



Journal de l'École polytechnique

Mathématiques

Camilla FELISETTI & Mirko MAURI

P=W conjectures for character varieties with symplectic resolution

Tome 9 (2022), p. 853-905.

http://jep.centre-mersenne.org/item/JEP_2022__9__853_0

© Les auteurs, 2022.

Certains droits réservés.



Cet article est mis à disposition selon les termes de la licence
LICENCE INTERNATIONALE D'ATTRIBUTION CREATIVE COMMONS BY 4.0.
<https://creativecommons.org/licenses/by/4.0/>

L'accès aux articles de la revue « Journal de l'École polytechnique — Mathématiques » (<http://jep.centre-mersenne.org/>), implique l'accord avec les conditions générales d'utilisation (<http://jep.centre-mersenne.org/legal/>).

Publié avec le soutien
du Centre National de la Recherche Scientifique



Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

P=W CONJECTURES FOR CHARACTER VARIETIES WITH SYMPLECTIC RESOLUTION

BY CAMILLA FELISETTI & MIRKO MAURI

ABSTRACT. — We establish $P=W$ and $PI=WI$ conjectures for character varieties with structural group GL_n and SL_n which admit a symplectic resolution, i.e., for genus 1 and arbitrary rank, and genus 2 and rank 2. We formulate the $P=W$ conjecture for a resolution, and prove it for symplectic resolutions. We exploit the topology of birational and quasi-étale modifications of Dolbeault moduli spaces of Higgs bundles. To this end, we prove auxiliary results of independent interest, like the construction of a relative compactification of the Hodge moduli space for reductive algebraic groups, and the projectivity of the compactification of the de Rham moduli space. In particular, we study in detail a Dolbeault moduli space which is a specialization of the singular irreducible holomorphic symplectic variety of type O'Grady 6.

RÉSUMÉ (Les conjectures $P=W$ pour les variétés de caractères ayant une résolution symplectique)

On établit les conjectures $P=W$ et $PI=WI$ pour les variétés de caractères avec groupe structurel GL_n et SL_n qui admettent une résolution symplectique, c'est-à-dire pour le genre 1 en rang arbitraire, et le genre 2 en rang 2. On formule la conjecture $P=W$ pour une résolution et on la prouve pour les résolutions symplectiques. Pour la démonstration on fait appel à la topologie des modifications birationnelles et quasi-étales des espaces de modules de fibrés de Higgs. Pour cela, on démontre des résultats auxiliaires d'intérêt indépendant, comme la construction d'une compactification relative de l'espace de modules de Hodge pour les groupes algébriques réductifs, ou la théorie de l'intersection de certains cycles lagrangiens singuliers. En particulier, on étudie en détail un espace de modules des fibrés de Higgs qui est une spécialisation de la variété symplectique holomorphe irréductible singulière de type O'Grady 6.

CONTENTS

| | |
|---|-----|
| 1. Introduction..... | 854 |
| 2. Preliminaries..... | 860 |
| 3. Lifting the non-abelian Hodge correspondence..... | 863 |
| 4. Moduli spaces for GL_n vs SL_n | 868 |
| 5. $P=W$ conjectures for genus 1..... | 871 |
| 6. The moduli space of Higgs bundles M and its alterations..... | 874 |

MATHEMATICAL SUBJECT CLASSIFICATION (2020). — 14D20, 53D30, 14D22, 14E15, 32S35, 55N33.

KEYWORDS. — $P=W$ conjecture, intersection cohomology, Higgs bundles.

C.F. is supported by the University of Geneva, the Swiss National Science Foundation grants 17599 and 156645, as well as the NCCR SwissMAP. M.M. is supported by Max Planck Institute for Mathematics in Bonn, and University of Michigan.

| | |
|---|-----|
| 7. P=W conjectures for M | 888 |
| 8. PI=WI conjecture for M | 890 |
| Appendix. Degenerations of hyperkähler varieties..... | 898 |
| References..... | 902 |

1. INTRODUCTION

Let X be a compact Riemann surface of genus g , and let G be a complex reductive algebraic group. The Betti and Dolbeault moduli spaces $M_B(X, G)$ and $M_{\text{Dol}}(X, G)$ are central objects in non-abelian Hodge theory. The *Betti* moduli space, or G -character variety of X , is the affine GIT quotient

$$(1) \quad M_B(X, G) := \text{Hom}(\pi_1(X), G) // G \\ = \{(A_1, B_1, \dots, A_g, B_g) \in G^{2g} \mid \prod_{j=1}^g [A_j, B_j] = 1_G\} // G.$$

It parametrizes isomorphism classes of semistable representations of the fundamental group of X with value in G .

The *Dolbeault* moduli space $M_{\text{Dol}}(X, G)$ instead parametrizes semistable principal G -Higgs bundles with vanishing Chern classes; see [78]. For example, we have that:

- a GL_n -Higgs bundle is a pair (E, ϕ) with E vector bundle of rank n and degree 0, and $\phi \in \text{Hom}(E, E \otimes K_X)$;
- a GL_n -Higgs bundle is an SL_n -Higgs bundle if in addition the determinant of E is trivial and the trace of ϕ vanishes;
- a PGL_n -Higgs bundle is an equivalence class of SL_n -Higgs bundles under tensorization by an n -torsion line bundle on C .

Despite the different origin of these moduli spaces, there exists a real analytic isomorphism

$$\Psi: M_{\text{Dol}}(X, G) \longrightarrow M_B(X, G)$$

called *non-abelian Hodge correspondence*; see [79] or Section 3. However, the map Ψ is not an algebraic isomorphism. Indeed, note that the Betti moduli space is an affine variety, while the Dolbeault moduli space admits a projective morphism with connected fibers

$$\chi: M_{\text{Dol}}(X, G) \longrightarrow \mathbb{A}^{\dim M_{\text{Dol}}(X, G)/2},$$

called the *Hitchin fibration*. The purpose of this paper is to study the behaviour in cohomology of the non-abelian Hodge correspondence in view of the P=W conjecture [11]. In the rest of the paper we will only consider reductive groups of type A , i.e., $G = \text{GL}_n, \text{SL}_n, \text{PGL}_n$, unless stated otherwise, e.g. in the formulation of the P=W conjectures or in Section 3.1.

One of the main difficulties while studying the cohomology of these moduli spaces is that they are generally singular. To circumvent this issue, it is customary to slightly

change the moduli problem as follows. Given an integer⁽¹⁾ d coprime with the rank n of the group, the twisted Betti moduli space is the GIT quotient

$$M_B^{tw}(X, G) := \{(A_1, B_1, \dots, A_g, B_g) \in G^{2g} \mid \prod_{j=1}^g [A_j, B_j] = e^{2\pi i d/n} 1_G\} / G.$$

On the other hand, the twisted version of Dolbeault moduli, denoted $M_{Dol}^{tw}(X, G)$, parametrizes semistable pairs (E, ϕ) , with E vector bundle of rank n and degree d ,⁽²⁾ and $\phi \in \text{Hom}(E, E \otimes K_X)$. The technical advantage of working with these twisted moduli spaces is that they are smooth varieties and satisfy a non-abelian Hodge theorem as in the untwisted case; see [44].

While studying the weight filtration on $H^*(M_B^{tw}(X, G), \mathbb{Q})$, Hausel and Rodriguez-Villegas discovered a surprising symmetry, that they called *curious hard Lefschetz* theorem: there exists a class $\alpha \in H^2(M_B^{tw}(X, G), \mathbb{Q})$ which induces the isomorphisms

$$(2) \quad \cup \alpha^k : G_{n-2k}^W H^*(M_B^{tw}(X, G), \mathbb{Q}) \xrightarrow{\simeq} G_{n+2k}^W H^{*+2k}(M_B^{tw}(X, G), \mathbb{Q}).$$

The theorem holds for $G = \text{GL}_2, \text{SL}_2$ and PGL_2 by [41], and for $G = \text{GL}_n$ by [61]. To explain this phenomenon, de Cataldo, Hausel and Migliorini conjectured that the non-abelian Hodge correspondence should exchange the weight filtration on the space $H^*(M_B^{tw}(X, G), \mathbb{Q})$ with the perverse (Leray) filtration associated to χ on the space $H^*(M_{Dol}^{tw}(X, G), \mathbb{Q})$; see Definition 2.8. In this way, the curious hard Lefschetz theorem would correspond to the classical relative hard Lefschetz theorem for χ ; see Theorem 2.10.

CONJECTURE 1.1 (P=W conjecture for twisted moduli spaces)

$$P_k H^*(M_{Dol}^{tw}(X, G), \mathbb{Q}) = \Psi^* W_{2k} H^*(M_B^{tw}(X, G), \mathbb{Q}).$$

The conjecture holds for $g > 2$ and $G = \text{GL}_2, \text{SL}_2$ and PGL_2 by [11], and for $g = 2$ and $G = \text{GL}_n, \text{SL}_p$ with p prime by [15, 14]. An enumerative approach has been proposed in [19], and other P=W phenomena have been studied in [74, 75, 27, 89, 82, 81, 66, 48, 37, 38, 59]. However, P=W phenomena for the original moduli spaces $M_B(X, G)$ and $M_{Dol}(X, G)$ have not been explored yet. This is then the goal of our paper.

In the singular case, relative and curious hard Lefschetz theorems fail in general for singular cohomology; see Remark 8.6. Nonetheless, it is known that the relative hard Lefschetz theorem for χ holds for *intersection cohomology* $IH^*(M_{Dol}(X, G))$; see Sections 2.2 and 2.4. Moreover, de Cataldo and Maulik proved in [13] that the perverse filtration on intersection cohomology is independent of the complex structure of the curve X , exactly as it happens for the weight filtration. Therefore, they conjectured [13, Quest. 4.1.7].

⁽¹⁾We omitted the dependence of $M_B^{tw}(X, G)$ and $M_{Dol}^{tw}(X, G)$ on the degree d not to burden the notation too much.

⁽²⁾Note that we recover the untwisted Dolbeault moduli space for $d = 0$.

CONJECTURE 1.2 (PI=WI conjecture). — *Let G be a complex reductive group. Then*

$$P_k IH^*(M_{\text{Dol}}(X, G), \mathbb{Q}) = \Psi^* W_{2k} IH^*(M_{\text{B}}(X, G), \mathbb{Q}).$$

It is also conceivable that one could obtain the P=W conjecture for the singular moduli spaces $M_{\text{Dol}}(X, G)$ from the previous conjectures.

CONJECTURE 1.3 (P=W conjecture for singular moduli spaces). — *Let G be a complex reductive group. Then*

$$P_k H^*(M_{\text{Dol}}(X, G), \mathbb{Q}) = \Psi^* W_{2k} H^*(M_{\text{B}}(X, G), \mathbb{Q}).$$

Alternatively, we may also opt for a desingularization of $M_{\text{Dol}}(X, G)$, and continue to work with singular cohomology. We show that a P=W conjecture for *symplectic resolution* does hold, i.e., for resolutions where a holomorphic symplectic form on the smooth locus of each moduli space extends to a symplectic form on the whole of the resolution.

To this end, we first show how to lift the non-abelian Hodge correspondence to resolutions of $M_{\text{Dol}}(X, G)$ and $M_{\text{B}}(X, G)$, up to isotopy, according to Theorem 3.8.

THEOREM 1.4 (Theorem 3.8). — *Let G be a complex reductive group. Then there exist resolutions of singularities $f_{\text{Dol}}: \widetilde{M}_{\text{Dol}}(X, G) \rightarrow M_{\text{Dol}}(X, G)$ and $f_{\text{B}}: \widetilde{M}_{\text{B}}(X, G) \rightarrow M_{\text{B}}(X, G)$, and a diffeomorphism $\widetilde{\Psi}: \widetilde{M}_{\text{Dol}}(X, G) \rightarrow \widetilde{M}_{\text{B}}(X, G)$, such that the following square commutes:*

$$\begin{array}{ccc} H^*(\widetilde{M}_{\text{Dol}}(X, G), \mathbb{Q}) & \xleftarrow{\widetilde{\Psi}^*} & H^*(\widetilde{M}_{\text{B}}(X, G), \mathbb{Q}) \\ f_{\text{Dol}}^* \uparrow & & \uparrow f_{\text{B}}^* \\ H^*(M_{\text{Dol}}(X, G), \mathbb{Q}) & \xleftarrow{\Psi^*} & H^*(M_{\text{B}}(X, G), \mathbb{Q}). \end{array}$$

The resolutions f_{Dol} and f_{B} can be taken functorial with respect to smooth algebraic or analytic morphisms, and symplectic if $G = \text{GL}_n$ or SL_n with $(g, n) = (1, n)$ or $(2, 2)$.

CONJECTURE 1.5 (P=W conjecture for symplectic resolution). — *Let G be a complex reductive group. Let $\widetilde{\Psi}, f_{\text{Dol}}$ and f_{B} be the diffeomorphism appearing in Theorem 1.4. If f_{Dol} is a symplectic resolution (if it exists!), or equivalently f_{B} is so, then*

$$P_k H^*(\widetilde{M}_{\text{Dol}}(X, G), \mathbb{Q}) = \widetilde{\Psi}^* W_{2k} H^*(\widetilde{M}_{\text{B}}(X, G), \mathbb{Q}).$$

In an earlier version of this paper, we stated the P=W conjecture for resolution without the assumption of the existence of a symplectic resolution, but later the second author proved that the hypothesis is indeed essential at least for $G = \text{GL}_n$ and SL_n , see [59, §5.6]. Recent results suggest that the existence of a holomorphic symplectic form should be a key ingredient for P=W phenomena, see [12, §4.4], [61] and [37, Th. 1.7].

In this paper, we provide the first evidence for the P=W conjectures in the singular context.

MAIN THEOREM. — *Let $G = \mathrm{GL}_n$ or SL_n . Suppose that $(g, n) = (1, n)$ or $(2, 2)$. Then the following conjectures hold:*

- (1) *the P=W conjecture;*
- (2) *the PI=WI conjecture;*
- (3) *the P=W conjecture for a symplectic resolution.*

Observe that $M_{\mathrm{Dol}}(X, \mathrm{GL}_n)$ and $M_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ admit a (unique) symplectic resolution if and only if $(g, n) = (1, n)$ or $(2, 2)$; see [5] and [29, Th. 2.2]. Under this assumption, the P=W and PI=WI conjectures for $M_{\mathrm{Dol}}(X, \mathrm{PGL}_n)$ hold too; see Remark 4.3.

The expectation is that the PI=WI conjecture holds even in the absence of a symplectic resolution. The second author has provided first evidence of this fact in [59, §5].

Proof of the main theorem. — We first reduce to $G = \mathrm{SL}_n$; see Theorem 4.1.

For $g = 1$, the P=W and PI=WI conjectures follow from Theorem 5.3 and Remark 5.4. Although not presented in these terms, the proof of the P=W conjecture for the symplectic resolution in $g = 1$ is due to [12].

The proof of the conjectures for $M := M_{\mathrm{Dol}}(C, \mathrm{SL}_2)$, with C a curve of genus 2, takes up most of the paper. We first reduce the P=W conjecture for M and \widetilde{M} to the PI=WI conjecture; see Theorems 7.1, 7.4 and 7.6. Finally, the PI=WI conjecture follows from Theorems 8.1, 8.8 and 8.17.

Symplectic resolutions. — The Dolbeault moduli spaces which admit a symplectic resolution appear as specialization of (a crepant contraction of) of compact hyperkähler manifolds as shown in the table.

| Special fiber | symplectic resolution of the general fiber |
|---|---|
| $M_{\mathrm{Dol}}(A, \mathrm{GL}_n)$ | Hilbert scheme of n points on a K3 surface containing the elliptic curve A |
| $M_{\mathrm{Dol}}(A, \mathrm{SL}_n)$ | generalized Kummer variety of dimension $2(n - 1)$ associated to the abelian surface $A \times A$ |
| $M_{\mathrm{Dol}}(C, \mathrm{GL}_2)$ | O’Grady 10-dimensional moduli space OG10 |
| $M := M_{\mathrm{Dol}}(C, \mathrm{SL}_2)$ | O’Grady 6-dimensional moduli space OG6 |

TABLE 1. Degenerations of compact hyperkähler manifolds to the space $M_{\mathrm{Dol}}(X, G)$; see the appendix. We denote by A and C a compact Riemann surface of genus 1 and 2 respectively.

Even if these degenerations are not strictly used in the proof of the main theorem, they have been our sources of inspiration. For instance, the proof of the P=W conjecture for $g = 1$ is inspired by the description of the cohomology of generalized Kummer varieties in [32], while the alterations in Section 6 are specializations of those exploited by [62] to determine the Hodge numbers of OG6. We included details about

the construction of the degenerations in the appendix for the interested reader. In the twisted case these degenerations have been exploited in [15, §4] and [14, §4]; see Proposition A.8 and Remark A.9 for a bizarre difference between the behaviour of the degenerations in the smooth and singular cases. Analogous degenerations on the Betti side for $g = 1$ have been considered in [60, §5.2 and §5.3] for the proof of the geometric P=W conjecture.

Twisted vs untwisted moduli spaces. — Let $G = \mathrm{GL}_n$ or SL_n . The known proofs [11] and [15] of the P=W conjecture for twisted moduli spaces crucially rely on the fact that $H^*(M_{\mathrm{Dol}}^{\mathrm{tw}}(X, G))$ is generated in degree not greater than 4. Further, the generators are Künneth components of the second Chern class of a universal Higgs bundle on $M_{\mathrm{Dol}}^{\mathrm{tw}}(X, G) \times X$, called *tautological classes*.

In the untwisted case this can fail.

– The cohomology ring of $M_{\mathrm{Dol}}(X, G)$ may not be generated in degree ≤ 4 . For instance, Theorem 6.14 and the second paragraph of the proof of Proposition 8.4 imply that

$$H^*(M, \mathbb{Q}) \simeq \mathbb{Q}[\alpha, \gamma_j] / (\alpha^3, \gamma_j^2, \alpha \cup \gamma_j),$$

with $\deg \alpha = 2$, $\deg \gamma_j = 6$, and $j = 1, \dots, 16$.

– A universal Higgs bundle E on $M_{\mathrm{Dol}}(X, G)^{\mathrm{sm}} \times X$ may not exist. Indeed, if E exists on $M^{\mathrm{sm}} \times C$, then its restriction to the moduli space of semistable vector bundles of rank 2 and degree 0 would be a universal vector bundle, which does not exist by [65].

If $g = 2$, we fix this problem by constructing a tautological class β on a quasi-étale cover of M , i.e., étale in codimension one; see Section 8.3. However, β does not descend in cohomology, but as an intersection cohomology class. More precisely, $IH^*(M, \mathbb{Q})$ is the $H^*(M, \mathbb{Q})$ -module

$$IH^*(M, \mathbb{Q}) \simeq H^*(M, \mathbb{Q})[1, \beta] / (\alpha \cup \beta - \sum_{j=1}^{16} \gamma_j, \alpha^2 \cup \beta, \gamma_j \cup \beta).$$

One can avoid constructing a universal bundle on a quasi-étale cover by appealing to the Dolbeault moduli stack, and the class β can be interpreted as a Chern class of an orbundle. However, the construction of the universal bundle on the quasi-étale cover is interesting in itself; cf. [45, §6.1]. Note also that the existence of this cover is a special feature of M : we show that when $g > 2$, M is the only Dolbeault moduli space which admits a non-trivial quasi-étale cover; see Section 6.2.7.

1.1. OUTLINE OF THE PAPER

– In Sections 2 and 4 we recall basic notions and theorems used throughout the paper.

– In Section 3.1 we lift the non-abelian Hodge correspondence Ψ to a diffeomorphism $\tilde{\Psi}$ between the resolutions of the Betti and Dolbeault moduli spaces; see Theorem 3.8. To this end, we describe an explicit compactification of the Hodge moduli space in Theorem 3.2. Note that $\tilde{\Psi}$ is the diffeomorphism which appears in the statement of the P=W conjecture for symplectic resolution. As a by-product, we answer a question by Simpson about the projectivity of the compactification of the de Rham moduli space, see Corollary 3.3.

| | | | | | | |
|-----|---|---|----|----|---|----------|
| p | | | | | | |
| 6 | 1 | | | | | |
| 5 | 0 | 1 | | | | |
| 4 | 1 | 0 | 16 | | | |
| 3 | 0 | 6 | 0 | 16 | ↔ | RHL |
| 2 | 1 | 0 | 16 | | | |
| 1 | 0 | 1 | | | | |
| 0 | 1 | | | | | |
| | 0 | 1 | 2 | 3 | | Δ |

FIGURE 1. The (p, Δ) -entry of the table is the dimension of the graded piece

$$Gr_p^P H^{\Delta-p}(\widetilde{M}_{Dol}(C, SL_2), \mathbb{Q}),$$

of the perverse Leray filtration on $H^*(\widetilde{M}_{Dol}(C, SL_2), \mathbb{Q})$, where C is a compact Riemann surface of genus 2. The sums along the northwest-southeast diagonals give the Betti numbers of $\widetilde{M}_{Dol}(C, SL_2)$. Relative hard Lefschetz accounts for a symmetry of this perverse diamond, namely a reflection about the horizontal axis placed at middle perversity. The P=W conjecture for resolution implies that the sums along the rows are the coefficients of the E-polynomial of $\widetilde{M}_B(C, SL_2)$, which is computed in (43).

- In Section 4.1 we show that the P=W conjecture for SL_n implies the P=W conjecture for GL_n .
- In Section 5 we prove the P=W conjectures for $g = 1$.
- The rest of the paper is devoted to the proof of the P=W conjectures for $M := M_{Dol}(C, SL_2)$, with C a curve of genus 2. We describe the geometry of M in great detail in Section 6: its singularities and its symplectic resolution \widetilde{M} in Section 6.1.1; the fixed loci of the G_m -action on M and \widetilde{M} in Section 6.1.4; the (universal) quasi-étale cover $q: M_t \rightarrow M$ in Section 6.2.1; a universal Higgs bundle on the smooth locus M_t^{sm} of M_t in Section 6.2.5; the zero fiber of the Hitchin fibration in Section 6.2.6.
- In Section 7 we explain the strategy of the proof of the P=W conjecture for M . Ultimately, we reduce the proof of the P=W conjectures for M and \widetilde{M} to the PI=WI conjecture for M .
- In Section 8.1 we compute the necessary intersection Poincaré and E-polynomials.
- In Section 8.3 we build a tautological class of perversity 2 and weight 4, out of the universal bundle on M_t^{sm} . This allows to conclude the proof of the PI=WI conjecture for M in Section 8.4.
- In the appendix we collected some information about degenerations of compact hyperkähler varieties to Dolbeault moduli spaces.

Acknowledgements. — We would like to acknowledge useful conversations with Thorsten Beckmann, Simone Chiarelllo, Mark de Cataldo, Peter Gothen, Isabell Hellmann, Daniel Huybrechts, Luca Migliorini, Giovanni Mongardi, Mircea Mustaș, André Oliveira, Johannes Schmitt, Junliang Shen, Andras Szenes, Michael Temkin, Marco Trozzo. In particular, we are grateful to Mark de Cataldo for useful discussions on the projectivity of the compactifications of de Rham and Dolbeault moduli spaces in Theorem 3.2, and to Michael Temkin for suggesting Lemma 3.5. Finally, we are grateful to the anonymous referees for many useful suggestions and comments.

We would like to thank the University of Geneva and Max Planck Institute for supporting our reciprocal visits during the preparation of this paper.

2. PRELIMINARIES

In this section we introduce preliminary notions and results which will be useful throughout the paper. For further details we refer to [4, 30, 31, 16].

When omitted, the coefficients of (intersection) cohomology are assumed to be rational.

