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SIMPLICITY OF VACUUM MODULES AND ASSOCIATED VARIETIES

by Tomoyuki Arakawa, Cuipo Jiang & Anne Moreau

Abstract. — In this note, we prove that the universal affine vertex algebra associated with a simple Lie algebra $\mathfrak{g}$ is simple if and only if the associated variety of its unique simple quotient is equal to $\mathfrak{g}^*$. We also derive an analogous result for the quantized Drinfeld-Sokolov reduction applied to the universal affine vertex algebra.

Résumé (Simplicité des algèbres vertex affines et variétés associées). — Dans cet article, nous démontrons que l'algèbre vertex affine universelle associée à une algèbre de Lie simple $\mathfrak{g}$ est simple si et seulement si la variété associée à son unique quotient simple est égale à $\mathfrak{g}^*$. Nous en déduisons un résultat analogue pour la réduction quantique de Drinfeld-Sokolov appliquée à l'algèbre vertex affine universelle.

Contents

1. Introduction

Let V be a vertex algebra, and let

$$V \longrightarrow (\operatorname{End} V)[\![z, z^{-1}]\!], \quad a \longmapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

be the state-field correspondence. The *Zhu* C_2*-algebra* [Zhu96] of V is by definition the quotient space $R_V = V/C_2(V)$, where $C_2(V) = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}$, equipped with the Poisson algebra structure given by

$$\overline{a}.\overline{b} = \overline{a_{(-1)}b}, \qquad \{\overline{a}, \overline{b}\} = \overline{a_{(0)}b},$$

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for $a, b \in V$ with $\overline{a} := a + C_2(V)$. The associated variety X_V of V is the reduced scheme $X_V = \operatorname{Specm}(R_V)$ corresponding to R_V. It is a fundamental invariant of V that captures important properties of the vertex algebra V itself (see, for example, [BFM, Zhu96, ABD04, Miy04, Ara12a, Ara15a, Ara15b, AM18a, AM17, AK18]). Moreover, the associated variety X_V conjecturally [BR18] coincides with the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$, if V corresponds to a theory $\mathcal{T}$ by the 4D/2D duality discovered in [BLL$^+$15]. Note that the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory is a hyperkähler cone, possibly singular.

In the case where V is the universal affine vertex algebra $V^k(\mathfrak{g})$ at level $k \in \mathbb{C}$ associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, the variety X_V is just the affine space $\mathfrak{g}^*$ with Kirillov-Kostant Poisson structure. In the case where V is the unique simple graded quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$, the variety X_V is a Poisson subscheme of $\mathfrak{g}^*$ which is G-invariant and conic, where G is the adjoint group of $\mathfrak{g}$.

Note that if the level k is irrational, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, and hence $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. More generally, if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple, then obviously $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.

In this article, we prove that the converse is true.

Theorem 1.1. — *The equality $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ holds, that is, $V^k(\mathfrak{g})$ is simple, if and only if $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.*

It is known by Gorelik and Kac [GK07] that $V^k(\mathfrak{g})$ is not simple if and only if

$$r^\vee(k + h^\vee) \in \mathbb{Q}_{\geqslant 0} \smallsetminus \{1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}, \tag{1.1}$$

where $h^\vee$ is the dual Coxeter number and $r^\vee$ is the lacing number of $\mathfrak{g}$. Therefore, Theorem 1.1 can be rephrased as

$$X_{L_k(\mathfrak{g})} \subsetneq \mathfrak{g}^* \iff (1.1) \text{ holds.} \tag{1.2}$$

Let us mention the cases when the variety $X_{L_k(\mathfrak{g})}$ is known for k satisfying (1.1).

First, it is known [Zhu96, DM06] that $X_{L_k(\mathfrak{g})} = \{0\}$ if and only if $L_k(\mathfrak{g})$ is integrable, that is, k is a nonnegative integer. Next, it is known that if $L_k(\mathfrak{g})$ is *admissible* [KW89], or equivalently, if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geqslant 1},\ (p,q) = 1,\ p \geqslant \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) \neq 1, \end{cases}$$

where h is the Coxeter number of $\mathfrak{g}$, then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit in $\mathfrak{g}$ ([Ara15a]). Further, it was observed in [AM18a, AM18b] that there are cases when $L_k(\mathfrak{g})$ is non-admissible and $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit. In fact, it was recently conjectured in physics [XY19] that, in view of the 4D/2D duality, there should be a large list of non-admissible simple affine vertex algebras whose associated varieties are the closures of some nilpotent orbits. Finally, there are also cases [AM17] where $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}^*$ nor contained in the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

In general, the problem of determining the variety $X_{L_k(\mathfrak{g})}$ is wide open.

Now let us explain the outline of the proof of Theorem 1.1. First, Theorem 1.1 is known for the critical level $k=-h^\vee$ ([FF92, FG04]). Therefore, since $R_{V^k(\mathfrak{g})}$ is a polynomial ring $\mathbb{C}[\mathfrak{g}^*]$, Theorem 1.1 follows from the following fact.

Theorem 1.2. — *Suppose that the level is non-critical, that is, $k\neq -h^\vee$. The image of any nonzero singular vector v of $V^k(\mathfrak{g})$ in the Zhu C_2-algebra $R_{V^k(\mathfrak{g})}$ is nonzero.*

The symbol $\sigma(w)$ of a singular vector w in $V^k(\mathfrak{g})$ is a singular vector in the corresponding vertex Poisson algebra $\mathrm{gr} V^k(\mathfrak{g})\cong S(t^{-1}\mathfrak{g}[t^{-1}])\cong \mathbb{C}[J_\infty\mathfrak{g}^*]$, where $J_\infty\mathfrak{g}^*$ is the arc space of $\mathfrak{g}^*$. Theorem 1.2 states that the image of $\sigma(w)$ of a non-trivial singular vector w under the projection

$$\mathbb{C}[J_\infty\mathfrak{g}^*]\longrightarrow \mathbb{C}[\mathfrak{g}^*]=R_{V^k(\mathfrak{g})} \tag{1.3}$$

is nonzero, provided that k is non-critical. Here the projection (1.3) is defined by identifying $\mathbb{C}[\mathfrak{g}^*]$ with the Zhu C_2-algebra of the commutative vertex algebra $\mathbb{C}[J_\infty\mathfrak{g}^*]$. Hence, Theorem 1.2 would follow if the image of any nontrivial singular vector in $\mathbb{C}[J_\infty\mathfrak{g}^*]$ under the projection (1.3) is nonzero. However, this is false as there are singular vectors in $\mathbb{C}[J_\infty\mathfrak{g}^*]$ that do not come from singular vectors of $V^k(\mathfrak{g})$ and that belong to the kernel of (1.3) (see Section 3.4). Therefore, we do need to make use of the fact that $\sigma(w)$ is the symbol of a singular vector w in $V^k(\mathfrak{g})$. We also note that the statement of Theorem 1.2 is not true if k is critical (see Section 3.4).

For this reason the proof of Theorem 1.2 is divided roughly into two parts. First, we work in the commutative setting to deduce a first important reduction (Lemma 3.1). Next, we use the Sugawara construction – which is available only at non-critical levels – in the non-commutative setting in order to complete the proof.

Now, let us consider the *W-algebra* $\mathscr{W}^k(\mathfrak{g},f)$ associated with a nilpotent element f of $\mathfrak{g}$ at the level k defined by the generalized quantized Drinfeld-Sokolov reduction [FF90, KRW03]:

$$\mathscr{W}^k(\mathfrak{g},f)=H^0_{\mathrm{DS},f}(V^k(\mathfrak{g})).$$

Here, $H^\bullet_{\mathrm{DS},f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f\in\mathcal{N}(\mathfrak{g})$ with coefficients in a $V^k(\mathfrak{g})$-module M.

By the Jacobson-Morosov theorem, f embeds into an $\mathfrak{sl}_2$-triple (e,h,f). The Slodowy slice $\mathscr{S}_f$ at f is the affine space $\mathscr{S}_f=f+\mathfrak{g}^e$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. It has a natural Poisson structure induced from that of $\mathfrak{g}^*$ (see [GG02]), and we have [DSK06, Ara15a] a natural isomorphism $R_{\mathscr{W}^k(\mathfrak{g},f)}\cong\mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$X_{\mathscr{W}^k(\mathfrak{g},f)}=\mathscr{S}_f.$$

The natural surjection $V^k(\mathfrak{g})\twoheadrightarrow L_k(\mathfrak{g})$ induces a surjection $\mathscr{W}^k(\mathfrak{g},f)\twoheadrightarrow H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ of vertex algebras ([Ara15a]). Hence the variety $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))}$ is a $\mathbb{C}^*$-invariant Poisson subvarieties of the Slodowy slice $\mathscr{S}_f$.

Conjecturally [KRW03, KW08], the vertex algebra $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ coincides the unique simple (graded) quotient $\mathscr{W}_k(\mathfrak{g},f)$ of $\mathscr{W}^k(\mathfrak{g},f)$ provided that $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))\neq 0$. (This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

As a consequence of Theorem 1.1, we obtain the following result.

