (see [4]), u_c is smooth up to the blow-up time as long as one stays outside of a sufficiently large ball in \mathbb{R}^3 . Step 2 thus implies that $u_c(x, T^*(u_{0,c})) \equiv 0$ for $|x| \geq R_0 \gg 1$. Then applying known backwards uniqueness and unique continuation results (see e.g. [5] where similar methods were used to complete their theorem) to the equation for the vorticity of u_c , one sees that the vorticity must vanish everywhere which, along with the fact $u_c(t) \in L^3$, implies that the solution itself is zero.

It is interesting to note that our proof of Step 3 (which is really an ‘‘ L^3 result’’) generalizes an important result of Nečas, Růžička and Šverák [21] ruling out the ‘‘self-similar’’ solutions conjectured by Leray in [19]. These have the form $u(x, t) = \frac{1}{\sqrt{T^*-t}} U\left(\frac{x}{\sqrt{T^*-t}}\right)$ for some given non-zero $U \in L^3$ and $T^* < \infty$. Of course, the result in [5] also implies this result, but Step 3 above is much simpler for that purpose.

6. An Alternative Proof of the $L_{3,\infty}$ Criterion $(R)_{L^3}$

In collaboration with I. Gallagher and F. Planchon [9], we build on the results above to give a critical element proof of the following mild solution version of [5]:

Theorem 7. *Suppose $u_0 \in L^3(\mathbb{R}^3)$ and u is the associated mild solution to (NSE) on $\mathbb{R}^3 \times [0, T^*(u_0))$, where $T^*(u_0)$ is the maximal time of existence of the solution. Suppose moreover that $\sup_{0 \leq t < T^*(u_0)} \|u(t)\|_{L^3(\mathbb{R}^3)} < \infty$. Then $T^*(u_0) = +\infty$.*

The main tool which makes such a proof possible is Theorem 3, the profile decomposition in $L^3(\mathbb{R}^3)$, which was not available at the time of writing [12] and allows us to establish (5.1) in L^3 with remainders small in $\dot{B}_{p,q}^{s_p}$. However one has the additional difficulty that L^3 lacks the Hilbert structure of $\dot{H}^{\frac{1}{2}}$ which was used heavily to establish Step 1 and Step 2 above in [12].

In particular, a statement such as (5.2) is likely untrue in L^3 , but we can still show that

$$\|u_n(\lambda_{j_0,n}^2 t)\|_{L^3}^3 \geq \|U_{j_0}(t)\|_{L^3}^3 + \|v_n(\lambda_{j_0,n}^2 t)\|_{L^3}^3 + o(1) \quad (6.1)$$

as $n \rightarrow \infty$ along some subsequence, where $v_n(x, t) = u_n(x, t) - \frac{1}{\lambda_{j_0,n}} U_{j_0} \left(\frac{x - x_{j_0,n}}{\lambda_{j_0,n}}, \frac{t}{\lambda_{j_0,n}^2} \right)$. This is proved using the decay result in [8] along with regularity properties of r_n^J and the heat-flow definition of Besov spaces in connection with w_n^J . This implies that $u_c := U_{j_0}$ is a critical element as in Step 1 above. For Step 2, we use (6.1) to show that $u_c(t) \rightarrow 0$ in distributions as $t \rightarrow T^*(u_c(0))$. This is sufficient to apply Step 3 exactly as above, and the theorem is proved.

7. Minimal Blow-up Initial Data

We briefly mention here the following result appearing as well in [9] as a simple application of decompositions of the form (5.1), which are proved for the spaces treated in Theorems 3 and 4 above:

Theorem 8. *If there exists a mild solution to NSE starting in any of the critical spaces $X = \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $L^3(\mathbb{R}^3)$ or $B_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ ($1 < p, q < \infty$) which develops a singularity in finite time, then there exists an initial datum leading to a singularity which has the minimal possible norm in X .*

This extends and gives a different proof of a recent result of Šverák and Rusin [25] for the $\dot{H}^{\frac{1}{2}}$ case, proved by other methods in the context of weak solutions. Moreover, due to the Hilbert structure of $\dot{H}^{\frac{1}{2}}$, we recover their statement of ‘‘compactness’’ of the set of minimal blow-up data in $\dot{H}^{\frac{1}{2}}$. In L^3 and $\dot{B}_{p,q}^{s_p}$, compactness is established in a weaker sense and the minimality is in terms of the equivalent wavelet norm used in [17].

8. Details of the Proof

For simplicity, we restrict ourselves here to the $\dot{H}^{\frac{1}{2}}$ setting. (For details in L^3 , see the upcoming work [9].) The key to establishing the critical element and compactness in the proof of Theorem 6 lies in the following three facts:

1. The profiles in the $\dot{H}^{\frac{1}{2}}$ decomposition of Gérard satisfy the following stability estimate:

$$\sum_{j=1}^{\infty} \|\phi_j\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \sup_n \|\varphi_n\|_{\dot{H}^{\frac{1}{2}}}^2 < \infty . \quad (8.1)$$

In particular, for large enough j the profiles are sufficiently small to generate global Navier-Stokes solutions.