2.1. PERVERSE SHEAVES. — An algebraic variety X is an irreducible separated scheme of finite type over \mathbb{C} . Denote by $D_c^b(X)$ the bounded derived category of \mathbb{Q} -constructible complexes on X . Let $D: D_c^b(X) \rightarrow D_c^b(X)$ be the Verdier duality functor. The full subcategories

$$\begin{aligned} {}^pD_{\leq 0}^b(X) &:= \{K^* \in D_c^b(X) \mid \dim \operatorname{Supp}(H^j(K^*)) \leq -j\}, \\ {}^pD_{> 0}^b(X) &:= \{K^* \in D_c^b(X) \mid \dim \operatorname{Supp}(H^j(DK^*)) \leq -j\}, \end{aligned}$$

determine a t -structure on $D_c^b(X)$, called perverse t -structure. The heart

$$\operatorname{Perv}(X) := {}^pD_{\leq 0}^b(X) \cap {}^pD_{> 0}^b(X)$$

of the t -structure is the abelian category of *perverse sheaves*. The truncation functors are denoted ${}^p\tau_{\leq k}: D_c^b(X) \rightarrow {}^pD_{\leq k}^b(X)$, ${}^p\tau_{> k}: D_c^b(X) \rightarrow {}^pD_{> k}^b(X)$, and the perverse cohomology functors are

$${}^pH^k := {}^p\tau_{\leq k} \circ {}^p\tau_{> k}: D_c^b(X) \longrightarrow \operatorname{Perv}(X).$$

DEFINITION 2.1. — Let K^* be a complex in $D_c^b(X)$. The cohomology $H^d(X, K^*)$ is endowed with the *perverse filtration* defined by

$$P_k H^d(X, K^*) = \operatorname{Im}\{H^d(X, {}^p\tau_{\leq k} K^*) \rightarrow H^d(X, K^*)\}.$$

2.2. INTERSECTION COHOMOLOGY. — The category $\operatorname{Perv}(X)$ is abelian, artinian, and its simple objects are the intersection cohomology complexes.

DEFINITION 2.2 (Intersection cohomology complex). — Let L be a local system on a smooth Zariski-dense open subset $U \subseteq X$. The *intersection cohomology complex* $IC_X(L)$ is a complex of sheaves in $D_c^b(X)$ which is uniquely determined up to isomorphism by the following conditions:

$$- IC_X(L)|_U \simeq L[\dim X];$$

- $\dim \text{Supp}^{H^j}(IC_X(L)) < -j$, for all $j > -\dim X$;
- $\dim \text{Supp}^{H^j}(DIC_X(L)) < -j$, for all $j > -\dim X$.

When $L = \underline{\mathbb{Q}}_{X^{\text{sm}}}$, i.e., the constant sheaf on the smooth locus of X , we just write IC_X for $IC_X(\underline{\mathbb{Q}}_{X^{\text{sm}}})$. Further, if X has at worst quotient singularities, then $IC_X \simeq \underline{\mathbb{Q}}_X[\dim X]$.

DEFINITION 2.3 (Intersection cohomology). — The *intersection cohomology* of X with coefficient in L is its (shifted) cohomology

$$IH^*(X, L) = H^{*-\dim X}(X, IC_X(L)).$$

Analogously, the intersection cohomology of X with compact support and coefficients in L is $IH_c^*(X, L) = H^{*-\dim X}(X, DIC_X(L))$. For further details, we refer the interested reader to [50].

There is a natural morphism

$$H^*(X) \longrightarrow IH^*(X),$$

which is an isomorphism when X has at worst quotient singularities. This morphism equips $IH^*(X)$ with the structure of $H^*(X)$ -module, but in general intersection cohomology has no ring structure or cup product.

Moreover, the groups $IH^*(X)$ are finite dimensional, satisfy Mayer-Vietoris theorem and Künneth formula. Although they are not homotopy invariant, they satisfy analogues of Poincaré duality, i.e., $IH^*(X) \simeq IH_c^{2\dim X-*}(X)^\vee$ and of the hard Lefschetz theorem. They also carry a mixed Hodge structures.

DEFINITION 2.4 (Mixed Hodge structure). — The mixed Hodge structure (V, F^*, W_*) is the datum of

- a \mathbb{Q} -vector space V ,
- an increasing filtration W_* on V , called *weight filtration*,
- a decreasing filtration F^* on $V \otimes \mathbb{C}$, called *Hodge filtration*,

such that the graded pieces $\text{Gr}_k^W V := W_k V / W_{k-1} V$ admit a pure Hodge structure of weight k , induced by F^* on $\text{Gr}_k^W V \otimes \mathbb{C}$.

An element $v \in V$ has *weight* k if $v \in W_k V$ but $v \notin W_{k-1} V$.

DEFINITION 2.5 (E-polynomial). — The *E-polynomial* of X is an additive function on the category of separated \mathbb{C} -schemes of finite type given by

$$E(X) = \sum_{p,q,d} (-1)^d \dim(\text{Gr}_{p+q}^W H_c^d(X, \mathbb{C}))^{p,q} u^p v^q.$$

Additivity means that if $Z \subset X$ is a closed subscheme, then $E(X) = E(X^{\text{red}}) = E(X \setminus Z) + E(Z)$.

Analogously, we define the *intersection E-polynomial* as

$$IE(X) = \sum_{p,q,d} (-1)^d \dim(\text{Gr}_{p+q}^W IH_c^d(X, \mathbb{C}))^{p,q} u^p v^q.$$

Note however that the intersection E -polynomial is not an additive function, due to the fact that in general the restriction to a closed subscheme $Z \subset X$ of IC_X is not isomorphic to IC_Z .

2.3. DECOMPOSITION THEOREM. — In this section we recall in brief the statement of the decomposition theorem for semismall maps.

DEFINITION 2.6. — A morphism of algebraic varieties $f: X \rightarrow Y$ is *semismall* if $\dim X \times_Y X \subset \dim X$.

A stratification of f is a collection of finitely many locally closed subsets Y_k such that $f^{-1}(Y_k) \rightarrow Y_k$ are topologically locally trivial fibrations. A stratum Y_k is *relevant* if $2 \dim f^{-1}(Y_k) - \dim(Y_k) = \dim X$.

THEOREM 2.7 (Decomposition theorem for semismall maps). — *Let $f: X \rightarrow Y$ be a proper algebraic semismall map from a smooth variety X . Then there exists a canonical isomorphism*

$$Rf_* \underline{\mathcal{O}}_X[\dim X] \simeq \bigoplus_{Y_k} IC_{\overline{Y}_k} (R^{\dim X - \dim Y_k} f_* \underline{\mathcal{O}}_{f^{-1}(Y_k)}),$$

where the summation index runs over all the relevant strata of a stratification of f .

2.4. PERVERSE LERAY FILTRATION. — Let $\chi: X \rightarrow Y$ be a projective morphism of algebraic varieties of relative dimension r . Set $r(\chi) := \dim X \times_Y X - \dim X$.

DEFINITION 2.8. — The *perverse Leray filtration associated to χ* is the (shifted) perverse filtration on the cohomology of the complex $R\chi_* IC_X$

$$P_k IH^*(X) = P_k H^{* - (\dim X - r(\chi))}(Y, R\chi_* IC_X[\dim X - r(\chi)]).$$

When Y is a line, de Cataldo and Migliorini provided an equivalent geometric description of the perverse Leray filtration. Assume for simplicity that $\dim X = 2 \dim Y = 2r(\chi)$. Let $\Lambda^k \subset Y$ be a general k -dimensional linear section of $Y \subset \mathbb{A}^N$.

THEOREM 2.9 (Flag filtration [17, Th. 4.1.1])

$$P_k IH^d(X) = \text{Ker}\{IH^d(X) \rightarrow IH^d(\chi^{-1}(\Lambda^{d-k-1}))\}.$$

This means that the class $\eta \in IH^d(X)$ belongs to $P_k IH^d(X)$ if and only if its restriction to $\chi^{-1}(\Lambda^{d-p-1})$ vanishes, i.e., $\eta|_{\chi^{-1}(\Lambda^{d-p-1})} = 0$.

Most remarkably, the perverse Leray filtration satisfies the relative hard Lefschetz theorem.

THEOREM 2.10 (Relative hard Lefschetz). — *Let $\chi: X \rightarrow Y$ be a proper map of algebraic varieties, and let $\alpha \in H^2(X)$ be the first Chern class of a relatively ample line bundle. Then there exists an isomorphism*

$$\alpha^i: \text{Gr}_{r-k}^P IH^*(X) \longrightarrow \text{Gr}_{r+k}^P IH^{*+2k}(X).$$

3. LIFTING THE NON-ABELIAN HODGE CORRESPONDENCE

Let X be a compact Riemann surface, and fix a complex reductive algebraic group G . The first cohomology group $H^1(X, G)$ comes in various incarnations (cf. [78] and [79]):

- the *Betti* moduli space $M_B(X, G)$, also named *character variety*, parametrizing semistable representations of the fundamental group of X with value in G ;
- the *Dolbeault* moduli space $M_{\text{Dol}}(X, G)$ of semistable principal G -Higgs bundles with vanishing Chern classes;
- the *De Rham* moduli space $M_{\text{DR}}(X, G)$ of semistable principal G -bundles with an integrable connection.

All these moduli spaces are homeomorphic to each other. The Riemann-Hilbert correspondence yields a complex analytic isomorphism

$$(3) \quad M_{\text{DR}}(X, G)^{\text{an}} \simeq M_B(X, G)^{\text{an}}.$$

There exists an algebraic fibration (real analytically trivialisable)

$$(4) \quad \lambda: M_{\text{Hod}}(X, G) \longrightarrow \mathbb{A}^1,$$

whose fibers are moduli spaces of semi-simple principal G -bundles with λ -connections; see [80]. Hence, the fiber over 0 is $M_{\text{Dol}}(X, G)$, and the fibers over $\lambda \neq 0$ are isomorphic to $M_{\text{DR}}(X, G)$. The space $M_{\text{Hod}}(X, G)$ is called *Hodge* moduli space. In particular, a continuous trivialization $M_{\text{Hod}}(X, G)^{\text{top}} \simeq M_{\text{Dol}}(X, G) \times \mathbb{A}^1$ gives the homeomorphism

$$(5) \quad M_{\text{Dol}}(X, G)^{\text{top}} \simeq M_{\text{DR}}(X, G)^{\text{top}}.$$

The *non-abelian Hodge correspondence*

$$\Psi: M_{\text{Dol}}(X, G)^{\text{top}} \longrightarrow M_B(X, G)^{\text{top}}$$

is the composition of the maps (3) and (5) for a choice of a preferred real analytic trivialization; see [80] for details.

3.1. COMPACTIFICATION OF HODGE MODULI SPACES

The Hodge moduli space $M_{\text{Hod}}(X, G)$ admits a partial compactification, relative to the morphism

$$\lambda: M_{\text{Hod}}(X, G) \longrightarrow \mathbb{A}^1.$$

We obtain it as a G_m -quotient of the total space of the degeneration of $M_{\text{Hod}}(X, G)$ to the normal cone of $\lambda^{-1}(0) \simeq M_{\text{Dol}}(X, G)$. The construction is an extension to the singular case of [43, Lem. 6.1] or [40, Th. 7.2.1].

To this end, we shall use the following results by Simpson.

PROPOSITION 3.1 ([80, Th. 11.2]). — *Let Z be a variety over the variety S , endowed with a G_m -action covering the trivial G_m -action on S . Assume that Z/S carries a relatively ample line bundle admitting a G_m -linearization. Assume that the fixed point set $\text{Fix}(Z) \subseteq Z$ is proper over S , and that for any $z \in Z$ the limit $\lim_{t \rightarrow 0} t \cdot z$ exist*

in Z . Let $U \subset Z$ be the subset of points z such that the limit $\lim_{t \rightarrow \infty} t \cdot z$ does not exist. Then U is open in Z and there exists a universal geometric quotient Z/G_m . This quotient is separated and proper over S .

THEOREM 3.2 (Partial compactification of the Hodge moduli space). — *There exists a projective morphism*

$$\bar{\lambda}: \bar{M}_{\text{Hod}}(X, G) \longrightarrow A^1$$

which is a relative compactification of the morphism λ .

Proof. — $M_{\text{Hod}}(X, G)$ is endowed with the G_m -action

$$t \cdot (E, \nabla_\lambda) \longrightarrow (E, t\nabla_\lambda)$$

covering the standard G_m -action on A^1 , namely $t \cdot \lambda = t\lambda$. Equip A^2 with the G_m -action given by $t \cdot (x, y) = (x, ty)$. The morphism $A^2 \rightarrow A^1$, given by $(x, y) \mapsto xy$, is G_m -equivariant. Therefore the fiber product $M_{\text{Hod}}(X, G) \times_{A^1} A^2$ is equipped with a G_m -action. We summarize the maps constructed in a diagram: note that the subscripts indicates the coordinatization chosen for the affine spaces.

$$\begin{array}{ccc} M_{\text{Hod}}(X, G) \times_{A^1} A^2 & \longrightarrow & M_{\text{Hod}}(X, G) \\ \downarrow & & \downarrow \lambda \\ A^2_{x,y} & \longrightarrow & A^1_\lambda \\ \downarrow & & \downarrow \\ A^1_x & & x \end{array} \quad \begin{array}{c} (x, y) \longmapsto xy \\ \downarrow \\ x \end{array}$$

Choose a λ' -ample line bundle L' on $M_{\text{Hod}}(X, G) \times_{A^1} A^2$ admitting a G_m -linearization (which exists since $M_{\text{Hod}}(X, G) \times_{A^1} A^2$ is normal and because of [63, Cor. 1.6]). Let $\chi(X, G): M_{\text{Dol}}(X, G) \rightarrow A^{\dim M_{\text{Dol}}(X, G)/2}$ be the Hitchin's proper map for $M_{\text{Dol}}(X, G)$; see [79, p. 22]. The fixed locus is contained in

$$\chi(X, G)^{-1}(0) \times \{y = 0\} \subset M_{\text{Dol}}(X, G) \times \{y = 0\} \subset M_{\text{Hod}}(X, G) \times_{A^1} A^2,$$

so it is proper over A^1_x . By Proposition 3.1, there exists a universal geometric quotient

$$\bar{M}_{\text{Hod}}(X, G) := (M_{\text{Hod}}(X, G) \times_{A^1} A^2 \cap (\chi(X, G)^{-1}(0) \times A^1_x))/G_m$$

and a proper morphism $\bar{\lambda}: \bar{M}_{\text{Hod}}(X, G) \rightarrow A^1_x$.

$\bar{M}_{\text{Hod}}(X, G)$ contains an open subset isomorphic to $M_{\text{Hod}}(X, G)$, given by the G_m -quotient of

$$(M_{\text{Hod}}(X, G) \times_{A^1} A^2) \times_{A^1_y} (A^1_y \cap \{0\}) \simeq M_{\text{Hod}}(X, G) \times G_m.$$

We show now that the morphism $\bar{\lambda}$ is projective. Let $\partial M_{\text{Hod}} := \bar{M}_{\text{Hod}}(X, G) \cap M_{\text{Hod}}(X, G)$ be the Cartier boundary divisor. By [24, Th. 2.3] (or [10, Proof of Prop. 3.2.2]), a power of the line bundle L' descends to a line bundle L on $\bar{M}_{\text{Hod}}(X, G)$. We claim that the line bundle $L \otimes O(m \cdot \partial M_{\text{Hod}})$ is ample for $m \gg 0$.

To this end, observe that $\bar{\lambda}^{-1}(0) = \bar{M}_{\text{Dol}}(X, G)$ coincides with the projective compactification of $M_{\text{Dol}}(X, G)$ constructed in [10, Th. 3.1.1(1)]. Let $\bar{\chi}: \bar{M}_{\text{Dol}}(X, G) \rightarrow \bar{A}$

be also the projective compactification of the Hitchin morphism constructed in [10, Th. 3.1.1(2)]. The restriction of L to $\bar{\lambda}^{-1}(0)$ is $\bar{\lambda}$ -ample by [10, Prop. 3.2.2], while the restriction of ∂M_{Hod} is the pullback of an ample divisor on \bar{A} . Therefore, the line bundle $L \otimes \mathcal{O}(m \cdot \partial M_{\text{Hod}})$ is ample for $m \gg 0$ when restricted to $\bar{M}_{\text{Dol}}(X, G)$. By the openness of ampleness [55, Th. 1.2.17], it is $\bar{\lambda}$ -ample in a neighbourhood of $\bar{\lambda}^{-1}(0)$. But $\bar{M}_{\text{Hod}}(X, G) \cap \bar{\lambda}^{-1}(0)$ is isomorphic to the trivial product $\bar{M}_{\text{DR}}(X, G) \times (A^1 \cap \{0\})$, where the first factor is Simpson's compactification of $M_{\text{DR}}(X, G)$; see [80, §11]. Therefore, we conclude that $L \otimes \mathcal{O}(m \cdot \partial M_{\text{Hod}})$ is $\bar{\lambda}$ -ample for $m \gg 0$.

Incidentally, note that Theorem 3.2 answers the question about the projectivity of the compactification of the de Rham moduli space risen in [80, p. 268] and [10, Rem. 3.1.2].

COROLLARY 3.3. — *Simpson's compactification $\bar{M}_{\text{DR}}(X, G)$ is projective.*

We now study the local geometry of the morphism of $\bar{\lambda}$.

PROPOSITION 3.4. — *The morphism $\bar{\lambda}$ is locally analytically trivial, i.e., for any $p \in \bar{M}_{\text{Hod}}(X, G)$ over $\lambda_p \in A^1$ there exist analytic neighborhoods $p \in U_p \subseteq \bar{M}_{\text{Hod}}(X, G)$ and $p \in V_p \subseteq \bar{M}_{\text{Hod}}(X, G)_\lambda$ such that $U_p \simeq V_p \times D$, with D a disk in A^1 , and λ corresponds to the second projection $V_p \times D \rightarrow D$.*

Proof. — The G_m -action on $M_{\text{Hod}}(X, G)$ extends to $\bar{M}_{\text{Hod}}(X, G)$, and so

$$\bar{M}_{\text{Hod}}(X, G)|_{A^1 \cap \{0\}} \simeq \bar{M}_{\text{DR}}(X, G) \times G_m;$$

see also [80, p. 232]. By [80, Th. 9.1], λ is locally analytically trivial. Therefore, it is enough to show that $\bar{\lambda}$ is locally analytically trivial at $p \in \bar{\lambda}^{-1}(0) \cap \lambda^{-1}(0)$.

Let $p' \in M_{\text{Hod}}(X, G) \times_{A^1} A^2$ be a lift of p . Since λ is locally analytically trivial, so λ' is. Following the proof of [10, Lem. 3.5.1], we can choose a transverse slice to the G_m -orbit through p' , locally isomorphic to an affine variety $N_{p'} \times A_x^1$, such that $\bar{M}_{\text{Hod}}(X, G)$ is locally isomorphic at p to $N_{p'}/\text{Stab}(p') \times A_x^1$, and $\bar{\lambda}$ is the projection onto the second factor. As a result, we obtain that $\bar{\lambda}$ is locally analytically trivial.

In Theorem 3.8 we show that there exists a diffeomorphism $\tilde{\Psi}$ which lifts the isomorphism

$$\Psi^*: H^*(M_{\text{B}}(X, G)) \longrightarrow H^*(M_{\text{Dol}}(X, G))$$

to an isomorphism between the cohomology of the resolution spaces.

To this purpose we recall that for any noetherian quasi-excellent generically reduced scheme X over $\text{Spec}(\mathbb{Q})$ there exists a resolution of singularities $R(X) \rightarrow X$ functorial with respect to regular morphism $X' \rightarrow X$, in the sense that $R(X')$ is isomorphic to $R(X) \times_X X'$. See [83] for further details and the definition of quasi-excellent schemes and regular morphisms. Here we just mention that by definition, if X is excellent, then the completion morphism $\hat{X}_x := \text{Spec } \hat{\mathcal{O}}_{X,x} \rightarrow X$ is regular for any closed point $x \in X$. In [83, Th. 5.2.2], Temkin showed also that quasi-compact analytic spaces admit functorial resolutions compatible with smooth analytic morphism. The

following lemma is implicit in [83], and it has been kindly communicated to us by Temkin. For clarity, we distinguish the complex algebraic variety X from its complex analytification X^{an} , but omit the difference elsewhere in the paper.

LEMMA 3.5. — *If X is a complex algebraic variety, then the analytification of the algebraic functorial resolution is biholomorphic to the analytic functorial resolutions of X^{an} , i.e., $R(X)^{\text{an}} \simeq R(X^{\text{an}})$.*

Proof. — Without loss of generality suppose that $X = \text{Spec}(B)$ is a line. We briefly recall Temkin's construction of the analytic functorial resolution; see [83, Th. 5.2.2]. Take a covering of $X^{\text{an}} = \bigcup_i X_i$ by Stein compact domains (e.g. embed locally X^{an} in a complex affine space and take intersections of X^{an} with closed polydiscs). The ring of functions $A_i := \mathcal{O}_X^{\text{an}}(X_i)$ is excellent, and the functorial resolution of $\text{Spec } A_i$ glue to the analytic functorial resolution $R(X^{\text{an}})$. Since B and A_i are excellent, the completion morphism $B \rightarrow \widehat{O}_{X,x}$ and $A_i \rightarrow \widehat{O}_{X_i,x}^{\text{an}}$ are regular, so the algebraic and the functorial resolutions $R(X)$ and $R(X)^{\text{an}}$ are compatible with completions. Now, since $\widehat{O}_{X,x} \simeq \widehat{O}_{X,x}^{\text{an}}$, we have $R(\widehat{X}_x) \simeq R(\widehat{X}_x^{\text{an}})$. By functoriality, we obtain that

$$R(X) \times_X \widehat{X}_x \simeq R(\widehat{X}_x) \simeq R(\widehat{X}_x^{\text{an}}) \simeq R(X^{\text{an}}) \times_X \widehat{X}_x$$

for any closed point $x \in X$. Hence, $R(X)^{\text{an}} \simeq R(X^{\text{an}})$.

COROLLARY 3.6. — *A biholomorphism $f: X' \rightarrow X$ between complex algebraic varieties (not necessarily algebraizable) lifts to a biholomorphism $R(f): R(X') \rightarrow R(X)$ between their functorial resolutions, which gives a fiber product square.*

$$\begin{array}{ccc} R(X') & \xrightarrow{R(f)} & R(X) \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

Proof. — By functoriality in the complex analytic category, $R(X) \times_X X'$ is an analytic functorial resolution, so biholomorphic to $R(X)^{\text{an}}$ by Lemma 3.5.

LEMMA 3.7. — *Let X be a normal locally \mathbb{Q} -factorial⁽³⁾ complex variety. Suppose that X admits a symplectic resolution $f: Y \rightarrow X$ with an irreducible exceptional divisor, obtained by blowing-up the singular locus. Then f is functorial.*

Proof. — By [29, Th. 2.2], any symplectic resolution of X is isomorphic to f . Let $h: X' \rightarrow X$ be any smooth morphism. The blow-up $Y' \rightarrow X'$ of the singular locus of X' is smooth and symplectic since h is smooth. Then X' satisfies all the hypotheses of Lemma 3.7 with symplectic resolution $Y'' := Y \times_X X'$, so $Y' = Y''$ by [29, Th. 2.2], i.e., the resolution is functorial for smooth morphisms, and also for regular morphism following [7, Th. 1.2, Cor. 4.6].

⁽³⁾This means that for any closed point $x \in X$ the analytic local ring $\mathcal{O}_{X,x}^{\text{an}}$ are \mathbb{Q} -factorial, that is some multiple of every Weil divisor is Cartier.

THEOREM 3.8 (Lift of the non-abelian Hodge correspondence Ψ). — *There exist resolutions of singularities $f_{\text{Dol}}: \widetilde{M}_{\text{Dol}}(X, G) \rightarrow M_{\text{Dol}}(X, G)$ and $f_{\text{B}}: \widetilde{M}_{\text{B}}(X, G) \rightarrow M_{\text{B}}(X, G)$, and a diffeomorphism*

$$\widetilde{\Psi}: \widetilde{M}_{\text{Dol}}(X, G) \longrightarrow \widetilde{M}_{\text{B}}(X, G),$$

such that the square

$$(6) \quad \begin{array}{ccc} \widetilde{M}_{\text{Dol}}(X, G) & \xrightarrow{\widetilde{\Psi}} & \widetilde{M}_{\text{B}}(X, G) \\ f_{\text{Dol}} \downarrow & & \downarrow f_{\text{B}} \\ M_{\text{Dol}}(X, G) & \xrightarrow{\Psi} & M_{\text{B}}(X, G) \end{array}$$

commutes up to an isotopy of $M_{\text{B}}(X, G)$. In particular, the following square in cohomology commutes

$$(7) \quad \begin{array}{ccc} H^*(\widetilde{M}_{\text{Dol}}(X, G)) & \xleftarrow{\widetilde{\Psi}^*} & H^*(\widetilde{M}_{\text{B}}(X, G)) \\ f_{\text{Dol}}^* \uparrow & & \uparrow f_{\text{B}}^* \\ H^*(M_{\text{Dol}}(X, G)) & \xleftarrow{\Psi^*} & H^*(M_{\text{B}}(X, G)). \end{array}$$

The resolutions f_{Dol} and f_{B} can be taken functorial with respect to smooth algebraic or analytic morphisms, and symplectic if $G = \text{GL}_n$ or SL_n with $(g, n) = (1, n)$ or $(2, 2)$.