Theorem 1.3. — *Let f be any nilpotent element of $\mathfrak{g}$. The following assertions are equivalent:*

(1) $V^k(\mathfrak{g})$ *is simple,*
(2) $\mathscr{W}^k(\mathfrak{g}, f) = H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$,
(3) $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = \mathscr{S}_f$.

Note that Theorem 1.3 implies that $V^k(\mathfrak{g})$ is simple if $X_{\mathscr{W}_k(\mathfrak{g},f)} = \mathscr{S}_f$ and $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \neq 0$ since $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} \supset X_{\mathscr{W}^k(\mathfrak{g},f)}$.

The remainder of the paper is structured as follows. In Section 2 we set up notation in the case of affine vertex algebras that will be the framework of this note. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we have compiled some known facts on Slodowy slices, W-algebras and their associated varieties. Theorem 1.3 is proved in this section.

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2. Universal affine vertex algebras and associated graded vertex Poisson algebras

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated with $\mathfrak{g}$, that is,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,$$

where the commutation relations are given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x|y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here,

$$(\ |\ ) = \frac{1}{2h^\vee} \times \text{Killing form of } \mathfrak{g}$$

is the usual normalized inner product. For $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we shall write $x(m)$ for $x \otimes t^m$.

2.1. Universal affine vertex algebras. — For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which K acts as multiplication by k and $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition, we have

$$V^k(\mathfrak{g}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) = U(t^{-1}\mathfrak{g}[t^{-1}]). \tag{2.1}$$

The space $V^k(\mathfrak{g})$ is naturally graded,

$$V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_\Delta,$$

where the grading is defined by

$$\deg(x^{i_1}(-n_1)\cdots x^{i_r}(-n_r)\mathbf{1}) = \sum_{i=1}^{r} n_i, \quad r \geqslant 0,\ x^{i_j} \in \mathfrak{g},$$

with $\mathbf{1}$ the image of $1\otimes 1$ in $V^k(\mathfrak{g})$. We have $V^k(\mathfrak{g})_0 = \mathbb{C}\mathbf{1}$, and we identify $\mathfrak{g}$ with $V^k(\mathfrak{g})_1$ via the linear isomorphism defined by $x \mapsto x(-1)\mathbf{1}$.

It is well-known that $V^k(\mathfrak{g})$ has a unique vertex algebra structure such that $\mathbf{1}$ is the vacuum vector,

$$x(z) := Y(x\otimes t^{-1}, z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n-1},$$

and

$$[T, x(z)] = \partial_z x(z)$$

for $x \in \mathfrak{g}$, where T is the translation operator. Here, $x(n)$ acts on $V^k(\mathfrak{g})$ by left multiplication, and so, one can view $x(n)$ as an endomorphism of $V^k(\mathfrak{g})$. The vertex algebra $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k [FZ92, Zhu96, LL04].

The vertex algebra $V^k(\mathfrak{g})$ is a vertex operator algebra, provided that $k+h^\vee \neq 0$, by the *Sugawara construction*. More specifically, set

$$S = \frac{1}{2}\sum_{i=1}^{d} x_i(-1)x^i(-1)\mathbf{1},$$

where $\{x_i \mid i=1,\dots,d\}$ is the dual basis of a basis $\{x^i \mid i = 1,\dots,\dim\mathfrak{g}\}$ of $\mathfrak{g}$ with respect to the bilinear form $(\ |\ )$, with $d = \dim\mathfrak{g}$. Then for $k \neq -h^\vee$, the vector $\omega = S/(k+h^\vee)$ is a conformal vector of $V^k(\mathfrak{g})$ with central charge

$$c(k) = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

Note that, writing $\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, we have

$$L_0 = \frac{1}{2(k+h^\vee)}\Big(\sum_{i=1}^{d} x_i(0)x^i(0) + \sum_{n=1}^{\infty}\sum_{i=1}^{d}(x_i(-n)x^i(n) + x^i(-n)x_i(n))\Big),$$

$$L_n = \frac{1}{2(k+h^\vee)}\Big(\sum_{m=1}^{\infty}\sum_{i=1}^{d} x_i(-m)x^i(m+n) + \sum_{m=0}^{\infty}\sum_{i=1}^{d} x^i(-m+n)x_i(m)\Big), \quad \text{if } n\neq 0.$$

Lemma 2.1 ([Kac90]). — *We have*

$$[L_n, x(m)] = -m x(m+n), \quad \textit{for } x\in\mathfrak{g},\ m,n\in\mathbb{Z},$$

and $L_n\mathbf{1} = 0$ *for* $n \geqslant -1$.

We have $V^k(\mathfrak{g})_\Delta = \{v \in V^k(\mathfrak{g}) \mid L_0 v = \Delta v\}$ and $T = L_{-1}$ on $V^k(\mathfrak{g})$, provided that $k+h^\vee \neq 0$.

Any graded quotient of $V^k(\mathfrak{g})$ as $\widehat{\mathfrak{g}}$-module has the structure of a quotient vertex algebra. In particular, the unique simple graded quotient $L_k(\mathfrak{g})$ is a vertex algebra, and is called the *simple affine vertex algebra associated with* $\mathfrak{g}$ *at level* k.

2.2. Associate graded vertex Poisson algebras of affine vertex algebras

It is known by Li [Li05] that any vertex algebra V admits a canonical filtration $F^\bullet V$, called the *Li filtration* of V. For a quotient V of $V^k(\mathfrak{g})$, $F^\bullet V$ is described as follows. The subspace F^pV is spanned by the elements

$$y_1(-n_1-1)\cdots y_r(-n_r-1)\mathbf{1}$$

with $y_i\in\mathfrak{g}$, $n_i\in\mathbb{Z}_{\geqslant 0}$, $n_1+\cdots+n_r\geqslant p$. We have

$$\begin{aligned}&V=F^0V\supset F^1V\supset\cdots,\quad \textstyle\bigcap_p F^pV=0,\\ &TF^pV\subset F^{p+1}V,\\ &a_{(n)}F^qV\subset F^{p+q-n-1}V \ \mathit{for}\ a\in F^pV,\quad n\in\mathbb{Z},\\ &a_{(n)}F^qV\subset F^{p+q-n}V\ \mathit{for}\ a\in F^pV,\quad n\geqslant 0.\end{aligned}\tag{2.2}$$

Here we have set $F^pV=V$ for $p<0$.

Let $\mathrm{gr}^FV=\bigoplus_p F^pV/F^{p+1}V$ be the associated graded vector space. The space gr^FV is a vertex Poisson algebra by

$$\begin{aligned}\sigma_p(a)\sigma_q(b)&=\sigma_{p+q}(a_{(-1)}b),\\ T\sigma_p(a)&=\sigma_{p+1}(Ta),\\ \sigma_p(a)_{(n)}\sigma_q(b)&=\sigma_{p+q-n}(a_{(n)}b)\end{aligned}$$

for $a,b\in V$, $n\geqslant 0$, where $\sigma_p\colon F^p(V)\to F^pV/F^{p+1}V$ is the principal symbol map. In particular, $\mathrm{gr}^F V$ is a $\mathfrak{g}[t]$-module by the correspondence

$$\mathfrak{g}[t]\ni x(n)\longmapsto \sigma_0(x)_{(n)}\in\mathrm{End}(\mathrm{gr}^F V)\tag{2.3}$$

for $x\in\mathfrak{g}$, $n\geqslant 0$.

The filtration $F^\bullet V$ is compatible with the grading: $F^pV=\bigoplus_{\Delta\in\mathbb{Z}_{\geqslant 0}}F^pV_\Delta$, where $F^pV_\Delta:=V_\Delta\cap F^pV$.

Let $U_\bullet(t^{-1}\mathfrak{g}[t^{-1}])$ be the PBW filtration of $U(t^{-1}\mathfrak{g}[t^{-1}])$, that is, $U_p(t^{-1}\mathfrak{g}[t^{-1}])$ is the subspace of $U(t^{-1}\mathfrak{g}[t^{-1}])$ spanned by monomials $y_1y_2\dots y_r$ with $y_i\in\mathfrak{g}$, $r\leqslant p$. Define

$$G_pV=U_p(t^{-1}\mathfrak{g}[t^{-1}])\mathbf{1}.$$

Then $G_\bullet V$ defines an increasing filtration of V. We have

$$F^pV_\Delta=G_{\Delta-p}G_\Delta,\tag{2.4}$$

where $G_pV_\Delta:=G_pV\cap V_\Delta$, see [Ara12a, Prop. 2.6.1]. Therefore, the graded space $\mathrm{gr}^GV=\bigoplus_{p\in\mathbb{Z}_{\geqslant 0}}G_pV/G_{p-1}V$ is isomorphic to $\mathrm{gr}^F V$. In particular, we have

$$\mathrm{gr}V^k(\mathfrak{g})\cong\mathrm{gr}\,U_\bullet(t^{-1}\mathfrak{g}[t^{-1}])\cong S(t^{-1}\mathfrak{g}[t^{-1}]).$$

The action of $\mathfrak{g}[t]$ on $\mathrm{gr}V^k(\mathfrak{g})=S(t^{-1}\mathfrak{g}[t^{-1}])$ coincides with the one induced from the action of $\mathfrak{g}[t]$ on $\mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t]\cong t^{-1}\mathfrak{g}[t^{-1}]$. More precisely, the element $x(m)$, for $x\in\mathfrak{g}$

and $m \in \mathbb{Z}_{\geqslant 0}$, acts on $S(t^{-1}\mathfrak{g}[t^{-1}])$ as follows:

$$
\begin{aligned}
&x(m)\cdot \mathbf{1} = 0,\\
(2.5)\qquad &x(m)\cdot v = \sum_{j=1}^{r} \sum_{n_j - m>0} y_1(-n_1)\cdots [x,y_j](m-n_j)\cdots y_r(-n_r),
\end{aligned}
$$

if $v = y_1(-n_1)\cdots y_r(-n_r)$ with $y_i \in \mathfrak{g}$, $n_1,\dots,n_r \in \mathbb{Z}_{>0}$.

