L'accès aux articles de la revue « Journal de l'École polytechnique — Mathématiques »
(http://jep.cedram.org/), implique l'accord avec les conditions générales d'utilisation
(http://jep.cedram.org/legal/).

ENERGY RELEASE RATE FOR NON-SMOOTH CRACKS IN PLANAR ELASTICITY

by Jean-François Babadjian, Antonin Chambolle
& Antoine Lemenant

Abstract. — This paper is devoted to the characterization of the energy release rate of a crack which is merely closed, connected, and with (length) density 1/2 at the tip, without further regularity assumptions. First, the blow-up limit of the displacement is analyzed, and the convergence to a (known) positively 1/2-homogenous function in the cracked plane is established. Then, the energy release rate, which is the derivative of the elastic energy with respect to an infinitesimal additional crack increment, is obtained as the solution of a variational problem.

Résumé (Taux de restitution d'énergie pour des fissures non régulières en élasticité plane)

Cet article est consacré à l'étude du taux de restitution d'énergie associé à une fissure fermée, connexe et de densité (de longueur) 1/2 en pointe de fissure, sans autre hypothèse de régularité. Tout d'abord, la limite de blow-up du déplacement à la pointe est analysée, ainsi que la convergence vers une certaine fonction, positivement 1/2-homogène, explicite. Le taux de restitution d'énergie, qui est la dérivée de l'énergie élastique par rapport à un incrément infinitésimal de fissure, est alors obtenu comme solution d'un problème variationnel.

Contents

1. Introduction

Griffith theory [18] is a model explaining the quasi-static crack growth in elastic bodies under the assumption that the crack set is preassigned. In a two-dimensional setting, let us denote by $\Omega \subset \mathbb{R}^2$ the reference configuration of a linearly elastic body allowing for cracks inside $\widehat{\Gamma}$. To fix the ideas, provided the evolution is sufficiently smooth, that $\widehat{\Gamma}$ is a simple curve, and that the evolution is growing only in one direction, then the crack is completely characterized by the position of its tip, and thus by its arc length. Denoting by $\Gamma(\ell)$ the crack of length ℓ inside $\widehat{\Gamma}$, the elastic energy associated to a given kinematically admissible displacement $u : \Omega \setminus \Gamma(\ell) \to \mathbb{R}^2$ satisfying $u = \psi(t)$ on $\partial\Omega \setminus \Gamma(\ell)$, is given by

$$E(t; u, \ell) := \frac{1}{2} \int_{\Omega \setminus \Gamma(\ell)} \mathbb{C}e(u) : e(u)\, dx,$$

where $\mathbb{C}$ is the fourth order Hooke's tensor, $e(u)$ is the symmetrized gradient of u, and $\psi(t) : \partial\Omega \to \mathbb{R}^2$ is a prescribed boundary datum depending on time, which is the driving mechanism of the process. If the evolution is slow enough, it is reasonable to neglect inertia and viscous effects so that the quasi-static assumption becomes relevant: at each time t, the body is in elastic equilibrium. It enables one to define the potential energy as

$$\mathscr{P}(t, \ell) := E(t; u(t, \ell), \ell) = \min E(t; \cdot, \ell),$$

where the minimum is computed over all kinematically admissible displacements at time t. Therefore, given a cracking state, the quasi-static assumption permits to find the displacement. In order to get the crack itself (or equivalently its length), Griffith introduced a criterion whose fundamental ingredient is the *energy release rate*. It is defined as the variation of potential energy along an infinitesimal crack increment, or in other words, the quantity of released potential energy with respect to a small crack increment. More precisely, it is given by

$$G(t, \ell) := -\frac{\partial \mathscr{P}}{\partial \ell}(t, \ell)$$

provided the previous expression makes sense. From a thermodynamical point of view, the energy release rate is nothing but the thermodynamic force associated to the crack length (the natural internal variable modeling the dissipative effect of fracture). Griffith's criterion is summarized into the three following items: for each $t > 0$

(i) $G(t, \ell(t)) \leqslant G_c$, where $G_c > 0$ is a characteristic material constant referred to as the toughness of the body;

(ii) $\dot{\ell}(t) \geqslant 0$;

(iii) $\big(G(t, \ell(t)) - G_c\big)\dot{\ell}(t) = 0$.

Item (i) is a threshold criterion which stipulates that the energy release rate cannot exceed the critical value G_c. Item (ii) is an irreversibility criterion which ensures that the crack can only grow. The third and last item is a compatibility condition

between (i) and (ii): it states that a crack will grow if and only if the energy release rate constraint is saturated.

In [17] (see also [3]), it has been observed that Griffith's criterion is nothing but the necessary first order optimality condition of a variational model. More precisely, if for every $t > 0$, $(u(t), \ell(t))$ satisfies:

(i) *Unilateral minimality:* for any $\widehat{\ell} \geqslant \ell(t)$, and any $v : \Omega \smallsetminus \Gamma(\widehat{\ell}) \to \mathbb{R}^2$ satisfying $v = \psi(t)$ on $\partial\Omega \smallsetminus \Gamma(\widehat{\ell})$, then

$$\mathscr{E}(t) := \frac{1}{2} \int_{\Omega \smallsetminus \Gamma(\ell(t))} \mathbb{C}e(u(t)) : e(u(t)) \, dx + G_c \, \ell(t) \leqslant \frac{1}{2} \int_{\Omega \smallsetminus \Gamma(\widehat{\ell})} \mathbb{C}e(v) : e(v) \, dx + G_c \, \widehat{\ell};$$

(ii) *Irreversibility:* $\dot{\ell}(t) \geqslant 0$;

(iii) *Energy balance:*

$$\dot{\mathscr{E}}(t) = \int_{\partial\Omega \smallsetminus \Gamma(\ell(t))} (\mathbb{C}e(u(t))\nu) \cdot \dot{\psi}(t) \, d\mathscr{H}^1,$$

then $(u(t), \ell(t))$ is a solution of Griffith's model. In the previous expression, $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure. The energy balance is nothing but a reformulation of the second law of thermodynamics which asserts the non-negativity of the mechanical dissipation. It states that the temporal variation of the total energy (the sum of the elastic and surface energies) is compensated by the power of external forces, which in our case reduces to the stress $(\mathbb{C}e(u(t))\nu)$ acting on $\partial\Omega \smallsetminus \Gamma(\ell(t))$ and generated by the boundary displacement $\psi(t)$. This new formulation relies on the constrained minimization of the total energy of Mumford-Shah type

$$\mathscr{E}(u, \Gamma) := \frac{1}{2} \int_{\Omega \smallsetminus \Gamma} \mathbb{C}e(u) : e(u) \, dx + G_c \mathscr{H}^1(\Gamma)$$

which puts in competition a bulk (elastic) energy and a surface (Griffith) energy. One of the main interests is that it makes it possible to get rid of the assumption of the a priori knowledge of the crack path. Following [17], a quasi-static evolution is defined as a mapping $t \mapsto (u(t), \Gamma(t))$ satisfying

(i) *Unilateral minimality:* for any $\overline{\Omega} \supset \widehat{\Gamma} \supset \Gamma(t)$, and any $v : \Omega \smallsetminus \widehat{\Gamma} \to \mathbb{R}^2$ satisfying $v = \psi(t)$ on $\partial\Omega \smallsetminus \widehat{\Gamma}$, then

$$\mathscr{E}(u(t), \Gamma(t)) \leqslant \mathscr{E}(v, \widehat{\Gamma});$$

(ii) *Irreversibility:* $\Gamma(s) \subset \Gamma(t)$ for every $s \leqslant t$;

(iii) *Energy balance:*

$$\mathscr{E}(u(t), \Gamma(t)) = \mathscr{E}(u(0), \Gamma(0)) + \int_0^t \int_{\Omega \smallsetminus \Gamma(s)} \mathbb{C}e(u(s)) : e(\dot{\psi}(s)) \, dx \, ds.$$

An existence result for this model has been given in [5] (see also [13, 16, 12] in other contexts) for cracks belonging to the class of compact and connected subsets of $\overline{\Omega}$. The main reason of this assumption was to ensure the lower semicontinuity of the Mumford-Shah type functional $(u, \Gamma) \mapsto \mathscr{E}(u, \Gamma)$ with respect to a reasonable notion of convergence. The lower semicontinuity of the surface energy with respect to the Hausdorff convergence of cracks is a consequence of Gołab's Theorem (see [15]), while

the continuity of the bulk energy is a consequence of continuity results of the Neumann problem with respect to the Hausdorff convergence of the boundary (see [4, 6]) together with a density result [5]. In any cases, all these results only hold in dimension 2 and in the class of compact and connected sets.

If one is interested into fine qualitative results such as crack initiation (see [8]) of kinking (see [7]) it becomes necessary to understand the nature of the singularity at the crack tip. Therefore one should be able to make rigorous a suitable notion energy release rate. The first proof of the differentiable character of the potential energy with respect to the crack length has been given in [14] (see also [22, 28, 27]). The generalized variational setting described above, a mathematical justification of the notions of energy release rate for any incremental crack attached to a given initial crack has been in [7] in the case where the crack is straight in a small neighborhood of its tip. In the footstep of that work, we attempt here weaken the regularity assumption on the initial crack, which is merely closed, connected, with density $1/2$ at the origin (that imply to blow up as a segment at the origin, up to rotations).

1.1. Main Results. — Our main results are contained in Theorem 6.4 and Theorem 7.1 respectively in Section 6 and Section 7.

1.1.1. *First Result.* — The first main result Theorem 6.4 is a purely P.D.E. result. We analyze the blow-up limit of the optimal displacement at the tip of the given initial crack. We prove that for some suitable subsequence, the blow-up limit converges to the classical crack-tip function in the complement of a half-line, i.e., of the form

$$(1.1) \qquad\qquad \kappa_1 \phi_1 + \kappa_2 \phi_2,$$

for some constants κ_1 and $\kappa_2 \in \mathbb{R}$, while ϕ_1 and ϕ_2 are positively $1/2$-homogenous functions which are explicitly given by (6.15) and (6.16) below.

This part can be seen as a partial generalization in planar elasticity of what was previously done in the anti-plane case [9]. Mathematically speaking, the corresponding function to be studied is now a vectorial function satisfying a Lamé type system, instead of being simply a scalar valued harmonic function. One of the key obstacles in the vectorial case is that no monotonicity property is known for such a problem, which leads to a slightly weaker result than in the scalar case: the convergence of the blow-up sequence only holds up to subsequences, and nothing is known for the whole sequence. Consequently, the constants κ_1 and κ_2 in (1.1) a priori depend on this particular subsequence. As a matter of fact, this prevents us to define properly the stress intensity factor analogously to what was proposed in [9]. On the other hand, we believe that the techniques employed in the proof and the results on their own are already interesting. In addition, the absence of monotonicity is not the only difference with the scalar case, which led us to find a new proof relying on a duality approach via the so-called Airy function in order to bypass some technical problems.

Another substantial difference with the scalar case appears while studying homogeneous solutions of the planar Lamé system in the complement of a half-line, which is crucial in the understanding of blow-up solutions at the crack tip. For harmonic

functions it is rather easy to decompose any solutions as a sum of spherical-harmonics directly by writing the operator in polar coordinates, and identify the degree of homogeneity of each term with the corresponding eigenvalue of the Dirichlet-Laplace-Beltrami operator on the circle minus a point. For the Lamé system, or alternatively for the biharmonic equation, a similar naive approach cannot work. The appropriate eigenvalue problem on the circle have a more complicate nature, and analogous results rely on an abstract theory developed first by Kondrat'ev which rests on pencil operators, weighted Sobolev spaces, the Fredholm alternative, and calculus of residues. We used this technology in the proof of Proposition 6.3 for which we could not find a more elementary argument.

1.1.2. *Second result.* — The second main result Theorem 7.1 concerns the energy release rate of an incremental crack Γ, which is roughly speaking the derivative of the elastic energy with respect to the crack increment (see (7.1) for the precise definition). We prove that the value of this limit is realized as an explicit minimization problem in the cracked-plane $\mathbb{R}^2 \smallsetminus \big((-\infty, 0] \times \{0\}\big)$. One can find a similar statement in [7, Th. 3.1], but with the additional assumption that the initial crack is a line segment close to the origin. We remove here this hypothesis, establishing the same result for any initial crack which is closed, connected and admits a line segment as blow-up limit at the origin. The starting point for this generalization is the knowledge of the blow-up limit at the origin for displacement associated to a general initial crack, namely our first result Theorem 6.4. Since this result holds only up to subsequences, the same restriction appears in the statement of Theorem 6.4 as well.

Therewith, it should be mentioned that Theorem 7.1 is new even for the scalar case, for which the conclusion is even more accurate. Indeed in this case, the monotonicity formula of [9] ensures that the convergence holds for the whole sequence and not only for a subsequence.

The paper is organized as follows: after introducing the main notation in Section 2, we describe precisely the mechanical model in Section 3. Section 4 is devoted to establish technical results related to the existence of the harmonic conjugate and the Airy function associated to the displacement in a neighborhood of the crack tip. In Section 5, we prove lower and upper bounds of the energy release rate. The blow-up analysis of the displacement around the crack tip is the object of Section 6. Section 7 is devoted to give a formula for the energy release rate as a global minimization problem. Finally, we shortly review Kondrat'ev theory of elliptic regularity vs singularity inside corner domains in an appendix.

2. Mathematical preliminaries

2.1. General notation. — The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathscr{L}^n$, and the k-dimensional Hausdorff measure by $\mathscr{H}^k$. If E is a measurable set, we will sometimes write $|E|$ instead of $\mathscr{L}^n(E)$. If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open ball of center x and radius ϱ is denoted by $B_\varrho(x)$. If $x = 0$, we simply write B_ϱ instead of $B_\varrho(0)$.

We write $\mathbb{M}^{n \times n}$ for the set of real $n \times n$ matrices, and $\mathbb{M}^{n \times n}_{\mathrm{sym}}$ for that of all real symmetric $n \times n$ matrices. Given a matrix $A \in \mathbb{M}^{n \times n}$, we let $|A| := \sqrt{\mathrm{tr}(AA^T)}$ (A^T is the transpose of A, and $\mathrm{tr}\, A$ is its trace) which defines the usual Euclidean norm over $\mathbb{M}^{n \times n}$. We recall that for any two vectors a and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{n \times n}$ stands for the tensor product, i.e., $(a \otimes b)_{ij} = a_i b_j$ for all $1 \leqslant i, j \leqslant n$, and $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a) \in \mathbb{M}^{n \times n}_{\mathrm{sym}}$ denotes the symmetric tensor product.

Given an open subset U of $\mathbb{R}^n$, we denote by $\mathscr{M}(U)$ the space of all real-valued Radon measures with finite total variation. We use standard notation for Lebesgue spaces $L^p(U)$ and Sobolev spaces $W^{k,p}(U)$ or $H^k(U) := W^{k,2}(U)$. If Γ is a closed subset of $\overline{U}$, we denote by $H^k_{0,\Gamma}(U)$ the closure of $\mathscr{C}^\infty_c(\overline{U} \smallsetminus \Gamma)$ in $H^k(U)$. In particular, if $\Gamma = \partial U$, then $H^k_{0,\partial U}(U) = H^k_0(U)$.

2.2. Capacities. — In the sequel, we will use the notion of capacity for which we refer to [1, 21]. We just recall the definition and several facts. The $(k, 2)$-capacity of a compact set $K \subset \mathbb{R}^n$ is defined by

$$\mathrm{Cap}_{k,2}(K) := \inf \left\{ \|\varphi\|_{H^k(\mathbb{R}^n)} : \varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n),\ \varphi \geqslant 1 \text{ on } K \right\}.$$

This definition is then extended to open sets $A \subset \mathbb{R}^n$ by

$$\mathrm{Cap}_{k,2}(A) := \sup \left\{ \mathrm{Cap}_{k,2}(K) : K \subset A,\ K \text{ compact} \right\},$$

and to arbitrary sets $E \subset \mathbb{R}^n$ by

$$\mathrm{Cap}_{k,2}(E) := \inf \left\{ \mathrm{Cap}_{k,2}(A) : E \subset A,\ A \text{ open} \right\}.$$

One of the interests of capacity is that it enables one to give an accurate sense to the pointwise value of Sobolev functions. More precisely, every $u \in H^k(\mathbb{R}^n)$ has a $(k, 2)$-quasicontinuous representative $\widetilde{u}$, which means that $\widetilde{u} = u$ a.e. and that, for each $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{k,2}(\mathbb{R}^n \smallsetminus A_\varepsilon) < \varepsilon$ and $\widetilde{u}|_{A_\varepsilon}$ is continuous on A_ε (see [1, Sec. 6.1]). The $(k, 2)$-quasicontinuous representative is unique, in the sense that two $(k, 2)$-quasicontinuous representatives of the same function $u \in H^k(\mathbb{R}^n)$ coincide $\mathrm{Cap}_{k,2}$-quasi-everywhere. In addition, if U is an open subset of $\mathbb{R}^n$, then $u \in H^k_0(U)$ if and only if for all multi-index $\alpha \in \mathbb{N}^n$ with length $|\alpha| \leqslant k$, $\partial^\alpha u$ has a $(k - |\alpha|, 2)$-quasicontinuous representative that vanishes $\mathrm{Cap}_{k-|\alpha|,2}$-quasi everywhere on ∂U, i.e., outside a set of zero $\mathrm{Cap}_{k-|\alpha|,2}$-capacity (see [1, Th. 9.1.3]). In the sequel, we will only be interested to the cases $k = 1$ or $k = 2$ in dimension $n = 2$.