Proof. — Let $f_{\text{Hod}}: R(\overline{M}_{\text{Hod}}(X, G)) \rightarrow \overline{M}_{\text{Hod}}(X, G)$ be the functorial resolution of $\overline{M}_{\text{Hod}}(X, G)$, equivalently in the analytic or algebraic category by Lemma 3.5. Since $\bar{\lambda}$ is locally analytically trivial by Proposition 3.4, f_{Hod} is a simultaneous resolution of $\overline{M}_{\text{Hod}}(X, G)_\lambda$; see for instance [33, Lem. 4.2]. Note also that any vector field on the smooth locus of $\overline{M}_{\text{Hod}}(X, G)$ can be lifted to a vector field on $R(\overline{M}_{\text{Hod}}(X, G))$ by [34, Cor. 4.7].

For such a resolution, Proposition 5.2 in [1] holds: the family $\bar{\lambda} \circ f_{\text{Hod}}$ admits a real analytic Ehresmann connection such that the corresponding flow of diffeomorphisms preserves the exceptional locus of f_{Hod} , and moreover it does so fiberwise over its image in $M_{\text{Hod}}(X, G)$. The same proof as that of [1, Prop. 5.2] shows that we can further suppose that the flow preserves $\partial M_{\text{Hod}}(X, G) \simeq \partial M_{\text{Dol}}(X, G) \times \mathbb{A}^1$ and its inverse image in $R(\overline{M}_{\text{Hod}}(X, G))$. Hence, there exists a resolution of singularities

$$\widetilde{M}_{\text{Hod}}(X, G) := f_{\text{Hod}}^{-1}(M_{\text{Hod}}(X, G))$$

of $M_{\text{Hod}}(X, G)$ such that the following square commutes

$$\begin{array}{ccc} \widetilde{M}_{\text{Dol}}(X, G) := \widetilde{M}_{\text{Hod}}(X, G)_0 & \rightarrow & \widetilde{M}_{\text{Hod}}(X, G)_\varepsilon := \widetilde{M}_{\text{DR}}(X, G) \\ f_{\text{Dol}} := f_{\text{Hod},0} \downarrow & & \downarrow f_{\text{DR}} := f_{\text{Hod},\varepsilon} \\ M_{\text{Dol}}(X, G) := M_{\text{Hod}}(X, G)_0 & \rightarrow & M_{\text{Hod}}(X, G)_\varepsilon := M_{\text{DR}}(X, G), \end{array}$$

where the horizontal arrows are stratified diffeomorphisms, and $\varepsilon \neq 0$.

Since the Riemann-Hilbert correspondence is a smooth analytic map, the map f_{DR} is obtained via base change from the functorial resolution $f_{\text{B}}: \widetilde{M}_{\text{B}}(X, G) := R(M_{\text{B}}(X, G)) \rightarrow M_{\text{B}}(X, G)$ by functoriality. Therefore, we obtain the commutative square

$$\begin{array}{ccc} \widetilde{M}_{\text{Dol}}(X, G) & \xrightarrow{\widetilde{\Psi}} & \widetilde{M}_{\text{B}}(X, G) \\ f_{\text{Dol}} \downarrow & & \downarrow f_{\text{B}} \\ M_{\text{Dol}}(X, G) & \xrightarrow{\Psi'} & M_{\text{B}}(X, G). \end{array}$$

Since Ψ' and the non-abelian Hodge correspondence Ψ are induced by trivialization of $M_{\text{Hod}}(X, G)$, the square (6) commutes up to a stratified isotopy of $M_{\text{Hod}}(X, G)$. Since stratified isotopy are trivial in cohomology, the square (7) commutes too.

We now show that the functorial resolutions f_{Dol} and f_{B} are symplectic if $G = \text{GL}_n$ or SL_n , with $(g, n) = (1, n)$ or $(2, 2)$. Indeed, in this case $M_{\text{Dol}}(X, G)$ and $M_{\text{B}}(X, G)$ are normal complex varieties which admit a symplectic resolution obtained by blowing-up the singular locus; see Sections 5.1 and 6.1.1, or [5, Th. 1.8]. Note that the results of [5] are stated for $M_{\text{B}}(X, G)$, but they extend to $M_{\text{Dol}}(X, G)$ by the isosingularity principle, see [79, Th. 10.6] or [59, §2.4 and the first paragraph of §3.2]. Further, the analytic neighborhoods of the singularities of these varieties are \mathbb{Q} -factorial. Indeed, the singularities of $M_{\text{Dol}}(X, G)$ and $M_{\text{B}}(X, G)$ are either quotient singularities or the nilpotent cone in $\mathfrak{sp}(4)$, which is a cone over a projective variety with quotient singularities and Picard number one; see the last paragraph of the proof of [5, Th. 1.3] and references therein, and [62, Lem. 1.3] or [59, §3.4]. By [53, Prop. 5.15] and [52, Prop. 7.4] these singularities are analytically \mathbb{Q} -factorial. Hence, the last statement of Theorem 3.8 follows from Lemma 3.7.

REMARK 3.9. — In this paper, functorial resolutions are used only for the following purposes: to lift vector fields and group actions to resolutions, and for the compatibility with respect to the Riemann–Hilbert correspondence; see proof of Theorem 3.8 and Section 4.1. If $G = \text{GL}_n$ or SL_n with $(g, n) = (1, n)$ or $(2, 2)$, the symplectic resolutions of $M_{\text{Dol}}(X, G)$ and $M_{\text{B}}(X, G)$ are indeed functorial by Lemma 3.7 but these properties can be shown more directly. The resolutions are obtained by blowing up the singular locus, which is invariant with respect to any group action on the varieties and preserved by the Riemann–Hilbert correspondence. Further, the liftability of vector fields follows easily for instance from [1, Lem. 5.3].

4. MODULI SPACES FOR GL_n VS SL_n

Let $\Gamma := \text{Pic}^0(X)[n] \simeq (Z/nZ)^{2g}$ be the group of n -torsion line bundles on the Riemann surface X of genus g and canonical line bundle K_X . We review the relation between the moduli spaces $M_{\text{Dol}}(X, G)$ and $M_{\text{B}}(X, G)$ for $G = \text{GL}_n$ and SL_n ; see also [46, 79, 78].

Recall that $M_{\text{Dol}}(X, \text{GL}_n)$ parametrizes semistable Higgs bundles (E, ϕ) , where E is a vector bundle on X of rank n and degree 0, and $\phi \in \text{Hom}(E, E \otimes K_X)$.

The fiber of the isotrivial morphism

$$(8) \quad \begin{aligned} \text{alb}: M_{\text{Dol}}(X, \text{GL}_n) &\longrightarrow \text{Pic}^0(X) \times H^0(X, K_X) \\ (E, \phi) &\longmapsto (\det E, \text{tr } \phi) \end{aligned}$$

is isomorphic to $M_{\text{Dol}}(X, \text{SL}_n)$. In particular, the monodromy of alb is the group Γ . Indeed, the étale cover

$$(9) \quad \begin{aligned} M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X) &\longrightarrow M_{\text{Dol}}(X, \text{GL}_n) \\ ((E, \phi), L, s) &\longmapsto (E \otimes L, \phi + (s/n)\text{id}_E) \end{aligned}$$

has Galois group Γ , which acts on the domain diagonally by tensorisation

$$\begin{aligned} \Gamma \times M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X) &\longrightarrow M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X) \\ (L_\gamma, (E, \phi), L, s) &\longmapsto (L_\gamma, (E \otimes L_\gamma, \phi), L \otimes L_\gamma^{-1}, s). \end{aligned}$$

Therefore, when we take cohomology, we obtain

$$(10) \quad \begin{aligned} H^*(M_{\text{Dol}}(X, \text{GL}_n)) &\simeq H^*(M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X))^\Gamma \\ &\simeq H^*(M_{\text{Dol}}(X, \text{SL}_n))^\Gamma \otimes H^*(\text{Pic}^0(X)), \end{aligned}$$

where the former equality follows from an observation of Grothendieck in [36], and the latter from the fact that Γ acts trivially on $H^*(\text{Pic}^0(X))$, since it is a restriction to a subgroup of the action of the connected group $\text{Pic}^0(X)$.

The *Hitchin map*

$$\chi(X, \text{GL}_n): M_{\text{Dol}}(X, \text{GL}_n) \longrightarrow \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$$

is a projective fibration sending (E, ϕ) to the characteristic polynomial of ϕ . It is Lagrangian with respect to ω , i.e., the holomorphic symplectic form of the canonical hyperkähler metric on the smooth locus of $M_{\text{Dol}}(X, \text{GL}_n)$; see [46, §6]. The map $\chi(X, \text{GL}_n)$ restricts on $M_{\text{Dol}}(X, \text{SL}_n)$ to

$$\chi(X, \text{SL}_n): M_{\text{Dol}}(X, \text{SL}_n) \longrightarrow \bigoplus_{i=2}^n H^0(X, K_X^{\otimes i}).$$

The map $\chi(X, \text{SL}_n)$ is Γ -equivariant, covering the trivial Γ -action of the codomain. In particular, there exists a commutative diagram

$$(11) \quad \begin{array}{ccc} M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X) & \longrightarrow & M_{\text{Dol}}(X, \text{GL}_n) \\ \downarrow (\chi(X, \text{SL}_n), S_{\text{Pic}^0(X)}, \text{id}_{H^0(X, K_X)}) & & \downarrow \chi(X, \text{GL}_n) \\ \bigoplus_{i=2}^n H^0(X, K_X^{\otimes i}) \times H^0(X, K_X) & \xrightarrow{=} & \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i}) \end{array}$$

with $S_{\text{Pic}^0(X)}: \text{Pic}^0(X) \rightarrow \text{pt}$.

Via the non-abelian Hodge correspondence Ψ , the action of Γ on $M_{\text{Dol}}(X, \text{SL}_n)$ corresponds to the algebraic action of the group of characters $\text{Hom}(\pi_1(C), \mathbb{Z}/n\mathbb{Z})$, which

acts on $M_B(X, \mathrm{SL}_n)$ by multiplication (changing the signs of the matrices A_j, B_j 's as in (1)).

The multiplication map $\mathrm{SL}_n \times G_m \rightarrow \mathrm{GL}_n$ induces the étale cover

$$M_B(X, \mathrm{SL}_n) \times (\mathbb{C}^*)^{2g} \longrightarrow M_{\mathrm{Dol}}(X, \mathrm{GL}_n)$$

with Galois group Γ . Therefore, the analogue of (10) holds

$$(12) \quad \begin{aligned} H^*(M_B(X, \mathrm{GL}_n)) &\simeq H^*(M_B(X, \mathrm{SL}_n) \times (\mathbb{C}^*)^{2g})^\Gamma \\ &\simeq H^*(M_{\mathrm{Dol}}(X, \mathrm{SL}_n))^\Gamma \otimes H^*((\mathbb{C}^*)^{2g}). \end{aligned}$$

4.1. **P=W FOR SL_n IMPLIES P=W FOR GL_n .** — In this section we show that the P=W conjectures for SL_n imply the corresponding statements for GL_n . In the twisted case, this is proved in [11, §2.4]; see also [14, §1]. In view of Theorem 4.1, starting from Section 5, we will focus our attention on the SL_n case exclusively.

Fix Γ -equivariant resolutions of singularities

$$\begin{aligned} f_{\mathrm{Dol}}(X, \mathrm{SL}_n): \widetilde{M}_{\mathrm{Dol}}(X, \mathrm{SL}_n) &\longrightarrow M_{\mathrm{Dol}}(X, \mathrm{SL}_n), \\ f_B(X, \mathrm{SL}_n): \widetilde{M}_B(X, \mathrm{SL}_n) &\longrightarrow M_B(X, \mathrm{SL}_n), \end{aligned}$$

which satisfy Theorem 3.8. Note that the functorial resolutions in the proof of Theorem 3.8 are actually $(\Gamma \times G_m)$ -equivariant; see [51, Prop. 3.9.1]. By the isotriviality of

$$\begin{aligned} \mathrm{alb}_{\mathrm{Hod}}: M_{\mathrm{Hod}}(X, \mathrm{GL}_n) &\longrightarrow M_{\mathrm{Hod}}(X, G_m) \\ (E, \nabla_\lambda) &\longmapsto (\det E, \mathrm{tr} \nabla_\lambda) \end{aligned}$$

(which extends the morphism alb defined in (8)), the resolutions $f_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ and $f_B(X, \mathrm{SL}_n)$ extend to resolutions

$$\begin{aligned} f_{\mathrm{Dol}}(X, \mathrm{GL}_n): \widetilde{M}_{\mathrm{Dol}}(X, \mathrm{GL}_n) &\longrightarrow M_{\mathrm{Dol}}(X, \mathrm{GL}_n), \\ f_B(X, \mathrm{GL}_n): \widetilde{M}_B(X, \mathrm{GL}_n) &\longrightarrow M_B(X, \mathrm{GL}_n), \end{aligned}$$

such that the square

$$(13) \quad \begin{array}{ccc} \widetilde{M}_{\mathrm{Dol}}(X, \mathrm{SL}_n) \times T^* \mathrm{Pic}^0(X) & \xrightarrow{\widetilde{\Psi}(X, \mathrm{SL}_n) \times \widetilde{\Psi}(X, G_m)} & \widetilde{M}_B(X, \mathrm{SL}_n) \times (\mathbb{C}^*)^{2g} \\ \downarrow / \Gamma & & \downarrow / \Gamma \\ \widetilde{M}_{\mathrm{Dol}}(X, \mathrm{GL}_n) & \xrightarrow{\widetilde{\Psi}(X, \mathrm{GL}_n)} & \widetilde{M}_B(X, \mathrm{GL}_n) \end{array}$$

and the diagrams in Theorem 3.8 commute.

THEOREM 4.1. — *In the notation above, if the P=W conjecture for the resolution $f_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ holds, then it holds for $f_{\mathrm{Dol}}(X, \mathrm{GL}_n)$.*

Proof. — Cohomologically, the Hitchin fibration

$$\chi(X, \mathrm{GL}_n) \circ f_{\mathrm{Dol}}(X, \mathrm{GL}_n): \widetilde{M}_{\mathrm{Dol}}(X, \mathrm{GL}_n) \longrightarrow \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$$

behaves like the product of the fibration $\chi(X, \mathrm{SL}_n) \circ f_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ and $S_{\mathrm{Pic}^0(X)} : \mathrm{Pic}^0(X) \rightarrow \mathrm{pt}$, by lifting (11) to the resolution. Hence, the perverse filtration associated to $\chi(X, \mathrm{GL}_n) \circ f_{\mathrm{Dol}}(X, \mathrm{GL}_n)$ is the convolution of the Γ -invariant part of the perverse filtrations associated to $\chi(X, \mathrm{SL}_n) \circ f_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ and $S_{\mathrm{Pic}^0(X)}$ (the latter being trivial); compare with [11, §2.4]. In symbols, we write

$$(14) \quad P_k H^d(\widetilde{M}_{\mathrm{Dol}}(X, \mathrm{GL}_n)) \simeq \bigoplus_{j>0} P_{k-j} H^{d-j}(\widetilde{M}_{\mathrm{Dol}}(X, \mathrm{SL}_n))^\Gamma \otimes H^j(\mathrm{Pic}^0(C)).$$

By the Γ -equivariance of $f_B(X, \mathrm{SL}_n)$, the map

$$M_B(X, \mathrm{SL}_n) \times (\mathbb{C}^*)^{2g} \longrightarrow M_{\mathrm{Dol}}(X, \mathrm{GL}_n)$$

lifts to the resolutions, and so there exists an isomorphism of mixed Hodge structures

$$(15) \quad H^*(\widetilde{M}_B(X, \mathrm{GL}_n)) \simeq H^*(\widetilde{M}_B(X, \mathrm{SL}_n))^\Gamma \otimes H^*((\mathbb{C}^*)^{2g}),$$

as in (12). Explicitly, we write

$$(16) \quad W_k H^d(\widetilde{M}_B(X, \mathrm{GL}_n)) \simeq \bigoplus_{j>0} W_{k-2j} H^{d-j}(\widetilde{M}_B(X, \mathrm{SL}_n))^\Gamma \otimes H^j((\mathbb{C}^*)^{2g}),$$

since $H^j((\mathbb{C}^*)^{2g})$ has weight $2j$.

Assume now that

$$P_k H^*(\widetilde{M}_{\mathrm{Dol}}(X, \mathrm{SL}_n)) = \widetilde{\Psi}(X, \mathrm{SL}_n)^* W_{2k} H^*(\widetilde{M}_B(X, \mathrm{SL}_n)).$$

Then by the commutativity of (13), together with (14) and (16), we conclude that

$$P_k H^*(\widetilde{M}_{\mathrm{Dol}}(X, \mathrm{GL}_n)) = \widetilde{\Psi}(X, \mathrm{GL}_n)^* W_{2k} H^*(\widetilde{M}_B(X, \mathrm{GL}_n)).$$

REMARK 4.2. — With obvious change, the analogues of Theorem 4.1 for the PI=W1 and P=W conjectures hold.

REMARK 4.3. — Since $M_{\mathrm{Dol}}(X, \mathrm{PGL}_n)$ is the quotient of $M_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ by the Γ -action, the PI=W1 conjecture for $M_{\mathrm{Dol}}(X, \mathrm{SL}_n)$ (or $M_{\mathrm{Dol}}(X, \mathrm{GL}_n)$) implies the PI=W1 conjecture for $M_{\mathrm{Dol}}(X, \mathrm{PGL}_n)$.

5. P=W CONJECTURES FOR GENUS 1

Let A be a compact Riemann surface of genus 1. The construction of the moduli spaces $M_{\mathrm{Dol}}(A, \mathrm{SL}_n)$ and $M_B(A, \mathrm{SL}_n)$ agrees formally with that of a generalized Kummer variety in [2, §7]. It is possible to make this analogy more precise by showing that $M_{\mathrm{Dol}}(A, \mathrm{SL}_n)$ and $M_B(A, \mathrm{SL}_n)$ are specializations of generalized Kummer varieties; see Example A.5 and also [60, §5.3].

Following [32], we describe a stratification of these Kummer-like varieties in Section 5.1, from which we deduce the P=W conjecture in genus 1 (Theorem 5.3).

5.1. **KUMMER-LIKE VARIETIES.** — Let X be a complex algebraic group of dimension 2. We denote by $X^{(n)}$ and $X^{[n]}$ the n -fold symmetric product of X and the Hilbert scheme of n -points on X ; see [2, §6] for an overview of their construction. Recall that the Hilbert-Chow morphism $f: X^{[n]} \rightarrow X^{(n)}$ is a desingularization of $X^{(n)}$.

Consider the addition map $a_n: X^{(n)} \rightarrow X$, given by $a_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i$. For any $g \in \mathbb{Z}_{>0}$, denote by $X(g)$ the set of g -torsion points in X . Let $P(n)$ be the set of partitions of n . We write $\alpha \in P(n)$ as $n = \alpha_1 \cdot 1 + \dots + \alpha_\ell \cdot \ell$, and put $|\alpha| = \sum \alpha_i$ and $g(\alpha) := \gcd\{\nu \mid \alpha_\nu \neq 0\}$.

Following [32], we describe a stratification of the fiber of a_n .

– The variety $K^{[n]}$ is the fiber $f^{-1} \circ a_n^{-1}(0)$ of the composition $X^{[n]} \xrightarrow{f} X^{(n)} \xrightarrow{a_n} X$. When necessary, we emphasize the dependence on X by writing $K^{[n]}(X)$.

– The fiber $K^{(n)} := a_n^{-1}(0)$ can be described as the set of maps from X to $\mathbb{Z}_{>0}$ of total sum n

$$K^{(n)} = \{h \in \text{Hom}_{\text{Sets}}(X, \mathbb{Z}_{>0}) \mid \sum_{x \in X} h(x) = n\}.$$

We say that $K^{(n)}$ is Kummer-like.

– There exists a stratification

$$K^{(n)} = \bigsqcup_{\alpha \in P(n)} K_\alpha^{(n)} \quad \text{with } K_\alpha^{(n)} = \{h \in K^{(n)} \mid \#h^{-1}(x) = \alpha_\nu, \forall \nu\}.$$

– The normalization of the closure of the stratum $K_\alpha^{(n)}$ in $K^{(n)}$, denoted $K^{(\alpha)}$, is the disjoint union

$$K^{(\alpha)} = \bigsqcup_{y \in X(g(\alpha))} K_y^{(\alpha)},$$

where

$$K_y^{(\alpha)} = \{h = (h_1, \dots, h_\ell) \in K^{(\alpha)} \mid \sum_{\nu, x} (\nu/g(\alpha)) h_\nu(x) \cdot x = y\}.$$

– Let $\tau_z: X \rightarrow X$ be the translation by $z \in X$. The finite map $q_y^{(\alpha)}: X \times K_y^{(\alpha)} \rightarrow X^{(\alpha)}$, given by $q_y^{(\alpha)}(z, h_1, \dots, h_\ell) = (h_1 \circ \tau_z, \dots, h_\ell \circ \tau_z)$, induces the isomorphism of mixed Hodge structures $H^*(X \times K_y^{(\alpha)}) \simeq H^*(X^{(\alpha)})$; see [32, p. 243].

All these facts implies the following theorem due to Göttsche and Soergel, that we state without proof.

THEOREM 5.1 ([32, Th. 7]). — Denote by f_0 the birational map $f_0 := (\text{id}_X, f|_{K^{[n]}}): X \times K^{[n]} \rightarrow X \times K^{(n)}$. Let $\kappa_y^{(\alpha)}: K_y^{(\alpha)} \rightarrow K^{(n)}$ be the composition

$$K_y^{(\alpha)} \hookrightarrow K^{(\alpha)} \longrightarrow \overline{K_\alpha^{(n)}} \hookrightarrow K^{(n)}.$$

Then there exists a distinguished splitting isomorphism

$$(17) \quad (f_0)_*(\underline{\mathbb{Q}}_{X \times K^{(n)}}[n]) \simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(\alpha))} (\text{id}_X \times \kappa_y^{(\alpha)})_*(\underline{\mathbb{Q}}_{X \times K_y^{(\alpha)}}[|\alpha|]).$$

The splitting induces a canonical isomorphism of mixed Hodge structures (recall that a Tate twist $(-k)$ increases the weights by $2k$):

$$(18) \quad H^{d+2n}(X \times K^{[n]})(n) \simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(\alpha))} H^{d+2|\alpha|}(X^{(\alpha)})(|\alpha|).$$

A morphism $\chi: X \rightarrow \mathbb{C}$ yields the commutative diagram

$$\begin{array}{ccccc} X \times K^{(n)} & \xleftarrow{\text{id}_X \times \kappa_y^{(\alpha)}} & X \times K_y^{(\alpha)} & \xrightarrow{q_y^{(\alpha)}} & X^{(\alpha)} \\ \chi_0 \downarrow & & \downarrow \chi_y^{(\alpha)} & & \downarrow \chi^{(\alpha)} \\ \mathbb{C} \times \mathbb{C}^{(n-1)} & \xleftarrow{\text{id}_{\mathbb{C}} \times \kappa_{\chi(y)}^{(\alpha)}} & \mathbb{C} \times \mathbb{C}^{(|\alpha|-1)} & \xrightarrow{q_{\chi(y)}^{(\alpha)}} & \mathbb{C}^{(\alpha)}. \end{array}$$

The perverse filtration associated with $\chi_0 := \text{id}_X \times \chi^{(n)}|_{K^{(n)}}$ can be written in terms of the perverse filtration associated with $\chi^{(\alpha)}$.