2.3. Zhu's C_2-algebras and associated varieties of affine vertex algebras

We have [Li05, Lem. 2.9]

$$F^pV = \operatorname{span}_{\mathbb{C}}\{a_{(-i-1)}b \mid a \in V,\, i \geqslant 1,\, b \in F^{p-i}V\}$$

for all $p \geqslant 1$. In particular,

$$F^1V = C_2(V),$$

where $C_2(V) = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a,b \in V\}$. Set

$$R_V = V/C_2(V) = F^0V/F^1V \subset \operatorname{gr}^F V.$$

It is known by Zhu [Zhu96] that R_V is a Poisson algebra. The Poisson algebra structure can be understood as the restriction of the vertex Poisson structure of $\operatorname{gr}^F V$. It is given by

$$\overline{a}\cdot\overline{b} = \overline{a_{(-1)}b}, \quad \{\overline{a},\overline{b}\} = \overline{a_{(0)}b},$$

for $a,b \in V$, where $\overline{a} = a + C_2(V)$.

By definition [Ara12a], the *associated variety* of V is the reduced scheme

$$X_V := \operatorname{Specm}(R_V).$$

It is easily seen that

$$F^1V^k(\mathfrak{g}) = C_2(V^k(\mathfrak{g})) = t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}).$$

The following map defines an isomorphism of Poisson algebras

$$
\begin{aligned}
\mathbb{C}[\mathfrak{g}^*] \cong S(\mathfrak{g}) &\longrightarrow R_{V^k(\mathfrak{g})}\\
\mathfrak{g} \ni x &\longmapsto x(-1)\mathbf{1} + t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}).
\end{aligned}
$$

Therefore, $R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*]$ and so, $X_{V^k(\mathfrak{g})} \cong \mathfrak{g}^*$.

More generally, if V is a quotient of $V^k(\mathfrak{g})$ by some ideal N, then we have

$$R_V \cong \mathbb{C}[\mathfrak{g}^*]/I_N \tag{2.6}$$

as Poisson algebras, where I_N is the image of N in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$. Then X_V is just the zero locus of I_N in $\mathfrak{g}^*$. It is a closed G-invariant conic subset of $\mathfrak{g}^*$.

Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the bilinear form $(\ |\ )$, one may view X_V as a subvariety of $\mathfrak{g}$.

2.4. PBW basis. — Let $\Delta_+ = \{\beta_1, \dots, \beta_q\}$ be the set of positive roots for $\mathfrak{g}$ with respect to a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $q = (d-\ell)/2$ and $\ell = \mathrm{rk}(\mathfrak{g})$.

Form now on, we fix a basis

$$\{u^i, e_{\beta_j}, f_{\beta_j} \mid i = 1, \dots, \ell,\ j = 1, \dots, q\}$$

of $\mathfrak{g}$ such that $\{u^i \mid i = 1, \dots, \ell\}$ is an orthonormal basis of $\mathfrak{h}$ with respect to $(\ |\ )$ and $(e_{\beta_i}|f_{\beta_i}) = 1$ for $i = 1, 2, \dots, q$. In particular, $[e_{\beta_i}, f_{\beta_i}] = \beta_i$ for $i = 1, \dots, q$ (see, for example, [Hum72, Prop. 8.3]), where $\mathfrak{h}^*$ and $\mathfrak{h}$ are identified through $(\ |\ )$. One may also assume that $\mathrm{ht}(\beta_i) \leqslant \mathrm{ht}(\beta_j)$ for $i < j$, where $\mathrm{ht}(\beta_i)$ stands for the height of the positive root β_i.

We define the structure constants $c_{\alpha,\beta}$ by

$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta},$$

provided that α, β and $\alpha+\beta$ are in Δ. Our convention is that $e_{-\alpha}$ stands for f_α if $\alpha \in \Delta_+$. If α, β and $\alpha+\beta$ are in Δ_+, then from the equalities,

$$c_{-\alpha,\alpha+\beta} = (f_\beta|[f_\alpha, e_{\alpha+\beta}]) = -(f_\beta|[e_{\alpha+\beta}, f_\alpha]) = -([f_\beta, e_{\alpha+\beta}]|f_\alpha) = -c_{-\beta,\alpha+\beta},$$

we get that

$$c_{-\alpha,\alpha+\beta} = -c_{-\beta,\alpha+\beta}. \tag{2.7}$$

By (2.1), the above basis of $\mathfrak{g}$ induces a basis of $V^k(\mathfrak{g})$ consisted of $\mathbf{1}$ and the elements of the form

$$z = z^{(+)} z^{(-)} z^{(0)} \mathbf{1}, \tag{2.8}$$

with

$$\begin{aligned} z^{(+)} &:= e_{\beta_1}(-1)^{a_{1,1}} \cdots e_{\beta_1}(-r_1)^{a_{1,r_1}} \cdots e_{\beta_q}(-1)^{a_{q,1}} \cdots e_{\beta_q}(-r_q)^{a_{q,r_q}}, \\ z^{(-)} &:= f_{\beta_1}(-1)^{b_{1,1}} \cdots f_{\beta_1}(-s_1)^{b_{1,s_1}} \cdots f_{\beta_q}(-1)^{b_{q,1}} \cdots f_{\beta_q}(-s_q)^{b_{q,s_q}}, \\ z^{(0)} &:= u^1(-1)^{c_{1,1}} \cdots u^1(-t_1)^{c_{1,t_1}} \cdots u^\ell(-1)^{c_{\ell,1}} \cdots u^\ell(-t_\ell)^{c_{\ell,t_\ell}}, \end{aligned}$$

where $r_1, \dots, r_q, s_1, \dots, s_q, , t_1, \dots, t_\ell$ are positive integers, and $a_{l,m}, b_{l,n}, c_{i,j}$, for $l = 1, \dots, q$, $m = 1, \dots, r_l$, $n = 1, \dots, s_l$, $i = 1, \dots, \ell$, $j = 1, \dots, t_i$ are nonnegative integers such that at least one of them is nonzero.

Definition 2.2. — Each element x of $V^k(\mathfrak{g})$ is a linear combination of elements in the above PBW basis, each of them will be called a *PBW monomial* of x.

Definition 2.3. — For a PBW monomial v as in (2.8), we call the integer

$$\mathrm{depth}(v) = \sum_{i=1}^{q} \left(\sum_{j=1}^{r_i} a_{i,j}(j-1) + \sum_{j=1}^{s_i} b_{i,j}(j-1) \right) + \sum_{i=1}^{\ell} \sum_{j=1}^{t_i} c_{i,j}(j-1)$$

the *depth* of v. In other words, a PBW monomial v has depth p means that $v \in F^p V^k(\mathfrak{g})$ and $v \notin F^{p+1} V^k(\mathfrak{g})$. By convention, $\mathrm{depth}(\mathbf{1}) = 0$.

For a PBW monomial v as in (2.8), we call *degree* of v the integer

$$\deg(v)=\sum_{i=1}^{q}\Big(\sum_{j=1}^{r_i}a_{i,j}+\sum_{j=1}^{s_i}b_{i,j}\Big)+\sum_{i=1}^{\ell}\sum_{j=1}^{t_i}c_{i,j},$$

In other words, v has degree p means that $v\in G_pV^k(\mathfrak{g})$ and $v\notin G_{p-1}V^k(\mathfrak{g})$ since the PBW filtration of $V^k(\mathfrak{g})$ coincides with the standard filtration $G_\bullet V^k(\mathfrak{g})$. By convention, $\deg(\mathbf{1})=0$.

Recall that a *singular vector* of a $\mathfrak{g}[t]$-representation M is a vector $m\in M$ such that $e_\alpha(0).m=0$, for all $\alpha\in\Delta_+$, and $f_\theta(1)\cdot m=0$, where θ is the highest positive root of $\mathfrak{g}$. From the identity

$$L_{-1}=\frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell}\sum_{m=0}^{\infty}u^i(-1-m)u^i(m)\\ +\sum_{\alpha\in\Delta_+}\sum_{m=0}^{\infty}(e_\alpha(-1-m)f_\alpha(m)+f_\alpha(-1-m)e_\alpha(m))\Big),$$

we deduce the following easy observation, which will be useful in the proof of the main result.