2.3. Kondrat'ev spaces. — Following [25, Sec. 6.1], if C is an open cone of $\mathbb{R}^n$ with vertex at the origin, we define for any $\beta \in \mathbb{R}$ and $\ell \geqslant 0$ the weighted Sobolev space $V_\beta^\ell(C)$ by the closure of $\mathscr{C}_c^\infty(\overline{C} \smallsetminus \{0\})$ with respect to the norm

$$\|u\|_{V_\beta^\ell(C)} := \left(\int_C \sum_{|\alpha| \leqslant \ell} |x|^{2(\beta - \ell + |\alpha|)} |\partial^\alpha u(x)|^2 dx \right)^{1/2}.$$

It will also be useful to introduce the spaces $V_\beta^\ell(C)$ for $\ell < 0$, which is defined as the dual space of $V_{-\beta}^{-\ell}(C)$, endowed with the usual dual norm.

Observe that when $\ell \geqslant 0$ then $u \in V_\beta^\ell(C)$ if and only if the function $x \mapsto |x|^{\beta - \ell + |\alpha|} \partial^\alpha u(x) \in L^2(C)$ for all $|\alpha| \leqslant \ell$. If one is interested in homogeneous functions, it turns out that the parameter β plays a different role regarding to the integrability at the origin or at infinity. To fix the ideas, one can check that in dimension 2, a function of the form $x \mapsto |x|^\gamma f(x/|x|)$ around the origin and with compact support belongs to $V_\beta^\ell(C)$ for every $\beta < 1 - \gamma$. On the other hand, a function having this behavior at infinity and vanishing around the origin will belong to a space $V_\beta^\ell(C)$ for every $\beta > 1 - \gamma$. For instance if $\gamma = 3/2$, then the corresponding space of critical exponent would be that with $\beta = -1/2$.

2.4. Functions with Lebesgue deformation. — Given a vector field (distribution) $u : U \to \mathbb{R}^n$, the symmetrized gradient of u is denoted by

$$e(u) := \frac{\nabla u + \nabla u^T}{2}.$$

In linearized elasticity, u stands for the displacement, while $e(u)$ is the elastic strain. The elastic energy of a body is given by a quadratic form of $e(u)$ so that it is natural to consider displacements such that $e(u) \in L^2(U; \mathbb{M}_{sym}^{n \times n})$. If U has Lipschitz boundary, it is well known that u actually belongs to $H^1(U; \mathbb{R}^n)$ as a consequence of Korn's inequality (see e.g. [10, 31]). However, when U is not smooth, we can only assert that $u \in L_{loc}^2(U; \mathbb{R}^n)$. This motivates the following definition of the space of Lebesgue deformations:

$$LD(U) := \{u \in L_{loc}^2(U; \mathbb{R}^n) : e(u) \in L^2(U; \mathbb{M}_{sym}^{n \times n})\}.$$

If U is connected and u is a distribution with $e(u) = 0$, then necessarily it is a rigid movement, i.e., $u(x) = Ax + b$ for all $x \in U$, for some skew-symmetric matrix $A \in \mathbb{M}^{n \times n}$ and some vector $b \in \mathbb{R}^n$. If, in addition, U has Lipschitz boundary, the following Poincaré-Korn inequality holds: there exists a constant $c_U > 0$ and a rigid movement r_U such that

(2.1) $\|u - r_U\|_{L^2(U)} \leqslant c_U \|e(u)\|_{L^2(U)}, \quad$ for all $u \in LD(U)$.

According to [2, Th. 5.2, Exam. 5.3], it is possible to make r_U more explicit in the following way: consider a measurable subset E of U with $|E| > 0$, then one can take

$$r_U(x) := \frac{1}{|E|} \int_E u(y) \, dy + \left(\frac{1}{|E|} \int_E \frac{\nabla u(y) - \nabla u(y)^T}{2} \, dy \right) \left(x - \frac{1}{|E|} \int_E y \, dy \right),$$

provided the constant c_U in (2.1) also depends on E.

2.5. Hausdorff convergence of compact sets. — Let K_1 and K_2 be compact subsets of a common compact set $K \subset \mathbb{R}^n$. The Hausdorff distance between K_1 and K_2 is given by

$$d_{\mathscr{H}}(K_1, K_2) := \max\left\{\sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1)\right\}.$$

We say that a sequence (K_n) of compact subsets of K converges in the Hausdorff distance to the compact set K_∞ if $d_{\mathscr{H}}(K_n, K_\infty) \to 0$. The Hausdorff convergence of compact sets turns out to be equivalent to the convergence in the sense of Kuratowski. Indeed $K_n \to K_\infty$ in the Hausdorff metric if and only if both following properties hold:

  a) any $x \in K_\infty$ is the limit of a sequence (x_n) with $x_n \in K_n$;
  b) if for all n, $x_n \in K_n$, any limit point of (x_n) belongs to K_∞.

Finally let us recall Blaschke's selection principle which asserts that from any bounded sequence (K_n) of compact subsets of K, one can extract a subsequence converging in the Hausdorff distance.

2.6. Convention. — If $f : \mathbb{R}^m \to \mathbb{R}^m$ is a vector field, the notation ∇f stands for the $m \times n$ matrix whose entries are $\partial f_i / \partial x_j$ (for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$). In particular, if $m = 1$, i.e., if f is scalar valued, ∇f is a row vector. If $n = 2$ and $m = 1$, we write

$$\nabla^\perp f := (-\partial_2 f_1, \partial_1 f_1),$$

while if $n = 2$ and $m = 2$, we write

$$\nabla^\perp f := \begin{pmatrix} -\partial_2 f_1 & \partial_1 f_1 \\ -\partial_2 f_2 & \partial_1 f_2 \end{pmatrix}.$$

3. Description of the model

Reference configuration. — We consider a homogeneous isotropic linearly elastic body occupying Ω in its reference configuration, a bounded and connected open subset of $\mathbb{R}^2$ with Lipschitz boundary. We suppose that the stress $\sigma \in \mathbb{M}^{2 \times 2}_{\text{sym}}$ is related to the strain $e \in \mathbb{M}^{2 \times 2}_{\text{sym}}$ thanks to Hooke's law

$$\sigma = \mathbb{C}e = \lambda(\operatorname{tr} e)I + 2\mu e,$$

where $\lambda, \mu \in \mathbb{R}$ are the Lamé coefficients satisfying $\lambda + \mu > 0$ and $\mu > 0$, and I is the identity matrix. This expression can be inverted into

$$(3.1) \qquad\qquad e = \mathbb{C}^{-1}\sigma = \frac{1 + \nu}{E}\sigma - \frac{\nu}{E}(\operatorname{tr} \sigma)I,$$

where $E := \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is the Young modulus and $\nu := \lambda/2(\lambda + \mu)$ is the Poisson coefficient.

External loads. — We suppose that the body is only subjected to a soft device loading, that is, to a prescribed displacement $\psi \in H^{1/2}(\partial\Omega; \mathbb{R}^2)$ acting on the entire boundary.

Admissible cracks. — We further assume that the body can undergo cracks which belong to the admissible class

$$\mathscr{K}(\overline{\Omega}) := \{\Gamma \subset \overline{\Omega} \text{ compact, connected, } 0 \in \Gamma \text{ and } \mathscr{H}^1(\Gamma) < \infty\}.$$

Admissible displacements. — For a given crack $\Gamma \in \mathscr{K}(\overline{\Omega})$, we define the space of admissible displacement by

$$LD(\Omega \smallsetminus \Gamma) := \{u \in L^2_{\text{loc}}(\Omega \smallsetminus \Gamma; \mathbb{R}^2) : e(u) \in L^2(\Omega \smallsetminus \Gamma; \mathbb{M}^{2\times 2}_{\text{sym}})\}.$$

Any point of $\partial\Omega \smallsetminus \Gamma$ is contained in a ball such that $\overline{B} \cap \Gamma = \varnothing$ and such that $\Omega \cap B$ has Lipschitz boundary so that Korn's inequality ensures that $u \in H^1(\Omega \cap B; \mathbb{R}^2)$. As a consequence, the trace of u is well defined on $\partial\Omega \cap B$. Since this property holds for any ball as above, then the trace of u is well defined on $\partial\Omega \smallsetminus \Gamma$.

Initial data. — We consider an initial crack $\Gamma_0 \in \mathscr{K}(\overline{\Omega})$ satisfying furthermore

$$(3.2) \qquad \lim_{\varrho \to 0} \frac{\mathscr{H}^1(\Gamma_0 \cap B_\varrho)}{2\varrho} = \frac{1}{2},$$

and an associated displacement $u_0 \in LD(\Omega \smallsetminus \Gamma_0)$ given as a solution of the minimization problem

$$(3.3) \qquad \min\left\{\frac{1}{2} \int_{\Omega \smallsetminus \Gamma_0} \mathbb{C}e(v) : e(v)\, dx : v \in LD(\Omega \smallsetminus \Gamma_0),\ v = \psi \text{ on } \partial\Omega \smallsetminus \Gamma_0\right\}.$$

Note that u_0 is unique up to an additive rigid movement in each connected component of $\Omega \smallsetminus \Gamma_0$ disjoint from $\partial\Omega \smallsetminus \Gamma_0$. However, the stress, which is given by Hooke's law

$$(3.4) \qquad \sigma_0 := \mathbb{C}e(u_0) \in L^2(\Omega \smallsetminus \Gamma_0; \mathbb{M}^{2\times 2}_{\text{sym}})$$

is unique and it satisfies the variational formulation

$$(3.5) \qquad \int_{\Omega \smallsetminus \Gamma_0} \sigma_0 : e(v)\, dx = 0$$

for any $v \in LD(\Omega \smallsetminus \Gamma_0)$ such that $v = 0$ on $\partial\Omega \smallsetminus \Gamma_0$. Note that standard results on elliptic regularity (see e.g. [10, Th. 6.3.6]) ensure that $u_0 \in \mathscr{C}^\infty(\Omega \smallsetminus \Gamma_0; \mathbb{R}^2)$.

Energy release rate. — To define the energy release rate, let us consider a crack increment $\Gamma_0 \cup \Gamma$, where $\Gamma \in \mathscr{K}(\overline{\Omega})$ and an associated displacement $u_\Gamma \in LD(\Omega \smallsetminus (\Gamma_0 \cup \Gamma))$ solving

$$\min\left\{\frac{1}{2} \int_{\Omega \smallsetminus (\Gamma_0 \cup \Gamma)} \mathbb{C}e(v) : e(v)\, dx : v \in LD(\Omega \smallsetminus (\Gamma_0 \cup \Gamma)),\ v = \psi \text{ on } \partial\Omega \smallsetminus (\Gamma_0 \cup \Gamma)\right\}.$$

We define

$$(3.6) \qquad \mathscr{G}(\Gamma) := \frac{1}{2} \int_\Omega \left[\mathbb{C}e(u_\Gamma) : e(u_\Gamma) - \mathbb{C}e(u_0) : e(u_0)\right] dx \leqslant 0,$$

and

$$(3.7) \qquad G_\varepsilon := \frac{1}{\varepsilon} \inf\left\{\mathscr{G}(\Gamma) : \Gamma \in \mathscr{K}(\overline{\Omega}),\ \mathscr{H}^1(\Gamma) \leqslant \varepsilon\right\}.$$

4. Construction of dual functions

The goal of this section is to construct the harmonic conjugate and the Airy function associated to the displacement u_0 in a neighborhood of the crack tip which is assumed to be the origin. Their construction rests on an abstract functional analysis result (Lemma 4.1 below) which puts in duality gradients and functions with vanishing divergence outside a (non-smooth) crack.

Let $B = B_{R_0}$ and $B' = B_{R'_0}$ be open balls centered at the origin with radii $R_0 < R'_0$, such that $\overline{B'} \subset \Omega$ and $\partial B' \cap \Gamma_0 \neq \varnothing$. By assumption, since $\Gamma_0 \in \mathscr{K}(\overline{\Omega})$ satisfies (3.2), this property certainly holds true provided R'_0 is small enough. Note in particular that the connectedness of Γ_0 ensures that $\partial B \cap \Gamma_0 \neq \varnothing$ as well.

The following result is a generalization of [5, Lem. 1].

Lemma 4.1. — *Consider the following subspaces of $L^2(B; \mathbb{R}^2)$:*

$$X := \{\sigma \in \mathscr{C}^\infty(\overline{B}; \mathbb{R}^2) : \mathrm{Supp}(\sigma) \cap \Gamma_0 = \varnothing, \ \mathrm{div}\,\sigma = 0 \ \textit{in } B\},$$

$$Y := \{\nabla v : v \in H^1(B \smallsetminus \Gamma_0), \ v = 0 \ \textit{on } \partial B \smallsetminus \Gamma_0\}.$$

Then $Y^\perp = \overline{X}$ (thus also $X^\perp = \overline{Y}$).

Proof. — Let $\sigma \in X$ and $v \in H^1(B \smallsetminus \Gamma_0)$ be such that $v = 0$ on $\partial B \smallsetminus \Gamma_0$. It follows that the product σv belongs to $H^1_0(B; \mathbb{R}^2)$ and that

$$(4.1) \qquad\qquad\qquad\qquad \mathrm{div}(\sigma v) = \sigma \nabla v,$$

from which we deduce that

$$\int_B \sigma \cdot \nabla v \, dx = \int_B \mathrm{div}(\sigma v) dx = 0.$$

Consequently, $X \subset Y^\perp$, and thus $\overline{X} \subset Y^\perp$.

We next establish the reverse inclusion. Let $\Psi \in X^\perp$. According to De Rham's Theorem (see [30, p. 20]), there exists $v \in H^1_{\mathrm{loc}}(B \smallsetminus \Gamma_0)$ such that $\nabla v = \Psi \in L^2(B \smallsetminus \Gamma_0)$. If further V is a smooth open set such that $\overline{V} \cap \Gamma_0 = \varnothing$ and $V \cap \partial B \neq \varnothing$, then the open set $V \cap B$ has Lipschitz boundary, and thus $v \in H^1(V \cap B)$. Therefore, the trace of v on $\partial B \smallsetminus \Gamma_0$ is well defined, and we claim that we can assume that $v = 0$ on $\partial B \smallsetminus \Gamma_0$.