THEOREM 5.2. — *The perverse filtration associated with χ_0 can be expressed as*

$$P_k H^{d+2n}(X \times K^{[n]})(n) \simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(\alpha))} P_k H^{d+2|\alpha|}(X^{(\alpha)})(|\alpha|),$$

where $P_k H^*(X^{(\alpha)})$ is the perverse filtration associated with $\chi^{(\alpha)}$.

Proof. — By Theorem 5.1 and the t -exactness of finite morphisms, we obtain

$$\begin{aligned} \mathbb{P}H^k((\chi_0 \circ f_0)_*(\underline{\mathcal{O}}_{X \times K^{(n)}}[n])) &\simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(\alpha))} (\text{id}_X \times \kappa_y^{(\alpha)})_* \mathbb{P}H^k(\chi_{y,*}^{(\alpha)} \underline{\mathcal{O}}_{X \times K_y^{(\alpha)}}[|\alpha|]), \\ q_{\chi(y),*}^{(\alpha)} \mathbb{P}H^k(\chi_{y,*}^{(\alpha)} \underline{\mathcal{O}}_{X \times K_y^{(\alpha)}}) &= \mathbb{P}H^k(\chi_*^{(\alpha)} q_{y,*}^{(\alpha)} \underline{\mathcal{O}}_{X \times K_y^{(\alpha)}}) \supset \mathbb{P}H^k(\chi_*^{(\alpha)} \underline{\mathcal{O}}_{X^{(\alpha)}}). \end{aligned}$$

This means that the isomorphism (18) is filtered strict with respect to the perverse filtration associated with χ_0 and $\chi^{(\alpha)}$.

5.2. THE PROOF OF THE CONJECTURE

THEOREM 5.3. — *The PI=WI conjectures for $M_{\text{Dol}}(A, \text{SL}_n)$ and the P=W conjectures for its symplectic resolutions hold.*

Proof. — The moduli space $M_{\text{Dol}}(A, \text{SL}_n)$ parametrizes semistable Higgs bundles on the elliptic curve A , and it is isomorphic to $K^{(n)}(A \times \mathbb{C})$; see for instance [28, Th. 4.27(v)], which actually holds for any n , not only for $n > 4$, or [35]. The character variety $M_{\text{B}}(A, \text{SL}_n)$ instead is isomorphic to $K^{(n)}(\mathbb{C}^* \times \mathbb{C}^*)$ (cf. [60, Proof of Th. 5.3.2]), and in suitable coordinates the non-abelian Hodge correspondence is induced by the symmetric product of the exponential map

$$\begin{aligned} A \times \mathbb{C} &\longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ (\theta_1, \theta_2, r_1, r_2) &\longmapsto (\exp(-2r_1 + i\theta_1), \exp(2r_2 + i\theta_2)); \end{aligned}$$

see [77, Ex. after Prop. 1.5]. By Theorem 5.1 and 5.2, the P=W conjecture for the symplectic resolution $K^{[n]}(A \times \mathbb{C})$ is equivalent to

$$(19) \quad P_k H^*((A \times \mathbb{C})^{(\alpha)}) = W_{2k} H^*((\mathbb{C}^* \times \mathbb{C}^*)^{(\alpha)})$$

for any partition $\alpha \in P(n)$. The identity (19) has already been proved in [12, Lem. 3.1.1 & 3.2.2].

REMARK 5.4. — Since $M_{\text{Dol}}(A, \text{SL}_n)$ has at worst quotient singularities, the P=W conjecture for $M_{\text{Dol}}(A, \text{SL}_n)$ is equivalent to the PI=WI conjecture for $M_{\text{Dol}}(A, \text{SL}_n)$.

6. THE MODULI SPACE OF HIGGS BUNDLES M AND ITS ALTERATIONS

Here and in the following C is a compact Riemann surface of genus 2. We denote by $\iota: C \rightarrow C$ the hyperelliptic involution, and by K_C the canonical bundle of C .

For the sake of notational simplicity, we denote

- the Dolbeault moduli space $M_{\text{Dol}}(C, \text{SL}_2)$ simply by M ;
- the desingularization $\widetilde{M}_{\text{Dol}}(C, \text{SL}_2)$ in Proposition 6.1 by \widetilde{M} ;
- the character variety $M_{\text{B}}(C, \text{SL}_2)$ by M_{B} ;
- the resolution $f_{\text{Dol}}(C, \text{SL}_2)$ by $f: \widetilde{M} \rightarrow M$;
- the Hitchin map $\chi(C, \text{SL}_2)$ by $\chi: M \rightarrow H^0(C, K^{\otimes 2})$.

6.1. SYMPLECTIC RESOLUTION OF M

6.1.1. *Singularities of M and its resolution.* — We briefly recall the description of the singular locus of M and the construction of the resolution. A key aspect is the local isomorphism between the singularities of M and those of the celebrated O’Grady six dimensional example of irreducible holomorphic symplectic variety. We refer to [26] for more details. Via the non-abelian Hodge correspondence, we obtain an analogous description of the singularities of M_{B} .

There exists a Whitney stratification of M

$$(20) \quad \Omega = \text{Sing}(\Sigma) \subset \Sigma = \text{Sing}(M) \subset M,$$

where

$$\begin{aligned} \Sigma &\simeq \{(E, \phi) \simeq (L, \varphi) \oplus (L^{-1}, -\varphi) \text{ with } L \in \text{Pic}^0(C), \text{ and } \varphi \in H^0(C, K_C)\}, \\ \Omega &\simeq \{(E, \phi) \simeq (L, 0) \oplus (L, 0) \text{ with } L \in \text{Pic}^0(C) \text{ s.t. } L^2 \simeq \mathcal{O}_C\}. \end{aligned}$$

Note that Σ is isomorphic to the quotient of $\text{Pic}^0(C) \times H^0(C, K_C)$ by the involution $(L, \varphi) \mapsto (L^{-1}, -\varphi)$, hence it has dimension 4. The locus Ω instead is the branch locus of the quotient map $\text{Pic}^0(C) \times H^0(C, K_C) \rightarrow \Sigma$, and consists of 16 points Ω_j , with $j = 1, \dots, 16$.

A transverse slice to Σ at a point in $\Sigma \setminus \Omega$ has a quotient surface singularity of type A_1 . An analytic neighbourhood of a point of Ω is more complicated, and it was described in detail in [56]. The singularities are symplectic, and a symplectic resolution can be constructed simply by blowing-up M along Σ .

PROPOSITION 6.1 ([26, Prop. 4.2]). — *Let $f: \widetilde{M} \rightarrow M$ be the blow-up of M along Σ . Then f is a symplectic resolution, and we have that:*

- f is an isomorphism over $M \setminus \Sigma$;
- $f^{-1}(p) \simeq \mathbb{P}^1$ for all $p \in \Sigma \setminus \Omega$;
- $f^{-1}(\Omega_j) \simeq \widetilde{\Omega}_j$, where $\widetilde{\Omega}_j$ is the Grassmannian of Lagrangian planes in a symplectic 4-dimensional vector space, which is isomorphic to a smooth quadric in \mathbb{P}^4 .

Via the non-abelian Hodge correspondence Ψ , the stratification of M in (20) induces the stratification of M_B given by

$$(21) \quad \Omega_B = \text{Sing}(\Sigma_B) = \Psi(\Omega) \subset \Sigma_B = \text{Sing}(M_B) = \Psi(\Sigma) \subset M_B,$$

where

$$(22) \quad \Sigma_B := \{(A_1, A_2, B_1, B_2) \in (\mathbb{C}^*)^4 \subset \text{SL}_2^4\} \quad \text{SL}_2 \simeq (\mathbb{C}^*)^4 / (\mathbb{Z}/2\mathbb{Z});$$

$$\Omega_B := \{(A_1, A_2, B_1, B_2) \in (\pm \text{id})^4 \subset \text{SL}_2^4\} = \bigcup_{j=1}^{16} \Omega_{B,j}.$$

By Theorem 1.4, M_B admits a symplectic resolution, and its fibers can be described as in Proposition 6.1.

6.1.2. *Attracting and repelling sets*

DEFINITION 6.2. — Let X be a complex variety with a G_m -action, and F be a subset of its fixed locus. We denote by

$$\text{Attr}(F) = \{x \in X \mid \lim_{\lambda \rightarrow 0} \lambda \cdot x \in F\}$$

the *attracting set* of F , and by

$$\text{Repell}(F) = \{x \in X \mid \lim_{\lambda \rightarrow \infty} \lambda \cdot x \in F\}$$

the *repelling set* of F .

The tangent space of any fixed point $p \in \text{Fix}(X)$ decomposes into the direct sum of weights spaces

$$T_p X = \bigoplus_{m \in \mathbb{Z}} T_p X_m,$$

where $T_p X_m = \{v \in T_p X \mid \lambda \cdot v = \lambda^m v \text{ for all } \lambda \in G_m\}$.

DEFINITION 6.3. — The sequences of integers m_1, m_2, \dots such that $\lambda^{m_1}, \lambda^{m_2}, \dots$ are eigenvalues of the linear operator induced by the G_m -action on $T_p X$ are called *weights* of the G_m -action at the fixed point p .

Let X^{sm} be the smooth locus of X , and denote a connected component of the fixed locus $\text{Fix}(X^{\text{sm}})$ simply by F . Note that the function of weights

$$\begin{aligned} \text{Fix}(X^{\text{sm}}) &\longrightarrow \mathbb{Z}^{(\dim X)} \\ p &\longmapsto (m_1(p), m_2(p), \dots) \end{aligned}$$

is locally constant.

In particular, the following identities hold:

$$(23) \quad T_p \text{Attr}(p) = \bigoplus_{m > 0} T_p X_m, \quad T_p \text{Repell}(p) = \bigoplus_{m < 0} T_p X_m,$$

$$(24) \quad T_p \text{Attr}(F) = \bigoplus_{m > 0} T_p X_m, \quad T_p \text{Repell}(F) = \bigoplus_{m < 0} T_p X_m.$$

6.1.3. *Białynicki–Birula decomposition.* — We briefly recall the celebrated Białynicki–Birula decomposition.

DEFINITION 6.4 ([42, Def. 1.1.1]). — A *semiprojective* variety is a complex quasi-projective algebraic variety X with a G_m -action such that:

- the fixed point set $\text{Fix}(X)$ is proper;
- for every $x \in X$ the limit $\lim_{\lambda \rightarrow 0} \lambda \cdot x$ exists.

THEOREM 6.5 (Białynicki–Birula decomposition). — *Let X be a normal semiprojective variety. Then the following facts hold:*

- (1) X admits a decomposition into G_m -invariant locally closed subsets

$$X = \bigsqcup_{F \in \pi_0(\text{Fix}(X))} \text{Attr}(F);$$

- (2) the limit map

$$\text{Attr}(F) \longrightarrow F : x \longmapsto \lim_{x \rightarrow 0} \lambda \cdot x$$

is an algebraic map, and it is an affine bundle if $F \subset X^{\text{sm}}$;

- (3) the connected components of the fixed locus $\text{Fix}(X^{\text{sm}})$ are smooth.

Proof. — See [6, Th. 4.3] in the smooth projective case; [42, §1.2] and [58, Lem. 3.2.4] in the smooth semiprojective case; [85, Cor. 4] in the normal complete case.

The cohomology of a semiprojective variety can be expressed in terms of the cohomology of the components of the fixed locus.

THEOREM 6.6 (Local-to-global spectral sequence, [86, §4.4]). — *Let X be a normal semiprojective variety. Fix an ordering F_0, F_1, \dots of the connected components of $\text{Fix}(X)$ such that if $F_i < F_j$ then $\dim \text{Attr } F_i > \dim \text{Attr } F_j$. Then the following facts hold.*

- The Białynicki–Birula decomposition yields the spectral sequence

$$(25) \quad E_1^{i,j} = H^{i+j}(\text{Attr}(F_i), u_i^! \underline{\mathbb{Q}}_X) \implies H^{i+j}(X, \mathbb{Q}),$$

where $u_i: \text{Attr}(F_i) \hookrightarrow X$ is the inclusion.

– If X is smooth and $\text{Attr}(F_i)$ are smooth subvarieties of codimension c_j , then we can rewrite the spectral sequence (25) as

$$(26) \quad E_1^{i,j} = H^{i+j-2c_j}(F_i, \mathbb{Q}) \implies H^{i+j}(X, \mathbb{Q})$$

– The spectral sequence (26) degenerates at the first page, and the Poincaré polynomial $P_t(X) := \sum_{k=0}^{2 \dim X} (-1)^n \dim H^k(X, \mathbb{Q}) t^k$ can be written

$$P_t(X) = \sum P_t(F_i) t^{2c_i}.$$

6.1.4. *Torus action on M and \widetilde{M} .* — The multiplicative group G_m acts on M by rescaling the Higgs field

$$\lambda \cdot (E, \phi) = (E, \lambda\phi).$$

The Hitchin map $\chi: M \rightarrow H^0(C, K_C^{\otimes 2})$ is G_m -equivariant, where G_m acts linearly on $H^0(C, K_C^{\otimes 2})$ with weight $(2, 2, 2)$. In particular, the fixed locus of M is contained in the nilpotent cone $\chi^{-1}(0)$. Therefore, M is semiprojective. Since the singular locus Σ of M is G_m -invariant, the action lifts to \widetilde{M} , and \widetilde{M} is semiprojective as well.

The goal of this section is to describe the fixed locus of the G_m -action on M , Ω_j and \widetilde{M} , and to compute the weights of the action.

PROPOSITION 6.7 ([46, Ex. 3.13]). — *A vector bundle E underlying a semistable Higgs bundle $(E, \phi) \in M$ satisfies one of the following property:*

- (1) E is a stable vector bundle;
- (2) $E \simeq L \oplus L^{-1}$ with $L \in \text{Pic}^0(C)$ and $L^2 \not\simeq O_C$, i.e., $(E, \phi) \in \Sigma \cap \Omega$;
- (3) $E \simeq L \oplus L^{-1}$ with $L^2 \simeq O_C$, i.e., $(E, \phi) \in \Omega$;
- (4) E is a non-trivial extension of L by L^{-1} with $L^2 \simeq O_C$;
- (5) E is an unstable vector bundle isomorphic to $\theta_j^{-1} \oplus \theta_j$, where θ_j is a theta-characteristic, i.e., a line bundle such that $\theta_j^2 = K_C$.

PROPOSITION 6.8 (Fixed locus of M). — *The fixed locus of the G_m -action on M is*

$$\text{Fix}(M) = N \sqcup \Theta = N \sqcup \bigsqcup_{j \in 1}^{16} \Theta_j,$$

where

- (1) N is the moduli space of semistable Higgs bundles (E, ϕ) with $\phi = 0$, equivalently the moduli space of semistable vector bundles of rank 2 and degree 0, which is isomorphic to \mathbb{P}^3 ;
- (2) Θ is the set of 16 points in M corresponding to the Higgs bundles

$$\Theta_j := \left(\theta_j^{-1} \oplus \theta_j, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Proof. — It is clear that N and Θ are fixed by the G_m -action. Hence, we just need to show that they are the only components of $\text{Fix}(M)$.

To this end, recall that by Proposition 6.7 the vector bundle E underlying a semistable Higgs bundle $(E, \phi) \in M$ is:

- (1) either a semistable vector bundle,
- (2) or an unstable vector bundle, isomorphic to $\theta_j^{-1} \oplus \theta_j$ for some θ_j .

In the former case, the limit of the one-parameter subgroup $(E, \lambda \cdot \phi)$ is $(E, 0)$ (or $(L \oplus L^{-1}, 0)$ in case (4) of Proposition 6.7), and so it lies in N , which is isomorphic to \mathbb{P}^3 by [64]. In the latter case, $(E, \lambda \cdot \phi)$ is isomorphic to

$$(27) \quad \left(\theta_j^{-1} \oplus \theta_j, \lambda \cdot \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \right) \simeq \left(\theta_j^{-1} \oplus \theta_j, \begin{pmatrix} 0 & 1 \\ \lambda^2 u & 0 \end{pmatrix} \right)$$

for some $u \in \text{Hom}(\theta_j^{-1}, \theta_j \otimes K_C)$, after normalizing with the group of diagonal automorphisms of E ; see [46, §11]. Therefore, the locus of G_m -fixed Higgs bundles with underlying unstable vector bundles is given by Θ (which corresponds to $u = 0$).

In Proposition 6.1 we mentioned that the G_m -invariant fiber $f^{-1}(\Omega_j) \subset \widetilde{M}$ over $\Omega_j \simeq (L \oplus L, 0)$, with $L^2 \simeq \mathcal{O}_C$, is the Grassmannian of Lagrangian subspaces of the 4-dimensional symplectic vector space (V, ω_V) .

The deformation theory of Higgs bundles gives the identification of (V, ω_V) with the space of Higgs bundles extensions of $(L, 0)$ by itself, namely

$$\text{Ext}_{\text{Higgs}}^1(L, L) \simeq H^0(C, K_C) \oplus H^1(C, \mathcal{O}_C) \simeq H^1(C, \mathbb{C}),$$

equipped with the symplectic form given by cup product. For further details, we refer the interested reader to [26, §3.2.2]. We just observe that $H^0(C, K_C)$ parametrizes deformations of L with fixed underlying line bundle, while $H^1(C, \mathcal{O}_C)$ parametrizes deformations of L with fixed underlying Higgs field. Therefore, the rescaling action of Higgs fields yields the G_m -action on $\text{Ext}_{\text{Higgs}}^1(L, L)$ defined by $\lambda \cdot (v, \bar{v}) = (\lambda v, \bar{v})$, where $v \in H^0(C, K_C)$ and $\bar{v} \in H^1(C, \mathcal{O}_C)$. This in turn induces the G_m -action on $\widetilde{\Omega}_j$, whose fixed loci are described in the next Proposition 6.9.

PROPOSITION 6.9 (Fixed locus of $\widetilde{\Omega}_j$). — *The fixed locus of the G_m -action on $\widetilde{\Omega}_j$ is*

$$\text{Fix}(\widetilde{\Omega}_j) = t_j \sqcup s_j^+ \sqcup T_j,$$

where

(1) *the points t_j and T_j correspond to the Lagrangian subspaces $H^0(C, K_C)$ and $H^1(C, \mathcal{O}_C)$;*

(2) *the curve s_j^+ parametrizes Lagrangian subspaces generated by $v_1 \in H^0(C, K_C)$ and $v_2 \in H^1(C, \mathcal{O}_C)$, and it is isomorphic to \mathbb{P}^1 .*

In particular, t_j , s_j^+ and T_j have weights $(1, 1, 1)$, $(-1, 0, 1)$ and $(-1, -1, -1)$ respectively.

Proof. — The Plücker polarization H_j embeds $\widetilde{\Omega}_j$ as a smooth quadric in the linear system $|H_j| = \mathbb{P}(W) \simeq \mathbb{P}^4 \subset \mathbb{P}(\wedge^2 V)$. The G_m -action on $\widetilde{\Omega}_j$ induces an action on W with weights $(0, 1, 1, 1, 2)$ in suitable coordinates (x_0, \dots, x_4) . In these coordinates, $\widetilde{\Omega}_j$ is defined by the equation $x_1^2 + x_2x_3 + x_0x_4 = 0$.

Since the Plücker embedding is G_m -equivariant, the fixed loci of $\widetilde{\Omega}_j$ are the intersections of $\widetilde{\Omega}_j$ with the isotypic components of the G_m -representation W , i.e.,

$$\begin{aligned} t_j &= [1 : 0 : 0 : 0 : 0], \\ s_j^+ &= [0 : x_1 : x_2 : x_3 : 0] \cap \widetilde{\Omega}_j = \{x_1^2 + x_2x_3 = 0\} \simeq \mathbb{P}^1, \\ T_j &= [0 : 0 : 0 : 0 : 1]. \end{aligned}$$

Moreover, the tangent space

$$T_{t_j} \widetilde{\Omega}_j \simeq T_{t_j} \mathbb{P}(W) / N_{\widetilde{\Omega}_j / \mathbb{P}(W), t_j} \simeq \text{Hom}(t_j, W / \langle t_j, T_j \rangle)$$

has weight $(1, 1, 1)$. Analogously, if $p = [0 : 0 : 1 : 0 : 0] \in s_j^+$, then $T_p \tilde{\Omega}_j \simeq \text{Hom}(p, \langle t_j, \partial_{x_1}, T_j \rangle)$ has weight $(-1, 0, 1)$, while $T_{T_j} \tilde{\Omega}_j \simeq \text{Hom}(T_j, W/\langle t_j, T_j \rangle)$ has weight $(-1, -1, -1)$.

PROPOSITION 6.10 (Fixed locus of \tilde{M}). — *The fixed locus of the G_m -action on \tilde{M} is*

$$\text{Fix}(\tilde{M}) = \tilde{N} \sqcup \tilde{S}^+ \sqcup \tilde{\Theta} \sqcup \bigsqcup_{j \in 1}^{16} T_j,$$

where

- (1) $\tilde{N} := f_*^{-1} N$ is the strict transform of N , isomorphic to \mathbb{P}^3 ;
- (2) \tilde{S}^+ is a Kummer surface;
- (3) $\tilde{\Theta} := f^{-1}(\Theta)$;
- (4) T_j are points lying on the Lagrangian Grassmannians $\tilde{\Omega}_j = f^{-1}(\Omega_j)$.

Proof. — First, observe that $\text{Fix}(\tilde{M})$ lies over $\text{Fix}(M)$, and so

$$\tilde{N} \sqcup \tilde{\Theta} = f_*^{-1} \text{Fix}(M) \subset \text{Fix}(\tilde{M}) \subset f^{-1}(\text{Fix}(M)).$$

The component of $\text{Fix}(\tilde{M})$ not contained in $f_*^{-1} \text{Fix}(M)$ lies over $\text{Fix}(M) \cap \Sigma := S \subset N$, which is isomorphic to $\text{Pic}^0(C)/(Z/2Z)$, i.e., the singular Kummer surface associated to $\text{Pic}^0(C)$.

The fiber of f over $p \in S \cap \Omega$ is isomorphic to \mathbb{P}^1 , and G_m acts with non-trivial weight on it by Proposition 6.11. Therefore, the \mathbb{P}^1 -bundle $f^{-1}(S \cap \Omega)$ has two G_m -fixed sections. We denote their closure by \tilde{S}^- and \tilde{S}^+ . Since the restriction of f to \tilde{N} is an isomorphism, one of the two sections, say \tilde{S}^- , lies in \tilde{N} . The same holds for one of the two fixed points in each $\tilde{\Omega}_j$, namely t_j because of the weight considerations in Proposition 6.9 and Proposition 6.11.

The other section \tilde{S}^+ must be the union of a copy of $S \cap \Omega$ and the rational curve s_j^+ , with $j = 1, \dots, 16$, thus isomorphic to the nonsingular Kummer surface associated to $\text{Pic}^0(C)$. Indeed, by construction $\tilde{S}^+ \cap \tilde{\Omega}_j$ is a non-empty component of $\text{Fix}(\tilde{\Omega}_j)$ different from a point; otherwise \tilde{S}^+ would be singular, which is a contradiction since \tilde{S}^+ is a fixed locus of a G_m -action on a smooth manifold. Therefore, $\tilde{S}^+ \cap \tilde{\Omega}_j = s_j^+$ by Proposition 6.9.

PROPOSITION 6.11 (Weights of \tilde{M})

- (1) \tilde{N} has weight $(0, 0, 0, 1, 1, 1)$;
- (2) \tilde{S}^+ has weight $(-1, 0, 0, 1, 1, 2)$;
- (3) $\tilde{\Theta}_j$ and Θ_j have weight $(-1, -1, -1, 2, 2, 2)$;
- (4) T_j has weight $(-1, -1, -1, 2, 2, 2)$.