Lemma 2.4. — *If w is a singular vector of $V^k(\mathfrak{g})$, then*

$$L_{-1}w=\frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell}u^i(-1)u^i(0)+\sum_{\alpha\in\Delta_+}e_\alpha(-1)f_\alpha(0)\Big)w.$$

2.5. Basis of associated graded vertex Poisson algebras. — Note that $\mathrm{gr}V^k(\mathfrak{g})=S(t^{-1}\mathfrak{g}[t^{-1}])$ has a basis consisting of $\mathbf{1}$ and elements of the form (2.8). Similarly to Definition 2.2, we have the following definition.

Definition 2.5. — Each element x of $S(t^{-1}\mathfrak{g}[t^{-1}])$ is a linear combination of elements in the above basis, each of them will be called a *monomial* of x.

As in the case of $V^k(\mathfrak{g})$, the space $S(t^{-1}\mathfrak{g}[t^{-1}])$ has two natural gradations. The first one is induced from the degree of elements as polynomials. We shall write $\deg(v)$ for the degree of a homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to this gradation.

The second one is induced from the Li filtration via the isomorphism $S(t^{-1}\mathfrak{g}[t^{-1}])\cong\mathrm{gr}^FV^k(\mathfrak{g})$. The degree of a homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to the gradation induced by Li filtration will be called the *depth* of v, and will be denoted by $\mathrm{depth}(v)$.

Notice that any element v of the form (2.8) is homogeneous for both gradations. By convention, $\deg(\mathbf{1})=\mathrm{depth}(\mathbf{1})=0$.

As a consequence of (2.5), we get that

$$\deg(x(m)\cdot v)=\deg(v)\quad\text{and}\quad\mathrm{depth}(x(m)\cdot v)=\mathrm{depth}(v)-m,\tag{2.9}$$

for $m\geqslant 0$, $x\in\mathfrak{g}$, and any homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to both gradations.

In the sequel, we will also use the following notation, for v of the form (2.8), viewed either as an element of $V^k(\mathfrak{g})$ or of $S(t^{-1}\mathfrak{g}[t^{-1}])$:

$$\deg_{-1}^{(0)}(v) := \sum_{j=1}^{\ell} c_{j,1}, \tag{2.10}$$

which corresponds to the degree of the element obtained from $v^{(0)}$ by keeping only the terms of depth 0, that is, the terms $u^i(-1)$, $i = 1, \dots, \ell$.

Notice that a nonzero depth-homogeneous element of $S(t^{-1}\mathfrak{g}[t^{-1}])$ has depth 0 if and only if its image in

$$R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g})$$

is nonzero.

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1.

3.1. Strategy. — Let N_k be the maximal graded submodule of $V^k(\mathfrak{g})$, so that $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k$. Our aim is to show that if $V^k(\mathfrak{g})$ is not simple, that is, $N_k \neq \{0\}$, then $X_{L_k(\mathfrak{g})}$ is strictly contained in $\mathfrak{g}^* \cong \mathfrak{g}$, that is, the image $I_k := I_{N_k}$ of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$ is nonzero.

For $k = -h^\vee$, it follows from [FG04] that I_k is the defining ideal of the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$, and so $X_{L_k(\mathfrak{g})} = \mathcal{N}(\mathfrak{g})$ (see [Ara12b] or Section 3.4 below). Hence, there is no loss of generality in assuming that $k + h^\vee \neq 0$.

Henceforth, we suppose that $k + h^\vee \neq 0$ and that $V^k(\mathfrak{g})$ is not simple, that is, $N_k \neq \{0\}$. Then there exists at least one non-trivial (that is, nonzero and different from $\mathbf{1}$) singular vector w in $V^k(\mathfrak{g})$. Theorem 1.2 states that the image of w in I_k is nonzero, and this proves Theorem 1.1. The rest of this section is devoted to the proof of Theorem 1.2.

Let w be a nontrivial singular vector of $V^k(\mathfrak{g})$. One can assume that $w \in F^pV^k(\mathfrak{g}) \smallsetminus F^{p+1}V^k(\mathfrak{g})$ for some $p \in \mathbb{Z}_{\geqslant 0}$.

The image

$$\overline{w} := \sigma(w)$$

of this singular vector in $S(t^{-1}\mathfrak{g}[t^{-1}]) \cong \mathrm{gr}^F V^k(\mathfrak{g})$ is a nontrivial singular vector of $S(t^{-1}\mathfrak{g}[t^{-1}])$. Here $\sigma\colon V^k(\mathfrak{g}) \to \mathrm{gr}^F V^k(\mathfrak{g})$ stands for the principal symbol map. It follows from (2.9) that one can assume that $\overline{w}$ is homogeneous with respect to both gradations on $S(t^{-1}\mathfrak{g}[t^{-1}])$. In particular $\overline{w}$ has depth p. It is enough to show that $p = 0$, that is, $\overline{w}$ has depth zero. Write

$$w = \sum_{j \in J} \lambda_j w^j,$$

where J is a finite index set, λ_j are nonzero scalar for all $j \in J$, and w_j are pairwise distinct PBW monomials of the form (2.8). Let $I \subset J$ be the subset of $i \in J$ such that depth $\overline{w}^i = p = \operatorname{depth} \overline{w}$. Since $w \in F^pV^k(\mathfrak{g}) \smallsetminus F^{p+1}V^k(\mathfrak{g})$, the set I is nonempty. Here, $\overline{w}^i$ stands for the image of w^i in $\operatorname{gr}^F V^k(\mathfrak{g}) \cong S(t^{-1}\mathfrak{g}[t^{-1}])$.

More specifically, for any $j \in I$, write

$$w^j = (w^j)^{(+)}(w^j)^{(-)}(w^j)^{(0)}\mathbf{1}, \tag{3.1}$$

with

$$\begin{aligned}(w^j)^{(+)} &:= e_{\beta_1}(-1)^{a^{(j)}_{1,1}} \cdots e_{\beta_1}(-r_1)^{a^{(j)}_{1,r_1}} \cdots e_{\beta_q}(-1)^{a^{(j)}_{q,1}} \cdots e_{\beta_q}(-r_q)^{a^{(j)}_{q,r_q}}\\ (w^j)^{(-)} &:= f_{\beta_1}(-1)^{b^{(j)}_{1,1}} \cdots f_{\beta_1}(-s_1)^{b^{(j)}_{1,s_1}} \cdots f_{\beta_q}(-1)^{b^{(j)}_{q,1}} \cdots f_{\beta_q}(-s_q)^{b^{(j)}_{q,s_q}},\\ (w^j)^{(0)} &:= u^1(-1)^{c^{(j)}_{1,1}} \cdots u^1(-t_1)^{c^{(j)}_{1,t_1}} \cdots u^\ell(-1)^{c^{(j)}_{\ell,1}} \cdots u^\ell(-t_\ell)^{c^{(j)}_{\ell,t_\ell}},\end{aligned}$$

where $r_1, \dots, r_q, s_1, \dots, s_q, , t_1, \dots, t_\ell$ are nonnegative integers, and $a^{(j)}_{l,m}, b^{(j)}_{l,n}, c^{(j)}_{i,p}$, for $l = 1, \dots, q$, $m = 1, \dots, r_l$, $n = 1, \dots, s_l$, $i = 1, \dots, \ell$, $p = 1, \dots, t_i$, are nonnegative integers such that at least one of them is nonzero.

The integers r_l's, for $l = 1, \dots, q$, are chosen so that at least one of the $a^{(j)}_{l,r_l}$'s is nonzero for j running through J if for some $j \in J$, $(w^j)^{(+)} \neq 1$. Otherwise, we just set $(w^j)^{(+)} := 1$. Similarly are defined the integers s_l's and t_m's, for $l = 1, \dots, q$ and $m = 1, \dots, \ell$. By our assumption, note that for all $i \in I$,

$$\begin{aligned}&\sum_{n=1}^{q}\Big(\sum_{l=1}^{r_n} a^{(i)}_{n,l} + \sum_{l=1}^{s_n} b^{(i)}_{n,l}\Big) + \sum_{n=1}^{\ell}\sum_{l=1}^{t_n} c^{(i)}_{n,l} = \deg(\overline{w})\\ &\sum_{n=1}^{q}\Big(\sum_{l=1}^{r_n} a^{(i)}_{n,l}(l-1) + \sum_{l=1}^{s_n} b^{(i)}_{n,l}(l-1)\Big) + \sum_{n=1}^{\ell}\sum_{l=1}^{t_n} c^{(i)}_{n,l}(l-1) = \operatorname{depth}(\overline{w}) = p.\end{aligned}$$

3.2. A technical lemma. — In this paragraph we remain in the commutative setting, and we only deal with $\overline{w} \in S(t^{-1}\mathfrak{g}[t^{-1}])$ and its monomials $\overline{w}^i$'s, for $i \in I$.

