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Measure equivalence classification of transvection-free right-angled Artin groups

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MEASURE EQUIVALENCE CLASSIFICATION OF TRANSECTION-FREE RIGHT-ANGLED ARTIN GROUPS

BY CAMILLE HORBEZ & JINGYIN HUANG

ABSTRACT. — We prove that if two transvection-free right-angled Artin groups are measure equivalent, then they have isomorphic extension graphs. As a consequence, two right-angled Artin groups with finite outer automorphism groups are measure equivalent if and only if they are isomorphic. This matches the quasi-isometry classification. However, in contrast with the quasi-isometry question, we observe that no right-angled Artin group is superrigid for measure equivalence in the strongest possible sense, for two reasons. First, a right-angled Artin group G is always measure equivalent to any graph product of infinite countable amenable groups over the same defining graph. Second, when G is nonabelian, the automorphism group of the universal cover of the Salvetti complex of G always contains infinitely generated (non-uniform) lattices.

RÉSUMÉ (Classification des groupes d'Artin à angles droits sans transvections pour l'équivalence mesurée)

Nous démontrons que si deux groupes d'Artin à angles droits sans transvections sont mesurablement équivalents, alors ils ont des graphes d'extension isomorphes. En conséquence, deux groupes d'Artin à angles droits ayant des groupes d'automorphismes extérieurs finis sont mesurablement équivalents si et seulement s'ils sont isomorphes. Ceci coïncide avec la classification pour la quasi-isométrie. Par contre, contrairement au cas de la quasi-isométrie, un groupe d'Artin à angles droits ne peut jamais être super-rigide pour l'équivalence mesurée, pour deux raisons. D'abord, un groupe d'Artin à angles droits G est toujours mesurablement équivalent à tout produit graphé de groupes moyennables infinis dénombrables sur le même graphe sous-jacent. Ensuite, lorsque G est non abélien, le groupe d'automorphismes du revêtement universel du complexe de Salvetti de G contient toujours des réseaux (non uniformes) qui ne sont pas de type fini.

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INTRODUCTION

Measured group theory studies groups through their ergodic actions on standard probability spaces. A central quest is to classify groups up to *measure equivalence*, a notion introduced by Gromov [Gro93] as a measure-theoretic analogue of that of quasi-isometry between finitely generated groups.

The definition is as follows: two countable groups G_1 and G_2 are *measure equivalent* if there exists a standard measure space Σ equipped with a measure-preserving action of $G_1 \times G_2$ by Borel automorphisms, such that for every $i \in \{1, 2\}$, the G_i -action on Σ is essentially free and has a finite measure fundamental domain. A typical example is that two lattices in the same locally compact second countable group are always measure equivalent.

A first striking result in the theory, due to Ornstein and Weiss [OW80] (building on previous work of Dye [Dye59, Dye63]) is that any two countably infinite amenable groups are measure equivalent. At the other extreme, strong rigidity results have been established from the viewpoint of measure equivalence, notably for lattices in higher rank Lie groups (Furman [Fur99a], building on earlier work of Zimmer [Zim80, Zim91]) and for mapping class groups of finite-type surfaces (Kida [Kid10]). In [MS06], Monod and Shalom used bounded cohomology techniques to obtain rigidity results for products of certain negatively curved groups. Other developments include superrigidity results for certain amalgamated free products [Kid11], for other groups related to mapping class groups [CK15], or classification results within certain classes of groups, like Baumslag–Solitar groups [Kid14, HR15]. See also [Sha05, Gab10, Fur11] for general surveys in measured group theory.

In previous work [HH20], we obtained measure equivalence superrigidity results for certain classes of two-dimensional Artin groups of hyperbolic type. In the present paper, we focus on right-angled Artin groups, which have a very simple definition and played a prominent role in geometric group theory recently. We obtain a complete measure equivalence classification of right-angled Artin groups with finite outer automorphism group. One motivation for studying these groups from the viewpoint of measured group theory is that they usually tend to be significantly less rigid than certain other classes of Artin groups, as will be evidenced in Corollary 5 and Theorem 6 below. With this in mind, one hopes to explore the subtle line between rigidity and flexibility phenomena in this setting.