To show this property, we consider a connected component U of $B \smallsetminus \Gamma_0$, and let $S = \partial B \cap U$. For any $g \in \mathscr{C}^\infty_c(S)$ with zero average, it is possible to find a smooth, bounded and connected open set $U' \subset U$ such that both $U_1 := U' \cap B$ and $U_2 := U' \smallsetminus \overline{B}$ are connected and have Lipschitz boundary, and with $\mathrm{Supp}\,g \subset U' \cap \partial B$. Next, we consider a smooth function with compact support in U', which coincides with g on $S' = S \cap U'$ (and which we still denote $g \in \mathscr{C}^\infty_c(U')$). We then denote by H_1 the subspace of $H^1(U_1)$ made of all functions $\varphi \in H^1(U_1)$ satisfying $\int_{U_1} \varphi \, dx = 0$, and introduce the following variational problem:

$$\inf_{\varphi \in H_1} \left\{ \frac{1}{2} \int_{U_1} |\nabla\varphi|^2 dx - \int_{S'} g\varphi \, d\mathscr{H}^1 \right\}.$$

It is standard that this problem has a unique minimizer $\varphi_1 \in H_1$, which is a solution of

$$\begin{cases} -\Delta\varphi_1 = 0 & \text{in } U_1, \\ \dfrac{\partial\varphi_1}{\partial\nu} = 0 & \text{on } \partial U_1 \smallsetminus S', \\ \dfrac{\partial\varphi_1}{\partial\nu} = g & \text{on } S' \cap \partial U_1. \end{cases}$$

Since $g \in \mathscr{C}_c^\infty(S')$ and S' is smooth, we deduce that $\varphi_1 \in \mathscr{C}^\infty(U_1 \cup (S' \cap \partial U_1))$ by elliptic regularity. In addition, since for any $\varphi \in H_0^1(B)$,

$$\int_{U_1} \nabla\varphi_1 \cdot \nabla\varphi \, dx = 0$$

one has $\mathrm{div}(\chi_{U_1}\nabla\varphi_1) = 0$ in $H^{-1}(B)$. Next, we argue similarly in U_2 to get a function $\varphi_2 \in H^1(U_2) \cap \mathscr{C}^\infty(U_2 \cup (S' \cap \partial U_2))$ satisfying $\int_{U_2} \varphi_2 \, dx = 0$, and

$$\begin{cases} -\Delta\varphi_2 = 0 & \text{in } U_2, \\ \dfrac{\partial\varphi_2}{\partial\nu} = 0 & \text{on } \partial U_2 \smallsetminus S', \\ \dfrac{\partial\varphi_2}{\partial\nu} = g & \text{on } S' \cap \partial U_2, \end{cases}$$

and in particular $\mathrm{div}(\chi_{U_2}\nabla\varphi_2) = 0$ in $H^{-1}(\mathbb{R}^2 \smallsetminus B)$. We finally consider the vector field defined in $\mathbb{R}^2$ by

$$\sigma = \chi_{U_1}\nabla\varphi_1 - \chi_{U_2}\nabla\varphi_2.$$

Since the normal trace of σ does not jump across S', σ is divergence free in $H^{-1}(\mathbb{R}^2)$. Therefore, taking a standard sequence of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$, and considering $\sigma_\varepsilon = \sigma * \rho_\varepsilon|_B$, we get an element of X, and since $\Psi \in X^\perp$ we infer that

$$\int_{B \smallsetminus \Gamma_0} \Psi \cdot \sigma_\varepsilon \, dx = 0.$$

Letting $\varepsilon \to 0$ it follows

$$0 = \int_{U_1} \Psi \cdot \nabla\varphi_1 \, dx = \int_{U_1} \nabla v \cdot \nabla\varphi_1 \, dx = \int_S gv \, d\mathscr{H}^1,$$

and since g was arbitrary (with zero average on S) we deduce that v is a constant on S. Since $S = \partial B \cap U$ and U is connected, we can remove this constant from v in the component U of $B \smallsetminus \Gamma_0$ and assume that $v = 0$ on S. Doing the same in all connected components yields a function v which vanishes on $\partial B \smallsetminus \Gamma_0$. Finally, considering the truncated function $v_k := (-k \vee v) \wedge k$, where $k \in \mathbb{N}$, we get that $v_k \in H^1(B \smallsetminus \Gamma_0)$, $v_k = 0$ on $\partial B \smallsetminus \Gamma_0$, and thus $\nabla v_k \in Y$. Moreover, since $\nabla v_k \to \nabla v = \Psi$ strongly in $L^2(B; \mathbb{R}^2)$ as $k \to \infty$ we get that $X^\perp \subset \overline{Y}$ and that $Y^\perp = (\overline{Y})^\perp \subset (X^\perp)^\perp = \overline{X}$. $\qquad\square$

4.1. The harmonic conjugate.

— We are now in position to construct the harmonic conjugate v_0 associated to u_0 in B. By construction, the displacement u_0 satisfies a Neumann condition on the crack Γ_0, while its associated stress σ_0 has zero divergence outside the crack, both in a weak sense. The harmonic conjugate v_0 is, roughly speaking, a dual function of u_0 in the sense that it satisfies a homogeneous

Dirichlet boundary condition on the crack Γ_0, and its rotated gradient coincides with the stress σ_0. The harmonic conjugate will be of use in the proof of Proposition 5.1 in order to prove a lower bound on the energy release rate. It will also appear in the construction of the Airy function.

Proposition 4.2. — *There exists a function $v_0 \in H^1_{0,\Gamma_0}(B;\mathbb{R}^2) \cap \mathscr{C}^\infty(\overline{B} \setminus \Gamma_0;\mathbb{R}^2)$ such that*

(4.2) $$-\nabla^\perp v_0 = \sigma_0 \quad in\ B \setminus \Gamma_0.$$

Proof. — According to the variational formulation (3.5), for any $v \in H^1(B \setminus \Gamma_0;\mathbb{R}^2)$ with $v = 0$ on $\partial B \setminus \Gamma_0$, we have

$$\int_B \sigma_0 : \nabla v\, dx = 0.$$

Consequently, both lines of σ_0, denoted by

$$\sigma^{(1)} := \big((\sigma_0)_{11}, (\sigma_0)_{12}\big), \quad \sigma^{(2)} := \big((\sigma_0)_{12}, (\sigma_0)_{22}\big),$$

belong to $Y^\perp$. Therefore, Lemma 4.1 ensures the existence of a sequence $(\sigma_n^{(1)}) \subset X$ such that $\sigma_n^{(1)} \to \sigma^{(1)}$ in $L^2(B;\mathbb{R}^2)$. Since $\operatorname{div}\sigma_n^{(1)} = 0$ in B and $\operatorname{Supp}(\sigma_n^{(1)}) \cap \Gamma_0 = \varnothing$, it follows that

$$\sigma_n^{(1)} = \nabla^\perp p_n^{(2)}$$

for some $p_n^{(2)} \in \mathscr{C}^\infty(\overline{B})$ with $\operatorname{Supp}(p_n^{(2)}) \cap \Gamma_0 = \varnothing$. Consequently, by the Poincaré inequality, we get that $p_n^{(2)} \to p^{(2)}$ in $H^1(B)$ for some $p^{(2)} \in H^1_{0,\Gamma_0}(B)$ satisfying $\nabla^\perp p^{(2)} = \sigma^{(1)}$. We prove similarly the existence of $p^{(1)} \in H^1_{0,\Gamma_0}(B)$ satisfying $\nabla^\perp p^{(1)} = -\sigma^{(2)}$. We then define

$$v_0 := \begin{pmatrix} -p^{(2)} \\ p^{(1)} \end{pmatrix} \in H^1_{0,\Gamma_0}(B;\mathbb{R}^2)$$

which satisfies (4.2). Finally, since $\sigma_0 \in \mathscr{C}^\infty(\overline{B}\setminus\Gamma_0;\mathbb{M}^{2\times2}_{\mathrm{sym}})$, then $v_0 \in \mathscr{C}^\infty(\overline{B}\setminus\Gamma_0;\mathbb{R}^2)$. □

4.2. **The Airy function.** — We next construct the Airy function w_0 associated to the displacement u_0 in B following an approach similar to [5]. This new function has the property to be a biharmonic function vanishing on the crack. Therefore, the original elasticity problem (3.3) can be recast into a suitable biharmonic equation whose associated natural energy (the L^2 norm of the Hessian) coincides with the original elastic energy. The Airy function will be useful in Section 6 in order to get an a priori bound on the rescaled elastic energy around the crack tip, as well as in our convergence result for the blow-up displacement.

Proposition 4.3. — *There exists a function $w_0 \in H^2_{0,\Gamma_0}(B)$ such that*

(4.3) $$\Delta^2 w_0 = 0 \ in\ \mathscr{D}'(B \setminus \Gamma_0)$$

and

(4.4) $$D^2 w_0 = \begin{pmatrix} (\sigma_0)_{22} & -(\sigma_0)_{12} \\ -(\sigma_0)_{12} & (\sigma_0)_{11} \end{pmatrix}.$$

Proof. — We reproduce the construction initiated in the proof of Proposition 4.2 with the larger ball B' instead of B. It ensures the existence of $p^{(1)}$ and $p^{(2)} \in H^1_{0,\Gamma_0}(B')$ such that

$$\nabla^\perp p^{(2)} = \big((\sigma_0)_{11}, (\sigma_0)_{12}\big), \qquad \nabla^\perp p^{(1)} = -\big((\sigma_0)_{12}, ((\sigma_0)_{22}\big).$$

By definition, there exists sequences $(p_n^{(1)})$ and $(p_n^{(2)}) \subset \mathscr{C}^\infty(\overline{B'})$ vanishing in a neighborhood of Γ_0 in $\overline{B'}$, and such that $p_n^{(1)} \to p^{(1)}$ and $p_n^{(2)} \to p^{(2)}$ in $H^1(B')$. For any $v \in H^1(B' \smallsetminus \Gamma_0)$ with $v = 0$ on $\partial B' \smallsetminus \Gamma_0$, we infer thanks to the integration by parts formula that

$$\int_{B'} \begin{pmatrix} -p^{(2)} \\ p^{(1)} \end{pmatrix} \cdot \nabla v \, dx = \int_{B'} (-p^{(2)} \partial_1 v + p^{(1)} \partial_2 v) \, dx = \lim_{n\to\infty} \int_{B'} (-p_n^{(2)} \partial_1 v + p_n^{(1)} \partial_2 v) \, dx$$

$$= \lim_{n\to\infty} \int_{B'} (-\partial_1 p_n^{(2)} + \partial_2 p_n^{(1)}) v \, dx = \int_{B'} (-\partial_1 p^{(2)} + \partial_2 p^{(1)}) v \, dx = 0.$$

Therefore, it follows that

$$\begin{pmatrix} -p^{(2)} \\ p^{(1)} \end{pmatrix} \in Y^\perp = \overline{X}$$

according again to Lemma 4.1. Arguing as in the proof of Proposition 4.2, we deduce the existence of some $w_0 \in H^1_{0,\Gamma_0}(B')$ such that

$$\nabla w_0 = (p^{(1)}, p^{(2)}).$$

By construction, the Airy function w_0 satisfies (4.4). Consequently, we have $w_0 \in H^1_{0,\Gamma_0}(B') \cap H^2(B')$ with $\nabla w_0 \in H^1_{0,\Gamma_0}(B'; \mathbb{R}^2)$. Observe that since $w_0 \in H^2(B')$, we can assume that it is continuous (that is, we consider its continuous representative), and even in $\mathscr{C}^{0,\alpha}(B)$ for any $\alpha < 1$.

Let us show that $w_0 \in H^2_{0,\Gamma_0}(B)$. This property rests on a capacity argument similar to that used in [5, Th. 1]. Thanks to [1, Th. 9.1.3], we just need to check that w_0 vanishes on $\Gamma_0 \cap B$, pointwise. First, we consider a cut-off function $\eta \in \mathscr{C}^\infty_c(B'; [0,1])$ satisfying $\eta = 1$ on $\overline{B}$. Denoting $z_0 := \eta w_0$, then one has $z_0 \in H^2(B' \smallsetminus \Gamma_0) \cap H^1_0(B' \smallsetminus \Gamma_0)$ and $\nabla z_0 \in H^1_0(B' \smallsetminus \Gamma_0; \mathbb{R}^2)$. In particular, $z_0 = 0$ on $\partial(B' \smallsetminus \Gamma_0)$ except in a set of zero $\mathrm{Cap}_{1,2}$-capacity (since being continuous it is obviously the quasicontinuous representative of z_0 in H^1). Let $K = \{x \in \partial(B' \smallsetminus \Gamma_0) : z_0(x) \neq 0\}$ (which is a relatively open subset of $\Gamma_0 \cap B'$) and consider $\gamma \subset K$ a connected component. Then the diameter of γ must be zero, otherwise γ would have positive $\mathrm{Cap}_{1,2}$-capacity (see [21, Cor. 3.3.25]). Therefore it must be at most an isolated point of Γ_0. However, $\partial(B \smallsetminus \Gamma_0)$ is connected and therefore has no isolated point, hence $K = \varnothing$.

We next show that w_0 is a biharmonic function. Indeed, according to (4.4), one has

$$\Delta^2 w_0 = \Delta((\sigma_0)_{11} + (\sigma_0)_{22}) \quad \text{in } \mathscr{D}'(B \smallsetminus \Gamma_0).$$

Denoting by $e_0 := e(u_0)$ the elastic strain, and using the compatibility condition

$$2\partial^2_{12}(e_0)_{12} = \partial^2_{11}(e_0)_{22} + \partial^2_{22}(e_0)_{11} \quad \text{in } \mathscr{D}'(B \smallsetminus \Gamma_0)$$

together with Hooke's law (3.1),

$$(e_0)_{11} = \frac{(\sigma_0)_{11}}{E} - \frac{\nu}{E}(\sigma_0)_{22},$$

$$(e_0)_{22} = \frac{(\sigma_0)_{22}}{E} - \frac{\nu}{E}(\sigma_0)_{11},$$

$$(e_0)_{12} = \frac{1+\nu}{E}(\sigma_0)_{12},$$

we infer that

$$\Delta^2 w_0 = (1+\nu)[\partial^2_{11}(\sigma_0)_{11} + \partial^2_{22}(\sigma_0)_{22} + 2\partial^2_{12}(\sigma_0)_{12}] \quad \text{in } \mathscr{D}'(B \smallsetminus \Gamma_0).$$

Finally, according to the variational formulation (3.5), we have

$$\operatorname{div}\sigma_0 = 0 \quad \text{in } \mathscr{D}'(B \smallsetminus \Gamma_0)$$

from which (4.3) follows. □

Remark 4.4. — The biharmonicity (4.3) of the Airy function w_0 is equivalent to the following local minimality property

$$\int_B |D^2 w_0|^2\, dx \leqslant \int_B |D^2 z|^2\, dx,$$

for all $z \in w_0 + H^2_0(B)$.

Remark 4.5. — According to the results of [24], we get the following estimate of the energy of w_0 around the origin: for every $2\varrho < R \leqslant R_0$,

$$\int_{B_\varrho} |D^2 w_0|^2\, dx \leqslant \frac{C_0\varrho}{R} \int_{B_R} |D^2 w_0|^2\, dx,$$

for some universal constant $C_0 > 0$ independent of R and ϱ. Indeed, it suffices to apply [24, Th. 2] in the open set $B \smallsetminus \Gamma_0$ with (in their notation) $\omega = 2\pi$ and $\delta = 1/2$. This is possible since, Γ_0 being connected, then for all $\varrho < R$ we have $\partial B_\varrho \cap \Gamma_0 \neq \varnothing$, $\mathscr{H}^1(\partial B_\varrho \smallsetminus \Gamma_0) \leqslant 2\pi\varrho$ and $\partial(B_R \smallsetminus \Gamma_0) \cap \partial(B_R \smallsetminus \Gamma_0) = \Gamma_0 \cap B_R \subset \Gamma_0 \cap B$.

Thanks to the reformulation of the elasticity problem as a biharmonic equation, and according to Remark 4.5 concerning the behavior of the energy of a biharmonic function in fractured domains, we get the following result about the elastic energy concentration around the crack tip. We observe that in [9] a stronger result has been obtained in the scalar (anti-plane) case where a monotonicity formula has been established.

Proposition 4.6. — *Let σ_0 be the stress defined in (3.4) and $R_0 > 0$ be such that $B_{R_0} \subset \Omega$ and $\partial B_{R_0} \cap \Gamma \neq \varnothing$. Then there exists a universal constant $C_0 > 0$ such that for all ρ, $R > 0$ satisfying $2\varrho < R \leqslant R_0$,*

$$\int_{B_\varrho} |\sigma_0|^2\, dx \leqslant \frac{C_0\varrho}{R} \int_{B_R} |\sigma_0|^2\, dx.$$

Proof. — The result is an immediate consequence of (4.4) together with Remark 4.5.
□

5. Bounds on the energy release rate

The goal of this section is to establish bounds on the energy release rate. This is the first step toward a more precise analysis and a characterization of the energy release rate as a limiting minimization problem (see Section 7). As in [7, Lem. 2.4], the proof of the upper bound relies on the construction of an explicit competitor for the minimization problem (3.7) defining G_ε. The lower bound rests in turn into a dual formulation (in term in the stress) of the minimization problem (3.6), and into the construction, for each crack increment, of an admissible stress competitor for this new dual variational problem. The construction we use is based on the harmonic conjugate v_0 associated to the displacement obtained in Proposition 4.2.