2. If $\|u_0\|_{L^3(\mathbb{R}^3)}$ is sufficiently small for a divergence-free $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with associated mild solution u , then one has the "energy-type" inequality

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 + \int_0^t \|\nabla u(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 ds \leq \|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 \quad \forall t > 0 .$$

3. For $u_0 \in \dot{H}^{\frac{1}{2}}$ generating a global solution u , one has (see [7]) the decay

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}}} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty .$$

The first fact implies that one can have at most a finite number of profiles generating solutions with finite maximal times of existence (i.e., which "blow up in finite time"). Moreover, the number of these must be positive in our case ($\varphi_n = u_{0,n}$, $T^*(u_{0,n}) < \infty$ as outlined above) due to a property of the decomposition (5.1) that the lifespans of the solutions u_n are bounded from below by the lifespan of the transformed profiles U_j . Specifically, for sufficiently large n , $T^*(u_{0,n}) \geq \lambda_{j_0,n}^2 T^*(\phi_{j_0})$ for $j_0 = j_0(n)$ which minimizes the right-hand side. The existence of blow-up profiles follows from this since we assume that $\{u_n\}$ consists of blow-up solutions which minimize A_c .

For sufficiently large n , one can re-order the profiles in (5.1) so that the first one has the shortest life-span (due to the finite number of blow-up profiles), and one can use the above properties to prove the following orthogonality property: for any s in the (finite) lifespan of U_1 , we have

$$A_c^2 \geq \|u_n(\lambda_{1,n}^2 s)\|_{\dot{H}^{\frac{1}{2}}}^2 \geq \|U_1(s)\|_{\dot{H}^{\frac{1}{2}}}^2 + \sum_{j=2}^J \left\| U_j \left(\frac{\lambda_{1,n}^2 s}{\lambda_{j,n}^2} \right) \right\|_{\dot{H}^{\frac{1}{2}}}^2 + o(1)$$

as J and n tend to infinity along subsequences. To prove this, one inserts $t = \lambda_{1,n}^2 s$ into (5.1), expands $\|u_n(\lambda_{1,n}^2 s)\|_{\dot{H}^{\frac{1}{2}}}^2$ as the square of a sum, and shows (essentially using (3.2)) that the "cross-terms" are small (while controlling the "tails" by (8.1)). This inequality implies that $\sup_s \|U_1(s)\|_{\dot{H}^{\frac{1}{2}}} = A_c$, due to the definition of A_c and the fact that U_1 is a blow-up solution. Hence $u_c := U_1$ is a critical element, and moreover the previous inequality also shows that for each $j \geq 2$, $U_j(\tau)$ is small in $\dot{H}^{\frac{1}{2}}$ for some $\tau > 0$ and therefore all other profiles are global solutions by "small data" results.

To show the compactness, one considers a new "minimizing sequence" of solutions for A_c with initial data given by $u_{0,n} := u_c(t_n)$, with t_n approaching the blow-up time of u_c . The profile decomposition for this sequence retains the properties of the original minimizing sequence as above. Then for any fixed T_1 in the lifespan of U_1

(the only blow-up profile), one can pick $\tau_n > 0$ such that (5.1) gives

$$\begin{aligned} u_n(\tau_n) - \frac{1}{\lambda_{1,n}} U_1 \left(\frac{\cdot - x_{1,n}}{\lambda_{1,n}}, T_1 \right) &= \\ &= \sum_{j=2}^J \frac{1}{\lambda_{j,n}} U_j \left(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}, \frac{\tau_n}{\lambda_{j,n}^2} \right) + w_n^{l,J}(\tau_n) + r_n^J(\tau_n) . \end{aligned}$$

One can essentially use the decay of the global profiles on the right plus the smallness of the remainders to show that this tends to zero in L^3 as $n \rightarrow \infty$ along a subsequence. Hence the transformational invariance of the L^3 norm implies that one can shift and re-scale $u_n(\tau_n)$ (by the inverse of the transformation on $U_1(T_1)$ above) so that the resulting sequence $\widetilde{u_n(\tau_n)}$ approaches $U_1(T_1)$ in L^3 as $n \rightarrow \infty$. Letting $s_n = t_n + \tau_n$ and noting that $u_n(\tau_n) = u_c(s_n)$ yields the result. The fact that s_n approaches the blow-up time of u_c follows from the relationship between the lifespan of the profiles and that of u_n .

