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Formation of Singularities in Fluid Interfaces

Charles Fefferman

In this expository paper, we discuss three problems about incompressible fluids with boundaries (interfaces), namely **Water Waves**, **the Muskat Equation** and **Alpha Patches**. We will see that singularities form in finite time for the first two problems, and likely for all three.

The results we present on water waves are joint work with Angel Castro, Diego Cordoba, Francisco Gancedo and Javier Gomez-Serrano [9]. Our work on Muskat is joint with Angel Castro, Diego Cordoba, Francisco Gancedo and Maria Lopez-Fernandez [7, 8]. Regarding alpha-patches, we discuss the work of Diego Cordoba, Marco Fontelos, Ana Mancho and José Rodrigo [16].

We will explain intuitive ideas, but not state precise theorems. Detailed proofs of precise theorems can be found in [7–9].

Let us first set up the problem of water waves in two space dimensions. As in Figure 1, we imagine the plane partitioned into a water region $\Omega(t)$ and a vacuum region $\mathbb{R}^2 \setminus \Omega(t)$, separated by an interface $\partial\Omega(t)$; here, t denotes the time. In the water region, the velocity at position x and time t is denoted $u(x, t) = (u_1(x, t), u_2(x, t)) \in \mathbb{R}^2$, and the pressure is denoted by $p(x, t) \in \mathbb{R}$. The water moves, influenced by gravity and pressure, but not surface tension. (We ignore surface tension until further notice.) Note that the velocity $u(x, t)$ and pressure $p(x, t)$ are defined only for $x \in \Omega(t)$. We don't know $\Omega(t)$ —we must solve for it.

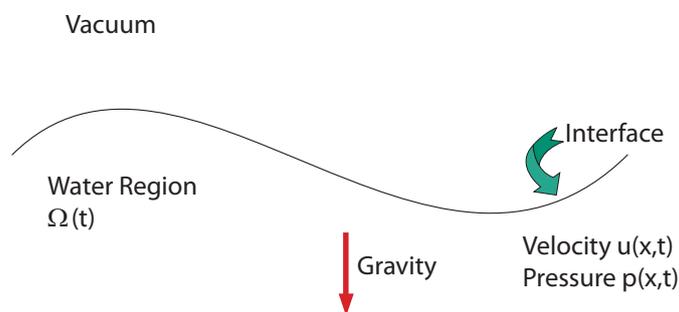


Fig. 1

We represent the interface $\partial\Omega(t)$ as a parametrized curve

$$\partial\Omega(t) = \{z(\alpha, t) : \alpha \in \mathbb{R}\}. \quad (0.1)$$

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The parametrization of $\partial\Omega(t)$ for fixed t has no physical meaning. The equations for 2D water waves are as follows:

- In $\Omega(t)$, the fluid satisfies the 2D incompressible, irrotational Euler equations

$$(\partial_t + u \cdot \nabla_x) u = -\nabla_x p - \begin{pmatrix} 0 \\ g \end{pmatrix} \quad (0.2)$$

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = 0. \quad (0.3)$$

Here, g is a non-negative constant. Recall that Euler's equation arises simply from Newton's law $F = ma$ for a fluid subject to pressure and gravity.

- At the interface, $\partial\Omega(t) = \{z(\alpha, t) = \alpha \in \mathbb{R}\}$, we have

$$p = 0, \quad \text{and} \quad (0.4)$$

$$\partial_t z(\alpha, t) = u(z(\alpha, t), t) + c(\alpha, t) \partial_\alpha z(\alpha, t). \quad (0.5)$$

This last equation just says that the interface moves with the fluid. Here, the function c may be picked arbitrarily. The choice of c affects the parametrization of the interface, but has no effect on anything physically significant.

We specify the interface $\partial\Omega$ and the velocity u at time $t = 0$. Initially, $\partial\Omega$ and u are smooth, and we have $\operatorname{div} u = 0, \operatorname{curl} u = 0$.

We solve equations (1) \cdots (5) for $t \geq 0$ with the above initial conditions, and we ask whether a singularity can form in finite time.

A lot of important work has been done on water waves. We mention here a few of the main results.

Solutions of the water wave equations exist and stay smooth for a short time. See S. Wu [27], as well as Christodoulou-Lindblad [12], Lindblad [24], Ambrose-Masmoudi [3], Coutand-Shkoller [18], Shatah-Zeng, [26], Lannes [23], Cordoba-Cordoba-Gancedo [15] and Alazard-Burq-Zuilly [1].