Proposition 5.1. — *There exist two constants $0 < G_* \leqslant G^* < \infty$ such that*

$$-G^* \leqslant \liminf_{\varepsilon \to 0} G_\varepsilon \leqslant \limsup_{\varepsilon \to 0} G_\varepsilon \leqslant -G_*.$$

Proof

Upper bound. — Since $0 \in \Omega$, one can choose $\varepsilon > 0$ small enough so that $\overline{B}_{\varepsilon/(2\pi+1)} \subset \Omega$. Let

$$\Gamma := \partial B_{\varepsilon/(2\pi+1)} \cup \{(t,0) : 0 \leqslant t \leqslant \varepsilon/(2\pi+1)\}.$$

This set clearly belongs to $\mathscr{K}(\overline{\Omega})$ and $\mathscr{H}^1(\Gamma) = \varepsilon$. Defining $v := u_0 \chi_{\Omega \smallsetminus \overline{B}_{\varepsilon/(2\pi+1)}}$, we infer that $v \in LD(\Omega \smallsetminus (\Gamma_0 \cup \Gamma))$ with $v = u_0 = \psi$ on $\partial\Omega \smallsetminus (\Gamma_0 \cup \Gamma)$. Consequently,

$$\mathscr{G}(\Gamma) \leqslant \frac{1}{2} \int_\Omega \left[\mathbb{C}e(v) : e(v) \, dx - \mathbb{C}e(u_0) : e(u_0) \right] dx = -\frac{1}{2} \int_{B_{\varepsilon/(2\pi+1)}} \mathbb{C}e(u_0) : e(u_0) \, dx.$$

We then apply Proposition 4.6 which shows that

$$\limsup_{\varepsilon \to 0} G_\varepsilon \leqslant -G_*,$$

for some $G_* > 0$.

Lower bound. — Let $\varepsilon > 0$ be small enough so that $2\varepsilon \leqslant R_0$, $\overline{B}_{2\varepsilon} \subset \Omega$. According to [8, p. 330], for any $\Gamma \in \mathscr{K}(\overline{\Omega})$ with $\mathscr{H}^1(\Gamma) \leqslant \varepsilon$, one has

$$(5.1) \quad \frac{1}{2} \int_\Omega \left[\mathbb{C}e(u_\Gamma) : e(u_\Gamma) - \mathbb{C}e(u_0) : e(u_0) \right] dx \geqslant -\frac{1}{2} \int_\Omega (\tau - \sigma_0) : \mathbb{C}^{-1}(\tau - \sigma_0) \, dx$$

for every statically admissible stress $\tau \in L^2(\Omega; \mathbb{M}^{2\times 2})$ satisfying

$$(5.2) \quad \int_\Omega \tau : e(v) \, dx = 0 \quad \text{for any } v \in LD(\Omega \smallsetminus (\Gamma_0 \cup \Gamma)) \text{ with } v = 0 \text{ on } \partial\Omega \smallsetminus (\Gamma_0 \cup \Gamma).$$

We now construct a convenient competitor τ for (5.2). Since Γ is connected, $0 \in \Gamma$ and $\mathscr{H}^1(\Gamma) \leqslant \varepsilon$ it follows that $\Gamma \subset B_\varepsilon$. Let $\eta \in \mathscr{C}_c^\infty(\Omega; [0,1])$ be a cut-off function satisfying

$$\begin{cases} \eta = 1 & \text{in } B_{5\varepsilon/4}, \\ \eta = 0 & \text{in } \Omega \smallsetminus B_{7\varepsilon/4}, \\ \|\nabla\eta\|_\infty \leqslant 3/\varepsilon. \end{cases}$$

We define $\tau \in L^2(\Omega; \mathbb{M}^{2\times 2})$ by

$$(5.3) \qquad \tau = \sigma_0 + \nabla^\perp(\eta v_0) = \begin{cases} 0 & \text{in } B_\varepsilon, \\ -\nabla^\perp((1-\eta)v_0) & \text{in } B_{2\varepsilon} \smallsetminus B_\varepsilon, \\ \sigma_0 & \text{in } \Omega \smallsetminus B_{2\varepsilon}, \end{cases}$$

where v_0 is the harmonic conjugate of u_0 in the ball $B = B_{R_0}$, which satisfies $-\nabla^\perp v_0 = \sigma_0$.

Let us check that τ satisfies (5.2). By the density result [5, Th. 1], it is enough to consider test functions $v \in H^1(\Omega \smallsetminus (\Gamma_0 \cup \Gamma); \mathbb{R}^2)$ with $v = 0$ on $\partial\Omega \smallsetminus (\Gamma_0 \cup \Gamma)$. We have

$$\int_\Omega \tau : e(v)\,dx = \int_\Omega (\sigma_0 + \nabla^\perp(\eta v_0)) : e(v)\,dx = \int_\Omega (\nabla^\perp(\eta v_0)) : e(v)\,dx.$$

We recall that there exists a sequence $(v_n)_n \subset \mathscr{C}^\infty(\overline{B_{2\varepsilon}}; \mathbb{R}^2)$ with $v_n = 0$ in a neighborhood of Γ_0 and such that $v_n \to v_0$ in $H^1(B_{2\varepsilon}; \mathbb{R}^2)$. Hence,

$$\lim_{n\to\infty} \int_\Omega (\nabla^\perp(\eta v_n)) : e(v)\,dx = \int_\Omega (\nabla^\perp(\eta v_0)) : e(v)\,dx.$$

On the other hand, since $\eta v_n \in \mathscr{C}_c^\infty(\Omega \smallsetminus \Gamma_0; \mathbb{R}^2)$

$$\int_{\Omega \smallsetminus \Gamma_0} (\nabla^\perp(\eta v_n)) : \nabla v\,dx = -\int_{\Omega \smallsetminus \Gamma_0} \operatorname{div}(\nabla^\perp(\eta v_n)) \cdot v\,dx = 0,$$

simply observing that $\operatorname{div}\nabla^\perp = 0$), showing that τ satisfies (5.2).

Taking τ defined by (5.3) as competitor in (5.1) and recalling that $\sigma_0 = -\nabla^\perp v_0$, we infer that

$$(5.4) \quad \frac{1}{2}\int_\Omega \left(\mathbb{C}e(u_\Gamma) : e(u_\Gamma) - \mathbb{C}e(u_0) : e(u_0)\right) dx$$
$$\geqslant -c\left(\int_{B_{2\varepsilon}} |\sigma_0|^2\,dx + \frac{1}{\varepsilon^2}\int_{B_{2\varepsilon} \smallsetminus B_\varepsilon} |v_0|^2\,dx\right),$$

for some constant $c > 0$ only depending on the Lamé constants λ and μ. We have

$$\int_{B_{2\varepsilon} \smallsetminus B_\varepsilon} |v_0|^2\,dx = \int_\varepsilon^{2\varepsilon} \int_{\partial B_r} |v_0|^2\,d\mathscr{H}^1\,dr.$$

We know that for a.e. r, $v_0 \in H^1(\partial B_r)$ with $v_0 = 0$ on $\Gamma_0 \cap \partial B_r$ (which is nonempty since Γ_0 is connected). For such r, if $x \in \partial B_r$ and $\xi \in \Gamma_0 \cap \partial B_r$, an integration along the circle shows that $v_0(x) = \int_{\overparen{(\xi,x)}} \partial_\tau v_0\,d\mathscr{H}^1$ so that (using the smallest arc from ξ to x)

$$v_0(x)^2 \leqslant \pi r \int_{\partial B_r} |\partial_\tau v_0|^2 d\mathscr{H}^1.$$

Hence

$$\int_{\partial B_r} |v_0|^2 d\mathscr{H}^1 \leqslant 2\pi^2 r^2 \int_{\partial B_r} |\partial_\tau v_0|^2 d\mathscr{H}^1$$

so that

$$\int_{B_{2\varepsilon} \smallsetminus B_\varepsilon} |v_0|^2 \, dx \leqslant 8\pi^2 \varepsilon^2 \int_\varepsilon^{2\varepsilon} \int_{\partial B_r} |\partial_\tau v_0|^2 \, d\mathscr{H}^1 \, dr$$

$$\leqslant 8\pi^2 \varepsilon^2 \int_{B_{2\varepsilon}} |\nabla v_0|^2 dx \leqslant 8\pi^2 \varepsilon^2 \int_{B_{2\varepsilon}} |\sigma_0|^2 dx.$$

Inserting this result into (5.4), it follows that

$$\frac{1}{2} \int_\Omega \left[\mathbb{C}e(u_\Gamma) : e(u_\Gamma) - \mathbb{C}e(u_0) : e(u_0) \right] dx \geqslant -c \int_{B_{2\varepsilon}} |\sigma_0|^2 \, dx$$

for some constant $c > 0$ only depending on λ and μ. Since this holds for all $\Gamma \in \mathscr{K}(\overline{\Omega})$ with $\mathscr{H}^1(\Gamma) \leqslant \varepsilon$, it follows

$$G_\varepsilon \geqslant -\frac{c}{\varepsilon} \int_{B_{2\varepsilon}} |\sigma_0|^2 \, dx.$$

Then Proposition 4.6 shows that

$$\liminf_{\varepsilon \to 0} G_\varepsilon \geqslant -G^*$$

for some $G^* > 0$. $\qquad\square$

6. Blow-up limit of the pre-existing crack

In this section we investigate the nature of the singularity of the displacement u_0 and the stress σ_0 at the origin, which is the tip of the crack Γ_0 having density $1/2$ at that point. We will prove, that along suitable subsequences of radius $\varepsilon_k \to 0$ of balls, the rescaled crack converges in the Hausdorff sense to a half-line (modulo a rotation), and the rescaled displacement converges in a certain sense to the usual crack-tip function in the complement of a half-line. Once again, the analysis strongly relies on the Airy function introduced in Proposition 4.3. Contrary to [9] where the scalar anti-plane was treated, we do not have any monotonicity formula on the energy (neither for the elastic problem nor for the biharmonic one) which prevents one to ensure the existence of the limit of the rescaled energy, and thus the uniqueness of the limit. Therefore, in contrast with [9], our result strongly depends upon the sequence (ε_n).

Let $R_0 > 0$ be such that $B_{R_0} \subset \Omega$, and $0 < \varepsilon \leqslant R_0/2$. According to [9, Prop. 1 & Rem. 2], there exists a sequence of rotations $\mathscr{R}_\varepsilon$ such that the rescaled crack

$$(6.1) \qquad\qquad \Sigma_\varepsilon := \varepsilon^{-1} \mathscr{R}_\varepsilon(\Gamma_0 \cap B_\varepsilon)$$

locally converges to the half line $\Sigma_0 := (-\infty, 0] \times \{0\}$ with respect to the Hausdorff distance.

In this section we are interested in the asymptotic behavior of the rescaled displacement $u_\varepsilon \in LD(B_{R_0/\varepsilon})$ defined by

$$(6.2) \qquad\qquad u_\varepsilon(y) := \varepsilon^{-1/2} u_0(\mathscr{R}_\varepsilon^{-1}(\varepsilon y)) \quad \text{for every } y \in B_{R_0/\varepsilon}.$$

To this aim, it will again be convenient to work on the Airy function. Let us consider the Airy function $w_0 \in H^2_{0,\Gamma_0}(B_{R_0})$ associated to u_0 in B_{R_0} given by Proposition 4.3 satisfying (4.3) and (4.4). The rescaled Airy function $w_\varepsilon \in H^2_{0,\Sigma_\varepsilon}(B_{R_0/\varepsilon})$ is defined by

$$(6.3) \qquad w_\varepsilon(y) := \varepsilon^{-3/2} w_0(\mathscr{R}_\varepsilon^{-1}(\varepsilon y)) \quad \text{for every } y \in B_{R_0/\varepsilon}.$$

6.1. Blow-up analysis of the Airy function. — We first show that the Airy function blows-up into a biharmonic function outside the half line limit crack, satisfying a homogeneous Dirichlet condition on the crack, and that its energy computed on a ball behaves like the radius.

Proposition 6.1. — *For every sequence $(\varepsilon_n) \searrow 0^+$, there exist a subsequence $(\varepsilon_k) \equiv (\varepsilon_{n_k}) \searrow 0^+$ and $w_{\Sigma_0} \in H^2_{\text{loc}}(\mathbb{R}^2)$ such that*

$$w_{\varepsilon_k} \longrightarrow w_{\Sigma_0} \text{ strongly in } H^2_{\text{loc}}(\mathbb{R}^2).$$

In addition, w_{Σ_0} is a solution of the following biharmonic problem with homogeneous Dirichlet boundary condition on the crack:

$$(6.4) \qquad \begin{cases} \Delta^2 w_{\Sigma_0} = 0 \text{ in } \mathscr{D}'(\mathbb{R}^2 \smallsetminus \Sigma_0), \\ w_{\Sigma_0} \in H^2_{0,\Sigma_0}(B_R) \text{ for any } R > 0, \end{cases}$$

and it satisfies the following energy bound

$$(6.5) \qquad \sup_{R>0} \frac{1}{R} \int_{B_R} |D^2 w_{\Sigma_0}|^2 \, dx < \infty.$$

Proof. — The proof is divided into several steps. We first derive weak compactness on the rescaled Airy function, according the energy bound of the original Airy function. We then derive a Dirichlet condition on the crack for the weak limit and its gradient. Using a cut-off function argument, we establish that the weak convergence is actually strong, which enables one to show that the limit Airy function is a biharmonic function outside the crack. In the sequel $R > 0$ is fixed, and $\varepsilon > 0$ is small enough such that $2R < R_0/\varepsilon$.

Weak compactness. — According to [24, Th. 2], we have

$$(6.6) \qquad \begin{aligned} \int_{B_{2R}} |D^2 w_\varepsilon(y)|^2 \, dy &= \varepsilon \int_{B_{2R}} |D^2 w_0(\mathscr{R}_\varepsilon^{-1}(\varepsilon y))|^2 \, dy \\ &= \frac{1}{\varepsilon} \int_{B_{2R\varepsilon}} |D^2 w_0(x)|^2 \, dx \leqslant C_0 R, \end{aligned}$$

where $C_0 > 0$ is independent of ε and R. Since $w_\varepsilon \in H^2_{0,\Sigma_\varepsilon}(B_{2R})$, Poincaré inequality implies that the sequence $(w_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $H^2(B_{2R})$. A standard diagonalisation argument shows that for each sequence $(\varepsilon_n) \searrow 0^+$, it is possible to extract a subsequence $(\varepsilon_k) \equiv (\varepsilon_{n_k}) \searrow 0^+$ and find $w_{\Sigma_0} \in H^2_{\text{loc}}(\mathbb{R}^2)$ such that $w_{\varepsilon_k} \rightharpoonup w_{\Sigma_0}$ weakly in $H^2_{\text{loc}}(\mathbb{R}^2)$. In particular, passing to the $\liminf$ in (6.6) yields (6.5). In addition, we can assume that, for the same subsequence, $w_{\varepsilon_k} \to w_{\Sigma_0}$ strongly in $H^1_{\text{loc}}(\mathbb{R}^2) \cap L^\infty_{\text{loc}}(\mathbb{R}^2)$, and that $|D^2 w_{\varepsilon_k}|^2 \mathscr{L}^2 \rightharpoonup \mu$ weakly* in $\mathscr{M}_{\text{loc}}(\mathbb{R}^2)$ for some nonnegative measure $\mu \in \mathscr{M}_{\text{loc}}(\mathbb{R}^2)$.

Condition on the crack. — Let us show that $w_{\Sigma_0} \in H^2_{0,\Sigma_0}(B_r)$ for any $r < 2R$. Consider a cut-off function $\eta \in \mathscr{C}^\infty_c(B_{2R}; [0,1])$ such that $\eta = 1$ on B_r, and let $z := \eta w_{\Sigma_0} \in H^2_0(B_{2R})$. Note that since $w_{\varepsilon_k} \to w_{\Sigma_0}$ uniformly on $\overline{B}_{2R}$ and $\Sigma_{\varepsilon_k} \to \Sigma_0$ in the sense of Hausdorff in B_{2R}, then $w_{\Sigma_0} = 0$ on Σ_0, and thus $z = 0$ on $\partial(B_{2R} \smallsetminus \Sigma_0)$. On the other hand, since $\nabla(\eta w_{\varepsilon_k}) \in H^1_0(B_{2R} \smallsetminus \Sigma_{\varepsilon_k}; \mathbb{R}^2)$ and $\nabla(\eta w_{\varepsilon_k}) \rightharpoonup \nabla z$ weakly in $H^1(B_{2R}; \mathbb{R}^2)$, it follows from [29] that $\nabla z \in H^1_0(B_{2R} \smallsetminus \Sigma_0; \mathbb{R}^2)$. Therefore, ∇z has a $\mathrm{Cap}_{1,2}$-quasicontinuous representative, denoted by $\widetilde{\nabla z}$, such that $\widetilde{\nabla z} = 0$ $\mathrm{Cap}_{1,2}$-q.e. on $\partial(B_{2R} \smallsetminus \Sigma_0)$. As a consequence of [1, Th. 9.1.3] (see also [21, Th. 3.8.3]), we get that $z \in H^2_0(B_{2R} \smallsetminus \Sigma_0)$, and thus that $w_{\Sigma_0} \in H^2_{0,\Sigma_0}(B_r)$.