To prove "rigidity," one notices that the proof of the previous step more specifically gives the following: there exists s_n approaching the blow-up time of the critical element u_c , $x_n \in \mathbb{R}^3$ and $\lambda_n \rightarrow +\infty$ such that

$$\frac{1}{\lambda_n} u_c \left(\frac{\cdot - x_n}{\lambda_n}, s_n \right) =: v_n \quad \longrightarrow \quad \bar{v}$$

in L^3 for some $\bar{v} \in L^3$. Using a change of variables, Hölder's inequality and a splitting up of the domain (essentially into a small region and a region tending to infinity due to the growth of λ_n on which $\|\bar{v}\|_{L^3} \rightarrow 0$), one can use this to show that for any ball $B \subset \subset \mathbb{R}^3$ one has

$$\lim_{n \rightarrow \infty} \int_B |u(x, s_n)|^2 dx = 0 . \quad (8.2)$$

The facts that $u_c \in L_{3,\infty}$ and $T^* := T^*(u_{0,c}) < \infty$ together imply, by known "partial-regularity" results for Navier-Stokes (see, e.g., [4]), that u_c is smooth up to T^* outside a large space-time cylinder. Hence (8.2) implies that $u_c(x, T^*) = 0$ for $|x| \geq R_0$ for some large $R_0 \gg 1$. In particular, the vorticity $\omega := \text{curl} u_c$ is also zero at time T^* outside a large ball. The vorticity satisfies the inequality $|\omega_t - \Delta \omega| \leq c_0(|\omega| + |\nabla \omega|)$, and known backwards uniqueness results followed by unique continuation properties for such a differential inequality show (similarly as in [5]) that $\omega \equiv 0$ on $\mathbb{R}^3 \times (0, T^*)$. Owing to the divergence-free property of u_c , this implies that $u_c(t) \in L^3$ is harmonic for each t and hence $u_c \equiv 0$.

This rigidity property (along with global existence for small data) immediately contradicts the fact that $T^* < \infty$, and the theorem is proved.

9. The Wavelet Profile Decomposition Method

In this final section, we give some historical context and describe the method used by the author to establish profile decompositions in critical Navier-Stokes spaces such as $L^3(\mathbb{R}^3)$.

In the spirit of such works as [20], [3] and [24], the profile decompositions established by Gérard in [10] describe the defect of compactness in the critical Sobolev

embeddings

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \quad s > 0, \quad \frac{1}{q} = \frac{1}{2} - \frac{s}{d} > 0 .$$

For bounded sequences in \dot{H}^s , he establishes a decomposition into profiles with remainders which are small in L^q . His methods rely heavily on the Hilbertian structure of \dot{H}^s , in particular Plancherel's Theorem, and he mentions in his introduction that they are therefore unsuitable for treating the more general embeddings

$$\dot{H}^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \quad \frac{1}{q} = \frac{1}{p} - \frac{s}{d} > 0, \quad p > 1 .$$

Soon afterwards, S. Jaffard [11] established profile decompositions in the general setting, $s > 0$ and $p > 1$, by use of wavelet methods. Only the "stability" properties are somewhat less specific than those in Gérard's decompositions, due to the non-Hilbertian structures. The key tool is the description of Besov spaces and Triebel-Lizorkin (e.g. L^q , $\dot{H}^{s,p}$) spaces using equivalent norms expressed in terms of the components of functions written in a wavelet basis.

Jaffard's method is actually quite general, and with a view towards the Navier-Stokes results mentioned above, the author used similar methods in [17] to establish decompositions for bounded sequences in $L^p(\mathbb{R}^d)$ and the homogeneous Besov spaces $\dot{B}_{r,q}^{-\frac{d}{p} + \frac{d}{r}}(\mathbb{R}^d)$, with remainders small in larger Besov spaces. Of course, when $p = d$ these are critical spaces for Navier-Stokes (see (2.1)), and as mentioned above the decompositions in [17] have been applied in that setting to give a critical element proof of Escauriaza-Seregin-Sverak. These specific cases still do not represent the full generality of the wavelet method of establishing profile decompositions, and the author is currently working in collaboration with H. Bahouri and A. Cohen (see [1]) to generalize and simplify the method to treat embeddings between a wide range of spaces including Besov spaces, Triebel-Lizorkin spaces and others.