Recall from (2.10) that,

$$\deg^{(0)}_{-1}(w^i) = \sum_{j=1}^{\ell} c^{(i)}_{j,1}$$

for $i \in I$. Set

$$d^{(0)}_{-1}(I) := \max\{\deg^{(0)}_{-1}(w^i) \mid i \in I\},$$

and

$$I^{(0)}_{-1} := \{i \in I \mid \deg^{(0)}_{-1}(w^i) = d^{(0)}_{-1}(I)\}.$$

If $(w^i)^{(0)} = 1$ for all $i \in I$, we just set $d^{(0)}_{-1}(I) = 0$ and then $I^{(0)}_{-1} = I$.

Lemma 3.1. — *If $i \in I^{(0)}_{-1}$, then $(\overline{w}^i)^{(-)} = 1$. In other words, for $i \in I^{(0)}_{-1}$, we have $\overline{w}^i = (\overline{w}^i)^{(0)}(\overline{w}^i)^{(+)}\mathbf{1}$.*

Proof. — Suppose the assertion is false. Then for some positive roots $\beta_{j_1}, \dots, \beta_{j_t} \in \Delta_+$, one can write for any $i \in I^{(0)}_{-1}$,

$$(\overline{w}^i)^{(-)} = f_{\beta_{j_1}}(-1)^{b^{(i)}_{j_1,1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b^{(i)}_{j_1,s_{j_1}}} \cdots f_{\beta_{j_t}}(-1)^{b^{(i)}_{j_t,1}} \cdots f_{\beta_{j_t}}(-s_{j_t})^{b^{(i)}_{j_t,s_{j_t}}}, \tag{3.2}$$

so that for any $l \in \{1, \dots, t\}$,

$$\{b^{(i)}_{j_l, s_{j_l}} \mid i \in I^{(0)}_{-1}\} \neq \{0\}.$$

Set

$$K^{(0)}_{-1} = \{i \in I^{(0)}_{-1} \mid b^{(i)}_{j_1, s_{j_1}} > 0\}.$$

Since $\overline{w}$ is a singular vector of $S(t^{-1}\mathfrak{g}[t^{-1}])$ and $s_{j_1} - 1 \in \mathbb{Z}_{\geqslant 0}$, we have

$$e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = 0.$$

On the other hand, using the action of $\mathfrak{g}[t]$ on $S(t^{-1}\mathfrak{g}[t^{-1}])$ as described by (2.5), we see that

$$0 = e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = \sum_{i \in K^{(0)}_{-1}} \lambda_i b^{(i)}_{j_1, s_{j_1}} v^i + v, \tag{3.3}$$

where for $i \in K^{(0)}_{-1}$,

$$v^i := (\overline{w}^i)^{(0)} \beta_{j_1}(-1) f_{\beta_{j_1}}(-1)^{b^{(i)}_{j_1,1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b^{(i)}_{j_1, s_{j_1}} - 1} \cdots f_{\beta_{j_t}}(-1)^{b^{(i)}_{j_t,1}} \cdots f_{\beta_{j_t}}(-s_{j_t})^{b^{(i)}_{j_t, s_{j_t}}} (w^i)^{(+)} \mathbf{1},$$

and v is a linear combination of monomials x such that

$$\deg^{(0)}_{-1}(x) \leqslant d^{(0)}_{-1}(I).$$

Indeed, for $i \in K^{(0)}_{-1}$, it is clear that

$$e_{\beta_{j_1}}(s_{j_1} - 1) \cdot w^i = b^{(i)}_{j_1, s_{j_1}} v^i + y^i,$$

where y^i is a linear combination of monomials y such that $\deg^{(0)}_{-1}(y) \leqslant d^{(0)}_{-1}(I)$ because $\mathrm{ht}(\beta_{j_1}) \leqslant \mathrm{ht}(\beta_{j_l})$ for all $l \in \{1, \dots, t\}$. Next, for $i \in I^{(0)}_{-1} \smallsetminus K^{(0)}_{-1}$, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}^i$ is a linear combination of monomials z such that $\deg^{(0)}_{-1}(z) \leqslant d^{(0)}_{-1}(I)$ because $b^{(i)}_{j_1, s_{j_1}} = 0$. Finally, for $i \in I \smallsetminus I^{(0)}_{-1}$, we have $\deg^{(0)}_{-1}(\overline{w}^i) < d^{(0)}_{-1}(I)$ and, hence, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}^i$ is a linear combination of monomials z such that $\deg^{(0)}_{-1}(z) \leqslant d^{(0)}_{-1}(I)$ as well.

Now, note that for each $i \in K^{(0)}_{-1}$,

$$\deg^{(0)}_{-1}(v^i) = \deg^{(0)}_{-1}(\overline{w}^i) + 1 = d^{(0)}_{-1}(I) + 1.$$

Hence by (3.3) we get a contradiction because all monomials v^i, for i running through $K^{(0)}_{-1}$, are linearly independent while $\lambda_i b^{(i)}_{j_1, s_{j_1}} \neq 0$, for $i \in K^{(0)}_{-1}$. This concludes the proof of the lemma. □

3.3. Use of Sugawara operators. — Recall that $w = \sum_{j \in J} \lambda_j w^j$. Let $J_1 \subseteq J$ be such that for $i \in J_1$, $(w^i)^{(-)} = 1$. Then by Lemma 3.1,

$$\varnothing \neq I^{(0)}_{-1} \subseteq J_1.$$

So $J_1 \neq \varnothing$. Set

$$d^{(0)}_{-1} := d^{(0)}_{-1}(J_1) = \max\{\deg^{(0)}_{-1}(w^i) \mid i \in J_1\},$$

and
$$J_{-1}^{(0)} := \{i \in J_1 \mid \deg_{-1}^{(0)}(w^i) = d_{-1}^{(0)}\}.$$
Then $d_{-1}^{(0)}(I) \leqslant d_{-1}^{(0)}$. Set
$$d^+ := \max\{\deg(w^i)^{(+)} \mid i \in J_{-1}^{(0)}\}$$
and let
$$J^+ = \{i \in J_{-1}^{(0)} \mid \deg(w^i)^{(+)} = d^+\} \subseteq J_{-1}^{(0)}.$$
Our next aim is to show that for $i \in J^+$, w^i has depth zero, whence $p = 0$ since p is by definition the smallest depth of the w^j's, and so the image of w in $R_{V^k(\mathfrak{g})} = F^0V^k(\mathfrak{g})/F^1V^k(\mathfrak{g})$ is nonzero.

This will be achieved in this paragraph through the use of the Sugawara construction.

Recall that by Lemma 2.4,
$$L_{-1}w = \widetilde{L}_{-1}w$$
since w is a singular vector of $V^k(\mathfrak{g})$, where
$$\widetilde{L}_{-1} := \frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell} u^i(-1)u^i(0) + \sum_{\alpha\in\Delta_+} e_\alpha(-1)f_\alpha(0)\Big).$$

Lemma 3.2. — *Let z be a PBW monomial of the form* (2.8). *Then $\widetilde{L}_{-1}z$ is a linear combination of PBW monomials x satisfying all the following conditions:*

(a) $\deg(x^{(+)}) \leqslant \deg(z^{(+)}) + 1$ *and* $\deg(x^{(0)}) \leqslant \deg(z^{(0)}) + 1$,
(b) *if $z^{(-)} \neq 1$, then $x^{(-)} \neq 1$.*
(c) *if $x^{(-)} = z^{(-)}$, then either* $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, *or $x^{(0)} = z^{(0)}$.*
(d) *if* $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, *then $x^{(-)} = z^{(-)}$ and* $\deg(x^{(+)}) \leqslant \deg(z^{(+)})$.

Proof. — Parts (a)–(c) are easy to see. We only prove (d). Assume that $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$. Either x comes from the term $\sum_{i=1}^{\ell} u^i(-1)u^i(0)z$, or it comes from a term $e_\alpha(-1)f_\alpha(0)z$ for some $\alpha \in \Delta_+$.

If x comes from the term $\sum_{i=1}^{\ell} u^i(-1)u^i(0)z$, then it is obvious that $x^{(-)} = z^{(-)}$ and $x^{(+)} = z^{(+)}$.

Assume that x comes from $e_\alpha(-1)f_\alpha(0)z$ for some $\alpha \in \Delta_+$. We have
$$\begin{aligned}e_\alpha(-1)f_\alpha(0)z = e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(-)}z^{(0)}\mathbf{1} + e_\alpha(-1)z^{(+)}[f_\alpha(0), z^{(-)}]z^{(0)}\mathbf{1}\\ + e_\alpha(-1)z^{(+)}z^{(-)}[f_\alpha(0), z^{(0)}]\mathbf{1}.\end{aligned}$$
Clearly, any PBW monomials x from
$$e_\alpha(-1)z^{(+)}[f_\alpha(0), z^{(-)}]z^{(0)}\mathbf{1} \quad \text{or} \quad e_\alpha(-1)z^{(+)}z^{(-)}[f_\alpha(0), z^{(0)}]\mathbf{1}$$
satisfies that $\deg(x^{(0)}) \leqslant \deg(z^{(0)})$. Then it is enough to consider PBW monomials in
$$e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(-)}z^{(0)}\mathbf{1}.$$
The only possibility for a PBW monomial x in $e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(-)}z^{(0)}\mathbf{1}$ to satisfy $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$ is that it comes from a term $[f_\alpha(0), e_\alpha(-n)] = -\alpha(-n)$ for

some $n \in \mathbb{Z}_{>0}$, where $e_\alpha(-n)$ is a term in $z^{(+)}$. But then, for PBW monomials x in $e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(0)}\mathbf{1}$ such that $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, we have $x^{(-)} = z^{(-)}$ and $\deg(x^{(+)}) \leqslant \deg(z^{(+)})$. $\square$

We now consider the action of $\widetilde{L}_{-1}$ on particular PBW monomials.