Strong convergence. — Our aim now is to prove that $w_{\varepsilon_k} \to w_{\Sigma_0}$ strongly in $H^2_{\mathrm{loc}}(\mathbb{R}^2)$. By the lower semicontinuity of the norm with respect to weak convergence, we already have for any $r < 2R$

$$(6.7) \qquad \int_{B_r} |D^2 w_{\Sigma_0}|^2 \, dx \leqslant \liminf_{k \to \infty} \int_{B_r} |D^2 w_{\varepsilon_k}|^2 \, dx,$$

so that it is enough to prove the converse inequality with a lim sup. To this aim we will use the minimality property of w_{ε_k}, and suitably modify w_{Σ_0} into an admissible competitor.

Let us select a radius $r \in (R, 2R)$ such that $\mu(\partial B_r) = 0$. Since $w_{\Sigma_0} \in H^2_{0,\Sigma_0}(B_r)$, for every $n \in \mathbb{N}$, there exists a function $h_n \in \mathscr{C}^\infty(\overline{B_r})$ such that $\mathrm{Supp}(h_n) \cap \Sigma_0 = \varnothing$ and $h_n \to w_{\Sigma_0}$ in $H^2(B_r)$ as $n \to \infty$. Note that, by Hausdorff convergence, one also has that $\mathrm{Supp}(h_n) \cap \Sigma_{\varepsilon_k} = \varnothing$ for $k \geqslant k_n$ large enough, for some integer $k_n \in \mathbb{N}$.

Let us consider a cut-off function $\eta_\delta \in \mathscr{C}^\infty_c(B_r; [0,1])$ satisfying

$$(6.8) \qquad \eta_\delta = 1 \text{ on } B_{r-\delta}, \quad |\nabla \eta_\delta| \leqslant \frac{C}{\delta}, \quad |D^2 \eta_\delta| \leqslant \frac{C}{\delta^2}.$$

We finally define

$$z_{\delta,n,k} := \eta_\delta h_n + (1 - \eta_\delta) w_{\varepsilon_k} = w_{\varepsilon_k} + \eta_\delta(h_n - w_{\varepsilon_k}).$$

Observe that $z_{\delta,n,k} \in H^2_{0,\Sigma_{\varepsilon_k}}(B_r)$ provided that $k \geqslant k_n$ is large enough. Consequently, since $z_{\delta,n,k} \in w_{\varepsilon_k} + H^2_0(B_r)$, we infer thanks to (4.3) and Remark 4.4 that

$$\int_{B_r} |D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_r} |D^2 z_{\delta,n,k}|^2 \, dx,$$

or still

$$\int_{B_r} |D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_r} |\eta_\delta D^2 h_n + (1 - \eta_\delta) D^2 w_{\varepsilon_k}|^2 \, dx$$

$$+ \int_{B_r} |(h_n - w_{\varepsilon_k}) D^2 \eta_\delta + 2 \nabla \eta_\delta \otimes (\nabla h_n - \nabla w_{\varepsilon_k})|^2 \, dx$$

$$+ 2 \int_{B_r} \left[\eta_\delta D^2 h_n + (1 - \eta_\delta) D^2 w_{\varepsilon_k} \right] : \left[(h_n - w_{\varepsilon_k}) D^2 \eta_\delta + 2 \nabla \eta_\delta \otimes (\nabla h_n - \nabla w_{\varepsilon_k}) \right] dx.$$

By convexity, we get that

$$\int_{B_r} |\eta_\delta D^2 h_n + (1-\eta_\delta)D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_r} \eta_\delta |D^2 h_n|^2 \, dx + \int_{B_r} (1-\eta_\delta)|D^2 w_{\varepsilon_k}|^2 \, dx,$$

and thanks to (6.8)

$$\int_{B_r} \eta_\delta |D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_r} \eta_\delta |D^2 h_n|^2 \, dx$$
$$+ C \int_{B_r \smallsetminus B_{r-\delta}} \left(\frac{1}{\delta^4}|h_n - w_{\varepsilon_k}|^2 + \frac{1}{\delta^2}|\nabla h_n - \nabla w_{\varepsilon_k}|^2 \right) dx$$
$$+ 2 \int_{B_r \smallsetminus B_{r-\delta}} \left[\eta_\delta D^2 h_n + (1-\eta_\delta)D^2 w_{\varepsilon_k} \right] : \left[(h_n - w_{\varepsilon_k})D^2 \eta_\delta + 2\nabla\eta_\delta \otimes (\nabla h_n - \nabla w_{\varepsilon_k}) \right] dx.$$

Letting first $k \to \infty$ and then $n \to \infty$, using that $w_{\varepsilon_k} \to w_{\Sigma_0}$ in $H^1(B_r)$ and that $h_n \to w_{\Sigma_0}$ in $H^2(B_r)$, we obtain

$$\limsup_{k\to\infty} \int_{B_r} \eta_\delta |D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_r} |D^2 w_{\Sigma_0}|^2 \, dx.$$

On the other hand

$$\lim_{k\to\infty} \int_{B_r} (1-\eta_\delta)|D^2 w_{\varepsilon_k}|^2 \, dx = \int_{\overline{B_r}} (1-\eta_\delta) d\mu \leqslant \mu(\overline{B_r} \smallsetminus B_{r-\delta}).$$

Therefore we can write that

$$\limsup_{k\to\infty} \int_{B_r} |D^2 w_{\varepsilon_k}|^2 \leqslant \limsup_{k\to\infty} \int_{B_r} \eta_\delta |D^2 w_{\varepsilon_k}|^2 \, dx + \limsup_{k\to\infty} \int_{B_r} (1-\eta_\delta)|D^2 w_{\varepsilon_k}|^2 \, dx$$

$$\text{(6.9)} \qquad \leqslant \int_{B_r} |D^2 w_{\Sigma_0}|^2 \, dx + \mu(\overline{B_r} \smallsetminus B_{r-\delta}).$$

Finally, letting $\delta \to 0$ in (6.9) and using the fact that $\mu(\partial B_r) = 0$, we get the desired bound

$$\limsup_{k\to\infty} \int_{B_r} |D^2 w_{\varepsilon_k}|^2 \leqslant \int_{B_r} |D^2 w_{\Sigma_0}|^2 \, dx,$$

which ensures together with (6.7) that w_{ε_k} converges strongly to w_{Σ_0} in $H^2(B_r)$.

Biharmonicity. — In order to show that w_{Σ_0} solves a biharmonic Dirichlet problem outside the crack Σ_0 is is enough to check that it satisfies the minimality property

$$\int_{B_R} |D^2 w_{\Sigma_0}|^2 \, dx \leqslant \int_{B_R} |D^2 w|^2 \, dx$$

for all $w \in w_{\Sigma_0} + H_0^2(B_R \smallsetminus \Sigma_0)$. Let $z \in H_0^2(B_R \smallsetminus \Sigma_0)$, by density, there exists a sequence of functions $(z_n) \subset \mathscr{C}_c^\infty(B_R \smallsetminus \Sigma_0)$ such that $z_n \to z$ strongly in $H^2(B_R \smallsetminus \Sigma_0)$. Since $z_n = 0$ in a neighborhood of Σ_0, it follows by Hausdorff convergence that $z_n = 0$ in a neighborhood of Σ_{ε_k} for $k \geqslant k_n$ large enough, for some integer $k_n \in \mathbb{N}$. Therefore, for any $k \geqslant k_n$, $w_{\varepsilon_k} + z_n \in w_{\varepsilon_k} + H_{0,\Sigma_{\varepsilon_k}}^2(B_R)$ is an admissible competitor for the minimality property satisfied by the Airy function (see Remark 4.4), and

$$\int_{B_R} |D^2 w_{\varepsilon_k}|^2 \, dx \leqslant \int_{B_R} |D^2 w_{\varepsilon_k} + D^2 z_n|^2 \, dx.$$

Letting first $k \to \infty$ and then $n \to \infty$, and using the strong convergence of (w_{ε_k}) established before yields

$$\int_{B_R} |D^2 w_{\Sigma_0}|^2 \, dx \leqslant \int_{B_R} |D^2 w_{\Sigma_0} + D^2 z|^2 \, dx.$$

The proof of the proposition is now complete. $\qquad\square$

Remark 6.2. — By elliptic regularity, it follows that w_{Σ_0} is smooth outside the origin up to both sides of Σ_0. In particular, for every $0 < r < R < \infty$ and for every $k \in \mathbb{N}$, $w_{\Sigma_0} \in H^k((B_R \smallsetminus B_r) \smallsetminus \Sigma_0)$ and is a solution for problem (6.4) in a stronger sense.

It turns out that w_{Σ_0} can be made explicit by showing that it is a positively $3/2$-homogeneous function. The proof of this result follows an argument given by Monique Dauge, relying on the theory introduced by Kondrat'ev in [23], that is briefly recalled in the appendix.

Proposition 6.3. — *The function w_{Σ_0} is positively $3/2$-homogeneous. More precisely, in polar coordinates, we have for all $(r,\theta) \in (0,+\infty) \times (0, 2\pi)$,*

$$w_{\Sigma_0}(r\cos\theta, r\sin\theta) = r^{3/2} \left[c_1 \psi_1(\theta) + c_2 \psi_2(\theta) \right],$$

where c_1 and $c_2 \in \mathbb{R}$ are constants, while ψ_1 and ψ_2 are given by

$$(6.10) \qquad \psi_1(\theta) := \left[\frac{3}{2} \cos(\theta/2) - \frac{1}{2} \cos(3\theta/2) \right],$$

$$(6.11) \qquad \psi_2(\theta) := \left[\frac{3}{2} \sin(\theta/2) + \frac{1}{2} \sin(3\theta/2) \right].$$

Proof. — Let w_{Σ_0} be the biharmonic function in $\mathbb{R}^2 \smallsetminus \Sigma_0$ with homogeneous Dirichlet boundary conditions given by Proposition 6.1, and let $\chi \in \mathscr{C}_c^\infty(\mathbb{R}^2; [0,1])$ be a cut-off function satisfying $\chi = 1$ in B_1 and $\chi = 0$ in $\mathbb{R}^2 \smallsetminus B_2$. We decompose w_{Σ_0} as follows:

$$w_{\Sigma_0} = w_0 + w_\infty$$

where $w_0 := \chi w_{\Sigma_0}$ and $w_\infty := (1-\chi) w_{\Sigma_0}$. Of course both w_0 and w_∞ still satisfy homogenous boundary Dirichlet conditions on Σ_0, and one can check that

$$\Delta^2 w_0 = f_0 \text{ and } \Delta^2 w_\infty = f_\infty \text{ in } \mathbb{R}^2 \smallsetminus \Sigma_0,$$

for some f_0 and f_∞ supported in the annulus $B_2 \smallsetminus B_1$. In addition, according to Remark 6.2, it follows that both f_0 and $f_\infty \in H^k(\mathbb{R}^2 \smallsetminus \Sigma_0)$ for every $k \in \mathbb{N}$, and consequently f_0 and $f_\infty \in V_\beta^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ for all $\ell \in \mathbb{Z}$ and all $\beta \in \mathbb{R}$ (we recall Section 2.3 for the definition of V_β^ℓ). We next intend to apply Theorem A.2 to w_0 and w_∞ separately.

Step 1: Analysis of w_0. — Since $w_0 \in H_0^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$, we get that $w_0 \in V_0^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$. To establish this property, it suffices to check that the functions $x \mapsto |x|^{-1}\partial^\alpha w_0(x)$ (with $|\alpha| = 1$) and $x \mapsto |x|^{-2}w_0(x)$ belong to $L^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$. Indeed,

$$
\int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{-2}|\partial^\alpha w_0|^2 \, dx = \sum_{j \in \mathbb{Z}} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |x|^{-2}|\partial^\alpha w_0|^2 \, dx
$$
$$
\leqslant \sum_{j \in \mathbb{Z}} 2^{-2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_0|^2 \, dx.
$$

Since all weak derivatives $\partial^\alpha w_0$ for $|\alpha| = 1$ belong to $H_{0,\Sigma_0}^1((B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0)$, Poincaré inequality yields

$$
\int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_0|^2 \, dx \leqslant C_0 2^{2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |D^2 w_0|^2 \, dx,
$$

for some constant $C_0 > 0$ independent of j, and thus

$$
(6.12) \quad \int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{-2}|\partial^\alpha w_0|^2 \, dx \leqslant C_0 \sum_{j \in \mathbb{Z}} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |D^2 w_0|^2 \, dx
$$
$$
= C_0 \int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |D^2 w_0|^2 \, dx < \infty.
$$

Similarly, we have

$$
\int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{-4}|w_0|^2 \, dx = \sum_{j \in \mathbb{Z}} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |x|^{-4}|w_0|^2 \, dx
$$
$$
\leqslant \sum_{j \in \mathbb{Z}} 2^{-4j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |w_0|^2 \, dx.
$$

Applying again Poincaré inequality to the function $w_0 \in H_{0,\Sigma_0}^1((B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0)$, we obtain

$$
\int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |w_0|^2 \, dx \leqslant C_0 2^{2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_0|^2 \, dx,
$$

and thus, according to (6.12),

$$
\int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{-4}|w_0|^2 \, dx \leqslant C_0 \sum_{j \in \mathbb{Z}} 2^{-2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_0|^2 \, dx
$$
$$
\leqslant 4C_0 \int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{-2}|\nabla w_0|^2 \, dx < \infty.
$$

Since in particular $f_0 \in V_\beta^{-2}(\mathbb{R}^2 \smallsetminus \Sigma_0) \cap V_0^{-2}(\mathbb{R}^2 \smallsetminus \Sigma_0)$ for any $\beta < 0$, applying Theorem A.2 yields that for any $\beta \in \mathbb{R}^- \smallsetminus \mathscr{S}$, there exists $z_0 \in V_\beta^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$ such that

$$
w_0 = z_0 + \sum_{\lambda \in \mathscr{S} \cap (1, 1-\beta)} r^\lambda \varphi_\lambda(\theta).
$$

Step 2: Analysis of w_∞. — We first observe that the growth condition (6.5) satisfied by w_{Σ_0} shows that

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |D^2 w_\infty|^2 \, dx < \infty$$

since w_∞ is supported in $\mathbb{R}^2 \smallsetminus B_1$. Let us check that this growth condition implies $w_\infty \in V_\beta^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$ with $\beta < -1/2$. Indeed, for $|\alpha| = 2$,

$$\int_{\mathbb{R}^2 \smallsetminus \Sigma_0} |x|^{2\beta} |\partial^\alpha w_\infty|^2 dx = \int_{\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup B_1)} |x|^{2\beta} |\partial^\alpha w_\infty|^2 dx$$

$$\leqslant \sum_{j \geqslant 0} \int_{B_{2^{j+1}} \smallsetminus B_{2^j}} |x|^{2\beta} |D^2 w_\infty|^2 dx$$

$$\leqslant \sum_{j \geqslant 0} 2^{2\beta j} \int_{B_{2^{j+1}} \smallsetminus B_{2^j}} |D^2 w_\infty|^2 dx$$

$$\leqslant \sum_{j \geqslant 0} C 2^{2\beta j} 2^{j+1} < +\infty$$

provided that $\beta < -1/2$. We next show that the functions $x \mapsto |x|^{\beta-1} \nabla w_\infty(x)$ and $x \mapsto |x|^{\beta-2} w_\infty(x)$ belong to $L^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$ arguing exactly as in Step 1. It again relies on a dyadic partition of $\mathbb{R}^2 \smallsetminus B_1$ together with the following Poincaré inequalities in each annulus $B_{2^{j+1}} \smallsetminus B_{2^j}$

$$\int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |w_\infty|^2 dx \leqslant C_0 2^{2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_\infty|^2 dx,$$

and

$$\int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \smallsetminus \Sigma_0} |\nabla w_\infty|^2 dx \leqslant C_0 2^{2j} \int_{(B_{2^{j+1}} \smallsetminus B_{2^j}) \Sigma_0} |D^2 w_\infty|^2 dx,$$

which hold since both w_∞ and ∇w_∞ vanish on Σ_0 allowing us to apply Poincaré inequality to them. Therefore it leads to $w_\infty \in V_{\beta_0}^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$ for $\beta_0 = -1/2 - \varepsilon$, where $\varepsilon > 0$ is small.