Proof. — Let $\tilde{\omega}$ be the holomorphic symplectic form on the symplectic resolution \tilde{M} extending the canonical holomorphic symplectic form ω on the smooth locus of M . As in [46, Prop. 7.1], the G_m -action rescales the holomorphic symplectic form $\tilde{\omega}$

$$\lambda^* \tilde{\omega} = \lambda \tilde{\omega}.$$

Let $p \in \text{Fix}(\widetilde{M})$, and W be a Lagrangian subspace of $T_p\widetilde{M}$ with weights (a, b, c) . Then the isotropy condition yields an isomorphism

$$W^* \simeq T_p\widetilde{M}/W,$$

and the weights of the action on W become

$$\lambda(\lambda^{-a}, \lambda^{-b}, \lambda^{-c}) = (\lambda^{-a+1}, \lambda^{-b+1}, \lambda^{-c+1})$$

on W^* . As a result the torus action at a fixed point has weights

$$(a, b, c, -a + 1, -b + 1, -c + 1).$$

In this way, the weights of the G_m -action at \widetilde{N} , \widetilde{S}^+ and \widetilde{T}_j follow immediately from the computations in Proposition 6.9, by observing that $t_j \in \widetilde{N}$ and $s_j^\pm \in \widetilde{S}^+$. For the weights at $\widetilde{\Theta}_j$, instead, note that the locus of semistable Higgs bundles with underlying vector bundle $\theta_j^{-1} \oplus \theta_j$ is Lagrangian by definition of ω (cf. [46, Lem. 6.8]), and has weight $(2, 2, 2)$ by (27).

COROLLARY 6.12. — \widetilde{N} and $\widetilde{\Omega}_j$ intersect transversely at the point $t_j = \widetilde{N} \cap \widetilde{\Omega}_j$.

Proof. — By Proposition 6.9, the tangent space $T_{t_j}\widetilde{\Omega}_j$ has weight one, while $T_{t_j}\widetilde{N}$ has weight zero.

COROLLARY 6.13. — The attracting sets $\text{Attr}(N)$, $\text{Attr}(\widetilde{S}^+)$, $\text{Attr}(\widetilde{\Theta}_j)$ and $\text{Attr}(\widetilde{T}_j)$ have codimension $0, 1, 3, 3$ respectively.

Proof. — It is an immediate corollary of (24) and Proposition 6.11.

6.1.5. Poincaré polynomials of M and \widetilde{M}

THEOREM 6.14 (Cohomology of M and \widetilde{M}). — The Poincaré polynomials of M and \widetilde{M} are

$$(28) \quad P_t(M) := \sum_k (-1)^k \dim H^k(M) t^k = 1 + t^2 + t^4 + 17t^6,$$

$$(29) \quad P_t(\widetilde{M}) := \sum_k (-1)^k \dim H^k(\widetilde{M}) t^k = 1 + 2t^2 + 23t^4 + 34t^6.$$

Proof. — Since $\text{Attr}(N)$ is an open subset of M , we have $H^*(\text{Attr}(N), u^! \underline{\mathbb{Q}}_M) = H^*(\text{Attr}(N)) = H^*(\mathbb{P}^3)$. The spectral sequence (25) gives

$$\begin{aligned} P_t(M) &= P_t(N) + P_t(\Theta)t^6 \\ &= 1 + t^2 + t^4 + 17t^6. \end{aligned}$$

See [21, Th. 1.5] for an alternative proof.

Similarly, by Theorem 6.6, Proposition 6.10 and Corollary 6.13, we obtain

$$\begin{aligned} P_t(\widetilde{M}) &= P_t(\widetilde{N}) + P_t(\widetilde{S}^+)t^2 + P_t(\widetilde{\Theta})t^6 + \sum_{j=1}^{16} P_t(T_j)t^6 \\ &= (1 + t^2 + t^4 + t^6) + (1 + 22t^2 + t^4)t^2 + 16t^6 + 16t^6 \\ &= 1 + 2t^2 + 23t^4 + 34t^6. \end{aligned}$$

6.2. MODULI SPACE OF EQUIVARIANT HIGGS BUNDLES M_t

6.2.1. *Equivariant Higgs bundle and the forgetful map q .* — Recall that $\iota : C \rightarrow C$ is the hyperelliptic involution of the curve C of genus 2.

DEFINITION 6.15. — A (ι -)equivariant Higgs bundle over C is a triple (E, h, ϕ) such that:

- (1) E is an ι -invariant vector bundle, i.e., $\iota^*E \simeq E$;
- (2) $h : E \rightarrow \iota^*E$ is a lift of the ι -action on E such that $\iota^*h \circ h = \text{id}_E$;
- (3) $\phi \in \text{Hom}(E, E \otimes K_C)$ is an ι -invariant Higgs field, i.e., a \mathcal{O}_C -linear morphism which makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes K_C \\ h \downarrow & & \downarrow h \otimes \text{id}_{K_C} \\ \iota^*E & \xrightarrow{\iota^*\phi} & \iota^*E \otimes K_C. \end{array}$$

A morphism between two equivariant Higgs bundles (E_1, h_1, ϕ_1) and (E_2, h_2, ϕ_2) is a homomorphism of vector bundles $\psi \in \text{Hom}(E_1, E_2)$ such that the following diagrams commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{h_1} & \iota^*E_1 \\ \psi \downarrow & & \downarrow \psi \\ E_2 & \xrightarrow{h_2} & \iota^*E_2, \end{array} \quad \begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_1 \otimes K_C \\ \psi \downarrow & & \downarrow \psi \otimes \text{id}_{K_C} \\ E_2 & \xrightarrow{\phi_2} & E_2 \otimes K_C. \end{array}$$

The slope of a vector bundle E over a curve C is defined by

$$\mu(E) := \text{deg}(E) / \text{rank}(E).$$

DEFINITION 6.16. — An equivariant Higgs bundle (E, h, ϕ) is *semistable* or *stable* if for any proper equivariant Higgs subbundle $F \subset E$, the inequality $\mu(F) \leq \mu(E)$ holds, respectively $\mu(F) < \mu(E)$.

Let $W = \{w_1, \dots, w_6\}$ be the set of all Weierstrass points, i.e., the fixed points of ι . For every $w \in W$, $h_w : E_w \rightarrow E_w$ is an involution of the fiber E_w .

DEFINITION 6.17. — The normal quasi-projective variety M_t (respectively M_t^s) is the coarse *moduli space of semistable* (respectively *stable*) *equivariant Higgs bundle* (E, h, ϕ) of rank 2 over C with trivial determinant and $\text{tr}(h_w) = 0$ for all $w \in W$.

The existence of M_ι and M_ι^s follows from the work of Seshadri [73] and Nitsure [67]. In Section 6.2.4 we review the construction. Here we first describe M_ι as a quasi-étale cover of M . This cover appears also in [45, §6.3] and references therein.

DEFINITION 6.18 (Quasi-étale morphism). — A morphism $f: X \rightarrow Y$ between normal varieties is *quasi-étale* if f is quasi-finite, surjective and étale in codimension one, i.e., there exists a closed, subset $Z \subseteq X$ of codimension $\text{codim } Z > 2$ such that $f|_{X \setminus Z}: X \setminus Z \rightarrow Y$ is étale.

REMARK 6.19. — By the purity of the branch locus, a quasi-étale morphism induces an étale cover of the smooth locus of the codomain.

PROPOSITION 6.20. — *The forgetful map*

$$\begin{aligned} q: M_\iota &\longrightarrow M \\ (E, h, \phi) &\longmapsto (E, \phi) \end{aligned}$$

is well-defined, quasi-étale of degree two, and branched along the singular locus Σ of M_ι .

Proof. — The forgetful map q is well-defined, because an equivariant Higgs bundle (E, h, ϕ) is semistable if and only if the Higgs bundle (E, ϕ) is semistable in the usual sense (the same proof of [8, Lem. 2.7] applies). The map q is also surjective: any semistable Higgs bundles (E, ϕ) admits a lift of the ι -action on E conjugating ϕ and $\iota^*\phi$ by [45, Chap. 6, p. 74, & Th. 2.1].

We show now that q is quasi-étale. To this end, we closely follow the proof of [54, Th. 2.1]. Given two equivariant Higgs bundles (E, h_1, ϕ) and (E, h_2, ϕ) , there exists an automorphism $A \in \text{Aut}(E)$ such that $h_2 = h_1 \circ A$ and $\phi = A^{-1}\phi A$.

If (E, ϕ) is stable, then the only automorphisms which fix the Higgs field are scalars. Then $h_2 = \pm h_1$, and so there are only two non-equivalent equivariant Higgs bundles (E, h_1, ϕ) and $(E, -h_1, \phi)$ over (E, ϕ) . Hence, q is generically $2 : 1$.

If (E, ϕ) is strictly semistable, i.e., $(E, \phi) \in \Sigma$, then $E \simeq L \oplus L^{-1}$ with $L \in \text{Pic}^0(C)$, and any two lifts are equivalent. Hence, q is quasi-finite and branched along Σ .

6.2.2. Non-abelian Hodge correspondence. — Let $C \rightarrow \mathbb{P}^1$ be the quotient of C via the hyperelliptic involution, and let \underline{W} be the critical divisors on \mathbb{P}^1 , i.e., the projection of the Weierstrass points.

The moduli space M_ι is isomorphic to the moduli space of parabolic Higgs bundle of rank 2 on \mathbb{P}^1 with parabolic weight $1/2$ at all points of \underline{W} and parabolic degree zero; see [9, Th. 3.5].

The topological space underlying M_ι parametrizes also representations of the orbifold fundamental group

$$\pi_1^{\text{orb}}(C/\iota) \simeq \langle \gamma_1, \dots, \gamma_6 \mid \gamma_1^2 = \dots = \gamma_6^2 = 1 \text{ and } \gamma_1 \dots \gamma_6 = 1 \rangle.$$

THEOREM 6.21 (Non-abelian Hodge correspondence). — *There exists a commutative square*

$$\begin{array}{ccc} M_\iota & \xrightarrow{\Psi_\iota} & M_B(2, \mathrm{SL}_2, \iota) := \mathrm{Hom}(\pi_1^{\mathrm{orb}}(C/\iota), \mathrm{SL}_2) & \mathrm{PGL}_2 \\ q \downarrow & & \downarrow q^{\mathrm{top}} & \\ M & \xrightarrow{\Psi} & M_B(2, \mathrm{SL}_2) = \mathrm{Hom}(\pi_1(C), \mathrm{SL}_2) & \mathrm{PGL}_2. \end{array}$$

where the horizontal arrows are real analytic isomorphisms, and the vertical arrows are quasi-étale covers.

Proof. — Identify M_ι with a moduli space of parabolic Higgs bundles as above. The correspondences Ψ and Ψ_ι have been constructed by Hitchin [46] and Simpson [76] respectively. By construction, the square commutes.

6.2.3. Singularities of M_ι

NOTATION 6.22. — We fix the following notation:

- $\mathrm{Bun}^{\mathrm{ss}}(C/\iota)$ is the moduli space of semistable ι -equivariant vector bundles (E, h) . It is the inverse image of the moduli space of semistable vector bundles N via q ;
- the inverse images of Ω via q consists of the 16 points Ω_ι ;
- the inverse images of Θ via q consists of the 32 points Θ_ι .

PROPOSITION 6.23 (Singularities of M_ι)

- (1) Ω_ι is the singular locus of M_ι .
- (2) The smooth locus of M_ι , denoted M_ι^{sm} , is the moduli space of stable equivariant Higgs bundles M_ι^s .

Proof. — The local isomorphism type of the singularities of M_ι coincides with the model described in [62, Lem. 3.1]. This yields the first statement. For the second statement, it is enough to show that

$$M_\iota^{\mathrm{sm}} = q^{-1}(M \cap \Sigma) \cup q^{-1}(\Sigma \cap \Omega) \subseteq M_\iota^s.$$

Any Higgs bundle $(E, \phi) \in M \cap \Sigma$ is stable, and so the equivariant Higgs bundles in $q^{-1}(M \cap \Sigma)$ are stable too. If $(E, \phi) \in \Sigma \cap \Omega$ with $E \simeq L \oplus L^{-1}$, then the only line subbundles of E are L and L^{-1} , but since they are not ι -invariant, $q^{-1}(\Sigma \cap \Omega) \subseteq M_\iota^s$.

6.2.4. Construction of M_ι . — The moduli space M_ι is constructed in the following way. All the ingredients have already appeared in [73, 67, 39].

Let (E, h, ϕ) be a stable equivariant Higgs bundle of rank 2 over C with trivial determinant and $\mathrm{tr}(h_w) = 0$ for all $w \in W$. Fix an equivariant ample line bundle $\mathcal{O}_C(1)$ on C . Choose an integer $m \in \mathbb{Z}$ such that $H^1(C, E(m)) = 0$ and $E(m)$ is globally generated.

The quot scheme Q parametrizes all quotient sheaves of $H^0(C, E(m)) \otimes \mathcal{O}_C$ with the Hilbert polynomial of $E(m)$. Let $H^0(C, E(m)) \otimes p_C^* \mathcal{O}_C \rightarrow E_Q \otimes p_C^* \mathcal{O}_C(m)$ be the universal quotient bundle on $Q \times C$, with the natural projection $p_C: Q \times C \rightarrow C$.

Let $R \subset Q$ be the subset of all $q \in Q$ for which E_q is locally free and the map $H^0(C, E(m)) \rightarrow H^0(C, E_q(m))$ is an isomorphism.

By [67, Prop. 3.6], there exists a locally universal family of semistable Higgs bundles $E_{ss} \xrightarrow{\Phi_{ss}} E_{ss} \otimes p_C^* K_C$ on $F_{ss} \times C$, where F_{ss} is an open subset of a linear R -scheme $F \rightarrow R$ together with a family of Higgs bundles $E_F \xrightarrow{\Phi_F} E_F \otimes p_C^* K_C$.

The involution ι^* on $H^0(C, E(m))$ induces a natural lift j_0 of the ι -action on the trivial bundle $C \times H^0(C, E(m))$, and so an ι -action on F_{ss} with fixed locus $\text{Fix}_\iota(F_{ss})$. In particular, j_0 descends to a lift h_q of the ι -action on E_q , for any $q \in \text{Fix}_\iota(F_{ss})$. Call $F_{ss,\iota}$ the connected component of $\text{Fix}_\iota(F_{ss})$ consisting of the equivariant Higgs bundles (E_q, h_q, Φ_q) with $\text{tr}(h_{q,w}) = 0$; see [73, Chap. II, Prop. 6(iv)] and [73, Chap. II, Prop. 5 & Rem. 2].

Let H be the group of automorphisms of the trivial bundle which commute with j_0 , and $PH := H/G_m$ the quotient of H modulo scalar matrices. The moduli spaces M_ι and M_ι^s are the quotients $F_{ss,\iota}/PH$ and $F_{s,\iota}/PH$ respectively, where $F_{s,\iota}$ is the subset of stable equivariant Higgs bundles in $F_{ss,\iota}$.

6.2.5. Universal bundles. — We show the existence of a universal bundle on $M_\iota^{\text{sm}} \times C$ (cf. [39, §5]).

DEFINITION 6.24. — Let Z be a subset of M_ι^{sm} . A *universal Higgs bundle* on $Z \times C$ is a rank two Higgs bundle (E, Φ) such $(E, \Phi)|_{\{(E,h,\phi)\} \times C} \simeq (E, \phi)$ for all $(E, h, \phi) \in Z$.

REMARK 6.25. — Let (E_1, Φ_1) and (E_2, Φ_2) be universal Higgs bundles on $Z \times C$. Then there exists a line bundle $L \in \text{Pic}(Z)$ such that $(E_1, \Phi_1) \simeq (E_2 \otimes p_C^* L, \Phi_2)$, with $p_C: Z \times C \rightarrow C$ the natural projection. In particular, $P(E_1) \simeq P(E_2)$ is canonical. See [44, 4.2].

We adopt the notation of Section 6.2.4. In addition, we define F_ι° as being the open subset of $F_{s,\iota}$ parametrizing stable equivariant Higgs bundle whose underlying vector bundle is either stable or isomorphic to $L \oplus \iota^* L$ with $L \in \text{Pic}^0(C)$ with $L^2 \neq \mathcal{O}_C$.

The quotient $M_\iota^\circ := F_\iota^\circ / PH$ is the attracting set of $\text{Bun}^{\text{ss}}(C/\iota) \curvearrowright \Omega_\iota$. Thus, according to Proposition 6.7, the complement $M_\iota \setminus M_\iota^\circ$ parametrizes stable equivariant Higgs bundles whose underlying vector bundle is unstable or a non-trivial extension of L by L with $L^2 \simeq \mathcal{O}_C$, and so it has codimension 2 by [46, Ex. 3.13(iv) & (v)]; see also [49, Lem. 3.4]. In particular, $F_{s,\iota} \setminus F_\iota^\circ$ has codimension 2.

PROPOSITION 6.26. — *A universal Higgs bundle on $M_\iota^{\text{sm}} \times C$ does exist.*

Proof. — Let E be the restriction of the universal Higgs bundle E_F to $F_{s,\iota} \times C$, and denote by $p_F: F_{s,\iota} \times C \rightarrow F_{s,\iota}$ and $p_C: F_{s,\iota} \times C \rightarrow C$ the two projections.

The natural lift of the H -action is such that the subgroup of scalar matrices acts by homotheties. Suppose that there exists an H -equivariant line bundle $\lambda(E)$ over $F_{s,\iota}$ with the same property, i.e., that the center of H acts by homotheties. Then, the center of H acts trivially on $E \otimes p_F^* \lambda(E)^{-1}$. By Kempf's descent lemma [24, Th. 2.3],

the PH -equivariant bundle $E \otimes p_F^* \lambda(E)^{-1}$ descends to a vector bundle on $M_l^{\text{sm}} \times C$, and since the section Φ is invariant, it also descends.

Here is how to construct $\lambda(E)$. For any $(E, h, \phi) \in F_l^\circ$, h acts on $H^0(C, E \otimes K_C)$, and induces a splitting

$$H^0(C, E \otimes K_C) = H^0(C, E \otimes K_C)^+ \oplus H^0(C, E \otimes K_C)^-$$

into one-dimensional eigenspaces (relative to eigenvalues ± 1 respectively); see [45, Prop. 4.1]. The lift j_0 induces an involution on $p_{F,*}(E \otimes p_C^* K_C)$. Hence, set

$$\lambda(E)^\circ := p_{F,*}(E \otimes p_C^* K_C)^+$$

as the j_0 -invariant subsheaf of $p_{F,*}(E \otimes p_C^* K_C)$. By semicontinuity, $\lambda(E)^\circ$ is a line bundle on F_l° with fiber $H^0(C, E \otimes K_C)^+$. The multiplication by a scalar in E induces multiplication in $H^0(C, E \otimes K_C)^+$ too, and so in $\lambda(E)^\circ$. Now let $i_{F_l^\circ} : F_l^\circ \hookrightarrow F_l$ be the natural inclusion, and define

$$\lambda(E) = i_{F_l^\circ,*} \lambda(E)^\circ.$$

Since $F_{s,\ell}$ is smooth and $F_{s,\ell} \cap F_l^\circ$ has codimension 2, $\lambda(E)$ is a line bundle on $F_{s,\ell}$ with the right H -linearization.

6.2.6. Nilpotent cone. — In this section we describe the components of the nilpotent cone of M , i.e., the zero fiber of the Hitchin fibration $\chi : M \rightarrow H^0(C, K_C^{\otimes 2})$.

We show that $\chi^{-1}(0)$ has 17 irreducible components, one of them being the moduli space of semistable vector bundle N . By [65, Main Th., §3] there is no universal Higgs bundle over any Zariski open set of N . On the other hand, we construct a universal bundle on the normalization of the other components; see Proposition 6.28 and Lemma 6.29.

PROPOSITION 6.27. — *The nilpotent cone of M is a compact union of 3-dimensional manifolds:*

$$\chi^{-1}(0) = N \sqcup \bigsqcup_{j=1}^{16} N_j,$$

where N_j is isomorphic to the vector space $\text{Ext}^1(\theta_j, \theta_j^{-1})$, where θ_j runs over the 16 theta-characteristics $\theta_j^2 = K_C$.

Proof. — We adapt the proof of [84, Prop. 19]; see also [68, §2]. Since $N \subset M$ is the locus of semistable Higgs bundles with trivial Higgs field, we see that $N \subset \chi^{-1}(0)$. However, there are also stable Higgs bundles $(E, \phi) \in \chi^{-1}(0)$ with $\phi \neq 0$.

Under this assumption, ϕ has generically rank one: denote by A the line bundle $\text{Im} \phi \subset E \otimes K_C$. Then E sits in the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{-1} & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \phi \downarrow & \swarrow & \downarrow u & & \\ 0 & \longrightarrow & A \otimes K_C & \longrightarrow & E \otimes K_C & \longrightarrow & A^{-1} \otimes K_C & \longrightarrow & 0. \end{array}$$

Since $\text{tr}(\phi) = 0$, the composition $A \rightarrow E \otimes K_C \rightarrow A \otimes K_C$ is zero, and the inclusion $A \rightarrow E \otimes K_C$ factors through $u: A \rightarrow A^{-1} \otimes K_C$. The stability of E implies $-\deg A < \deg E = 0$, and since $u \in H^0(C, K_C \otimes A^{\otimes(-2)})$ is non-zero, we conclude that A is a theta-characteristic.

Therefore, the Higgs bundle (E, ϕ) is determined by the triple (θ_j, v, u) given by

- the theta-characteristic θ_j ,
- the extension class $v \in \text{Ext}^1(\theta_j, \theta_j^{-1})$ giving the exact sequence $\theta_j^{-1} \rightarrow E \rightarrow \theta_j$,
- the non-zero scalar $u \in H^0(C, \text{Hom}(\theta_j, \theta_j^{-1} \otimes K_C)) \simeq H^0(C, \mathcal{O}_C)$,

modulo the G_m -action

$$c \cdot (\theta_j, v, u) = (\theta_j, cv, cu).$$

The equivalence class (θ_j, v, u) under rescaling is denoted $[\theta_j, v, u]$, and we identify the Higgs bundle $(E, \phi) \in N_j$ with $[\theta_j, v, u]$. In particular, the irreducible components of $\chi^{-1}(0)$ different from N are

$$N_j := \mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1}) \oplus H^0(C, \mathcal{O}_C)) \setminus \{u = 0\} \simeq \text{Ext}^1(\theta_j, \theta_j^{-1}).$$

Alternative proof. — The nilpotent cone on M is the union of the repelling sets of all the fixed loci

$$\chi^{-1}(0) = \text{Repell}(N) \cup \text{Repell}(\Theta) = N \cup \bigcup_{j=1}^{16} \text{Repell}(\Theta_j).$$

By Theorem 6.5(2), $\text{Repell}(\Theta_j)$ is isomorphic to a 3-dimensional vector space. However, we rely on the previous proof for a modular interpretation of $\text{Repell}(\Theta_j)$.

Let R_j be the total space of the projective bundle $\mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1}) \oplus H^0(C, \mathcal{O}_C))$ with hyperplane bundle $\mathcal{O}_{R_j}(1)$. As we observed above, there is a natural decomposition $R_j = N_j \cup \mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1}))$. The inclusion $N_j \hookrightarrow \chi^{-1}(0)$ extends to a bijective and algebraic morphism

$$\begin{aligned} r_j: R_j &\hookrightarrow \chi^{-1}(0) \\ [v : u] &\longmapsto [\theta_j, v, u] = (E, \phi), \end{aligned}$$

whose image is the closure \overline{N}_j of N_j in $\chi^{-1}(0)$; see Theorem 6.5(2) and also [84, Prop. 24].

PROPOSITION 6.28. — *There exists a universal bundle E_{R_j} on $R_j \times C$ which sits in the following exact sequence*

$$0 \longrightarrow p_{R_j}^* \mathcal{O}_{R_j}(1) \otimes p_C^* \theta_j^{-1} \longrightarrow E_{R_j} \longrightarrow p_C^* \theta_j \longrightarrow 0,$$

where $p_{R_j}: R_j \times C \rightarrow R_j$ and $p_C: R_j \times C \rightarrow C$ are the natural projections.

Proof. — Mutatis mutandis, the same argument as in [84, p. 22] works.