For small initial data, solutions remain smooth for an exponentially long time; see S. Wu [29].

For 3D water waves, S. Wu [28] proved short-time existence and smoothness of solutions; moreover, for small initial data there is global existence; see S. Wu [30] and Germain-Masmoudi-Shatah [22], as well as Alvarez-Lannes [2].

For local existence, one can drop the restriction to irrotational flows. See Christodoulou-Lindblad [12] and Zhang-Zhang [32].

With A. Castro et al [9], we have proven that solutions of the water wave equations may become singular in finite time by a simple natural scenario. We now describe the singularity. We first explain what we believe based on numerical simulation, then say what we can prove rigorously.

Our water wave starts as in Figure 2 at time $t = 0$. Note that u and $\partial\Omega$ are smooth, and the interface $\partial\Omega$ is a graph.

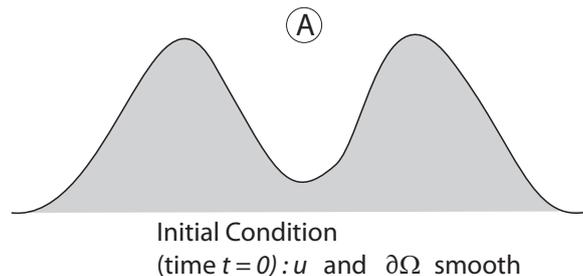
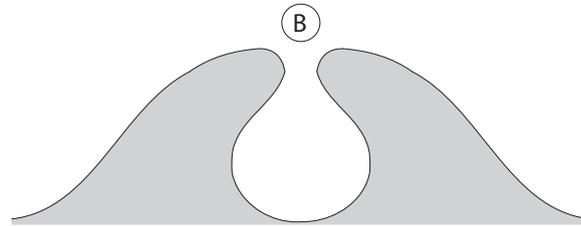


Fig. 2

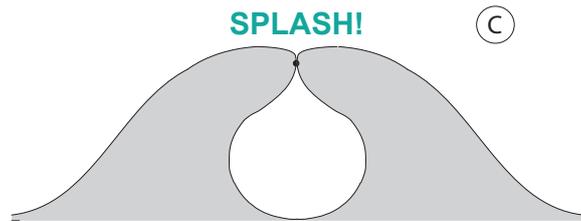
At a later time $t_1 > 0$, our water wave is pictured in Figure 3. The velocity u and the interface $\partial\Omega$ are still smooth, but the water wave has “turned over”; the interface $\partial\Omega$ is no longer the graph of a function $x_2 = \varphi(x_1)$.



The water wave has “turned over” at time $t_1 > 0$, but u and $\partial\Omega$ still smooth

Fig. 3

At a time $t_2 > t_1$, the singularity occurs; see Figure 4. The interface self-intersects at a single point, but u and $\partial\Omega$ are otherwise smooth. We call such a Singularity a SPLASH.

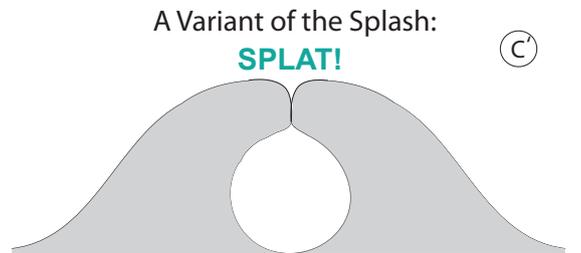


The interface self-intersects at a single point at time $t_2 > t_1$, but u and $\partial\Omega$ are otherwise smooth.

Fig. 4

Beyond time t_2 , there is no longer any physically meaningful solution of the water wave equation.

In a variant of the SPLASH, the singularity forms at time t_2 in such a way that the interface $\partial\Omega$ self-intersects along an arc, as in Figure 5. We call this scenario a SPLAT. Again, no physically meaningful solution of the water wave equation exists after the moment depicted in Figure 5.



At time t_2 , the interface self-intersects along an arc, but u and $\partial\Omega$ are otherwise smooth.

Fig. 5

A numerical simulation (modified from Beale-Hou-Lowengrub [4] to remain accurate up to the moment of the SPLASH) indicates that a water wave may start as in Figure 2, then turn over as in Figure 3, and finally form a SPLASH as in Figure 4.

We can prove rigorously that a water wave may start as in Figure 2, and then turn over as in Figure 3; see [7].

We can prove also that a water wave may start as in Figure 3, and then form a SPLASH as in Figure 4 or a SPLAT as in Figure 5.