Conclusion. — We finally gather all the results established so far by taking the same β_0 for the above functions w_0 and w_∞. Observing that $\mathscr{S} \cap (1, 1 - \beta_0) = \{3/2\}$, we get that, in polar coordinates,

$$w(r\cos\theta, r\sin\theta) = r^{3/2} \varphi_{3/2}(\theta) + z(r\cos\theta, r\sin\theta) \quad \text{for a.e. } (r, \theta) \in (0, +\infty) \times (0, 2\pi),$$

for some $z \in V_{-1/2-\varepsilon}^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$. We finally complete the proof of the proposition by establishing that $z = 0$. To this aim, we recall that the function $(r, \theta) \mapsto r^{3/2} \phi_{3/2}(\theta)$ is biharmonic on $\mathbb{R}^2 \smallsetminus \Sigma_0$, and that it vanishes together with its gradient on the crack Σ_0. In other words it is a solution of (P_1) with $f = 0$. We deduce that $z \in V_{-1/2-\varepsilon}^2(\mathbb{R}^2 \smallsetminus \Sigma_0)$ must be a solution of (P_1) with $f = 0$ as well. But since $-1/2 - \varepsilon \notin \mathscr{S}$, Theorem A.1 (with $\beta = 3/2$ and $\ell = 2$) ensures that $z = 0$. $\qquad \square$

6.2. **Blow-up analysis of the displacement.** — We are now in position to study the blow-up of the displacement. We show that, up to a subsequence and rigid movement, it converges to the usual positively 1/2-homogeneous function satisfying the Lamé system outside a half-line.

Theorem 6.4. — *For every sequence $(\varepsilon_n) \searrow 0^+$, there exist a subsequence $(\varepsilon_k) \equiv (\varepsilon_{n_k}) \searrow 0^+$, a sequence (m_k) of rigid movements and a function $u_{\Sigma_0} \in LD_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Sigma_0)$ such that the blow-up sequence of displacements satisfies*

$$(6.13) \qquad \begin{cases} u_{\varepsilon_k} - m_k \to u_{\Sigma_0} & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2), \\ e(u_{\varepsilon_k})\chi_{\mathbb{R}^2 \setminus \Sigma_{\varepsilon_k}} \to e(u_{\Sigma_0}) & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times2}_{\mathrm{sym}}). \end{cases}$$

In addition, the function u_{Σ_0} is positively 1/2-homogeneous and it is given in polar coordinates by

$$(6.14) \quad u_{\Sigma_0}(r\cos\theta, r\sin\theta) = \sqrt{r}[\kappa_1\phi_1(\theta) + \kappa_2\phi_2(\theta)] \quad \text{for all } (r,\theta) \in (0, +\infty) \times (0, 2\pi),$$

where κ_1 and $\kappa_2 \in \mathbb{R}$ are constants, while ϕ_1 and ϕ_2 are defined by

$$(6.15) \qquad \phi_1(\theta) := \begin{pmatrix} \dfrac{\lambda+\mu}{2}\cos(3\theta/2) + \dfrac{\lambda-3\mu}{2}\cos(\theta/2) \\ \dfrac{\lambda+\mu}{2}\sin(3\theta/2) + \dfrac{5\lambda+9\mu}{2}\sin(\theta/2) \end{pmatrix},$$

and

$$(6.16) \qquad \phi_2(\theta) := \begin{pmatrix} -\dfrac{\lambda+\mu}{2}\sin(3\theta/2) - \dfrac{3\lambda+7\mu}{2}\sin(\theta/2) \\ \dfrac{\lambda+\mu}{2}\cos(3\theta/2) + \dfrac{\lambda+5\mu}{2}\cos(\theta/2) \end{pmatrix}.$$

Proof. — A scalar version of that theorem is contained in [9, Th. 1.1], but the proof does not extend directly to the vectorial case. This is why we present here an alternative argument based on the Airy function.

Let (ε_k) be the subsequence given by Proposition 6.1. As in the proof of that result, $R > 1$ is fixed, and $k \in \mathbb{N}$ is large enough such that $2R < R_0/\varepsilon_k$.

Compactness. — Let us denote by $C := B_{1/4}(1/2, 0)$ the ball of center $(1/2, 0)$ and radius $1/4$. We consider the following sequence of rigid displacements

$$\overline{u}_k(x) := \frac{1}{|C|}\int_C u_{\varepsilon_k}(y)\,dy + \left(\frac{1}{|C|}\int_C \frac{\nabla u_{\varepsilon_k}(y) - \nabla u_{\varepsilon_k}(y)^T}{2}\,dy\right)\left(x - \frac{1}{|C|}\int_C y\,dy\right).$$

Thanks to (4.4), (6.2) and (6.3), the stress is given by

$$(6.17) \qquad\qquad \mathbb{C}e(u_{\varepsilon_k}) = \begin{pmatrix} D_{22}w_{\varepsilon_k} & -D_{12}w_{\varepsilon_k} \\ -D_{12}w_{\varepsilon_k} & D_{11}w_{\varepsilon_k} \end{pmatrix}.$$

Therefore, according to (6.6), we deduce that the sequence $(e(u_{\varepsilon_k}))_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(B_R; \mathbb{M}^{2\times2}_{\mathrm{sym}})$. Consequently, up to a subsequence (not relabeled), there exists $e \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times2}_{\mathrm{sym}})$ such that $e(u_{\varepsilon_k}) \rightharpoonup e$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times2}_{\mathrm{sym}})$. In addition, the strong $H^2_{\mathrm{loc}}(\mathbb{R}^2)$-convergence of the Airy function established in Theorem 6.1

together with (4.4), (6.2) and (6.3) shows that actually $e(u_{\varepsilon_k}) \to e$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$.

We next show that e is the symmetrized gradient of some displacement. To this aim, we consider, for any $0 < \delta < 1/10$, the Lipschitz domain

$$U_\delta := \{x \in B_R : \mathrm{dist}(x, \Sigma_0) > \delta\}.$$

Note that for such δ, $C \Subset U_\delta$, while $\Sigma_{\varepsilon_k} \cap U_\delta = \varnothing$ for k large enough (depending on δ). By virtue of the Poincaré-Korn inequality [2, Th. 5.2 and Exam. 5.3] we get that

$$\tag{6.18} \|u_{\varepsilon_k} - \overline{u}_k\|_{H^1(U_\delta)} \leqslant c_\delta \|e(u_{\varepsilon_k})\|_{L^2(U_\delta)},$$

for some constant $c_\delta > 0$ depending on δ. Thanks to a diagonalisation argument, we obtain for a subsequence (not relabeled) a function $\widehat{u}_{\Sigma_0} \in LD_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Sigma_0)$ such that $u_{\varepsilon_k} - \overline{u}_k \to \widehat{u}_{\Sigma_0}$ strongly in $H^1(U_\delta; \mathbb{R}^2)$, for any $0 < \delta < 1/10$. Necessarily we must have that $e = e(\widehat{u}_{\Sigma_0})$ and

$$\begin{cases} u_{\varepsilon_k} - \overline{u}_k \to \widehat{u}_{\Sigma_0} & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2), \\ e(u_{\varepsilon_k})\chi_{\mathbb{R}^2 \setminus \Sigma_{\varepsilon_k}} \to e(\widehat{u}_{\Sigma_0}) & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \end{cases}$$

Minimality. — We next show that $\widehat{u}_{\Sigma_0}$ satisfies the minimality property

$$\int_{B_R} \mathbb{C}e(\widehat{u}_{\Sigma_0}) : e(\widehat{u}_{\Sigma_0})\, dx \leqslant \int_{B_R} \mathbb{C}e(\widehat{u}_{\Sigma_0} + v) : e(\widehat{u}_{\Sigma_0} + v)\, dx$$

for all $v \in LD(B_R \setminus \Sigma_0)$ such that $v = 0$ on $\partial B_R \setminus \Sigma_0$. According to [5, Th. 1], it is enough to consider competitors $v \in H^1(B_R \setminus \Sigma_0; \mathbb{R}^2)$ with $v = 0$ on $\partial B_R \setminus \Sigma_0$. Moreover, since $\{0\}$ has zero $\mathrm{Cap}_{1,2}$-capacity, we can also assume without loss of generality that $v = 0$ in a neighborhood of the origin.

Denoting by $C_k^{\pm}$ the connected component of $(B_R \setminus \Sigma_{\varepsilon_k}) \cap \{x_1 \leqslant 0\}$ which contains the point $(-1/2, \pm 1/2)$, we define v_k as follows:

$$v_k(x_1, x_2) = \begin{cases} v(x_1, x_2) & \text{if } (x_1, x_2) \in [B_R \cap \{x_1 > 0\}] \\ & \qquad \cup [C_k^+ \cap \{x_2 \geqslant 0\}] \cup [C_k^- \cap \{x_2 \leqslant 0\}]; \\ v(x_1, -x_2) & \text{if } (x_1, x_2) \in [C_k^+ \cap \{x_2 < 0\}] \cup [C_k^- \cap \{x_2 > 0\}]; \\ 0 & \text{elsewhere.} \end{cases}$$

Then, one can check that $v_k \in H^1(B_R \setminus \Sigma_{\varepsilon_k}; \mathbb{R}^2)$ and $v_k = 0$ on $\partial B_R \setminus \Sigma_{\varepsilon_k}$. Moreover, $v_k \to v$ strongly in $L^2(B_R; \mathbb{R}^2)$ and $(\nabla v_k)\chi_{B_R \setminus \Sigma_{\varepsilon_k}} \to \nabla v$ strongly in $L^2(B_R; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. Therefore, thanks to the minimality property (3.3) satisfied by u_0, we infer that

$$\int_{B_R} \mathbb{C}e(u_{\varepsilon_k}) : e(u_{\varepsilon_k})\, dx \leqslant \int_{B_R} \mathbb{C}e(u_{\varepsilon_k} + v_k) : e(u_{\varepsilon_k} + v_k)\, dx,$$

so that passing to the limit as $k \to \infty$, and invoking the strong convergences (6.13) yields the desired minimality property.

Explicit expression of the displacement. — According to [20, Th. I and Rem. 1.2], (see also [7, Rem. 2.1]), there exist constants κ_1 and $\kappa_2 \in \mathbb{R}$, and a function $g \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\widehat{u}_{\Sigma_0}(r\cos\theta, r\sin\theta) = \sqrt{r}[\kappa_1\phi_1(\theta) + \kappa_2\phi_2(\theta)] + g(r\cos\theta, r\sin\theta)$$
$$\text{for a.e. } (r,\theta) \in (0, +\infty) \times (0, 2\pi).$$

The previous expression of the displacement shows that

$$(6.19) \qquad\qquad \mathbb{C}e(\widehat{u}_{\Sigma_0}) = \Phi + \mathbb{C}e(g),$$

where Φ is a positively $-1/2$-homogeneous function. On the other hand, passing to the limit in (6.17) as $k \to \infty$ and using Proposition 6.1 yields

$$(6.20) \qquad\qquad \mathbb{C}e(\widehat{u}_{\Sigma_0}) = \begin{pmatrix} D_{22}w_{\Sigma_0} & -D_{12}w_{\Sigma_0} \\ -D_{12}w_{\Sigma_0} & D_{11}w_{\Sigma_0} \end{pmatrix}.$$

According to Proposition 6.3 the right hand side of the previous equality is positively $-1/2$-homogeneous as well. Therefore gathering (6.19) and (6.20) ensures that $e(g) = 0$ which shows that $g = m$ is a rigid movement. We finally define the rigid displacement $m_k := \overline{u}_k + m$ which fulfills the conclusions of the proposition. $\qquad\square$

7. Energy release rate

Following the approach of [7], our aim is to give a definition of energy release rate by studying the convergence of the blow-up functional $\frac{1}{\varepsilon}\mathscr{G}(\varepsilon\Gamma)$. The following statement is the same as [7, Th. 3.1], but with the substantial difference that now Γ_0 is not assumed to be a straight line segment near the origin, but only blowing-up to such a segment for the Hausdorff distance.

Theorem 7.1. — *Let $(\Gamma_\varepsilon)_{\varepsilon>0}$ be a sequence of crack increments in $\mathscr{K}(\overline{\Omega})$ be such that $\sup_\varepsilon \mathscr{H}^1(\Gamma_\varepsilon) < \infty$, and $\Gamma_\varepsilon \to \Gamma$ in the sense of Hausdorff in $\overline{\Omega}$. Let us consider the rescaled crack Σ_ε and displacement u_ε defined, respectively by (6.1) and (6.2). Then for every sequence $(\varepsilon_n) \searrow 0^+$, there exist a subsequence $(\varepsilon_k) \equiv (\varepsilon_{n_k}) \searrow 0^+$ and a rotation $\mathscr{R} \in SO(2)$ such that*

$$(7.1) \qquad\qquad \lim_{k\to\infty} \frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) = \mathscr{F}(\Gamma)$$

where $\mathscr{F}$ is defined by

$$(7.2) \quad \mathscr{F}(\Gamma) := \min_{w \in LD(\mathbb{R}^2 \setminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))} \left\{ \frac{1}{2}\int_{\mathbb{R}^2} \mathbb{C}e(w) : e(w)\,dx + \int_{B_R} \mathbb{C}e(u_{\Sigma_0}) : e(w)\,dx \right.$$
$$\left. - \int_{\partial B_R} \mathbb{C}e(u_{\Sigma_0}) : (w \odot \nu)d\mathscr{H}^1 \right\},$$

where $R > 0$ is any radius such that $\Gamma \subset B_R$, and u_{Σ_0} is the function introduced in Theorem 6.4.

Remark 7.2. — The proof of Theorem 7.1 follows the scheme of [7, Th. 3.1], but some technical issues arise at two main points: 1) the explicit expression for the blow-up at the origin does not come directly from the literature but now follows from our first main result Theorem 6.4, and 2) the construction of a recovery sequence of functions in the moving domains that converges in a strong sense to prove the minimality of the limit is more involved, since now after rescaling everything in B_1 our sequence of domains also moves on ∂B_1.

Remark 7.3. — In the scalar case (antiplane) the limit does actually not depend on the subsequence due to the existence of blow-up limit for the whole sequence [9].

Proof of Theorem 7.1. — Let $(\varepsilon_n) \searrow 0^+$ and $(\varepsilon_k) \equiv (\varepsilon_{n_k}) \subset (\varepsilon_n)$ be the subsequence given by Theorem 6.4. Let us consider the rotation $\mathscr{R}_\varepsilon$ be introduced at the beginning of Section 6. It is not restrictive to assume that $\mathscr{R}_{\varepsilon_k}$ converges to some limit rotation $\mathscr{R}$. In particular $\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k})$ converges to $\mathscr{R}(\Gamma)$ in the sense of Hausdorff.