Consider now the quasi-étale cover $q: M_t \rightarrow M$. Since N_j is simply connected, $q^{-1}(N_j)$ breaks into two irreducible components, say N_j^+ and N_j^- . In particular,

q restricts to an isomorphism between $M_i^\circ \cap N_j^+$ (equivalently $M_i^\circ \cap N_j^-$) and $\overline{N}_j \cap (\Omega \cup \Theta_j)$. If we set $R_j^\circ := R_j \cap (r_j^{-1}(\Omega) \cup r_j^{-1}(\Theta_j))$, then the product map

$$(30) \quad \tau_j := (r_j \circ q^{-1}, \text{id}): R_j^\circ \times C \longrightarrow (M_i^\circ \cap N_j^+) \times C$$

is an algebraic bijection. Let E be the universal bundle on $M_i^\circ \times C$.

LEMMA 6.29. — *The \mathbb{P}^1 -bundles $P(\tau_j^*E)$ and $P(E_{R_j})$ on $R_j^\circ \times C$ are isomorphic.*

Proof. — The vector bundles τ_j^*E and E_{R_j} are both universal on $R_j^\circ \times C$. The result follows from Remark 6.25.

6.2.7. *Quasi-étale covers of $M_{\text{Dol}}(X, \text{SL}_n)$.* — In this section we show that the quasi-étale cover ι is a special feature of the moduli space $M = M_{\text{Dol}}(C, \text{SL}_2)$, which is not shared by any other space $M_{\text{Dol}}(X, \text{SL}_n)$, $g > 2$.

PROPOSITION 6.30. — *The smooth locus $M_{\text{Dol}}^{\text{sm}}(X, \text{SL}_n)$ of $M_{\text{Dol}}(X, \text{SL}_n)$ is simply-connected for $g > 2$ and $(g, n) \neq (2, 2)$. In particular,*

$$\pi_1(M^{\text{sm}}) = \mathbb{Z}/2\mathbb{Z}.$$

Proof. — $M_{\text{Dol}}^{\text{sm}}(X, \text{SL}_n)$ contains a Zariski open subset which can be identified with the cotangent bundle of the moduli space $\text{Bun}^s(X, n)$ of stable vector bundles of rank r and trivial determinant over X . Therefore, the fundamental group of $M_{\text{Dol}}^{\text{sm}}(X, \text{SL}_n)$ is a quotient of $\pi_1(\text{Bun}^s(X, n))$, which is trivial by [20, Th. 3.2(i)], for $g > 2$ and $(g, n) \neq (2, 2)$.

Consider now M . The forgetful map q induces the following exact sequence in homotopy

$$(31) \quad 1 \longrightarrow \pi_1(M_i^{\text{sm}}) \longrightarrow \pi_1(M^{\text{sm}}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

As before, M_i^{sm} contains a Zariski open subset isomorphic to the cotangent bundle of the moduli space $\text{Bun}^s(C/\iota)$ of stable ι -equivariant bundles of rank 2 over C with trivial determinant and $\text{tr}(h_w) = 0$ for all $w \in W$; see for instance [45, Chap. 6, p. 73]. Thus, we obtain that $\pi_1(M_i^{\text{sm}})$ is a quotient of $\pi_1(\text{Bun}^s(C/\iota))$.

The space $\text{Bun}^s(C/\iota)$ is the smooth locus of the double cover $\text{Bun}^{\text{ss}}(C/\iota)$ of \mathbb{P}^3 branched along a singular Kummer quartic. The singular locus of $\text{Bun}^{\text{ss}}(C/\iota)$ consists of 16 ordinary double points, which are known to admit a small resolution, i.e., the exceptional locus has codimension > 2 . This implies that $\pi_1(\text{Bun}^s(C/\iota))$ coincides with the fundamental group of a (small) resolution of $\text{Bun}^{\text{ss}}(C/\iota)$. Further, $\text{Bun}^{\text{ss}}(C/\iota)$ is rational by [54, Th. 2.2] or [18, Th. 1.3]; see also [45, §5.4.2 & §5.5], where a small resolution of $\text{Bun}^{\text{ss}}(C/\iota)$ is denoted $\text{Bun}_{1/4,1/2}^{\text{ss}}(C/i)$. Since the fundamental group is a birational invariant of smooth proper varieties, we observe that $\text{Bun}_{1/4,1/2}^{\text{ss}}(C/i)$ is simply-connected, since the projective space is so.

To summarize, we have shown that

$$1 = \pi_1(\mathbb{P}^3) \simeq \pi_1(\text{Bun}_{1/4,1/2}^{\text{ss}}(C/\iota)) \simeq \pi_1(\text{Bun}^s(C/\iota)) \twoheadrightarrow \pi_1(M_i^{\text{sm}}).$$

By the exact sequence (31), we conclude that $\pi_1(M^{\text{sm}}) \simeq \mathbb{Z}/2\mathbb{Z}$.

The following corollary is an immediate consequence of Remark 6.19 and Proposition 6.30.

COROLLARY 6.31. — *There are no non-trivial quasi-étale cover of $M_{\text{Dol}}(X, \text{SL}_n)$ for $g > 2$ and $(g, n) \neq (2, 2)$. The forgetful map q is the only non-trivial quasi-étale cover of M .*

7. P=W CONJECTURES FOR M

In this section we reduce the proof of the P=W conjecture for M and \widetilde{M} to P=W phenomena for the summands of the decomposition theorem for $f: \widetilde{M} \rightarrow M$; see Theorem 7.1 and Theorem 7.4. The exchange of the perverse and weight filtrations for the summands supported on a subvariety strictly contained in M is proved in Theorem 7.6. Therefore, the ultimate goal of this section is to reduce the proof of the P=W conjecture for M and \widetilde{M} to the PI=WI conjecture.

We first show that the PI=WI conjecture for M implies the P=W conjecture for M . Actually, this first statement does not require the decomposition theorem.

THEOREM 7.1. — *If the PI=WI conjecture for M holds, then the P=W conjecture for M holds.*

Proof. — The fixed locus of the G_m -action on M can be identified with the (disjoint) union of connected components of the fixed locus of the G_m -action on \widetilde{M} ; see Proposition 6.8 and Proposition 6.10. This induces an injective morphism between the local-to-global spectral sequences (25) for M and \widetilde{M} . Therefore, $f^*: H^*(M) \rightarrow H^*(\widetilde{M})$ is an injective map, and so is the natural map $H^*(M) \rightarrow IH^*(M)$, since $f^*: H^*(M) \rightarrow H^*(\widetilde{M})$ factors as $H^*(M) \rightarrow IH^*(M) \rightarrow H^*(\widetilde{M})$. The statement now follows from the fact that the injective map $H^*(M) \rightarrow IH^*(M)$ preserves the perverse and weight filtrations.

With a slight abuse of notation, we denote by f both the symplectic resolutions $f_{\text{Dol}}(C, \text{SL}_2): \widetilde{M} \rightarrow M$ and $f_B(C, \text{SL}_2): \widetilde{M}_B \rightarrow M_B$. By [47, Lem.2.11], any symplectic resolution is semismall. Therefore, the decomposition theorem (Theorem 2.7) provides canonical isomorphisms:

$$(32) \quad \mathbf{R}f_* \underline{\mathcal{O}}_{\widetilde{M}}[6] \simeq IC_M \oplus \underline{\mathcal{O}}_{\Sigma}[4](-1) \oplus \underline{\mathcal{O}}_{\Omega}(-3),$$

$$(33) \quad \mathbf{R}f_* \underline{\mathcal{O}}_{\widetilde{M}_B}[6] \simeq IC_{M_B} \oplus \underline{\mathcal{O}}_{\Sigma_B}[4](-1) \oplus \underline{\mathcal{O}}_{\Omega_B}(-3).$$

Thus, in cohomology we have:

$$(34) \quad H^*(\widetilde{M}) \simeq IH^*(M) \oplus H^{*-2}(\Sigma)(-1) \oplus H^{*-6}(\Omega)(-3),$$

$$(35) \quad H^*(\widetilde{M}_B) \simeq IH^*(M_B) \oplus H^{*-2}(\Sigma_B)(-1) \oplus H^{*-6}(\Omega_B)(-3).$$

These decompositions split the perverse and weight filtration, as shown in the following lemmas.

LEMMA 7.2. — We have

$$P_k H^*(\widetilde{M}) = P_k IH^*(M) \oplus P_{k-1} H^{*-2}(\Sigma) \oplus P_{k-3} H^{*-6}(\Omega),$$

where $P_k H^*(\widetilde{M})$, $P_k IH^*(M)$, $P_k H^*(\Sigma)$ and $P_k H^*(\Omega)$ denote the pieces of the perverse filtration associated to the maps $\chi \circ f$, χ , $\chi|_\Sigma$ and $\chi|_\Omega$ respectively.

Proof. — Apply χ_* to the splitting (32) and notice that perverse truncation functors ${}^p\tau_{\leq i}$ are exact.

LEMMA 7.3

$$W_{2k} H^*(\widetilde{M}_B) = W_{2k} IH^*(M_B) \oplus W_{2k-2} H^{*-2}(\Sigma_B) \oplus W_{2k-6} H^{*-6}(\Omega_B).$$

Proof. — As the decomposition theorem is an isomorphism of mixed Hodge structures, we have

$$W_{2k} H^*(\widetilde{M}_B) = W_{2k} IH^*(M_B) \oplus W_{2k} H^{*-2}(\Sigma_B)(-1) \oplus W_{2k} H^{*-6}(\Omega_B)(-3).$$

Recalling that Tate shifts $(-k)$ increase weights of $2k$, the result follows by including them in the grading of the weight filtration.

THEOREM 7.4. — The $P=W$ conjecture for \widetilde{M} is equivalent to the following two statements:

- (1) $PI=WI$ conjecture for M ;
- (2) $P=W$ conjecture for Σ and Ω , i.e.,

$$P_k H^*(\Sigma) = \Psi|_\Sigma^* W_{2k} H^*(\Sigma_B), \quad P_k H^*(\Omega) = \Psi|_\Omega^* W_{2k} H^*(\Omega_B).$$

Proof. — Let $\Psi: M \rightarrow M_B$ be the non-abelian Hodge correspondence, and $\widetilde{\Psi}: \widetilde{M} \rightarrow \widetilde{M}_B$ be the diffeomorphism lifting Ψ in the sense of Theorem 3.8. By the commutativity of the diagram (7), and since the map Ψ preserves the stratifications (20) and (21), the map $\widetilde{\Psi}^*: H^*(\widetilde{M}_B) \rightarrow H^*(\widetilde{M})$ splits on the summands of the decomposition theorem. More precisely, $\widetilde{\Psi}^*$ is given by the product map

$$(36) \quad (\Psi^*, \Psi|_\Sigma^{*-2}, \Psi|_\Omega^{*-6}): IH^*(M_B) \oplus H^{*-2}(\Sigma_B)(-1) \oplus H^{*-6}(\Omega_B)(-3) \\ \longrightarrow IH^*(M) \oplus H^{*-2}(\Sigma)(-1) \oplus H^{*-6}(\Omega)(-3).$$

The statement then follows by Lemma 7.2 and Lemma 7.3.

REMARK 7.5. — The product map (36) suggests that it is possible to define the isomorphism in cohomology $\widetilde{\Psi}^*$ without constructing the diffeomorphism $\widetilde{\Psi}$. This is indeed the approach of [12]. However, the virtue of Theorem 3.8 is to establish that the isomorphism between the cohomology rings of \widetilde{M} and \widetilde{M}_B which realizes the exchange of perverse and weight filtration has a geometric origin.

THEOREM 7.6 ($P=W$ for singular loci). — The $P=W$ conjecture for Σ and Ω holds.

Proof. — Since Ω is a collection of points, the perverse and the weight filtrations are all concentrated in degree zero, and so the P=W conjecture for Ω trivially holds.

We show now that the P=W conjecture for Σ holds. To this end, note that the map $\chi|_{\Sigma}$ factors as follows:

$$\chi|_{\Sigma}: \Sigma \simeq (\text{Pic}^0(C) \times H^0(K_C))/(Z/2Z) \twoheadrightarrow H^0(K_C)/(Z/2Z) \subset H^0(K_C^{\otimes 2}).$$

Equivalently, $\chi|_{\Sigma}$ can be identified with the quotient of the projection

$$\text{Pic}^0(C) \times H^0(C, K_C) \longrightarrow H^0(C, K_C)$$

via the involution $(L, s) \mapsto (L^{-1}, -s)$. Therefore, the general fiber $\chi|_{\Sigma}^{-1}(s)$, with $s \in H^0(K_C)/(Z/2Z)$, is isomorphic to $\text{Pic}^0(C)$. The zero fiber $\chi|_{\Sigma}^{-1}(0)$ instead is isomorphic to the singular Kummer surface associated to $\text{Pic}^0(C)$, denoted by S as in the proof of Proposition 6.10. Since Σ is attracted by S via the flow of the G_m -action, Σ retracts on S . In particular, we obtain that

$$H^*(\Sigma) \simeq H^*(S) \simeq H^*(\chi|_{\Sigma}^{-1}(s))^{Z/2Z},$$

and the restriction $H^*(\Sigma) \rightarrow H^*(\chi|_{\Sigma}^{-1}(s))$ is injective. Hence, by Theorem 2.9, we conclude that $H^d(\Sigma)$ has top perversity d .

On the Betti side, Σ_B is isomorphic to $(C^*)^4/(Z/2Z)$ (cf. (22)). This means that

$$H^*(\Sigma_B) \simeq H^*((C^*)^4)^{Z/2Z} \subset H^*((C^*)^4).$$

In particular, $H^d(\Sigma_B)$ has only even cohomology of top weight $2d$, since $H^d((C^*)^4)$ does. Since both the perverse and the weight filtrations are supported in top degree, the P=W conjecture for Σ holds.

Having proved the second item in Theorem 7.4, Section 8 will be devoted to the proof of the PI=WI conjecture.

8. PI=WI CONJECTURE FOR M

8.1. ACTION OF THE 2-TORSION OF THE JACOBIAN. — The action of $\Gamma = \text{Pic}^0(C)[2]$ induces the splitting

$$(37) \quad IH^*(M) = IH^*(M)^{\Gamma} \oplus IH_{\text{var}}^*(M),$$

where $IH^*(M)^{\Gamma}$ is fixed by the action of Γ , and $IH_{\text{var}}^*(M)$ is the variant part, i.e., the unique Γ -invariant complement of $IH^*(M)^{\Gamma}$ in $IH^*(M)$. Note that the decomposition (37) induces a splitting of the perverse filtration. This follows from the exactness of the perverse truncation functors ${}^p\tau_{\leq i}$ applied to the character decomposition $\chi_* \underline{\mathbb{Q}}_M \simeq \chi_* \underline{\mathbb{Q}}_M^{\Gamma} \oplus \chi_* \underline{\mathbb{Q}}_{M, \text{var}}$.

In a similar way there exists an isomorphism of mixed Hodge structures

$$IH^*(M_B) = IH^*(M_B)^{\Gamma} \oplus IH_{\text{var}}^*(M_B).$$

This implies the following theorem.

THEOREM 8.1. — *The PI=WI conjecture for M is equivalent to the following two statements:*

(1) (PI = WI conjecture for the invariant intersection cohomology)

$$(38) \quad P_k IH^*(M)^\Gamma = \Psi^* W_{2k} IH^*(M_B)^\Gamma, \quad k > 0.$$

(2) (PI = WI conjecture for the variant intersection cohomology)

$$(39) \quad P_k IH^*(M)_{\text{var}} = \Psi^* W_{2k} IH^*(M_B)_{\text{var}}, \quad k > 0.$$

We continue with the computation of the intersection Poincaré polynomial of M and the intersection E-polynomial of M_B .

PROPOSITION 8.2. — *The intersection Poincaré polynomials are*

$$\begin{aligned} IP_t(M) &:= \sum_k \dim IH^k(M) t^k = 1 + t^2 + 17t^4 + 17t^6, \\ IP_t(M)^\Gamma &:= \sum_k \dim IH^k(M)^\Gamma t^k = 1 + t^2 + 2t^4 + 2t^6, \\ IP_{t,\text{var}}(M) &:= \sum_k \dim IH^k_{\text{var}}(M) t^k = 15t^4 + 15t^6. \end{aligned}$$

Proof. — By (29) and (35) we have

$$\begin{aligned} IP_t(M) &= P_t(\widetilde{M}) - P_t(\Sigma)t^2 - P_t(\Omega)t^6 \\ &= (1 + 2t^2 + 23t^4 + 34t^6) - (1 + 6t^2 + 1)t^2 - 16t^6 \\ &= 1 + t^2 + 17t^4 + 17t^6; \end{aligned}$$

see also [26, Th. 6.1].

Since the differentials of the local-to-global spectral sequence (26) are Γ -equivariant, we obtain

$$P_{t,\text{var}}(\widetilde{M}) = P_{t,\text{var}}(\widetilde{N}) + P_{t,\text{var}}(\widetilde{S}^+)t^2 + P_{t,\text{var}}(\widetilde{\Theta})t^6 + P_{t,\text{var}}(\bigcup_{j=1}^{16} T_j)t^6,$$

in the notation of Theorem 6.14. The group Γ acts trivially on $H^*(\widetilde{N})$ and $H^*(\Sigma) \simeq H^*(S) \subset H^*(\widetilde{S}^+)$, and as the regular representation on the 16-dimensional vector spaces

$$\bigoplus_{j=1}^{16} \mathbb{Q}[s_j^+] \subset H^*(\widetilde{S}^+), \quad H^0(\widetilde{\Theta}), \quad \bigoplus_{j=1}^{16} \mathbb{Q}[T_j], \quad H^0(\Omega).$$

Again by (35), we get

$$\begin{aligned} IP_{t,\text{var}}(M) &= P_{t,\text{var}}(\widetilde{M}) - P_{t,\text{var}}(\Sigma)t^2 - P_{t,\text{var}}(\Omega)t^6 \\ &= (\dim(\bigoplus_{j=1}^{16} \mathbb{Q}[s_j^+]) - 1)t^4 + (\dim H^0(\bigcup_{j=1}^{16} T_j) - 1)t^6 \\ &\quad + (\dim H^0(\widetilde{\Theta}) - 1)t^6 - (\dim H^0(\Omega) - 1)t^6 \\ &= 15t^4 + 15t^6. \end{aligned}$$

Finally, $IP_t(M)^\Gamma = IP_t(M) - IP_{t,\text{var}}(M) = 1 + t^2 + 2t^4 + 2t^6$.

PROPOSITION 8.3. — *The intersection E-polynomial of M_B is*

$$\begin{aligned} IE(M_B) &:= \sum_{p,q,d} (-1)^d \dim(\mathrm{Gr}_{p+q}^W IH_c^d(M_B, \mathbb{C}))^{p,q} u^p v^q \\ &= \sum_{k,d} \dim \mathrm{Gr}_{2k}^W IH^d(M_B) q^k = 1 + 17q^2 + 17q^4 + q^6, \end{aligned}$$

with $q = uv$. In particular, $\dim \mathrm{Gr}_{2k+1}^W IH^d(M_B) = 0$ for all $k, d \in \mathbb{N}$.

Proof. — The analogue of Lemma 7.3 for compactly supported cohomology yields

$$(40) \quad E(M_B) = E(\widetilde{M}_B) - E(\Sigma_B)q - E(\Omega_B)q^3.$$

In order to compute $E(\widetilde{M}_B)$, consider the stratification of \widetilde{M}_B :

$$\widetilde{M}_B = M_B^{\mathrm{sm}} \sqcup \widetilde{\Sigma}_B \sqcup \widetilde{\Omega}_B \sqcup \widetilde{\Omega}_B,$$

where $\widetilde{\Sigma}_B \sqcup \widetilde{\Omega}_B := f^{-1}(\Sigma_B \sqcup \Omega_B)$ and $\widetilde{\Omega}_B := f^{-1}(\Omega_B)$. It is proved in [57, §8.2.3] that the E-polynomial of M_B is $E(M_B) = 1 + q^2 + 17q^4 + q^6$. This implies that

$$(41) \quad \begin{aligned} E(M_B^{\mathrm{sm}}) &= E(M_B) - E(\Sigma_B) \\ &= (1 + q^2 + 17q^4 + q^6) - (1 + 6q^2 + q^4) = -5q^2 + 16q^4 + q^6, \end{aligned}$$

where the second equality follows from the fact that the weight filtration on $H_c^*(\Sigma_B)$ is concentrated in top degree; see Theorem 7.6. Since $\widetilde{\Sigma}_B \sqcup \widetilde{\Omega}_B$ is a \mathbb{P}^1 -bundle over $\Sigma_B \sqcup \Omega_B$, we obtain

$$E(\widetilde{\Sigma}_B \sqcup \widetilde{\Omega}_B) = E(\mathbb{P}^1) \cdot E(\Sigma_B \sqcup \Omega_B) = (q + 1)(1 + 6q^2 + q^4 - 16).$$

Observe that $\widetilde{\Omega}_B$ is the disjoint union of 16 smooth quadric 3-folds $\widetilde{\Omega}_{B,j}$, so that

$$(42) \quad E(\widetilde{\Omega}_B) = \sum_{j=1}^{16} E(\widetilde{\Omega}_{B,j}) = 16(1 + q + q^2 + q^3).$$

Adding up the E-polynomials (41), (8.1) and (42), we get

$$(43) \quad E(\widetilde{M}_B) = 1 + q + 17q^2 + 22q^3 + 17q^4 + q^5 + q^6.$$

Finally, from (43) and (40) we obtain

$$(44) \quad IE(M_B) = 1 + 17q^2 + 17q^4 + q^6.$$

By the vanishing of the odd intersection cohomology (cf. Proposition 8.2), every non-trivial component $(\mathrm{Gr}_{p+q}^W IH_c^d(M_B, \mathbb{C}))^{p,q}$ will contribute with non-negative coefficient to $IE(M_B)$. Therefore, there is no cancellation and by (44) any non-trivial $(\mathrm{Gr}_{p+q}^W IH_c^d(M_B, \mathbb{C}))^{p,q}$ has type (p, p) , i.e., the mixed Hodge structure on $IH_c^d(M_B, \mathbb{C})$ is of Hodge-Tate type. In symbols, we write

$$\begin{aligned} IE(M_B) &= \sum_{p,q,d} \dim(\mathrm{Gr}_{p+q}^W IH_c^d(M_B, \mathbb{C}))^{p,q} u^p v^q \\ &= \sum_{k,d} \dim(\mathrm{Gr}_{2k}^W IH_c^d(M_B, \mathbb{C}))^{k,k} q^k = \sum_{k,d} \dim \mathrm{Gr}_{2k}^W IH^d(M_B, \mathbb{C}) q^k, \end{aligned}$$

where the last equality follows from Poincaré duality and the fact that the polynomial in (44) is palindromic.

PROPOSITION 8.4. — *The intersection E-polynomials are*

$$IE(M_B)^\Gamma := \sum_{k,d} \dim \text{Gr}_{2k}^W IH^d(M_B)^\Gamma q^k = 1 + 2q^2 + 2q^4 + q^6$$

$$IE_{\text{var}}(M_B) := \sum_{k,d} \dim \text{Gr}_{2k}^W IH_{\text{var}}^d(M_B) q^k = 15q^2 + 15q^4.$$

Proof. — The solution of the linear system

$$\begin{cases} \dim \text{Gr}_k^W IH^d(M_B) \subset \dim \text{Gr}_k^W H^d(\widetilde{M}_B) = 0 \text{ for } k < d & [71, \text{Prop. 4.20}] \\ \dim \text{Gr}_{2k+1}^W IH^d(M_B) = 0 & \text{Proposition 8.3} \\ \sum_{k,d} \dim \text{Gr}_{2k}^W IH^d(M_B) q^k = 1 + 17q^2 + 17q^4 + q^6 & \text{Proposition 8.3} \\ \sum_k \dim IH^k(M_B) t^k = 1 + t^2 + 17t^4 + 17t^6 & \text{Proposition 8.2} \end{cases}$$

is given by

$$(45) \quad \dim \text{Gr}_{4d}^W IH^{2d}(M_B) = 1 \quad \text{for } d = 0, 1, 2, 3,$$

$$(46) \quad \dim \text{Gr}_4^W IH^4(M_B) = \dim \text{Gr}_8^W IH^6(M_B) = 16.$$

The terms in this list are all the non-zero graded pieces of the mixed Hodge structure on $IH^*(M_B)$.