We are working to produce a rigorous, computer-assisted proof that a water wave may begin as in Figure 2, then turn over as in Figure 3, and finally form a SPLASH as in Figure 4.

Our work implies easily that a variant of the SPLASH and SPLAT occur for water waves in dimension 3. Another variant of the SPLASH in dimension 3 is considered by Coutand-Shkoller [19].

We have shown [10] that the SPLASH and SPLAT occur also when we include surface tension in the statement of the problem. It would be very interesting to understand what happens if we replace the vacuum in Figure 1 by an incompressible irrotational fluid of low density.

We do not assert that the SPLASH and the SPLAT are the only possible singularities for water waves. It would be very interesting to exhibit another type of singular solution.

Let us now turn our attention to the Muskat equation, which governs the motion of oil and water in sand. We again consider a two-dimensional case.

At time t , the plane \mathbb{R}^2 is partitioned into the oil region $\Omega_{OIL}(t)$ and the water region $\Omega_{WATER}(t)$, separated by an interface, as in Figure 6.

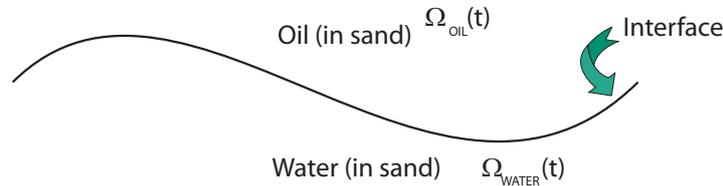


Fig. 6

Remarkably, the same equations govern a “Hele-Shaw cell,” consisting of two parallel vertical plates separated by a thin region filled with oil and water as in Figure 7.

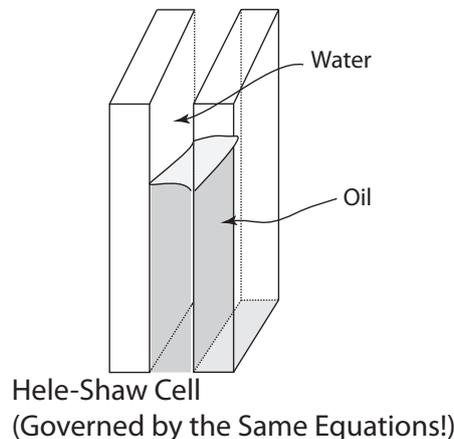


Fig. 7

Oil and water in sand, or in a Hele-Shaw cell, do not satisfy Newton's law $F = ma$ (unless we include the large and complicated forces exerted by the sand in Figure 6, and the walls in Figure 7). Rather, the system is governed by an experimental fact called "Darcy's law":

$$u = -\nabla_x p - \begin{pmatrix} 0 \\ g\rho \end{pmatrix}, \text{ and } \operatorname{div} u = 0 \quad (0.6)$$

Here, u and p denote the velocity and pressure; they are defined throughout \mathbb{R}^2 . The constant g is the acceleration due to gravity, and ρ denotes the fluid density:

$$\rho(x, t) = \begin{cases} \rho_{OIL} & \text{if } x \in \Omega_{OIL}(t) \\ \rho_{WATER} & \text{if } x \in \Omega_{WATER}(t) \end{cases}. \quad (0.7)$$

In (0.7), ρ_{OIL} and ρ_{WATER} are positive constants (the densities of oil and water, respectively), with $\rho_{OIL} < \rho_{WATER}$.

We are assuming here that "OIL" and "WATER" have the same viscosity but different densities. It would of course be natural to relax the assumption of equal viscosities. Darcy's law would then look more complicated than (0.6).

At the interface

$$\partial\Omega_{WATER}(t) = \partial\Omega_{OIL}(t) = \{z(\alpha, t) : \alpha \in \mathbb{R}\},$$

we have:

$$U_{WATER} - U_{OIL} \text{ is tangential to the interface;} \quad (0.8)$$

and

$$p_{WATER} = p_{OIL}. \quad (0.9)$$

Here, U_{WATER} and p_{WATER} are the limiting values of U and p (respectively), as we approach the interface from within the region $\Omega_{WATER}(t)$. Similarly, U_{OIL} , p_{OIL} are the limiting values of U , p as we approach the interface from within $\Omega_{OIL}(t)$. Equation (0.9) neglects surface tension.