In [17], the following critical Sobolev-type embeddings were considered: setting $s_{p,r} := \frac{d}{r} - \frac{d}{p}$, one has

$$L^p(\mathbb{R}^d) \hookrightarrow \dot{B}_{r,q}^{s_{p,r}}(\mathbb{R}^d), \quad 2 \leq p < q, r \leq \infty \quad (s_{p,r} < 0) \quad (9.1)$$

and, for any $p \in [-\infty, +\infty] \setminus \{0\}$,

$$\dot{B}_{a,q}^{s_{p,a}}(\mathbb{R}^d) \hookrightarrow \dot{B}_{b,r}^{s_{p,b}}(\mathbb{R}^d), \quad 1 \leq a < b \leq +\infty, \quad 1 \leq q \leq r \leq +\infty . \quad (9.2)$$

Up to a subsequence, bounded sequences $\{\varphi_n\}$ in the smaller source space are shown to have a profile decomposition of the form

$$\varphi_n(x) = \sum_{l=1}^L (2^{j_n^l})^{d/p} \phi_l(2^{j_n^l} x - k_n^l) + r_n^L(x) \quad (9.3)$$

with $\{(j_n^l, k_n^l)\}_{n=1}^\infty \subset \mathbb{Z} \times \mathbb{Z}^d$ and remainders small in the larger target space for large L and n . The orthogonality condition on the scales and cores (compare to (3.2)) takes the form

$$\left| \log \left(2^{(j_n^l - j_n^{l'})} \right) \right| + \left| 2^{(j_n^l - j_n^{l'})} k_n^{l'} - k_n^l \right| \xrightarrow{n \rightarrow \infty} +\infty \quad (9.4)$$

for $l \neq l'$. In each case the stability property of the profiles in Statement 2 is established as well (in an appropriate norm).

The reason for the particular form of these decompositions becomes clear when one writes functions in these spaces in terms of an unconditional wavelet basis:

$$f(x) = \sum a_{ijk} \cdot (2^j)^{\frac{d}{p}} \psi^{(i)}(2^j x - k) ,$$

where the sum ranges over $(i, j, k) \in \{1, \dots, 2^d - 1\} \times \mathbb{Z} \times \mathbb{Z}^d$ and one has a finite collection of "mother wavelets" $\psi^{(i)} \in C_0^m$ for a fixed, arbitrarily large, $m \in \mathbb{Z}$. One then has a norm for the space (which is equivalent to the usual one) in terms of the coefficients. For example, one can characterize the Besov spaces by

$$\|f\|_{\dot{B}_{\alpha, \beta}^s} = \left\| 2^{j(s+d(\frac{1}{p}-\frac{1}{\alpha}))} \|(a_{ijk})\|_{\ell_{i,k}^\alpha} \right\|_{\ell_j^\beta} . \quad (9.5)$$

The proof relies on two main ingredients. Writing the embedding in either (9.1) or (9.2) generically as $X \hookrightarrow Y$, it is noted that one always has $Y \hookrightarrow Z$ where $Z = \dot{B}_{\infty, \infty}^{-d/p}$ (and using (9.5) one may write $\|f\|_Z = \sup |a_{ijk}|$). The first ingredient is an "improved Sobolev inequality" of the form

$$\|f\|_Y \leq C \|f\|_X^\alpha \|f\|_Z^{1-\alpha}$$

for fixed $C > 0$ and $\alpha \in (0, 1)$. The second ingredient is a decay of "nonlinear wavelet projections." Abbreviating the wavelet expansion as $f = \sum_\lambda a_\lambda \psi_\lambda$ one establishes the following decay property:

$$\sup_{\|f\|_X \leq M} \left\| \sum_{\substack{\text{all but} \\ N \text{ largest} \\ |a_\lambda|}} a_\lambda \psi_\lambda \right\|_Z \xrightarrow{N \rightarrow \infty} 0 \quad \forall M > 0.$$

The proof of the profile decomposition can be described roughly as follows. Consider a bounded sequence $\{\varphi_n\} \subset X$. As a first step, one notes that one can use the unconditionality of the basis to rearrange the components of each function φ_n so that the coefficients are decreasing in modulus. In an iterative manner, starting with the largest wavelet components, one "extracts" the components into groups (roughly "profiles") in a pre-determined way.

The norm of the remainder (the part of φ_n which has not yet been categorized) in the largest space Z (consisting simply of functions with bounded wavelet coefficients) decreases while the norm in the smallest space X remains bounded. In fact, the remainder must tend to zero in Z due to the decay of the nonlinear projections, and finally the decay of the remainder in the space Y is established by the improved Sobolev interpolation inequality. In the end (and after passing to appropriate subsequences), each profile is a limit of a sum of wavelet components of φ_n from which one can "pull out" a common norm-invariant transformation (i.e., scales and cores) for each n . Moreover, the procedure is orchestrated so that the scales and cores of distinct profiles will be orthogonal.

The fact that one can pull out a common transformation from a sum of certain wavelet components comes from the fact that the scales and cores take values on a lattice, which significantly simplifies the proof. Essentially, one uses the fact that a bounded sequence in \mathbb{Z}^n must have a constant subsequence, and this principle is used to pull out a *fixed* common transformation after passing to an appropriate subsequence. The interested reader is referred to [17] for more details.

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