Rescaling. — We denote by u_k a solution of the minimization problem

$$(7.3) \quad \min\left\{\frac{1}{2}\int_\Omega \mathbb{C}e(v) : e(v)\,dx \;:\; v \in LD(\Omega \smallsetminus (\Gamma_0 \cup \varepsilon_k\Gamma_{\varepsilon_k}))\right.$$
$$\left. \text{and } v = \psi \text{ on } \partial\Omega \smallsetminus (\Gamma_0 \cup \varepsilon_k\Gamma_{\varepsilon_k})\right\}.$$

Recalling (3.6) and (3.7), we can write

$$\frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) = \frac{1}{2\varepsilon_k}\int_\Omega \left[\mathbb{C}e(u_k):e(u_k) - \mathbb{C}e(u_0):e(u_0)\right]dx,$$

and setting $\widehat{w}_k := u_k - u_0$, we obtain that

$$\frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) = \frac{1}{2\varepsilon_k}\int_\Omega \mathbb{C}e(\widehat{w}_k):e(\widehat{w}_k)\,dx + \frac{1}{\varepsilon_k}\int_\Omega \mathbb{C}e(\widehat{w}_k):e(u_0)\,dx.$$

Since $\widehat{w}_k = 0$ on $\partial\Omega \smallsetminus (\Gamma_0 \cup \varepsilon_k\Gamma_{\varepsilon_k})$, the variational formulation of (7.3) ensures that

$$\int_\Omega \mathbb{C}e(u_k):e(\widehat{w}_k)\,dx = 0,$$

and it follows, writing $u_0 = u_k - \widehat{w}_k$,

$$(7.4) \qquad\qquad \frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) = -\frac{1}{2\varepsilon_k}\int_\Omega \mathbb{C}e(\widehat{w}_k):e(\widehat{w}_k)\,dx.$$

On the other hand, from (7.3) it is easy to see that $\frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k})$ is also resulting from a minimization problem with homogeneous boundary condition. Indeed, for any $\widehat{w} \in LD(\Omega \smallsetminus (\Gamma_0 \cup \varepsilon_k\Gamma_{\varepsilon_k}))$ with $\widehat{w} = 0$ on $\partial\Omega \smallsetminus (\Gamma_0 \cup \varepsilon_k\Gamma_{\varepsilon_k})$, denoting $v = u_0 + \widehat{w}$, we obtain that

$$\frac{1}{2}\int_\Omega \mathbb{C}e(v):e(v)\,dx$$
$$= \frac{1}{2}\int_\Omega \mathbb{C}e(u_0):e(u_0)\,dx + \frac{1}{2}\int_\Omega \mathbb{C}e(\widehat{w}):e(\widehat{w})\,dx + \int_\Omega \mathbb{C}e(u_0):e(\widehat{w})\,dx,$$

which implies

$$(7.5) \quad \frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) = \frac{1}{\varepsilon_k} \min \left\{ \frac{1}{2} \int_\Omega \mathbb{C}e(\widehat{w}) : e(\widehat{w}) \, dx + \int_\Omega \mathbb{C}e(u_0) : e(\widehat{w}) \, dx \ : \right.$$

$$\left. \widehat{w} \in LD(\Omega \setminus (\Gamma_0 \cup \varepsilon_k \Gamma_{\varepsilon_k})) \text{ and } \widehat{w} = 0 \text{ on } \partial\Omega \setminus (\Gamma_0 \cup \varepsilon_k \Gamma_{\varepsilon_k}) \right\}$$

$$= \frac{1}{2\varepsilon_k} \int_\Omega \mathbb{C}e(\widehat{w}_k) : e(\widehat{w}_k) \, dx + \frac{1}{\varepsilon_k} \int_\Omega \mathbb{C}e(u_0) : e(\widehat{w}_k) \, dx.$$

According to the assumptions done on Γ_ε, there exists $R > 0$ such that if ε is small enough, then $\Gamma_\varepsilon \subset \overline{B}_R \subset \Omega$, and $\mathscr{H}^1(\varepsilon\Gamma_\varepsilon) \leqslant C\varepsilon$ for some constant $C > 0$ independent of ε. In addition, thanks to the lower bound in Proposition 5.1, we get again for ε small enough,

$$-\frac{1}{\varepsilon} \mathscr{G}(\varepsilon\Gamma_\varepsilon) \leqslant C,$$

which implies from (7.4)

$$(7.6) \qquad\qquad \frac{1}{\varepsilon_k} \int_\Omega \mathbb{C}e(\widehat{w}_k) : e(\widehat{w}_k) \, dx \leqslant C.$$

We now proceed to the following change of variable:

$$\Omega_k := \varepsilon_k^{-1} \mathscr{R}_{\varepsilon_k}(\Omega), \quad \Sigma_{\varepsilon_k} := \varepsilon_k^{-1} \mathscr{R}_{\varepsilon_k}(\Gamma_0),$$

and for $y \in \Omega_k$,

$$w_k(y) := \varepsilon_k^{-1/2} \widehat{w}_k(\mathscr{R}_{\varepsilon_k}^{-1}(\varepsilon_k y)), \quad u_{\varepsilon_k}(y) := \varepsilon_k^{-1/2} u_0(\mathscr{R}_{\varepsilon_k}^{-1}(\varepsilon_k y)).$$

We easily deduce from (7.6) that

$$(7.7) \qquad\qquad \int_{\Omega_k} \mathbb{C}e(w_k) : e(w_k) \, dx \leqslant C.$$

We can also recast the minimization problem in (7.5) in terms of w_k, which now writes as

$$(7.8) \quad \frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) = \min \left\{ \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w) : e(w) \, dx + \int_{\Omega_k} \mathbb{C}e(u_{\varepsilon_k}) : e(w) \, dx \ : \right.$$

$$\left. w \in LD(\Omega_k \setminus (\Sigma_{\varepsilon_k} \cup \mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))) \text{ and } w = 0 \text{ on } \partial\Omega_k \setminus (\Sigma_{\varepsilon_k} \cup \mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k})) \right\}$$

$$= \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w_k) : e(w_k) \, dx + \int_{\Omega_k} \mathbb{C}e(u_{\varepsilon_k}) : e(w_k) \, dx$$

where we used (7.4) in the last equality.

Compactness. — We now extend w_k by 0 outside Ω_k in such a way that $w_k \in LD(\mathbb{R}^2 \setminus (\Sigma_{\varepsilon_k} \cup \mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k})))$. Defining

$$e_k := \begin{cases} e(w_k) & \text{in } \Omega_k \\ 0 & \text{otherwise,} \end{cases}$$

and using (7.7) together with the coercivity of $\mathbb{C}$, we infer that the sequence $(e_k)_{k \in \mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{R}^2; \mathbb{M}_{\text{sym}}^{2 \times 2})$. Consequently, up to a new subsequence (not relabeled), we can assume that $e_k \rightharpoonup e$ weakly in $L^2(\mathbb{R}^2; \mathbb{M}_{\text{sym}}^{2 \times 2})$ for some function $e \in L^2(\mathbb{R}^2; \mathbb{M}_{\text{sym}}^{2 \times 2})$.

Let us recall that $\Sigma_{\varepsilon_k} \to \Sigma_0 := (-\infty, 0] \times \{0\}$ locally in the sense of Hausdorff in $\mathbb{R}^2$, and that $\Gamma_{\varepsilon_k} \to \Gamma$ in the sense of Hausdorff in $\overline{\Omega}$. Let us denote by $\widehat{B} := B_{1/2}((R+1, 0))$ the ball of $\mathbb{R}^2$ centered at the point $(R+1, 0)$ and of radius $1/2$. Since $\Gamma \subset B_R$ and thus $\mathscr{R}(\Gamma) \subset B_R$, we deduce that $(\Sigma_0 \cup \mathscr{R}(\Gamma)) \cap \widehat{B} = \varnothing$. Therefore, for k large enough, $\widehat{B} \subset \Omega_k \smallsetminus (\Sigma_{\varepsilon_k} \cup \mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))$. Let us consider a bounded and smooth open set $U \subset \mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma))$ containing $\widehat{B}$. Then for all k large enough, we have $\overline{U} \subset \Omega_k \smallsetminus (\Sigma_{\varepsilon_k} \cup \mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))$, and we denote by r_k the rigid movement defined by

$$r_k(x) := \frac{1}{|\widehat{B}|} \int_{\widehat{B}} w_k(y) \, dy + \left(\frac{1}{|\widehat{B}|} \int_{\widehat{B}} \frac{\nabla w_k(y) - \nabla w_k(y)^T}{2} \, dy \right) \left(x - \frac{1}{|\widehat{B}|} \int_{\widehat{B}} y \, dy \right).$$

By Korn's inequality, we obtain that

$$\|w_k - r_k\|_{H^1(U)} \leqslant C_U,$$

for some constant $C_U > 0$ depending on U but independent of k. This implies that, up to a subsequence, $w_k - r_k \rightharpoonup w$ weakly in $H^1(U; \mathbb{R}^2)$ for some $w \in H^1(U; \mathbb{R}^2)$. By exhausting $\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma))$ with countably many open sets, extracting successively many subsequences and using a diagonal argument, we obtain that $w \in H^1_{\text{loc}}(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)); \mathbb{R}^2)$ and

$$w_k - r_k \rightharpoonup w \text{ weakly in } H^1_{\text{loc}}(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)); \mathbb{R}^2).$$

Moreover by uniqueness of the limit we infer that $e(w) = e$ a.e. in $\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma))$, therefore that $e(w) \in L^2(\mathbb{R}^2; \mathbb{M}_{\text{sym}}^{2 \times 2})$ and $w \in LD(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))$.

Lower bound inequality. — Let $\zeta \in W^{1,\infty}(\mathbb{R}^2; [0, 1])$ be a cut-off function such that $\zeta = 1$ on B_R and $\zeta = 0$ on $\mathbb{R}^2 \smallsetminus B_{R'}$ for some given $R' > R$. Recalling (7.8) we can write

$$\frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) = \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w_k) : e(w_k) \, dx$$

$$+ \int_{B_{R'}} \zeta \, \mathbb{C}e(u_{\varepsilon_k}) : e(w_k) \, dx + \int_{\Omega_k \smallsetminus B_R} (1 - \zeta)\mathbb{C}e(u_{\varepsilon_k}) : e(w_k) \, dx$$

$$= \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w_k - r_k) : e(w_k - r_k) \, dx$$

$$+ \int_{B_{R'}} \zeta \, \mathbb{C}e(u_{\varepsilon_k}) : e(w_k - r_k) \, dx + \int_{\Omega_k \smallsetminus B_R} (1 - \zeta)\mathbb{C}e(u_{\varepsilon_k}) : e(w_k - r_k) \, dx.$$

Since $\Gamma_{\varepsilon_k} \subset B_R$, then $(1-\zeta)w_k + \zeta r_k = r_k$ in B_R. On the other hand, $(1-\zeta)w_k + \zeta r_k = w_k = 0$ on $\partial \Omega_k \smallsetminus \Sigma_{\varepsilon_k}$ so that it is and admissible variation for the minimization

problem defining u_{ε_k}. We deduce

$$\int_{\Omega_k \smallsetminus B_R} \mathbb{C}e(u_{\varepsilon_k}) : e\big((1-\zeta)(w_k - r_k)\big)\,dx = \int_{\Omega_k} \mathbb{C}e(u_{\varepsilon_k}) : e\big((1-\zeta)w_k + \zeta r_k\big)\,dx = 0,$$

and it follows

$$\frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) = \frac{1}{2}\int_{\Omega_k} \mathbb{C}e(w_k - r_k) : e(w_k - r_k)\,dx$$
$$+ \int_{B_{R'}} \zeta\,\mathbb{C}e(u_{\varepsilon_k}) : e(w_k - r_k)\,dx + \int_{\Omega_k \smallsetminus B_R} (\nabla\zeta \odot (w_k - r_k)) : \mathbb{C}e(u_{\varepsilon_k})\,dx.$$

Recalling from Theorem 6.4 that $u_{\varepsilon_k} \to u_{\Sigma_0}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, and $e(u_{\varepsilon_k}) \to e(u_{\Sigma_0})$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{M}^{2\times2}_{\mathrm{sym}})$, while $w_k - r_k \to w$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, and $e(w_k - r_k) \rightharpoonup e(w)$ weakly in $L^2(\mathbb{R}^2; \mathbb{M}^{2\times2}_{\mathrm{sym}})$, we infer that

$$(7.9) \quad \liminf_{j\to\infty} \frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) \geqslant \frac{1}{2}\int_{\mathbb{R}^2} \mathbb{C}e(w) : e(w)\,dx$$
$$+ \int_{B_{R'}} \zeta\,\mathbb{C}e(u_{\Sigma_0}) : e(w)\,dx + \int_{\Omega_k \smallsetminus B_R} (\nabla\zeta \odot w) : \mathbb{C}e(u_{\Sigma_0})\,dx.$$

We now let ζ be the Lipschitz and radial function defined by

$$(7.10) \qquad \zeta(x) = \begin{cases} 1 & \text{if } x \in B_R, \\ \dfrac{|x| - R}{R' - R} & \text{if } x \in B_{R'} \smallsetminus B_R, \\ 0 & \text{if } x \in \mathbb{R}^2 \smallsetminus B_{R'}. \end{cases}$$

Letting $R' \to R$ in the right-hand side of (7.9) we finally get that, for $\mathscr{L}^1$-a.e. $R > 0$,

$$\liminf_{j\to\infty} \frac{1}{\varepsilon_k}\mathscr{G}(\varepsilon_k\Gamma_{\varepsilon_k}) \geqslant \frac{1}{2}\int_{\mathbb{R}^2} \mathbb{C}e(w) : e(w)\,dx$$
$$+ \int_{B_R} \mathbb{C}e(u_{\Sigma_0}) : e(w)\,dx + \int_{\partial B_R} w \cdot (\mathbb{C}e(u_{\Sigma_0}\nu))\,d\mathscr{H}^1.$$

Reduction to competitors in $H^1(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)); \mathbb{R}^2)$ with compact support. — In order to show that w is a minimizer of the limit problem (7.2), we start by establishing that, without loss of generality, competitors in (7.2) can be taken in $H^1(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)); \mathbb{R}^2)$ with compact support. First we reduce to the case where the competitor belong to $LD(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))$ have compact support. To this purpose, let us show that any $z \in LD(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))$ can be approximated strongly in $LD(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))$ by functions with compact support. To this aim we consider $\varphi \in \mathscr{C}^\infty_c(B_2; [0,1])$ satisfying $\varphi = 1$ on B_1, and define

$$\varphi_R(x) := \varphi(x/R).$$

We assume that R is large enough so that $\Gamma \subset B_R$. Then we set $z_R := (z - m_R)\varphi_R$ where m_R is a suitable rigid movement associated to the Poincaré-Korn inequality in the domain $B_{2R} \smallsetminus (B_R \cup \Sigma_0)$ (note that this domain is not Lipschitz but it is a

connected finite union of Lipschitz domains in which Korn's inequality remains valid), namely

$$(7.11) \qquad \int_{B_{2R}\smallsetminus(B_R\cup\Sigma_0)} |z - m_R|^2\, dx \leqslant CR^2 \int_{B_{2R}\smallsetminus(B_R\cup\Sigma_0)} |e(z)|^2\, dx$$

Moreover a immediate computation yields

$$e(z_R) = \varphi_R e(z) + \frac{1}{R}\nabla\varphi\left(\frac{\cdot}{R}\right)\odot(z - m_R).$$

The first term converges strongly to $e(z)$ in $L^2(\mathbb{R}^2;\mathbb{M}_{\mathrm{sym}}^{2\times2})$, while the second term converges to 0 strongly in $L^2(\mathbb{R}^2;\mathbb{M}_{\mathrm{sym}}^{2\times2})$ due to (7.11). As a consequence $z_R \to z$ strongly in $LD(\mathbb{R}^2\smallsetminus(\Sigma_0\cup\mathscr{R}(\Gamma)))$.

Next, we reduce to the case where z lies in the Sobolev space

$$H^1(\mathbb{R}^2\smallsetminus(\Sigma_0\cup\mathscr{R}(\Gamma));\mathbb{R}^2).$$

Let D and D' be bounded open sets such that $\mathrm{Supp}(z)\subset D'\Subset D$. According to the density result [5, Th. 1], we get the existence of a sequence

$$(z_n)\subset H^1(D\smallsetminus(\Sigma_0\cup\mathscr{R}(\Gamma));\mathbb{R}^2)$$

such that $z_n \to z$ strongly in $L^2(D;\mathbb{R}^2)$ and $e(z_n) \to e(z)$ both strongly in $L^2(D;\mathbb{M}_{\mathrm{sym}}^{2\times2})$. This implies in particular that $z_n \to 0$ in $L^2(D\smallsetminus D';\mathbb{R}^2)$. Let $\varphi\in\mathscr{C}_c^\infty(D;[0,1])$, $\varphi = 1$ on D', and set $\widehat{z}_n = \varphi z_n\in H^1(\mathbb{R}^2\smallsetminus(\Sigma_0\cup\mathscr{R}(\Gamma)))$ with $\mathrm{Supp}(\widehat{z}_n)\subset D$, and satisfying $\widehat{z}_n \to z$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$, and $e(\widehat{z}_n) \to e(z)$ strongly in $L^2(\mathbb{R}^2;\mathbb{M}_{\mathrm{sym}}^{2\times2})$.