Lemma 3.3. — *Let z be a PBW monomial of the form* (2.8) *such that $z^{(-)} = 1$ and $\operatorname{depth}(z^{(+)}) = 0$, that is, either $z^{(+)} = 1$, or for some $j_1, \dots, j_t \in \{1, \dots, q\}$ (with possible repetitions),*

$$z = e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Then $\widetilde{L}_{-1}z$ is a linear combination of PBW monomials y satisfying one of the following conditions:

(1) $y^{(-)} = 1$, $\operatorname{depth}(y^{(+)}) \geqslant 1$, $\deg(y^{(+)}) \leqslant \deg(z^{(+)})$, $y^{(0)} = z^{(0)}$,

(2) $y^{(-)} = 1$, $\operatorname{depth}(y^{(+)}) = 0$, $\deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1$, *and* $\deg(y^{(0)}) > \deg(z^{(0)})$, $\deg^{(0)}_{-1}(y) = \deg^{(0)}_{-1}(z)$,

(3) $y^{(-)} = 1$, $\operatorname{depth}(y^{(+)}) \geqslant 1$, $\deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1$, *and* $\deg^{(0)}_{-1}(y) = \deg^{(0)}_{-1}(z) + 1$,

(4) $y^{(-)} \neq 1$.

Proof. — First, we have

$$\sum_{i=1}^{\ell} u^i(-1)u^i(0)z = \sum_{r=1}^{t} e_{\beta_{j_1}}(-1)\cdots\Big[\sum_{i=1}^{\ell} u^i(-1)u^i(0), e_{\beta_{j_r}}(-1)\Big]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1},$$

and

$$\begin{aligned}\sum_{i=1}^{\ell} u^i(-1)u^i(0), e_{\beta_{j_r}}(-1) &= \sum_{i=1}^{\ell} \big(u^i(-1)[u^i(0), e_{\beta_{j_r}}(-1)] + [u^i(-1), e_{\beta_{j_r}}(-1)]u^i(0)\big)\\ &= \beta_{j_r}(-1)e_{\beta_{j_r}}(-1) + e_{\beta_{j_r}}(-2)\beta_{j_r}(0).\end{aligned}$$

So

$$\begin{aligned}(3.4)\quad \sum_{i=1}^{\ell} u^i(-1)u^i(0)z\\ = \sum_{r=1}^{t} e_{\beta_{j_1}}(-1)\cdots(\beta_{j_r}(-1)e_{\beta_{j_r}}(-1) + e_{\beta_{j_r}}(-2)\beta_{j_r}(0))\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.\end{aligned}$$

Second, we have

$$\begin{aligned}\sum_{\alpha\in\Delta_+} e_\alpha(-1)f_\alpha(0)z = \sum_{\alpha\in\Delta_+}\sum_{r=1}^{t} e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots[f_\alpha(0), e_{\beta_{j_r}}(-1)]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\ + \sum_{\alpha\in\Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)[f_\alpha(0), z^{(0)}]\mathbf{1}.\end{aligned}$$

It is clear that any PBW monomial y in

$$\sum_{\alpha\in\Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)[f_\alpha(0), z^{(0)}]\mathbf{1}$$

satisfies

$$(3.5)\qquad y^{(-)} \neq 1.$$

We now consider

$$u_r := \sum_{\alpha\in\Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots[f_\alpha(0),e_{\beta_{j_r}}(-1)]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1},\ \text{for } 1\leqslant r\leqslant t.$$

– If $\beta_{j_r}=\alpha+\beta$ for some $\alpha,\beta\in\Delta_+$, then there is a partial sum of two terms in u_r:

$$c_{-\alpha,\alpha+\beta}e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots e_\beta(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1} + c_{-\beta,\alpha+\beta}e_\beta(-1)e_{\beta_{j_1}}(-1)\cdots e_\alpha(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Rewriting the above sum to a linear combination of PBW monomials, and noticing that

$$c_{-\alpha,\alpha+\beta}e_\alpha(-1)e_\beta(-1)+c_{-\beta,\alpha+\beta}e_\beta(-1)e_\alpha(-1)=c_{-\alpha,\alpha+\beta}c_{\alpha,\beta}e_{\alpha+\beta}(-2),$$

due to (2.7), we deduce that it is a linear combination of PBW monomials y such that

$$y^{(-)}=z^{(-)}=1,\ y^{(0)}=z^{(0)},\ \operatorname{depth}(y^{(+)})\geqslant 1,\ \deg(y^{(+)})\leqslant\deg(z^{(+)}), \tag{3.6}$$

where $c_{-\alpha,\alpha+\beta},c_{-\beta,\alpha+\beta},c_{\alpha,\beta}\in\mathbb{R}^*$.

– If $\alpha-\beta_{j_r}\in\Delta_+$ for some $\alpha\in\Delta_+$, then there is a term in u_r:

$$c_{-\alpha,\beta_{j_r}}e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)f_{\alpha-\beta_{j_r}}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}. \tag{3.7}$$

It is easy to see that (3.7) is a linear combination of PBW monomials y such that y satisfies one of the following:

$$y^{(-)}=1,\ \operatorname{depth}(y^{(+)})\geqslant 1,\ \deg(y^{(+)})\leqslant\deg(z^{(+)}),\ y^{(0)}=z^{(0)}, \tag{3.8}$$

$$\begin{gathered}y^{(-)}=1,\ \operatorname{depth}(y^{(+)})=0,\ \deg(y^{(+)})\leqslant\deg(z^{(+)})-1,\\ \deg(y^{(0)})>\deg(z^{(0)}),\ \deg^{(0)}_{-1}(y)=\deg^{(0)}_{-1}(z),\end{gathered} \tag{3.9}$$

$$y^{(-)}\neq 1. \tag{3.10}$$

Notice also that with $\alpha=\beta_{j_r}$, there is a term in u_r:

$$-e_{\beta_{j_r}}(-1)e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Together with (3.4), we see that

$$\begin{aligned}\sum_{i=1}^{\ell}u^i(-1)u^i(0)z+\sum_{r=1}^{t}e_{\beta_{j_r}}(-1)e_{\beta_{j_1}}(-1)\cdots[f_{\beta_{j_r}}(0),e_{\beta_{j_r}}(-1)]\cdots\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\ =\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots(\beta_{j_r}(-1)e_{\beta_{j_r}}(-1)+e_{\beta_{j_r}}(-2)\beta_{j_r}(0))\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\ -\sum_{r=1}^{t}\sum_{s=1}^{r-1}e_{\beta_{j_1}}(-1)\cdots[e_{\beta_{j_r}}(-1),e_{\beta_{j_s}}(-1)]\\ \cdots e_{\beta_{j_{r-1}}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\ -\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)e_{\beta_{j_r}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\end{aligned}$$

is a linear combination of PBW monomials y satisfying one of the following:

$$y^{(-)} = 1,\ \operatorname{depth}(y^{(+)}) \geqslant 1, \deg(y^{(+)}) \leqslant \deg(z^{(+)}),\ y^{(0)} = z^{(0)}, \tag{3.11}$$

$$\begin{aligned} & y^{(-)} = 1,\ \operatorname{depth}(y^{(+)}) \geqslant 1, \deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1, \\ & \deg_{-1}^{(0)}(y) = \deg_{-1}^{(0)}(z) + 1. \end{aligned} \tag{3.12}$$

Then the lemma follows from (3.5), (3.6), (3.8)–(3.12). □

Lemma 3.4. — *Let z be a PBW monomial of the form* (2.8) *such that $z^{(-)} = 1$. Then*

$$\widetilde{L}_{-1} z = c z^{(+)} (\gamma - \sum_{j=1}^{q} a_{j,1} \beta_j)(-1) z^{(0)} + y^1,$$

where c is a nonzero constant, $\gamma = \sum_{j=1}^{q} \sum_{s=1}^{r_j} a_{j,s} \beta_j$, and y^1 is a linear combination of PBW monomials y such that

$$\deg_{-1}^{(0)}(y) = \deg_{-1}^{(0)}(z) + 1,\ \deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1,$$

or

$$\deg_{-1}^{(0)}(y) \leqslant \deg_{-1}^{(0)}(z).$$

Proof. — Since the proof is similar to that of Lemma 3.3, we left the verification to the reader. □

Lemma 3.5. — *For $i \in J^+$, we have that* $\operatorname{depth}((w^i)^{(+)}) = 0$.