Main classification theorem. — We recall that given a finite simple graph Γ (i.e., with no edge loops and no multiple edges between vertices), the right-angled Artin group G_Γ is the group defined by the following presentation: the generators are the vertices of Γ , and two generators commute if and only if the corresponding vertices of Γ are joined by an edge. Our main theorem is the following.

THEOREM 1. — *Let G_1 and G_2 be two right-angled Artin groups such that $\text{Out}(G_1)$ and $\text{Out}(G_2)$ are finite. Then G_1 and G_2 are measure equivalent if and only if they are isomorphic.*

Finiteness of $\text{Out}(G_\Gamma)$ can easily be checked on the defining graph Γ ; by work of Laurence [Lau95] and Servatius [Ser89], this amounts to the two conditions below. For a vertex $v \in \Gamma$, the star of v , denoted by $\text{st}(v)$, is the subgraph made of all vertices which are either equal or adjacent to v , together with edges of Γ between these vertices.

(1) Γ does not contain two distinct vertices v and w such that the link of v is contained in the star of w – this prevents the existence of transvections $v \mapsto vw$ in $\text{Aut}(G_\Gamma)$ (we then say that G_Γ is *transvection-free*);

(2) Γ contains no separating star, i.e., there does not exist any vertex $v \in \Gamma$ such that $\Gamma \setminus \text{st}(v)$ is disconnected. This prevents the existence of partial conjugations.

Every right-angled Artin group has an associated *extension graph*, introduced by Kim and Koberda in [KK13] as the graph whose vertices are the rank one parabolic subgroups of G_Γ (i.e., conjugates of the cyclic subgroups generated by the standard generators of G_Γ), two of these being joined by an edge exactly when they commute. This graph can be viewed as an analogue of the curve graph in the context of right-angled Artin groups [KK14]. Theorem 1 is in fact obtained as a consequence of the following theorem, using the additional fact that two right-angled Artin groups with finite outer automorphism groups have isomorphic extension graphs if and only if they are isomorphic [Hua17a].

THEOREM 2. — *Let G_1 and G_2 be two transvection-free right-angled Artin groups. If G_1 and G_2 are measure equivalent, then they have isomorphic extension graphs.*

In fact, in Theorem 1, if we only assume that $\text{Out}(G_1)$ is finite and that G_2 is transvection-free, then using [Hua17a, Th. 1.2], we deduce that G_1 and G_2 are measure equivalent if and only if they have isomorphic extension graphs, and in this case this happens if and only if G_2 is a finite-index subgroup of G_1 .

To our knowledge, right-angled Artin groups have not been systematically studied from the point of view of measured group theory before; however, a few things were already known concerning their measure equivalence classification. As already mentioned, all countably infinite free abelian groups (and more generally amenable groups) are measure equivalent [Dye63, OW80] (and not measure equivalent to non-amenable groups, see e.g. [Fur11, Cor. 3.2]). The behavior of measure equivalence under free products was thoroughly studied by Alvarez and Gaboriau in [AG12]. The invariance of the canonical decomposition of a right-angled Artin group as a direct product (corresponding to the maximal decomposition of Γ as a join) can be proved by combining work of Monod and Shalom [MS06, Th 1.16 & 1.17] together with a theorem of Chatterji, Fernós and Iozzi [CFI16, Cor. 1.8] stating that non-cyclic right-angled Artin groups that do not split as direct products belong to the class \mathcal{C}_{reg} . Also, Gaboriau proved that ℓ^2 -Betti numbers can be used to obtain measure equivalence invariants [Gab02a]; they have been computed by Davis and Leary for many Artin groups including all the right-angled ones [DL03]. Finally, we comment that Theorem 1 was proved in our previous work on Artin groups under the extra assumption that the

underlying graphs of G_1 and G_2 have girth at least 5 [HH20, Cor.10.3]. However, our proof in [HH20] relies on fact that, under this extra assumption, G_1 and G_2 are special instances of 2-dimensional Artin groups of hyperbolic type. Our methods in [HH20] are specific to this class of Artin groups, and therefore cannot be used to prove Theorem 1 in general.