Finally, the interface $\{z(\alpha, t) : \alpha \in \mathbb{R}\}$ moves with the fluid:

$$\begin{aligned} \partial_t z(\alpha, t) &= U_{OIL}(z(\alpha, t), t) + C_{OIL}(\alpha, t)\partial_\alpha z(\alpha, t) \\ &= U_{WATER}(z(\alpha, t), t) + C_{WATER}(\alpha, t)\partial_\alpha z(\alpha, t), \end{aligned} \quad (0.10)$$

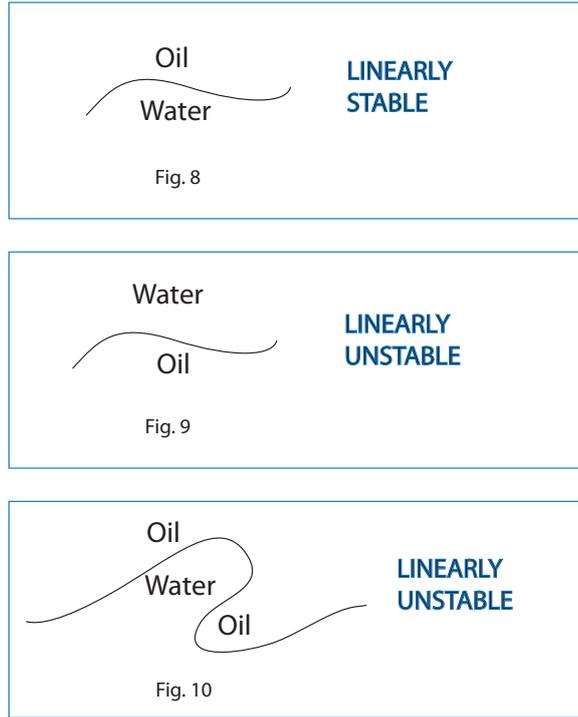
where the choice of the functions C_{OIL}, C_{WATER} affects only the parametrization of the interface and thus has no physical meaning.

The Muskat problem is to solve (6) \cdots (10) for times $t \geq 0$, given the initial domains $\Omega_{OIL}(t), \Omega_{WATER}(t)$ at time $t = 0$.

It makes a crucial difference whether the heavier fluid ("water") is on top or underneath.

In Figure 8, the heavier fluid lies underneath the lighter one. When we give initial conditions at time 0 and try to solve Muskat for times $t > 0$, configurations like the one in Figure 8 are linearly stable.

On the other hand, in Figures 9 and 10, the lighter fluid lies underneath the heavier fluid, at least somewhere in the picture. Such configurations are linearly unstable when we give initial conditions at time 0 and try to solve Muskat for times $t > 0$.



In the linearly stable case depicted in Figure 8, the Muskat equation is a nonlinear version of the equation

$$\partial_t F = - \wedge_x F \quad \left(\wedge_x = \sqrt{-\Delta_x} \right)$$

in one space dimension.

In the linearly unstable case in Figure 9, the Muskat equation is a nonlinear version of the bad equation

$$\partial_t F = + \wedge_x F.$$

In the bad case shown in Figure 10, the Muskat equation is a nonlinear version of the nasty equation

$$\partial_t F(x, t) = \sigma(x, t) \wedge_x F(x, t),$$

where the coefficient $\sigma(x, t)$ changes sign for fixed t .

Let us recall some of the previous work on the Muskat equation.

For small initial data in the stable regime (Figure 8), global smooth solutions exist. See Constantin-Pugh [14], Yi [31], Caffisch-Howison-Siegel [6], Cordoba-Gancedo [17] and Escher-Matioc [20].

Constantin-Cordoba-Gancedo-Strain [13] show that global solutions exist if the initial interface has the form $\{(x_1, x_2) : x_2 = f(x_1)\}$ with $|f'| < 1$ everywhere (and the “oil” is on top of the “water”).

Next we explain how a Muskat solution may become singular.

At time $t = 0$, the picture is as in Figure 8. The interface is smooth, and we are in the linearly stable regime. At a later time $t_1 > 0$, the Muskat solution looks like Figure 10. The interface has “turned over”, yet it remains smooth for a while, even though it has entered a linearly unstable regime. Finally, at some time $t_2 > t_1$, the picture is still as in Figure 10, but the interface is no longer smooth. Rather, at a single point, the interface is C^3 but not C^4 . At all other points, the interface is real-analytic.

The papers [7, 8] by Castro et al prove that the above scenario occurs for some choice

of initial data for the Muskat problem. Presumably, there is no meaningful Muskat solution past the breakdown time t_2 , but this has not yet been proven.