Proof. — First we have

$$w = \sum_{j \in J^+} \lambda_j w^j + \sum_{j \in J_{-1}^{(0)} \smallsetminus J^+} \lambda_j w^j + \sum_{j \in J_1 \smallsetminus J_{-1}^{(0)}} \lambda_j w^j + \sum_{j \in J \smallsetminus J_1} \lambda_j w^j.$$

Then by Lemma 3.2(b) and Lemma 3.4, we have

$$\begin{aligned} (k+h^\vee)\widetilde{L}_{-1} w = \sum_{i \in J^+} (w^i)^{(+)} \Big(\gamma_i - \sum_{j=1}^{q} a_{j,1}^{(i)} \beta_j\Big)(-1)(w^i)^{(0)} \\ + \sum_{i \in J_1 \smallsetminus J^+} (w^i)^{(+)} \Big(\gamma_i - \sum_{j=1}^{q} a_{j1}^{(i)} \beta_j\Big)(-1)(w^i)^{(0)} + y^1, \end{aligned}$$

where $\gamma_i = \sum_{j=1}^{q} \sum_{s=1}^{r_j^{(i)}} a_{j,s}^{(i)} \beta_i$, for $i \in J_1$, and y^1 is a linear combination of PBW monomials y satisfying one of the following conditions:

$$\begin{aligned} & \deg_{-1}^{(0)}(y) = d_{-1}^{(0)} + 1,\ \deg(y^{(+)}) \leqslant d^+ - 1, \\ & \deg_{-1}^{(0)}(y) \leqslant d_{-1}^{(0)}, \\ & y^{(-)} \neq 1. \end{aligned}$$

On the other hand, by Lemma 2.4

$$L_{-1} w = \widetilde{L}_{-1} w.$$

By Lemma 2.1, there is no PBW monomial y in $L_{-1}w$ such that $\deg(y^{(+)}) = d^+$, $y^{(-)} = 1$, and $\deg^{(0)}_{-1}(y) = d^{(0)}_{-1} + 1$. Then we deduce that

$$\sum_{i\in J^+} (w^i)^{(+)}\bigg(\gamma_i - \sum_{j=1}^{q} a^{(i)}_{j,1}\beta_j\bigg)(-1)(w^i)^{(0)} = 0,$$

which means that $(\gamma_i - \sum_{j=1}^{q} a^{(i)}_{j,1}\beta_j) = 0$, for $i \in J^+$, that is, $\operatorname{depth}((w^i)^{(+)}) = 0$. $\square$

As explained at the beginning of §3.3, Theorem 1.1 will be a consequence of the following lemma.

Lemma 3.6. — *For each $i \in J^+$, we have* $\operatorname{depth}(w^i) = 0$.

Proof. — By definition, for $i \in J^+$, $(w^i)^{(0)} = 1$. Moreover, by Lemma 3.5, $\operatorname{depth}((w^i)^{(+)}) = 0$. Hence it suffices to prove that for $i \in J^+$,

$$(w^i)^{(0)} = u^1(-1)^{c^{(i)}_{1,1}} \cdots u^\ell(-1)^{c^{(i)}_{\ell,1}}.$$

Suppose the contrary. Then there exists $i \in J^+$ such that

$$\begin{aligned} w^i = e_{\beta_1}(-1)^{a^{(i)}_{1,1}} \cdots e_{\beta_q}(-1)^{a^{(i)}_{q,1}} u^1(-1)^{c^{(i)}_{1,1}} \cdots u^1(-m_1)^{c^{(i)}_{1,m_1}} \\ \cdots u^\ell(-1)^{c^{(i)}_{\ell,1}} \cdots u^\ell(-m_\ell)^{c^{(i)}_{\ell,m_\ell}} \mathbf{1}, \end{aligned}$$

with at least one of the m_j's, for $j = 1, \ldots, \ell$, strictly greater than 1 and $c^{(i)}_{j,m_j} \neq 0$ for such a j. Without loss of generality, one may assume that $1 \in J^+$, that

$$m_1 = \max\{m_j \mid j = 1, \ldots, \ell\} \quad \text{and} \quad 0 \neq c^{(1)}_{1,m_1} \geqslant c^{(i)}_{1,m_1}, \text{ for } i \in J^+.$$

Writing $L_{-1}w$ as

$$L_{-1}w = \sum_{i\in J^+} L_{-1}w^i + \sum_{i\in J^{(0)}_{-1}\smallsetminus J^+} L_{-1}w^i + \sum_{i\in J_1\smallsetminus J^{(0)}_{-1}} L_{-1}w^i + \sum_{i\in J\smallsetminus J_1} L_{-1}w^i,$$

we see by Lemma 2.1 that

$$L_{-1}w = \lambda_1 m_1 c^{(1)}_{1,m_1} v^1 + \sum_{i\in J^+, i\neq 1} \lambda_i m_1 c^{(i)}_{1,m_1} v^i + v + v', \tag{3.13}$$

where for $i \in J^+$, v^i is the PBW monomial defined by:

$$(v^i)^{(-)} = (w^i)^{(-)} = 1, \tag{3.14}$$

$$(v^i)^{(+)} = (w^i)^{(+)} = e_{\beta_1}(-1)^{a^{(i)}_{1,1}} \cdots e_{\beta_q}(-1)^{a^{(i)}_{q,1}}, \tag{3.15}$$

$$(v^i)^{(0)} = u^1(-1)^{c^{(i)}_{1,1}} \cdots u^1(-m_1)^{c^{(i)}_{1,m_1}-1} u^1(-m_1-1) \cdots u^\ell(-m_\ell)^{c^{(i)}_{\ell,m_\ell}}, \tag{3.16}$$

and so, by definition of $J^+ \subset J^{(0)}_{-1}$,

$$\deg^{(0)}_{-1}(v^i) = d^{(0)}_{-1}, \tag{3.17}$$

v is a linear combination of PBW monomials x such that

$$x^{(0)} = u^1(-1)^{c^{(x)}_{1,1}} \cdots u^1(-n^{(x)}_1)^{c^{(x)}_{1,n^{(x)}_1}} \cdots u^\ell(-1)^{c^{(x)}_{\ell,1}} \cdots u^\ell(-n^{(x)}_\ell)^{c^{(x)}_{\ell,n^{(x)}_\ell}}$$

and

$$\text{either } n_1^{(x)} \leqslant m_1, \quad \text{or } \deg(x^{(+)}) \leqslant d^+ - 1, \quad \text{or } \deg_{-1}^{(0)}(x) \leqslant d_{-1}^{(0)} - 1,$$

and v' is a linear combination of PBW monomials x such that $x^{(-)} \neq 1$. Note that the assumption that $m_1 \geqslant 2$ makes sure that (3.17) holds, and that $\operatorname{depth}(v^i) = \operatorname{depth}(w^i) + 1$ for all $i \in J^+$.

On the other hand, by Lemma 2.4,

$$L_{-1}w = \widetilde{L}_{-1}w,$$

since w is a singular vector of $V^k(\mathfrak{g})$. Hence v^1 must be a PBW monomial of $\widetilde{L}_{-1}w$. Our strategy to obtain the expected contradiction is to show that there is no PBW monomial v^1 in $\widetilde{L}_{-1}w^i$ for each $i \in J$.

– Assume that $i \in J^+$, and suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. First of all, $\deg((w^i)^{(+)}) = d^+$ because $i \in J^+$. Moreover, by the definition of J_1 and Lemma 3.5, we have $(w^i)^{(-)} = 1$ and $\operatorname{depth}((w^i)^{(+)}) = 0$. Hence by Lemma 3.3(2),

$$\deg((v^1)^{(+)}) < \deg((w^i)^{(+)}) = d^+$$

because $(v^1)^{(-)} = 1$ and $\operatorname{depth}((v^1)^{(+)}) = 0$ by (3.14) and (3.15). But $d^+ = \deg((v^1)^{(+)})$ by (3.15), whence a contradiction.

– Assume that $i \in J_{-1}^{(0)} \smallsetminus J^+$. By the definition of J^+ and (3.15),

$$\deg((w^i)^{(+)}) < d^+ = \deg((v^1)^{(+)}). \tag{3.18}$$

Suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. Then

$$(w^i)^{(-)} = 1 = (v^1)^{(-)} \tag{3.19}$$

by Lemma 3.1 since $i \in J_{-1}^{(0)}$. The last equality follows from (3.14). Then by Lemma 3.2(c), either $\deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1$, or $(v^1)^{(0)} = (w^i)^{(0)}$. But it is impossible that $\deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1$, by (d) of Lemma 3.2 because $\deg((v^1)^{(+)}) > \deg((w^i)^{(+)})$. Therefore,

$$(v^1)^{(0)} = (w^i)^{(0)}.$$

Computing $\widetilde{L}_{-1}w^i$, we deduce from

$$(v^1)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(1)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(1)}},$$

that

$$(w^i)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(j)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(j)}}.$$

Since $(v^1)^{(-)} = (w^i)^{(-)} = 1$, it results from Lemma 3.3 that $\deg((v^1)^{(+)}) \leqslant \deg((w^i)^{(+)})$, which contradicts (3.18).