Note that the top graded pieces $\text{Gr}_{2d}^W IH^d(M_B)$ are generated by α^d , where α is a (Γ -invariant) generator of $IH^2(M_B)$. The class α corresponds via the non-abelian Hodge correspondence to the first Chern class of a χ -ample (or χ -anti-ample) divisor on M . In particular, α^2 and α^3 are non-zero and Γ -invariant. This implies that

$$IH_{\text{var}}^4(M_B) \subset W_4 IH^4(M_B) \simeq \text{Gr}_4^W IH^4(M_B),$$

$$IH_{\text{var}}^6(M_B) \subset W_8 IH^6(M_B) \simeq \text{Gr}_8^W IH^6(M_B).$$

Together with Proposition 8.2 and Proposition 8.3, we conclude that

$$IE_{\text{var}}(M_B) = \dim \text{Gr}_4^W IH_{\text{var}}^4(M_B) q^2 + \dim \text{Gr}_8^W IH_{\text{var}}^6(M_B) q^4$$

$$= \dim IH_{\text{var}}^4(M_B) q^2 + \dim IH_{\text{var}}^6(M_B) q^4 = 15q^2 + 15q^4$$

$$IE(M_B)^\Gamma = IE(M_B) - IE_{\text{var}}(M_B)$$

$$= (1 + 17q^2 + 17q^4 + q^6) - (15q^2 + 15q^4) = 1 + 2q^2 + 2q^4 + q^6.$$

As a result, an analogue of [41, Cor. 4.5.1] holds for M_B .

COROLLARY 8.5. — *The intersection form on $H_c^6(M_B) = IH_c^6(M_B)$ is trivial. Equivalently, the forgetful map $H_c^6(M_B) \rightarrow H^6(M_B)$ is zero.*

Proof. — By (45), (46) and Poincaré duality, the weight filtrations on $IH^6(M_B)$ and $IH_c^6(M_B)$ are concentrated in degree $[8, 12]$ and $[0, 4]$. Since the forgetful map is a morphism of mixed Hodge structures, it has to vanish.

REMARK 8.6 (Failure of curious hard Lefschetz). — By (28) and the proof of Proposition 8.4, we have

$$(47) \quad \sum_{k,d} \dim \operatorname{Gr}_{2k}^W H^d(M_B) q^k = 1 + q^2 + 17q^4 + q^6.$$

The fact that the polynomial (47) is not palindromic implies that curious hard Lefschetz (2) fails for $H^*(M_B)$. Analogously, one can show that relative hard Lefschetz fails for $H^*(M)$.

8.2. THE VARIANT PART OF $IH^*(M)$. — The goal of this section is to show that the PI = WI conjecture for the variant part of $IH^*(M)$ holds. As we will explain in the proof of Theorem 8.8, it is enough to prove it in degree 4 and 6.

PROPOSITION 8.7. — $IH_{\text{var}}^4(M) \subset P_2 IH^4(M)$.

Proof. — The argument of [11, §4.4] and [14, Prop. 1.4] works with few changes.

The endoscopic locus $A_e \subset H^0(C, K_C^{\otimes 2})$ is the subset of sections $s' \in H^0(C, K_C^{\otimes 2})$ such that the Prym variety associated to the corresponding spectral curve $C_{s'}$ is not connected (cf. [11, §4.4]). It is the union of 15 lines, obtained as images of the squaring map

$$i_L: H^0(C, K_C \otimes L) \longrightarrow H^0(C, K_C^{\otimes 2}), \quad i_L(a) = a \otimes a,$$

where $L \in \Gamma \setminus \{0\}$. In particular, a general line Λ^1 in $H^0(C, K_C^{\otimes 2})$ does not intersect A_e . It is important to remark that Γ acts trivially on $H^*(\chi^{-1}(s))$ for any $s \in \Lambda^1$: the proof in [11, §4.4] is independent of the choice of the degree of the Higgs bundles, and so it holds also in the untwisted case. This implies

$$H^*(\chi^{-1}(\Lambda^1))^\Gamma = H^*(\chi^{-1}(\Lambda^1)) = IH^*(\chi^{-1}(\Lambda^1)),$$

where the last equality follows from the fact that $\chi^{-1}(\Lambda^1)$ has quotient singularities. We conclude by Theorem 2.9 that

$$\begin{aligned} IH_{\text{var}}^4(M) &\subset \operatorname{Ker}\{IH^4(M) \rightarrow IH^4(\chi^{-1}(\Lambda^1)) = IH^4(\chi^{-1}(\Lambda^1))^\Gamma\} \\ &= P_2 IH^4(M), \end{aligned}$$

because the restriction map is Γ -equivariant.

THEOREM 8.8. — *The PI = WI conjecture for the variant intersection cohomology of M (39) holds.*

Proof. — The variant Poincaré polynomial in Proposition 8.2 shows that $IH_{\text{var}}^*(M)$ is concentrated in degree 4 and 6.

By relative hard Lefschetz, we can write

$$\operatorname{Gr}_0^P IH^4(M) \simeq \operatorname{Gr}_6^P IH^{10}(M), \quad \operatorname{Gr}_1^P IH^4(M) \simeq \operatorname{Gr}_5^P IH^8(M),$$

which both vanish by Proposition 8.2. Together with Proposition 8.7 and the proof of Proposition 8.4, this implies

$$P_2 IH_{\text{var}}^4(M) = IH_{\text{var}}^4(M) = \Psi^* IH_{\text{var}}^4(M_B) = \Psi^* W_4 IH_{\text{var}}^4(M_B).$$

This proves the PI=WI conjecture for the variant part in degree 4.

Again by relative hard Lefschetz, there exists a χ -ample $\alpha \in H^2(M)$ such that the cup product $\cup\alpha$ induces the isomorphism

$$\cup\alpha: IH_{\text{var}}^4(M) \simeq \text{Gr}_2^P IH_{\text{var}}^4(M) \longrightarrow \text{Gr}_4^P IH_{\text{var}}^6(M).$$

By Proposition 8.2 we obtain that

$$15 = \dim IH_{\text{var}}^4(M) = \dim \text{Gr}_4^P IH_{\text{var}}^6(M) \subset \dim IH_{\text{var}}^6(M) = 15.$$

This implies that the cup product

$$\cup\alpha: IH_{\text{var}}^4(M) \longrightarrow IH_{\text{var}}^6(M)$$

is an isomorphism, which preserves the perverse and weight filtrations; see [11, Lem. 1.4.4]. Therefore, the PI=WI conjecture for the variant part holds in degree 6 as well.

8.3. A TAUTOLOGICAL CLASS. — We show now that $IH^4(M)^\Gamma$ is generated by the square of the relatively ample class α , and a class of perversity 2 and weight 4. As usual, we adopt the notation of the previous sections, and in particular of Section 6.2.

Consider the forgetful map $q: M_\iota \rightarrow M$. The action of Γ on M lifts to M_ι , and together with the deck transformation of q , we obtain a group of symmetries of order 32, denoted Γ_ι .

PROPOSITION 8.9. — $IH^4(M)^\Gamma = H^4(M_\iota^{\text{sm}})^{\Gamma_\iota}$.

Proof. — Since M_ι has isolated singularities by Proposition 6.23, we have that $IH^4(M_\iota) = H^4(M_\iota^{\text{sm}})$; see [31, §1.7] or [25, Lem. 1]. The proof of [32, Prop. 3] implies that

$$IH^4(M)^\Gamma = IH^4(M_\iota)^{\Gamma_\iota} = H^4(M_\iota^{\text{sm}})^{\Gamma_\iota}.$$

Fix a base point $c \in C$. Recall that E is a universal bundle on $M_\iota^{\text{sm}} \times C$; see Section 6.2.5.

DEFINITION 8.10. — The space R is the total space of the projective bundle $P(E|_{M_\iota^{\text{sm}} \times \{c\}})$. Its associated principal PGL_2 -bundle parametrizes equivariant Higgs bundles (E, h, ϕ) together with a frame for the fiber E_c , up to rescaling.

The second Chern class of a \mathbb{P}^1 -bundle is the pull-back of a generator of $H^4(\text{BPGL}_2) \simeq \mathbb{Q}$ via the classifying map. In particular, if the \mathbb{P}^1 -bundle is a projectivization of the rank-two vector bundle E , then

$$c_2(P(E)) = c_1^2(E) - 4c_2(E).$$

PROPOSITION 8.11. — *The second Chern class $c_2(R)$ of the projective bundle R and the square of the χ -ample class α generate $IH^4(M)^\Gamma$*

$$IH^4(M)^\Gamma = \mathbb{Q}\alpha^2 \oplus \mathbb{Q}c_2(R).$$

Proof. — The proposition is a consequence of the following facts:

- (1) $c_2(R) \in H^4(M_\iota^{\text{sm}})^{\Gamma_\iota} = IH^4(M)^\Gamma$, since the Γ_ι -action lifts to R .
- (2) $\alpha^2 \in H^4(M)^\Gamma \subset IH^4(M)^\Gamma$.

- (3) $c_2(R) \neq 0$ by Lemma 8.12.
- (4) α^2 and $c_2(R)$ are linearly independent, because α^2 has top perversity by Theorem 2.9, while $c_2(R) \in P_2IH^4(M)$; see Lemma 8.14.
- (5) $\dim IH^4(M)^\Gamma = 2$ by Proposition 8.2.

We now prove the lemmas used in the proof above.

LEMMA 8.12. — $c_2(R) \neq 0$.

Proof. — Let $\tau_j: R_j^\circ \times C \rightarrow (M_v^\circ \cap N_j^+) \times C$ be the algebraic bijection defined in 6.2.6(30). Lemma 6.29 and Proposition 6.28 give

$$\begin{aligned} \tau_j^*c_2(R) &= c_2(\mathrm{P}(\mathrm{E}|_{R_j^\circ \times \{c\}})) \\ &= (c_1(p_{R_j}^* \mathcal{O}_{R_j}(1) \otimes p_C^* \theta_j^{-1}) - c_1(p_C^* \theta_j))^2|_{R_j^\circ \times \{c\}} = c_1(\mathcal{O}_{R_j^\circ}(1))^2. \end{aligned}$$

In particular, $0 \neq c_1(\mathcal{O}_{R_j^\circ}(1))^2 \in H^4(R_j^\circ) \simeq H^4(\mathbb{P}^3)$.

LEMMA 8.13. — $c_2(R) \in P_3IH^4(M)$.

Proof. — Fix s a generic point in $H^0(C, K_C^{\otimes 2})$, and let $p_s: C_s \rightarrow C$ the corresponding spectral curve, i.e., the double cover of C ramified along the zeroes of s ; see for instance [3, §3]. We denote the product map $p_s \times \mathrm{id}: C_s \times \mathrm{Pic}^0(C_s) \rightarrow C \times \mathrm{Pic}^0(C_s)$ simply by p_s . A universal bundle on $\chi^{-1}(s) \times C \simeq C \times \mathrm{Pic}^0(C_s)$ does exist, and it is isomorphic to $p_{s,*}P$, where P is the Poincaré line bundle over $C_s \times \mathrm{Pic}^0(C_s)$.

The abelian variety $(\chi \circ q)^{-1}(s)$ parametrizes line bundles of C_s decorated with a lift of the hyperelliptic involution $\iota: C \rightarrow C$. This implies that the restriction of E to $(\chi \circ q)^{-1}(s) \times C$ is isomorphic to $q^*(p_s, \mathrm{id})_*P$, up to tensorization by a line bundle in $\mathrm{Pic}((\chi \circ q)^{-1}(s))$.

As a result, we have that

$$\begin{aligned} c_2(R)|_{\chi^{-1}(s)} &= c_1^2(\mathrm{E}|_{(\chi \circ q)^{-1}(s) \times \{c\}}) - 4c_2(\mathrm{E}|_{(\chi \circ q)^{-1}(s) \times \{c\}}) \\ &= q^* \left(c_1^2((p_s, *)P)|_{\chi^{-1}(s) \times \{c\}} - 4c_2((p_s, *)P)|_{\chi^{-1}(s) \times \{c\}} \right) = 0, \end{aligned}$$

where the last equality follows from [84, §4] or [11, Eq. (5.1.10) & (5.1.11)]. This implies that $c_2(R)$ does not have top perversity by Theorem 2.9.

LEMMA 8.14. — $c_2(R) \in P_2IH^4(M)$.

Proof. — By Lemma 8.13, it is enough to show that the projection $[c_2(R)]$ in the graded piece $\mathrm{Gr}_3^P IH^4(M)^\Gamma$ vanishes. Suppose on the contrary that $[c_2(R)] \neq 0$. Then Proposition 8.2 would imply

$$\dim \mathrm{Gr}_2^P IH^4(M)^\Gamma \triangleleft \dim IH^4(M)^\Gamma - \dim(\mathbb{Q} \alpha^2 \oplus \mathbb{Q} c_2(R)) = 2 - 2 = 0.$$

By relative hard Lefschetz, $\mathrm{Gr}_4^P IH^6(M)^\Gamma$ would be trivial. Analogously,

$$\mathrm{Gr}_5^P IH^6(M)^\Gamma \simeq \mathrm{Gr}_1^P IH^2(M)^\Gamma = 0.$$

Again by Proposition 8.2 we would conclude that

$$\begin{aligned} \dim \operatorname{Gr}_3^P IH^6(M)^\Gamma &= \dim IH^6(M)^\Gamma - \sum_{k=4}^6 \dim \operatorname{Gr}_k^P IH^6(M)^\Gamma \\ &= \dim IH^6(M)^\Gamma - \dim \mathbb{Q}\alpha^3 = 2 - 1 = 1. \end{aligned}$$

However, this is a contradiction by Corollary 8.5.

LEMMA 8.15. — $P_3 IH^6(M)^\Gamma = 0$.

Proof. — Lemma 7.2 gives the splitting

$$P_3 H^6(\widetilde{M})^\Gamma = P_3 IH^6(\widetilde{M})^\Gamma \oplus P_2 H^4(\Sigma)^\Gamma \oplus H^0(\Omega)^\Gamma.$$

The P=W conjecture for Σ gives $P_2 H^4(\Sigma) = 0$. Moreover, we have that $H^0(\Omega)^\Gamma \simeq \mathbb{Q}[\Omega]$; see the proof of Proposition 8.2. Therefore, we get

$$P_3 H^6(\widetilde{M})^\Gamma = P_3 IH^6(\widetilde{M})^\Gamma \oplus \mathbb{Q}[\Omega].$$

Up to a different numbering convention, [16, Th. 2.1.10] says that the dimension of $P_3 H^6(\widetilde{M})^\Gamma$ is not greater than the rank of the intersection form on $H^6(\widetilde{M})^\Gamma$. Therefore, Corollary 8.5 implies

$$\dim P_3 H^6(\widetilde{M})^\Gamma \leq 1.$$

We conclude that $\dim P_3 IH^6(M)^\Gamma = 0$.

We conclude the section by showing that the class $c_2(R)$ has weight 4.

LEMMA 8.16. — $c_2(R) \in W_4 IH^4(M_B)$.

Proof. — The principal PGL_2 -bundle $\mathcal{S} \rightarrow M_{B,\iota}^{\text{sm}} := \Psi_\iota(M_\iota^{\text{sm}})$ is the restriction of the quotient $\operatorname{Hom}(\pi_1^{\text{orb}}(C/\iota), \operatorname{SL}_2) \rightarrow M_B(2, \operatorname{SL}_2, \iota)$. It parametrizes ι -equivariant local systems E' on C together with a frame for the fiber E'_c over $c \in C$, i.e., the base point of $\pi_1^{\text{orb}}(C/\iota) = \pi_1^{\text{orb}}(C/\iota, c)$, up to rescaling.

By construction, the non-abelian Hodge correspondence $\Psi_\iota : M_\iota^{\text{sm}} \rightarrow M_{B,\iota}^{\text{sm}}$ (Theorem 6.21) extends to a diffeomorphism between the principal PGL_2 -bundle associate to R and \mathcal{S} . This implies that $c_2(R) = (\Psi_\iota^{-1})^* c_2(\mathcal{S})$, and $c_2(\mathcal{S})$ has weight 4 by [22, Th. 9.1.1, Prop. 9.1.2].

8.4. THE INVARIANT PART OF $IH^*(M)$

THEOREM 8.17. — *The invariant PI=WI conjecture (38) holds for M .*

Proof. — The statement is obvious in degree 0 and 2, because $IH^0(M)^\Gamma$ and $IH^2(M)^\Gamma$ have dimension one by Proposition 8.2 and Proposition 8.4.

Now we have

$$\begin{aligned} P_2 IH^4(M)^\Gamma &\simeq W_4 IH^4(M_B)^\Gamma, \\ \operatorname{Gr}_3^P IH^4(M)^\Gamma &\simeq \operatorname{Gr}_6^W IH^4(M_B)^\Gamma = 0, \\ P_4 IH^4(M)^\Gamma &= IH^4(M)^\Gamma \simeq IH^4(M_B)^\Gamma = W_8 IH^4(M_B)^\Gamma, \end{aligned}$$

due to Proposition 8.11, Lemma 8.16 and Proposition 8.4. This proves the invariant PI=WI conjecture in degree 4.

By relative hard Lefschetz, the cup product with the χ -ample $\alpha \in H^2(M)$ induces the isomorphisms

$$\begin{aligned} \cup\alpha: \mathrm{Gr}_2^P IH^4(M)^\Gamma &\longrightarrow \mathrm{Gr}_4^P IH^6(M)^\Gamma, \\ \cup\alpha: \mathrm{Gr}_4^P IH^4(M) \simeq \mathbb{Q}[\alpha^2] &\longrightarrow \mathrm{Gr}_6^P IH^6(M) \simeq \mathbb{Q}[\alpha^3]. \end{aligned}$$

Note that $\mathrm{Gr}_2^P IH^4(M)^\Gamma$ and $\mathrm{Gr}_4^P IH^4(M)$ are the only non-trivial pieces of the perverse filtration on $IH^4(M)^\Gamma$, and by Proposition 8.2 we have that $\dim IH_{\mathrm{var}}^4(M)^\Gamma = \dim IH_{\mathrm{var}}^6(M)^\Gamma$. This implies that the cup product

$$\cup\alpha: IH^4(M)^\Gamma \longrightarrow IH^6(M)^\Gamma$$

is an isomorphism which preserves both perverse and weight filtration (cf. [11, Lem. 1.4.4]). Therefore, the invariant PI=WI conjecture holds in degree 6, as well.

APPENDIX. DEGENERATIONS OF HYPERKÄHLER VARIETIES

In this appendix we describe degenerations of compact hyperkähler manifolds to (non-compact) symplectic resolutions of Dolbeault moduli spaces. Instances of these constructions can be found in [23], [15], [14]. Here a degeneration is a flat (not necessarily proper) morphism of normal algebraic varieties, typically over a curve.

The compact hyperkähler manifolds appearing in these degenerations are Mukai moduli spaces of sheaves on a K3 surface or an abelian surface S . Given an effective Mukai vector⁽⁴⁾ $v \in H_{\mathrm{alg}}^*(S, \mathbb{Z})$, we denote by $M(S, v)$ the moduli space of Gieseker semistable sheaves on S with Mukai vector v for a sufficiently general polarization H (which we will typically omit in the notation); see [78, §1]. Further, if S is an abelian variety with dual \widehat{S} , and $\dim M(S, v) > 6$, then the Albanese morphism $\mathrm{alb}_S: M(S, v) \rightarrow \widehat{S} \times S$ is isotrivial, and we set $K(S, v) := \mathrm{alb}_S^{-1}(0_S, 0_S)$. By [70], the moduli space $M(S, v)$ of sheaves on the K3 surface S and the moduli space $K(S, v)$ of sheaves on the abelian surface S are irreducible holomorphic symplectic varieties, in brief IHSv.

A.1. DEFORMATION TO THE NORMAL CONE: GL_n CASE. — Let $j: X \hookrightarrow S$ be the embedding of a smooth projective curve⁽⁵⁾ of genus g into a K3 surface S . The degeneration to the normal cone of $j: X \hookrightarrow S$ is the family

$$\mathcal{S} = (\mathrm{Bl}_{X \times 0} S \times \mathbb{A}^1) \cap (S \times 0) \longrightarrow \mathbb{A}^1.$$

The central fiber \mathcal{S}_0 is isomorphic to T^*X , while the restriction to $\mathbb{A}^1 \cap \{0\}$ is a trivial fibration $S \times (\mathbb{A}^1 \cap \{0\}) \rightarrow \mathbb{A}^1 \cap \{0\}$.

⁽⁴⁾i.e., there exists a coherent sheaf F on S such that $v = (\mathrm{rk}(F), c_1(F), \chi(F) - \varepsilon(S) \mathrm{rk}(F))$, with $\varepsilon(S) := 1$ if S is K3, and 0 if S is abelian.

⁽⁵⁾In [23] X is a very ample divisor, but this assumption can be dropped.

For all $t \in A^1$, let $\beta_t = n[X] \in H_2(S_t, \mathbb{Z})$ with $n > 0$. Take a relative compactification $S \subset \overline{S}$ over A^1 . Then

$$M \longrightarrow A^1$$

is the coarse relative moduli space of one-dimensional Gieseker semistable sheaves F whose support is proper and contained in $S_t \subseteq \overline{S}_t$ with $\chi(F) = n(1 - g)$ and $[\text{Supp} F] = \beta_t$; see [78, Th. 1.21]. The central fiber recovers the Dolbeault moduli space

$$M_0 \simeq M_{\text{Dol}}(X, \text{GL}_n).$$

Indeed, the moduli space of Higgs bundles on X of rank n and degree 0 can be realized as the moduli space of one-dimensional Gieseker-semistable sheaves F on T^*X with $\chi(F) = n(1 - g)$ and $[\text{Supp} F] = \beta_0$, via the BNR-correspondence [3]. The general fiber is isomorphic to

$$M_t \simeq M(S, v)$$

with Mukai vector $v = (0, nX, n(g - 1))$.

EXAMPLE A.1 (genus one: $K3^{[n]}$). — If $g = 1$, then the degeneration $M \rightarrow A^1$ is the relative n -fold symmetric product of S . The relative Hilbert-Chow morphism $\widetilde{M} \rightarrow M$ is a desingularization of M . The composition $\widetilde{M} \rightarrow M \rightarrow A^1$ is a family whose general fiber is the compact hyperkähler manifold $S^{[n]}$ and whose central fiber is $(T^*X)^{[n]}$, i.e., the symplectic resolution of $M_{\text{Dol}}(X, \text{GL}_n) \simeq (T^*X)^{(n)}$.

EXAMPLE A.2 (genus two and rank two: O’Grady 10). — If $(g, n) = (2, 2)$, then the blow-up \widetilde{M}_t of the singular locus of $M_t \simeq M(S, v)$ is a smooth compact hyperkähler manifold deformation equivalent to OG10; see for instance [69]. Analogously, the blow-up \widetilde{M}_0 of the singular locus of $M_0 \simeq M_{\text{Dol}}(X, \text{GL}_2)$ gives the symplectic resolution of M_0 . Note that the proof of [69, Prop. 2.16] shows that the degeneration $M \rightarrow T$ is locally analytically trivial. Therefore, the blow-up \widetilde{M} of the singular locus of M is a smooth family over A^1 whose general member is deformation equivalent to OG10 and whose central fiber is the symplectic resolution of $M_{\text{Dol}}(X, \text{GL}_2)$.

REMARK A.3. — Taking schematic supports via Fitting ideals defines a Lagrangian morphism $M(S, v) \rightarrow |nX|$, called the Mukai system. It is classically known that the Mukai system degenerates to the Hitchin fibration, see [23].

REMARK A.4. — If the Mukai vector $(0, X, g - 1)$ is primitive (e.g. if $\text{Pic}(S) = \mathbb{Z}X$), then the second author observed in [59, Rem. 2.5] that the degeneration $M \rightarrow A^1$ is locally analytically trivial. Therefore, the functorial resolution $R(M) \rightarrow M$ of M gives a simultaneous resolution of M_t for any $t \in A^1$; see for instance [33, Lem. 4.2].