Upper bound and minimality. — We now assume that $z\in H^1(\mathbb{R}^2\smallsetminus(\Sigma_0\cup\mathscr{R}(\Gamma));\mathbb{R}^2)$ with compact support, contained in some bounded open set D. Clearly the number of connected components of $\partial D\cup((\Sigma_{\varepsilon_k}\cup\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))\cap D)$ is bounded. Hence by [4] or [6] we get the existence of $z_k\in H^1(D\smallsetminus(\Sigma_{\varepsilon_k}\cup\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}));\mathbb{R}^2)$ such that $z_k \to z$ strongly in $L^2(D;\mathbb{R}^2)$ and $(\nabla z_k)\chi_{D\smallsetminus(\Sigma_{\varepsilon_k}\cup\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))} \to \nabla z$ strongly in $L^2(D;\mathbb{M}_{\mathrm{sym}}^{2\times2})$. Multiplying by the same cut-off function φ as in the previous step, we can also assume that $z_k = 0$ in a neighborhood of ∂D. In this way we have obtained

$$z_k\in H^1(\mathbb{R}^2\smallsetminus(\Sigma_{\varepsilon_k}\cup\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}));\mathbb{R}^2)$$

satisfying

$$\mathrm{Supp}(z_k)\subset D\subset\Omega_k \quad\text{(for k large enough),}$$
$$z_k \longrightarrow z \text{ strongly in } L^2(\mathbb{R}^2;\mathbb{R}^2),$$
$$(\nabla z_k)\chi_{\mathbb{R}^2\smallsetminus(\Sigma_{\varepsilon_k}\cup\mathscr{R}_{\varepsilon_k}(\Gamma_{\varepsilon_k}))} \longrightarrow \nabla z \text{ strongly in } L^2(\mathbb{R}^2;\mathbb{M}_{\mathrm{sym}}^{2\times2}).$$

According to the minimality property of w_k (see (7.8)), we have

$$(7.12) \quad \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w_k) : e(w_k) \, dx + \int_{\Omega_k} \mathbb{C}e(u_{\varepsilon_k}) : e(w_k) \, dx$$

$$\leqslant \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(z_k) : e(z_k) \, dx + \int_{\Omega_k} \mathbb{C}e(u_{\varepsilon_k}) : e(z_k) \, dx.$$

Let ζ be the cut-off function defined in (7.10), then performing an integration by parts exactly as we did in step 3 (with z_k instead of $w_k - r_k$) we arrive at the following

$$\frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) = \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(w_k) : e(w_k) \, dx$$

$$+ \int_{\Omega_k} \zeta \mathbb{C}e(u_{\varepsilon_k}) : e(w_k) \, dx + \int_{\Omega_k} [\nabla\zeta \odot w_k] : \mathbb{C}e(u_{\varepsilon_k}) \, dx$$

$$\leqslant \frac{1}{2} \int_{\Omega_k} \mathbb{C}e(z_k) : e(z_k) \, dx + \int_{\Omega_k} \zeta \mathbb{C}e(u_{\varepsilon_k}) : e(z_k) \, dx + \int_{\Omega_k} [\nabla\zeta \odot z_k] : \mathbb{C}e(u_{\varepsilon_k}) \, dx.$$

The convergences established so far for the sequences (z_k) and (u_{ε_k}) enable one to pass to the limit in the previous expression, first as $k \to \infty$ and then $R' \to R$. We finally get that

$$(7.13) \quad \limsup_{j \to \infty} \frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) \leqslant \frac{1}{2} \int_{\mathbb{R}^2} \mathbb{C}e(z) : e(z) \, dx$$

$$+ \int_{B_R} \mathbb{C}e(u_{\Sigma_0}) : e(z) \, dx + \int_{\partial B_R} z \cdot (\mathbb{C}e(u_{\Sigma_0})\nu) \, d\mathscr{H}^1$$

for almost every $R > 0$. By the density result established in step 4, inequality (7.13) holds for any $z \in LD(\mathbb{R}^2 \smallsetminus (\Sigma_0 \cup \mathscr{R}(\Gamma)))$. Taking $z = w$, and gathering with (7.9) yields

$$\lim_{j \to \infty} \frac{1}{\varepsilon_k} \mathscr{G}(\varepsilon_k \Gamma_{\varepsilon_k}) = \frac{1}{2} \int_{\mathbb{R}^2} \mathbb{C}e(w) : e(w) \, dx$$

$$+ \int_{B_R} \mathbb{C}e(u_{\Sigma_0}) : e(w) \, dx + \int_{\partial B_R} w \cdot (\mathbb{C}e(u_{\Sigma_0})\nu) \, d\mathscr{H}^1,$$

and using again (7.13), we deduce that w is a solution of the minimization problem (7.2) for a.e. $R > 0$ with $\Gamma \subset B_R$. Finally, an integration by parts ensures that the value of $\mathscr{F}(\Gamma)$ is independent of $R > 0$ and *a fortiori* holds for every $R > 0$. $\quad\square$

Appendix. A short review of Kondrat'ev theory

We follow the notations and statements of the book [25, Sec. 6.1] that we briefly recall here in the case of the bilaplacian in the cracked plane $\mathbb{R}^2 \smallsetminus \Sigma_0$. Let us consider weak solutions of the problem

$$(P_1) \qquad \begin{cases} \Delta^2 w = f & \text{in } \mathbb{R}^2 \smallsetminus \Sigma_0, \\ w = 0 \text{ and } \partial w/\partial\nu = 0 & \text{on } \Sigma_0, \end{cases}$$

in weighted Sobolev spaces of type $V_\beta^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ (see the definition in Section 2.3) which is the core of Kondrat'ev's Theory. It is easily seen that Δ^2 (associated with homogenous Dirichlet conditions) maps $w \in V_\beta^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ to $f \in V_\beta^{\ell-4}(\mathbb{R}^2 \smallsetminus \Sigma_0)$. For $\ell \geqslant 4$ this fact is quite obvious from the definition, and for $\ell < 4$, it follows from a standard extension argument (see [25, Th. 6.1.2]). Kondrat'ev theory ensures that this operator is actually of Fredholm type, and that it defines an isomorphism provided $\beta \in \mathbb{R} \smallsetminus \mathscr{S}$ and $\ell \in \mathbb{Z}$, where $\mathscr{S}$ is an exceptional countable set. In our special case it turns out to be contained in the set of half integers $\frac{1}{2}\mathbb{Z}$, as for most elliptic operators (see [11]). Indeed, this set appears as the *spectrum* of the Mellin transform of the operator written in polar coordinates, with corresponding boundary conditions. In the language of [25] this will be called the Pencil operator, denoted by $\mathfrak{A}(\lambda)$ and studied in [25, Chap. 5] (and defined pp. 197 in [25] in the case that we are interested in). The exact computations in the special case of the bilaplacian are quite standard, and can be found for instance in [26, Chap. 7.1] (see also [19, Sec. 7.2.1], but with different notations and conventions leading to slightly different characteristic equations). Let us recall here those computations, still using the language of [25].

First we recall that the Mellin transform of a function $g \in \mathscr{C}_c^\infty(\mathbb{R}^+)$ is given by

$$(\text{A.1}) \qquad \widehat{u}(\lambda) = \int_0^{+\infty} r^{-\lambda-1} g(r) dr, \quad \text{for all } \lambda \in \mathbb{C}.$$

Another way to understand this transformation is by taking the Laplace transform of the function $t \mapsto g(e^t)$. Relevant properties are recalled in [25, Lem. 6.1.3], and one of the most important is probably

$$(\text{A.2}) \qquad \widehat{r\partial_r g} = \lambda \widehat{g}.$$

Now let us look for the pencil operator. Since it is obtained via the Mellin transform of Δ^2 (up to a factor r^4), we need to write it in polar coordinates (r, θ) which gives

$$\Delta^2 = \partial_r^4 + \frac{2\partial_r^3}{r} - \frac{\partial_r^2}{r^2} + \frac{\partial_r}{r^3} + \frac{\partial_\theta^4}{r^4} + \frac{4\partial_\theta^2}{r^4} - \frac{2\partial_\theta^2 \partial_r}{r^3} + \frac{2\partial_\theta^2 \partial_r^2}{r^2} \ .$$

We then identify the terms of the form $(r\partial_r)^k$, and for this purpose we shall use the following elementary formulas

$$(r\partial_r)^2 = r\partial_r + r^2\partial_r^2$$
$$(r\partial_r)^3 = r\partial_r + 3r^2\partial_r^2 + r^3\partial_r^3$$
$$(r\partial_r)^4 = r\partial_r + 7r^2\partial_r^2 + 6r^3\partial_r^3 + r^4\partial_r^4$$

which imply

$$\Delta^2 = r^{-4} \left([(r\partial_r)^4 - 4(r\partial_r)^3 + 4(r\partial_r)^2] + [2(r\partial_r)^2 - 4r\partial_r + 4]\partial_\theta^2 + \partial_\theta^4 \right)$$
$$=: r^{-4} \mathscr{L}(\partial_\theta, r\partial_r)$$

The pencil operator $\mathfrak{A}(\lambda)$ is then obtained by taking the Mellin transform (A.1) in the r variable of the operator $\mathscr{L}(\partial_\theta, r\partial_r)$ defined above. Using (A.2) we therefore

obtain

$$(A.3) \quad \begin{aligned} \mathfrak{A}(\lambda) &= (\lambda^4 - 4\lambda^3 + 4\lambda^2) + (2\lambda^2 - 4\lambda + 4)\partial_\theta^2 + \partial_\theta^4 \\ &= (\partial_\theta^2 + (\lambda - 2)^2)(\partial_\theta^2 + \lambda^2), \end{aligned}$$

and the boundary conditions in the variable θ are still zero (i.e., acting on functions φ with the boundary conditions $\varphi(0) = \varphi(2\pi) = \varphi'(0) = \varphi'(2\pi) = 0$). The set $\mathscr{S}$ is then the *spectrum* of $\mathfrak{A}(\lambda)$, and according to the terminology of Operator Pencils this means the set of λ for which the operator is non invertible [25, Chap. 5]. By [26, Chap. 7.1] (see in particular the last paragraph before Section 7.2 for the special case $\alpha = 2\pi$), this set is real and

$$\mathscr{S} = \left\{ 1 \pm \frac{k}{2} ; \; k \in \mathbb{N} \smallsetminus \{0\} \right\}.$$

All of them, except $\lambda = 0$ and $\lambda = 2$, have geometric and algebraic multiplicities equal to 2. The associated eigenfunctions are given by explicit functions that one can find in [26, Eq. (7.1.14) and (7.1.15)]. We shall only give the ones corresponding to $\lambda = 3/2$, which are the functions defined in (6.10) and (6.11).

According to all the above facts, a direct application of [25, Th. 6.1.3] yields

Theorem A.1. — *If $\beta \in \mathbb{R}$ and $\ell \in \mathbb{Z}$ are such that*

$$-\beta + \ell - 1 \notin \mathscr{S},$$

then for every $f \in V_\beta^{\ell-4}(\mathbb{R}^2 \smallsetminus \Sigma_0)$, there exists a unique solution $w \in V_\beta^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ of (P_1).

In addition, a direct application of [25, Th. 6.1.5] implies that

Theorem A.2. — *Let $\beta_2 < \beta_1$ be two real numbers, $\ell \in \mathbb{Z}$, and assume that*

$$-\beta_i + \ell - 1 \notin \mathscr{S}, \quad \text{for all } i \in \{1, 2\}.$$

If $w \in V_{\beta_1}^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ is a solution of (P_1) with $f \in V_{\beta_1}^{\ell-4}(\mathbb{R}^2 \smallsetminus \Sigma_0) \cap V_{\beta_2}^{\ell-4}(\mathbb{R}^2 \smallsetminus \Sigma_0)$, then there exists $z \in V_{\beta_2}^\ell(\mathbb{R}^2 \smallsetminus \Sigma_0)$ such that

$$w - z = \sum_{\lambda \in \mathscr{S} \cap (1-\beta_1, 1-\beta_2)} r^\lambda \varphi_\lambda(\theta),$$

where the φ_λ are linear combinations of eigenfunctions of $\mathfrak{A}(\lambda)$. In particular $\varphi_{3/2} = c_1\psi_1 + c_2\psi_2$ where ψ_1 and ψ_2 are defined in (6.10) and (6.11).

References

[1] D. R. Adams & L. I. Hedberg – *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer-Verlag, Berlin, 1996.

[2] G. Alessandrini, A. Morassi & E. Rosset – "The linear constraints in Poincaré and Korn type inequalities", *Forum Math.* **20** (2008), no. 3, p. 557–569.

[3] B. Bourdin, G. A. Francfort & J.-J. Marigo – *The variational approach to fracture*, Springer, New York, 2008.

[4] D. Bucur & N. Varchon – "A duality approach for the boundary variation of Neumann problems", *SIAM J. Math. Anal.* **34** (2002), no. 2, p. 460–477.

[5] A. Chambolle – "A density result in two-dimensional linearized elasticity, and applications", *Arch. Rational Mech. Anal.* **167** (2003), no. 3, p. 211–233.

[6] A. Chambolle & F. Doveri – "Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets", *Comm. Partial Differential Equations* **22** (1997), no. 5-6, p. 811–840.

[7] A. Chambolle, G. A. Francfort & J.-J. Marigo – "Revisiting energy release rates in brittle fracture", *J. Nonlinear Sci.* **20** (2010), no. 4, p. 395–424.

[8] A. Chambolle, A. Giacomini & M. Ponsiglione – "Crack initiation in brittle materials", *Arch. Rational Mech. Anal.* **188** (2008), no. 2, p. 309–349.

[9] A. Chambolle & A. Lemenant – "The stress intensity factor for non-smooth fractures in antiplane elasticity", *Calc. Var. Partial Differential Equations* **47** (2013), no. 3-4, p. 589–610.

[10] Ph. G. Ciarlet – *Mathematical elasticity. Vol. I*, Studies in Mathematics and its Applications, vol. 20, North-Holland Publishing Co., Amsterdam, 1988, Three-dimensional elasticity.

[11] M. Costabel & M. Dauge – "Crack singularities for general elliptic systems", *Math. Nachr.* **235** (2002), p. 29–49.

[12] G. Dal Maso, G. A. Francfort & R. Toader – "Quasistatic crack growth in nonlinear elasticity", *Arch. Rational Mech. Anal.* **176** (2005), no. 2, p. 165–225.

[13] G. Dal Maso & R. Toader – "A model for the quasi-static growth of brittle fractures: existence and approximation results", *Arch. Rational Mech. Anal.* **162** (2002), no. 2, p. 101–135.

[14] P. Destuynder & M. Djaoua – "Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile", *Math. Methods Appl. Sci.* **3** (1981), no. 1, p. 70–87.

[15] K. J. Falconer – *The geometry of fractal sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986.

[16] G. A. Francfort & Ch. J. Larsen – "Existence and convergence for quasi-static evolution in brittle fracture", *Comm. Pure Appl. Math.* **56** (2003), no. 10, p. 1465–1500.

[17] G. A. Francfort & J.-J. Marigo – "Revisiting brittle fracture as an energy minimization problem", *J. Mech. Phys. Solids* **46** (1998), no. 8, p. 1319–1342.

[18] A. A. Griffith – "The phenomena of rupture and flow in solids", *Philos. Trans. Roy. Soc. London* **221A** (1920), p. 163–198.

[19] P. Grisvard – *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman, Boston, MA, 1985.

[20] _____, "Singularités en elasticité", *Arch. Rational Mech. Anal.* **107** (1989), no. 2, p. 157–180.

[21] A. Henrot & M. Pierre – *Variation et optimisation de formes. une analyse géométrique*, Mathématiques & Applications, vol. 48, Springer, Berlin, 2005.

[22] D. Knees & A. Mielke – "Energy release rate for cracks in finite-strain elasticity", *Math. Methods Appl. Sci.* **31** (2008), no. 5, p. 501–528.

[23] V. A. Kondrat'ev – "Boundary value problems for elliptic equations in domains with conical or angular points", *Trudy Moskov. Mat. Obšč.* **16** (1967), p. 209–292.

[24] V. A. Kondrat'ev, I. Kopachek, D. M. Lekveishvili & O. A. Oleĭnik – "Sharp estimates in Hölder spaces and the exact Saint-Venant principle for solutions of the biharmonic equation", *Trudy Mat. Inst. Steklov.* **166** (1984), p. 91–106, Modern problems of mathematics. Differential equations, mathematical analysis and their applications.

[25] V. A. Kozlov, V. G. Maz'ya & J. Rossmann – *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.

[26] _____, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, vol. 85, American Mathematical Society, Providence, RI, 2001.

[27] G. Lazzaroni & R. Toader – "Energy release rate and stress intensity factor in antiplane elasticity", *J. Math. Pures Appl. (9)* **95** (2011), no. 6, p. 565–584.

[28] M. Negri & Ch.. Ortner – "Quasi-static crack propagation by Griffith's criterion", *Math. Models Methods Appl. Sci.* **18** (2008), no. 11, p. 1895–1925.

[29] V. Šverák – "On optimal shape design", *J. Math. Pures Appl. (9)* **72** (1993), no. 6, p. 537–551.

[30] R. Temam – *Navier-Stokes equations. Theory and numerical analysis*, Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[31] _____, *Problèmes mathématiques en plasticité*, Méthodes Mathématiques de l'Informatique, vol. 12, Gauthier-Villars, Montrouge, 1983.

Jean-François Babadjian, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie–Paris 6, CNRS
Boîte courrier 187, 75252 Paris Cedex 05, France
E-mail : `jean-francois.babadjian@upmc.fr`
Url : `https://www.ljll.math.upmc.fr/babadjian/`

Antonin Chambolle, CMAP, École Polytechnique, CNRS
91128 Palaiseau, France
E-mail : `antonin.chambolle@cmap.polytechnique.fr`
Url : `http://www.cmap.polytechnique.fr/~antonin/`

Antoine Lemenant, Laboratoire Jacques-Louis Lions, Université Paris Diderot–Paris 7, CNRS
Bâtiment Sophie Germain, 75205 Paris Cedex 13, France
E-mail : `lemenant@ljll.univ-paris-diderot.fr`
Url : `https://www.ljll.math.upmc.fr/~lemenant/`