Finally, we turn our attention to a third problem regarding fluid interfaces, for which (we think) a singularity develops in finite time, namely “ α -patches”. This is an equation for an active scalar

$$\theta(x, t) = \begin{cases} 1 & \text{if } x \in \Omega(t) \\ 0 & \text{if } x \notin \Omega(t) \end{cases} \quad (x \in \mathbb{R}^2), \quad (0.11)$$

where $\Omega(t) \subset \mathbb{R}^2$. The scalar θ is advected by an incompressible fluid velocity

$$U(x, t) = \nabla_x^\perp \psi(x, t) = \left(-\frac{\partial}{\partial x_2} \psi, \frac{\partial}{\partial x_1} \psi \right). \quad (0.12)$$

Thus,

$$(\partial_t + U \cdot \nabla_x) \theta = 0. \quad (0.13)$$

To close the equations, we suppose that the stream function ψ in (0.12) is obtained from the scalar θ in (0.11), (0.13) by the formula

$$\psi = (-\Delta_x)^{\frac{\alpha}{2}-1} \theta. \quad (0.14)$$

Here, $0 \leq \alpha \leq 1$ is a parameter.

We want to solve equations (0.10) \dots (0.14) with initial condition obtained by specifying the region $\Omega(t)$ at time $t = 0$. We suppose that $\Omega(t = 0)$ has a smooth boundary, and we ask whether $\partial\Omega(t)$ can lose smoothness in finite time.

The case $\alpha = 0$ is the vortex patch problem for the 2D incompressible Euler equation. In this case, the boundary $\partial\Omega(t)$ remains smooth for all time; this was proven by Chemin [11], and a simple proof was given by Bertozzi and Constantin [5].

The case $\alpha = 1$ is the analogue of the vortex patch problem for the surface quasi-geostrophic (SQG) equation.

For $0 < \alpha \leq 1$, Rodrigo [25] proved that (10) \dots (14) may be solved for a short time when $\partial\Omega(t = 0)$ is smooth; and $\partial\Omega(t)$ remains smooth. See also Gancedo [21]. However, we don't have rigorous results to tell us whether $\partial\Omega(t)$ may become singular in finite time. Numerical simulations suggest that the scenario shown in Figure 11 may occur, at least for $\alpha > 2/3$.

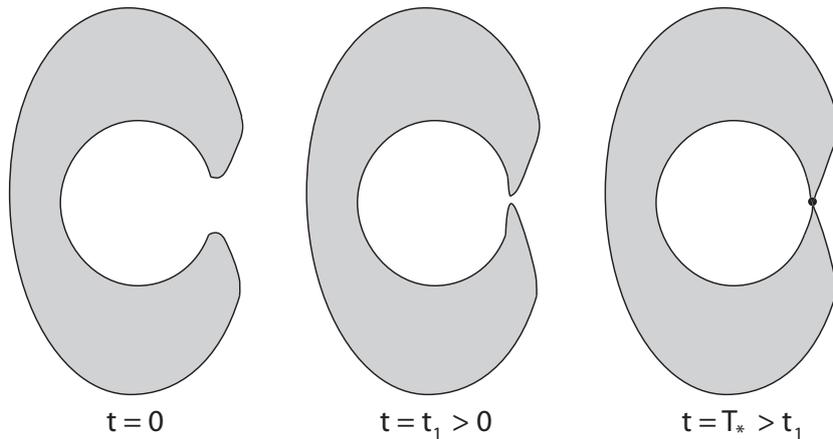


Fig. 11

At time $t = T_*$, a singularity forms at the point 0.

The boundary $\partial\Omega(T_*)$ includes four arcs that meet at 0. Moreover, the singularity

appears to be asymptotically self-similar as we approach the time T_* when the breakdown occurs.

The above numerical results appear in the paper of Cordoba-Fontelos-Mancho-Rodrigo [16].

It would be very interesting to prove or disprove that solutions to the α -patch equations (10) \cdots (14) can become singular by the scenario in [16].

We hope the reader is convinced that the study of fluid interfaces leads to several interesting breakdown results, with more to come in the future.