– Assume that $i \in J_1 \smallsetminus J_{-1}^{(0)}$. Then

$$\deg_{-1}^{(0)}(w^i) < d_{-1}^{(0)} = \deg_{-1}^{(0)}(v^1) \tag{3.20}$$

by (3.17). Suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. By Lemma 3.2(b) and (c),

$$(w^i)^{(-)} = 1, \quad \deg_{-1}^{(0)}(v^1) = \deg_{-1}^{(0)}(w^i) + 1, \tag{3.21}$$

because $(v^1)^{(-)} = 1$ by (3.14). Remember that

$$(v^1)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(1)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(1)}}. \tag{3.22}$$

Computing $\widetilde{L}_{-1}w^i$, we deduce that

$$(w^i)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(i)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(i)}}.$$

Since $v^{(-)} = 1$ and $\deg_{-1}^{(0)}(v^1) = \deg_{-1}^{(0)}(w^i) + 1$, it results from Lemma 3.3(3) that $\operatorname{depth}((v^1)^{(+)}) \geqslant 1$, which contradicts (3.22).

– Finally, if $j \in J \smallsetminus J_1$, then by Lemma 3.2(b), any PBW monomial y in $\widetilde{L}_{-1}w^j$ satisfies that $y^{(-)} \neq 1$. So v^1 cannot be a PBW monomial in $\widetilde{L}_{-1}w^j$.

This concludes the proof of the lemma. $\square$

As already explained, Lemma 3.6 implies that w has zero depth and so its image in $R_{V^k(\mathfrak{g})}$ is nonzero, achieving the proof of Theorem 1.1.

3.4. Remarks. — The statement of Theorem 1.2 is not true at the critical level. Also, it is not true that the depth of a depth-homogeneous singular vector of $S(\mathfrak{g}[t^{-1}]t^{-1})$ is always zero. Indeed, the $\mathfrak{g}[\![t]\!]$-module $S(\mathfrak{g}[t^{-1}]t^{-1})$ can be naturally identified with $\mathbb{C}[J_\infty\mathfrak{g}^*]$, where $J_\infty X$ is the arc space of X, and so $S(\mathfrak{g}[t^{-1}]t^{-1})^{\mathfrak{g}[t]} \cong \mathbb{C}[J_\infty\mathfrak{g}^*]^{J_\infty G}$. It is known [RT92, BD, EF01] that

$$\mathbb{C}[J_\infty\mathfrak{g}^*]^{J_\infty G} \cong \mathbb{C}[J_\infty(\mathfrak{g}^*/\!/G)].$$

This means that the invariant ring is a polynomial ring with infinitely many variables $\partial^j p_i$, $i = 1, \ldots, \ell$, $j \geqslant 0$, where $p_1, \ldots, p_\ell$ is a set of homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ considered as elements of $S(\mathfrak{g}[t^{-1}]t^{-1})$ via the embedding $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}[t^{-1}]t^{-1})$, $\mathfrak{g} \ni x \mapsto x(-1)$. We have $\operatorname{depth}(\partial^j p_i) = j$ although each $\partial^j p_i$ is a singular vector of $S(\mathfrak{g}[t^{-1}]t^{-1})$.

For $k = -h^\vee$, the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by Feigin-Frenkel center ([FG04]). Hence [FF92, Fre05], $\operatorname{gr} N_k$ is exactly the argumentation ideal of $S(\mathfrak{g}[t^{-1}]t^{-1})^{\mathfrak{g}[t]}$. Therefore, the above argument shows that the statement of Theorem 1.2 is false at the critical level.

4. W-algebras and proof of Theorem 1.3

Let f be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov theorem, it embeds into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Recall that the Slodowy slice $\mathscr{S}_f$ is the affine space $f + \mathfrak{g}^e$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. It has a natural Poisson structure induced from that of $\mathfrak{g}^*$ ([GG02]).

The embedding $\operatorname{span}_{\mathbb{C}}\{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $\mathrm{SL}_2 \to G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho\colon \mathbb{C}^* \to G$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\widetilde{\rho}(t)x := t^2\rho(t)(x).$$

We have $\widetilde{\rho}(t)f = f$, and the $\mathbb{C}^*$-action of $\widetilde{\rho}$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is, for all $x \in \mathfrak{g}^e$,

$$\lim_{t\to 0} \widetilde{\rho}(t)(f + x) = f.$$

The following proposition is well-known. Since its proof is short, we give below the argument for the convenience of the reader.

Proposition 4.1 ([Slo80, Pre02, CM16]). — *The morphism*

$$\theta_f : G \times \mathscr{S}_f \longrightarrow \mathfrak{g}, \quad (g, x) \longmapsto g \cdot x$$

is smooth onto a dense open subset of $\mathfrak{g}^*$.

Proof. — Since $\mathfrak{g} = \mathfrak{g}^e + [f, \mathfrak{g}]$, the map θ_f is a submersion at $(1_G, f)$. Therefore, θ_f is a submersion at all points of $G \times (f + \mathfrak{g}^e)$ because it is G-equivariant for the left multiplication in G, and

$$\lim_{t\to\infty} \rho(t) \cdot x = f$$

for all x in $f+\mathfrak{g}^e$. So, by [Har77, Ch. III, Prop. 10.4], the map θ_f is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G \cdot f$. □

As in the introduction, let $\mathscr{W}^k(\mathfrak{g}, f)$ be the affine W-algebra associated with a nilpotent element f of $\mathfrak{g}$ defined by the generalized quantized Drinfeld-Sokolov reduction:

$$\mathscr{W}^k(\mathfrak{g}, f) = H^0_{\mathrm{DS},f}(V^k(\mathfrak{g})).$$

Here, $H^\bullet_{\mathrm{DS},f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f \in \mathcal{N}(\mathfrak{g})$ with coefficients in a $V^k(\mathfrak{g})$-module M. Recall that we have [DSK06, Ara15a] a natural isomorphism $R_{\mathscr{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$X_{\mathscr{W}^k(\mathfrak{g},f)} = \mathscr{S}_f.$$

We write $\mathscr{W}_k(\mathfrak{g}, f)$ for the unique simple (graded) quotient of $\mathscr{W}^k(\mathfrak{g}, f)$. Then $X_{\mathscr{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant Poisson subvariety of the Slodowy slice $\mathscr{S}_f$.

Let $\mathscr{O}_k$ be the category $\mathscr{O}$ of $\widehat{\mathfrak{g}}$ at level k. We have a functor

$$\mathscr{O}_k \longrightarrow \mathscr{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \longmapsto H^0_{\mathrm{DS},f}(M),$$

where $\mathscr{W}^k(\mathfrak{g}, f)$-Mod denotes the category of $\mathscr{W}^k(\mathfrak{g}, f)$-modules.

The full subcategory of $\mathscr{O}_k$ consisting of objects M on which $\mathfrak{g}$ acts locally finitely will be denoted by KL_k. Note that both $V^k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ are objects of KL_k.

Theorem 4.2 ([Ara15a])

(1) $H^i_{\mathrm{DS},f}(M) = 0$ *for all* $i \neq 0$, $M \in \mathrm{KL}_k$. *In particular, the functor*

$$\mathrm{KL}_k \longrightarrow \mathscr{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \longmapsto H^0_{\mathrm{DS},f}(M),$$

is exact.

(2) *For any quotient V of $V^k(\mathfrak{g})$,*

$$X_{H^0_{\mathrm{DS},f}(V)} = X_V \cap \mathscr{S}_f.$$

In particular $H^0_{\mathrm{DS},f}(V) \neq 0$ if and only if $\overline{G \cdot f} \subset X_V$.

By Theorem 4.2(1), $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ is a quotient vertex algebra of $\mathscr{W}^k(\mathfrak{g}, f)$ if it is nonzero. Conjecturally [KRW03, KW08], we have

$$\mathscr{W}_k(\mathfrak{g}, f) \cong H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \quad \text{provided that } H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \neq 0.$$

(This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

Proof of Theorem 1.3. — The directions (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. Let us show that (3) implies (1). So suppose that $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = \mathscr{S}_f$. By Theorem 1.1, it is enough to show that $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. Assume the contrary. Then $X_{L_k(\mathfrak{g})}$ is contained in a proper G-invariant closed subset of $\mathfrak{g}$. On the other hand, by Theorem 4.2 and our hypothesis, we have

$$\mathscr{S}_f = X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = X_{L_k(\mathfrak{g})} \cap \mathscr{S}_f.$$

Hence, $\mathscr{S}_f$ must be contained in a proper G-invariant closed subset of $\mathfrak{g}$. But this contradicts Proposition 4.1. The proof of the theorem is completed. $\square$

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Tomoyuki Arakawa, Research Institute for Mathematical Sciences, Kyoto University
Kyoto, 606-8502, Japan
E-mail : arakawa@kurims.kyoto-u.ac.jp
Url : http://www.kurims.kyoto-u.ac.jp/~arakawa/

Cuipo Jiang, School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai, 200240, China
E-mail : cpjiang@sjtu.edu.cn
Url : http://math.sjtu.edu.cn/faculty/cpjiang/

Anne Moreau, Faculté des Sciences d'Orsay, Université Paris-Saclay
91405 Orsay, France
E-mail : anne.moreau@universite-paris-saclay.fr
Url : https://www.imo.universite-paris-saclay.fr/~moreau/