A.2. DEFORMATION TO THE NORMAL CONE: SL_n CASE. — Suppose now that X is a smooth projective curve embedded in an abelian surface S . To avoid confusion, we re-label S by A .⁽⁶⁾ As in the previous section, there exists a degeneration $M \rightarrow A^1$

⁽⁶⁾In this section we denote by A an abelian surface, and not a curve of genus one as in the rest of the paper.

from the moduli space $M(A, v)$ to $M_{\text{Dol}}(X, \text{GL}_n)$. In this case, however, $M(A, v)$ is no longer an IHSv because of the Albanese morphism $\text{alb}_S: M(A, v) \rightarrow \widehat{A} \times A$.

In genus one and two it is possible to slice \mathcal{M} to obtain a degeneration of the IHSv $K(A, v)$ to $M_{\text{Dol}}(X, \text{SL}_n)$.

EXAMPLE A.5 (genus one: $K^{[n]}(A)$). — If $g = 1$, then the family

$$A = (\text{Bl}_{C \times 0} A \times A^1) \cap (A \times 0) \longrightarrow A^1$$

is a group scheme, and the degeneration $\mathcal{M} \rightarrow A^1$ is the relative n -fold symmetric product $A^{(n)} \rightarrow A^1$ whose general fiber is $A^{(n)}$, and whose central fiber is $(T^*C)^{(n)}$.

Consider now the relative addition map $a_n: A^{(n)} \rightarrow A$, given by $a_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i$. The inverse image of the identity section of $A \rightarrow A^1$ under the addition map is a degeneration

$$K \longrightarrow A^1$$

whose general fiber is the singular generalized Kummer variety $K(A, v) \simeq K^{(n)}(A)$ and whose central fiber is $M_{\text{Dol}}(X, \text{SL}_n) \simeq K^{(n)}(T^*C)$. The inverse image of the identity section of $A \rightarrow A^1$ under the composition $A^{[n]} \rightarrow A^{(n)} \rightarrow A$ is a degeneration

$$\widetilde{K} \longrightarrow K \longrightarrow A^1$$

whose general fiber is the generalized Kummer manifold $K^{[n]}(A)$ and whose central fiber is the symplectic resolution of $M_{\text{Dol}}(X, \text{SL}_n)$.

EXAMPLE A.6 (genus two). — If $g = 2$, the Albanese map [88]

$$\text{alb}_S: M(A, v) \longrightarrow \widehat{A} \times A$$

degenerates to the map

$$\text{alb}: M_{\text{Dol}}(X, \text{GL}_n) \longrightarrow \text{Pic}^0(X) \times H^0(X, K_X) \simeq \widehat{A} \times A^g,$$

defined in (8); see [14, §4]. Taking fibers over the identity, one obtains a family $\widetilde{K} \rightarrow A^1$ such that the central fiber is $M_{\text{Dol}}(X, \text{SL}_n)$ and the general fiber is the IHSv $K(A, v)$.

EXAMPLE A.7 (genus two and rank two: O'Grady 6). — The symplectic resolution $f_A: \widetilde{K}(A, v) \rightarrow K(A, v)$, with $v = (0, 2X, 2)$ and $g = 2$, is a compact hyperkähler manifold of OG6 type. Let \widetilde{K} be the blow-up of the singular locus of the variety K obtained in Example A.6, with $(g, n) = (2, 2)$. Then $\widetilde{K} \rightarrow A^1$ is a degeneration of $K(A, v)$ to the Dolbeault moduli space \mathcal{M} in §6. Further, as in Example A.2, \widetilde{K} is a smooth family over A^1 whose general member is the compact hyperkähler manifold $\widetilde{K}(A, v)$ of OG6 type and whose central fiber is the symplectic resolution \widetilde{M} of M .

We observe that the cohomology of $\widetilde{K}(A, v)$ governs the cohomology of \widetilde{M} in the following sense.

PROPOSITION A.8. — *The specialization morphism [15, (86)]*

$$\text{sp}^1: H^*(\widetilde{K}(A, v)) \longrightarrow H^*(\widetilde{M})$$

is a surjection.

Proof. — The following facts hold:

- The Mukai system $\chi_A: \widetilde{K}(A, v) \rightarrow |2X|$ specializes to the Hitchin fibration $\chi \circ f: \widetilde{M} \rightarrow H^0(X, K_X^{\otimes 2})$. In particular, a χ_A -ample line bundle on $\widetilde{K}(A, v)$ specializes to a generator of $H^2(\widetilde{M})$.

- The fiber $\chi_A^{-1}(2X)$ consists of 34 irreducible components which specialize to the irreducible components of the nilpotent cone of \widetilde{M} that generate $H^6(\widetilde{M})$; see [87, Prop. 3.0.3].

- Denote by Σ_A and Σ the singular locus of $K(A, v)$ and M , isomorphic to $(A \times \widehat{A})/\pm 1$ and $(A^2 \times \widehat{A})/\pm 1$ respectively. As in (35), $H^{*-2}(\Sigma_A)$ is a direct summand of $H^*(K(A, v))$. By definition of sp^1 in [15, (86)], the restriction of the specialization map to $H^{*-2}(\Sigma_A)$ is the pullback

$$\begin{aligned} (A^2 \times \widehat{A})/\pm 1 &\hookrightarrow P(T^*\widehat{A} \oplus \mathcal{O}_{\widehat{A}})/\pm 1 \hookrightarrow (\text{Bl}_{0 \times \widehat{A} \times 0} A \times \widehat{A} \times A^1)/\pm 1 \\ &\longrightarrow (A \times \widehat{A} \times A^1)/\pm 1 \longrightarrow (A \times \widehat{A})/\pm 1. \end{aligned}$$

So, given the inclusion $j: (0 \times \widehat{A})/\pm 1 \hookrightarrow (A \times \widehat{A})/\pm 1$, we have

$$\text{sp}^1(\gamma) = j^* \gamma \in H^*(\widehat{A}/\pm 1) \simeq H^*((A^2 \times \widehat{A})/\pm 1)$$

for $\gamma \in H^*((A \times \widehat{A})/\pm 1)$, which is a surjection.

We conclude that

$$\text{Im}(\text{sp}^1) \supset H^2(\widetilde{M}) \oplus H^6(\widetilde{M}) \oplus H^{*-2}(\Sigma_A).$$

By the description of $H^*(\widetilde{M})$ (cf. Fig. 1) and relative hard Lefschetz, this suffices to show that $\text{Im}(\text{sp}^1)$ equals the whole $H^*(\widetilde{M})$.

REMARK A.9. — Recall that for any odd number d the twisted Dolbeault moduli space $M^{\text{tw}}(X, \text{SL}_2, d)$ parametrizes semistable SL_2 -Higgs bundles of degree d on the curve X . It is curious that the analogue of Proposition A.8 fails for $M^{\text{tw}}(X, \text{SL}_2, d)$ and $g = 2$: there is no degeneration of compact hyperkähler manifolds to $M^{\text{tw}}(X, \text{SL}_2, d)$ such that the specialization map sp^1 is surjective; see [14, Prop. 4.3].

EXAMPLE A.10 (genus > 2). — There is no degeneration from $K(A, v)$ with Mukai vector $v = (0, nX, n(g - 1))$ with $g > 2$ to $M_{\text{Dol}}(X, \text{SL}_n)$ for dimensional reason. However, $K(A, v)$ and $M_{\text{Dol}}(X, \text{SL}_n)$ have the same type of singularities: they are stably isosingular in the sense of [59, Def. 2.6 & Th. 2.11]. Therefore, it is natural to ask the following.

QUESTION. — *Does there exist a degeneration of compact symplectic varieties equipped with a Lagrangian fibration in Prym varieties to the Hitchin fibration*

$$\chi(X, \text{SL}_n): M_{\text{Dol}}(X, \text{SL}_n) \longrightarrow \bigoplus_{i=2}^n H^0(X, K_X^{\otimes i})$$

for $g > 2$?

Note that the question is answered positively in [72] if we replace the special linear group SL_n with the symplectic group Sp_n .

REFERENCES

- [1] E. AMERIK & M. VERBITSKY – “Contraction centers in families of hyperkähler manifolds”, *Selecta Math. (N.S.)* **27** (2021), no. 4, article no. 60 (26 pages).
- [2] A. BEAUVILLE – “Variétés kähleriennes dont la première classe de Chern est nulle”, *J. Differential Geom.* **18** (1983), no. 4, p. 755–782 (1984).
- [3] A. BEAUVILLE, M. S. NARASIMHAN & S. RAMANAN – “Spectral curves and the generalised theta divisor”, *J. reine angew. Math.* **398** (1989), p. 169–179.
- [4] A. A. BEILINSON, J. N. BERNSTEIN & P. DELIGNE – “Faisceaux pervers”, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, Astérisque, vol. 100, Société Mathématique de France, Paris, 1982, p. 5–171.
- [5] G. BELLAMY & T. SCHEDLER – “Symplectic resolutions of character varieties”, 2019, arXiv: 1909.12545.
- [6] A. BIALYNICKI-BIRULA – “Some theorems on actions of algebraic groups”, *Ann. of Math. (2)* **98** (1973), p. 480–497.
- [7] E. BIERSTONE, P. D. MILMAN & M. TEMKIN – “ \mathbb{Q} -universal desingularization”, *Asian J. Math.* **15** (2011), no. 2, p. 229–249.
- [8] I. BISWAS – “Orbifold principal bundles on an elliptic fibration and parabolic principal bundles on a Riemann surface. II”, *Collect. Math.* **56** (2005), no. 3, p. 235–252.
- [9] I. BISWAS, S. MAJUMDER & M. L. WONG – “Parabolic Higgs bundles and Γ -Higgs bundles”, *J. Austral. Math. Soc.* **95** (2013), no. 3, p. 315–328.
- [10] M. A. DE CATALDO – “Projective compactification of Dolbeault moduli spaces”, *Internat. Math. Res. Notices* (2021), no. 5, p. 3543–3570.
- [11] M. A. DE CATALDO, T. HAUSEL & L. MIGLIORINI – “Topology of Hitchin systems and Hodge theory of character varieties: the case A_1 ”, *Ann. of Math. (2)* **175** (2012), no. 3, p. 1329–1407.
- [12] ———, “Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces”, *J. Singul.* **7** (2013), p. 23–38.
- [13] M. A. DE CATALDO & D. MAULIK – “The perverse filtration for the Hitchin fibration is locally constant”, *Pure Appl. Math. Q* **16** (2020), no. 5, p. 1441–1464.
- [14] M. A. DE CATALDO, D. MAULIK & J. SHEN – “On the $P=W$ conjecture for SL_n ”, 2020, arXiv: 2002.03336.
- [15] ———, “Hitchin fibrations, abelian surfaces, and the $P=W$ conjecture”, *J. Amer. Math. Soc.* (2021), online, 47 p.
- [16] M. A. DE CATALDO & L. MIGLIORINI – “The Hodge theory of algebraic maps”, *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 5, p. 693–750.
- [17] ———, “The perverse filtration and the Lefschetz hyperplane theorem”, *Ann. of Math. (2)* **171** (2010), no. 3, p. 2089–2113.
- [18] I. CHELTSOV, V. PRZYJALKOWSKI & C. SHRAMOV – “Which quartic double solids are rational?”, *J. Algebraic Geom.* **28** (2019), no. 2, p. 201–243.
- [19] S. M. CHIARELLO, T. HAUSEL & A. SZENES – “An enumerative approach to $P = W$ ”, 2020, arXiv: 2002.08929.
- [20] G. D. DASKALOPOULOS & K. K. UHLENBECK – “An application of transversality to the topology of the moduli space of stable bundles”, *Topology* **34** (1995), no. 1, p. 203–215.
- [21] G. D. DASKALOPOULOS, R. A. WENTWORTH & G. WILKIN – “Cohomology of $SL(2, \mathbb{C})$ character varieties of surface groups and the action of the Torelli group”, *Asian J. Math.* **14** (2010), no. 3, p. 359–383.
- [22] P. DELIGNE – “Théorie de Hodge. III”, *Publ. Math. Inst. Hautes Études Sci.* (1974), no. 44, p. 5–77.
- [23] R. DONAGI, L. EIN & R. LAZARUSFELD – “Nilpotent cones and sheaves on $K3$ surfaces”, in *Birational algebraic geometry (Baltimore, MD, 1996)*, Contemp. Math., vol. 207, American Mathematical Society, Providence, RI, 1997, p. 51–61.
- [24] J.-M. DREZET & M. S. NARASIMHAN – “Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques”, *Invent. Math.* **97** (1989), no. 1, p. 53–94.
- [25] A. H. DURFEE – “Intersection homology Betti numbers”, *Proc. Amer. Math. Soc.* **123** (1995), no. 4, p. 989–993.

- [26] C. FELISETTI – “Intersection cohomology of the moduli space of Higgs bundles on a genus 2 curve”, *J. Inst. Math. Jussieu* (2021), p. 1–50.
- [27] C. FELISETTI, J. SHEN & Q. YIN – “On intersection cohomology and Lagrangian fibrations of irreducible symplectic varieties”, *Trans. Amer. Math. Soc.* **375** (2022), no. 4, p. 2987–3001.
- [28] E. FRANCO, O. GARCIA-PRADA & P. E. NEWSTEAD – “Higgs bundles over elliptic curves”, *Illinois J. Math.* **58** (2014), no. 1, p. 43–96.
- [29] B. FU & Y. NAMIKAWA – “Uniqueness of crepant resolutions and symplectic singularities”, *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 1, p. 1–19.
- [30] M. GORESKY & R. MACPHERSON – “Intersection homology theory”, *Topology* **19** (1980), no. 2, p. 135–162.
- [31] ———, “Morse theory and intersection homology theory”, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque, vol. 101, Société Mathématique de France, Paris, 1983, p. 135–192.
- [32] L. GÖTTSCHE & W. SOERGEL – “Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces”, *Math. Ann.* **296** (1993), no. 2, p. 235–245.
- [33] P. GRAF & M. SCHWALD – “On the Kodaira problem for uniruled Kähler spaces”, *Ark. Mat.* **58** (2020), no. 2, p. 267–284.
- [34] D. GREB, S. KEBEKUS & S. KOVÁCS – “Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties”, *Compositio Math.* **146** (2010), no. 1, p. 193–219.
- [35] M. GROECHENIG – “Hilbert schemes as moduli of Higgs bundles and local systems”, *Internat. Math. Res. Notices* (2014), no. 23, p. 6523–6575.
- [36] A. GROTHENDIECK – “Sur quelques points d’algèbre homologique”, *Tôhoku Math. J.* **9** (1957), p. 119–221.
- [37] A. HARDER – “Torus fibers and the weight filtration”, 2019, arXiv: 1908.05110.
- [38] A. HARDER, L. ZHIYUAN, J. SHEN & Q. YIN – “P=W for Lagrangian fibrations and degenerations of hyper-Kähler manifolds”, *Forum Math. Sigma* **9** (2021), article no. e50.
- [39] T. HAUSEL – “Vanishing of intersection numbers on the moduli space of Higgs bundles”, *Adv. Theo. Math. Phys.* **2** (1998), no. 5, p. 1011–1040.
- [40] T. HAUSEL, E. LETELLIER & F. RODRIGUEZ-VILLEGAS – “Arithmetic harmonic analysis on character and quiver varieties”, *Duke Math. J.* **160** (2011), no. 2, p. 323–400.
- [41] T. HAUSEL & F. RODRIGUEZ-VILLEGAS – “Mixed Hodge polynomials of character varieties”, *Invent. Math.* **174** (2008), no. 3, p. 555–624, With an appendix by Nicholas M. Katz.
- [42] ———, “Cohomology of large semiprojective hyperkähler varieties”, in *De la géométrie algébrique aux formes automorphes*, Astérisque, vol. 370, Société Mathématique de France, Paris, 2015, p. 113–156.
- [43] T. HAUSEL & M. THADDEUS – “Mirror symmetry, Langlands duality, and the Hitchin system”, *Invent. Math.* **153** (2003), no. 1, p. 197–229.
- [44] ———, “Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles”, *Proc. London Math. Soc. (3)* **88** (2004), no. 3, p. 632–658.
- [45] V. HEU & F. LORAY – *Flat rank two vector bundles on genus two curves*, Mem. Amer. Math. Soc., vol. 259, no. 1247, American Mathematical Society, Providence, RI, 2019.
- [46] N. J. HITCHIN – “The self-duality equations on a Riemann surface”, *Proc. London Math. Soc. (3)* **55** (1987), no. 1, p. 59–126.
- [47] D. KALEDIN – “Symplectic singularities from the Poisson point of view”, *J. reine angew. Math.* **600** (2006), p. 135–156.
- [48] L. KATZARKOV, V. V. PRYZJALKOWSKI & A. HARDER – “P=W phenomena”, *Mat. Zametki* **108** (2020), no. 1, p. 33–46.
- [49] Y.-H. KIEM & S.-B. YOO – “The stringy E -function of the moduli space of Higgs bundles with trivial determinant”, *Math. Nachr.* **281** (2008), no. 6, p. 817–838.
- [50] F. KIRWAN & J. WOOLF – *An introduction to intersection homology theory*, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [51] J. KOLLÁR – *Lectures on resolution of singularities*, Annals of Math. Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.
- [52] ———, *Singularities of the minimal model program*, Cambridge Tracts in Math., vol. 200, Cambridge University Press, Cambridge, 2013.

- [53] J. KOLLÁR & S. MORI – *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. 134, Cambridge University Press, Cambridge, 1998.
- [54] C. KUMAR – “Invariant vector bundles of rank 2 on hyperelliptic curves”, *Michigan Math. J.* **47** (2000), no. 3, p. 575–584.
- [55] R. LAZARSFELD – *Positivity in algebraic geometry. I*, *Ergeb. Math. Grenzgeb. (3)*, vol. 48, Springer-Verlag, Berlin, 2004.
- [56] M. LEHN & C. SORGER – “La singularité de O’Grady”, *J. Algebraic Geom.* **15** (2006), no. 4, p. 753–770.
- [57] M. LOGARES, V. MUÑOZ & P. NEWSTEAD – “Hodge polynomials of $SL(2, \mathbb{C})$ -character varieties for curves of small genus”, *Rev. Mat. Univ. Complut. Madrid* **26** (2013), no. 2, p. 635–703.
- [58] D. MAULIK & A. OKOUNKOV – *Quantum groups and quantum cohomology*, Astérisque, vol. 408, Société Mathématique de France, Paris, 2019.
- [59] M. MAURI – “Intersection cohomology of rank 2 character varieties of surface groups”, *J. Inst. Math. Jussieu* (2021), p. 1–40.
- [60] M. MAURI, E. MAZZON & M. STEVENSON – “On the geometric $P=W$ conjecture”, *Selecta Math. (N.S.)* **28** (2022), no. 3, article no. 65 (45 pages).
- [61] A. MELLIT – “Cell decompositions of character varieties”, 2019, arXiv: 1905.10685.
- [62] G. MONGARDI, A. RAPAGNETTA & G. SACCÀ – “The Hodge diamond of O’Grady’s six-dimensional example”, *Compositio Math.* **154** (2018), no. 5, p. 984–1013.
- [63] D. MUMFORD, J. FOGARTY & F. KIRWAN – *Geometric invariant theory*, third ed., *Ergeb. Math. Grenzgeb. (2)*, vol. 34, Springer-Verlag, Berlin, 1994.
- [64] M. S. NARASIMHAN & S. RAMANAN – “Moduli of vector bundles on a compact Riemann surface”, *Ann. of Math. (2)* **89** (1969), p. 14–51.
- [65] ———, “Vector bundles on curves”, in *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, Oxford Univ. Press, London, 1969, p. 335–346.
- [66] A. NÉMETHI & S. SZABÓ – “The geometric $P=W$ conjecture in the Painlevé cases via plumbing calculus”, *Internat. Math. Res. Notices* (2022), no. 5, p. 3201–3218.
- [67] N. NITSURE – “Moduli space of semistable pairs on a curve”, *Proc. London Math. Soc. (3)* **62** (1991), no. 2, p. 275–300.
- [68] S. PAL & C. PAULY – “The wobbly divisors of the moduli space of rank-2 vector bundles”, *Adv. Geom.* **21** (2021), no. 4, p. 473–482.
- [69] A. PEREGO & A. RAPAGNETTA – “Deformation of the O’Grady moduli spaces”, *J. reine angew. Math.* **678** (2013), p. 1–34.
- [70] ———, “The moduli spaces of sheaves on K3 surfaces are irreducible symplectic varieties”, 2018, arXiv: 1802.01182.
- [71] C. PETERS & J. H. M. STEENBRINK – *Mixed Hodge structures*, *Ergeb. Math. Grenzgeb. (3)*, vol. 52, Springer-Verlag, Berlin, 2008.
- [72] J. SAWON & C. SHEN – “Deformations of compact Prym fibrations to Hitchin systems”, 2021, arXiv: 2103.04274.
- [73] C. S. SESHADRI – “Moduli of π -vector bundles over an algebraic curve”, in *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, Edizioni Cremonese, Rome, 1970, p. 139–260.
- [74] J. SHEN & Z. ZHANG – “Perverse filtrations, Hilbert schemes, and the $P=W$ conjecture for parabolic Higgs bundles”, *Algebraic Geom.* **8** (2021), no. 4, p. 465–489.
- [75] J. SHEN & Q. YIN – “Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds”, *Duke Math. J.* **171** (2022), no. 1, p. 209–241.
- [76] C. T. SIMPSON – “Harmonic bundles on noncompact curves”, *J. Amer. Math. Soc.* **3** (1990), no. 3, p. 713–770.
- [77] ———, “Higgs bundles and local systems”, *Publ. Math. Inst. Hautes Études Sci.* (1992), no. 75, p. 5–95.
- [78] ———, “Moduli of representations of the fundamental group of a smooth projective variety. I”, *Publ. Math. Inst. Hautes Études Sci.* (1994), no. 79, p. 47–129.
- [79] ———, “Moduli of representations of the fundamental group of a smooth projective variety. II”, *Publ. Math. Inst. Hautes Études Sci.* (1994), no. 80, p. 5–79 (1995).

- [80] ———, “The Hodge filtration on nonabelian cohomology”, in *Algebraic geometry (Santa Cruz 1995)*, Proc. Sympos. Pure Math., vol. 62, American Mathematical Society, Providence, RI, 1997, p. 217–281.
- [81] S. SZABÓ – “Simpson’s geometric P=W conjecture in the Painlevé VI case via abelianization”, 2019, arXiv:1906.01856.
- [82] ———, “Perversity equals weight for Painlevé spaces”, *Adv. Math.* **383** (2021), article no. 107667 (45 pages).
- [83] M. TEMKIN – “Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case”, *Duke Math. J.* **161** (2012), no. 11, p. 2207–2254.
- [84] M. THADDEUS – “Topology of the moduli space of stable bundles on a Riemann surface”, Master’s thesis, University of Oxford, 1989.
- [85] A. WEBER – “Hirzebruch class and Białynicki-Birula decomposition”, *Transform. Groups* **22** (2017), no. 2, p. 537–557.
- [86] G. WILLIAMSON – “Modular representations and reflection subgroups”, in *Current developments in math., 2019*, International Press, Somerville, MA, 2021, p. 113–184.
- [87] B. WU – “Hodge numbers of O’Grady 6 via Ngô strings”, 2021, arXiv:2105.08545.
- [88] K. YOSHIOKA – “Moduli spaces of stable sheaves on abelian surfaces”, *Math. Ann.* **321** (2001), no. 4, p. 817–884.
- [89] Z. ZHANG – “The P=W identity for cluster varieties”, *Math. Res. Lett.* **28** (2021), no. 3, p. 925–944.

Manuscript received 24th October 2020

accepted 14th May 2022

CAMILLA FELISETTI, University of Trento, Department of mathematics
Via Sommarive 14, 38123 Povo (TN), Italy
E-mail : camilla.feliseti@uni.tn.it
Url : <https://camillafeliseti.wixsite.com/home>

MIRKO MAURI, University of Michigan
East Hall, 530 Church St, Ann Arbor 48109, Michigan, USA
E-mail : mirkom@umich.edu
Url : <https://sites.google.com/view/mirko-mauri>