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References

- [1] T. Alazard, N. Burq, and C. Zuily, *On the water-wave equations with surface tension*, Duke Math. J. **158** (2011), no. 3, 413–499.
- [2] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water waves and asymptotics*, Invent. Math. **171** (2008), no. 3, 485–541.
- [3] D. M. Ambrose and N. Masmoudi, *The zero surface tension limit of two-dimensional water waves*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1287–1315.
- [4] J. T. Beale, T. Y. Hou, and J. Lowengrub, *Convergence of a boundary integral method for water waves*, SIAM J. Numer. Anal. **33** (1996), no. 5, 1797–1843.
- [5] A. Bertozzi and P. Constantin, *Global regularity for vortex patches*, Comm. Pure Appl. Math. **152** (1993), no. 1, 19–28.
- [6] R. Caflisch, S. Howison, and M. Siegel, *Global existence, singular solutions and ill-posedness for the Muskat problem*, Comm. Pure Appl. Math. **57** (2004), 1374–1411.
- [7] A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, and M. Lopez-Fernandez, *Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves*, Annals of Math. **175** (2012), 909–948.
- [8] A. Castro, D. Cordoba, C. Fefferman, and F. Gancedo, *Breakdown of smoothness for the Muskat problem*, (preprint), available at [arXiv:1201.2525](https://arxiv.org/abs/1201.2525).
- [9] A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, and J. Gomez-Serrano, *Finite time singularities for the free boundary incompressible Euler equations*, (preprint), available at [arXiv:1112.2170v2](https://arxiv.org/abs/1112.2170v2).
- [10] A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, and J. Gomez-Serrano, *Finite time singularities for water waves with surface tension*, (preprint), available at [arXiv:1204.6633](https://arxiv.org/abs/1204.6633).
- [11] J. Y. Chemin, *Persistence de structures géométriques dans les fluides incompressibles bidimensionnels*, Ann. École Norm. Sup. **26** (1993), 1–16.
- [12] D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid*, Comm. Pure Appl. Math. **53** (2000), no. 12, 1536–1602.
- [13] P. Constantin, D. Cordoba, F. Gancedo, and R. Strain, *On the global existence for the Muskat problem*, JEMS.
- [14] P. Constantin and M. Pugh, *Global solutions for small data to the Hele-Shaw problem*, Nonlinearity **6** (1993), 393–415.
- [15] A. Cordoba, D. Cordoba, and F. Gancedo, *Interface evolution: Water waves in 2D*, Adv. Math. **223** (2010), no. 1, 120–173.
- [16] D. Cordoba, M. Fontelos, A. Mancho, and J. L. Rodrigo, *Evidence of singularities for a family of contour dynamics equations*, Proc. Nat. Acad. Sci. USA **102** (2005), 5949–5952.
- [17] D. Cordoba and F. Gancedo, *Contour dynamics of incompressible 3D fluids in a porous medium with different densities*, Comm. Math. Phys. **273** (2007), no. 2, 445–471.
- [18] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. **20** (2007), no. 3, 829–930.
- [19] ———, *On the finite-time splash singularity for the 3D free surface Euler equations*, (preprint).
- [20] J. Escher and B-V. Matioc, *On the parabolicity of the Muskat problem: Well-posedness, fingering and stability results*, Z. Anal. Anwend. **30** (2011), 193–218.

- [21] F. Gancedo, *Existence for the α -patch model and the QG sharp front in Sobolev spaces*, Adv. Math. **217** (2008), 2569–2598.
- [22] P. Germain, N. Masmoudi, and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Annals of Math. (2011).
- [23] D. Lannes, *Well-posedness of the water-waves equation*, J. Amer. Math. Soc. **18** (2005), no. 3, 605–654.
- [24] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Annals of Math. **162** (2005), 109–194.
- [25] J. Rodrigo, *On the evolution of sharp fronts for the quasigeostrophic equation*, Comm. Pure Appl. Math. **58** (2005), no. 6, 821–866.
- [26] J. Shatah and C. Zeng, *Geometry and a-priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math **61** (2008), no. 5, 698–744.
- [27] S. Wu, *Well-posedness in Sobolev spaces of the full water-wave problem in 2D*, Invent. Math. **177** (1997), 39–72.
- [28] ———, *Well-posedness in Sobolev spaces of the full water-wave problem in 3D*, J. Amer. Math. Soc. **12** (1999), no. 2, 445–495.
- [29] ———, *Almost global well-posedness of the 2D full water wave problem*, Invent. Math. **177** (2009), no. 1, 45–135.
- [30] ———, *Global well-posedness of the 3D full water-wave problem*, Invent. Math. **184** (2011), no. 1, 125–220.
- [31] F. Yi, *Global classical solution of Muskat free boundary problem*, J. Math. Anal. Appl. **288** (2003), 442–461.
- [32] P. Zhang and Z. Zhang, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math **61** (2008), 877–940.