Orbit equivalence and W^ -equivalence.* — Our results can be reinterpreted in the language of orbit equivalence: indeed, by a result of Furman [Fur99b] (with an additional argument by Gaboriau [Gab02b, Th. 2.3]), two groups are measure equivalent if and only if they admit stably orbit equivalent essentially free measure-preserving ergodic actions on standard probability spaces. As will be explained with more details later in this introduction, the way we prove Theorem 2 is actually through this viewpoint – see Theorem 3.19 for a precise statement phrased in terms of measured groupoids, also including the case of non-free actions.

As right-angled Artin groups are cubical, our results can in fact also be phrased using the weaker notion of (stable) W^* -equivalence (we refer to [PV10, §6.2] for a detailed discussion of the notion of stable W^* -equivalence).

COROLLARY 3. — *Let G_1 and G_2 be two right-angled Artin groups with finite outer automorphism groups. Assume that G_1 and G_2 have essentially free ergodic probability measure-preserving actions on standard probability spaces which are stably W^* -equivalent (i.e., their associated von Neumann algebras are isomorphic, or more generally one is isomorphic to an amplification of the other).*

Then G_1 and G_2 are isomorphic.

The version for W^* -equivalence (i.e., assuming that the two von Neumann algebras are isomorphic, without passing to amplifications) can be seen directly as a consequence of [HHL20, Cor.4.2]; we will explain in Section 3.9 how the full statement follows from work of Popa and Vaes [PV14].

REMARK. — In Corollary 3, one cannot expect to reach the stronger conclusion that the given actions of G_1 and G_2 are conjugate. See indeed Remark 4.4 for a construction of free, ergodic, probability measure-preserving actions that are orbit equivalent but not conjugate.

Comparison with quasi-isometry; and failure of superrigidity. — Incidentally, the classification results given by Theorems 1 and 2 both match the quasi-isometry classification obtained by the second named author in [Hua17a], generalizing earlier work of Bestvina, Kleiner and Sageev [BKS08]. A lot of work has revolved around the quasi-isometry classification problem for right-angled Artin groups, starting from work of Behrstock and Neumann [BN08] for right-angled Artin groups whose defining graph is a tree, with a higher dimensional generalization in [BJN10], and followed more recently by [Hua16, Mar20], for instance.

However, apart from the analogy with mapping class groups and Artin groups suggesting that having a finite outer automorphism group is often the source of further

rigidity phenomena, there was *a priori* no reason to expect that the measure equivalence and quasi-isometry classification should match. By contrast, they already fail to match on the most basic class of right-angled Artin groups (with infinite outer automorphism group), namely the free abelian groups.

In fact, this fundamental difference on the most basic examples is the source of an extremely different situation between the quasi-isometry and measure equivalence classifications of right-angled Artin groups. This is evidenced by our next proposition (and its corollary), that we prove by adapting an argument of Gaboriau [Gab00, Gab05], who dealt with the case of free products.

We now recall the notion of graph products introduced by Green in [Gre90]. Given a finite simple graph Γ and an assignment of a group G_v to every vertex v of Γ , the *graph product* over Γ with vertex groups $\{G_v\}_{v \in V\Gamma}$ is the group obtained from the free product of the groups G_v by adding as only extra relations that every element of G_v commutes with every element of G_w whenever v and w are adjacent in Γ .

PROPOSITION 4. — *Let G and H be two groups that split as graph products over the same finite simple graph Γ , with countable vertex groups $\{G_v\}_{v \in V\Gamma}$ and $\{H_v\}_{v \in V\Gamma}$, respectively. Assume that for every vertex $v \in V\Gamma$, the groups G_v and H_v admit orbit equivalent free measure-preserving actions on standard probability spaces.*

Then G and H admit orbit equivalent free measure-preserving actions on standard probability spaces; in particular they are measure equivalent.

We mention that orbit equivalence cannot be replaced by measure equivalence in the above statement, even for free products (see [Gab05, §2.2]). Since right-angled Artin groups are graph products where the vertex groups are isomorphic to \mathbb{Z} , combining the above proposition with the aforementioned theorem of Ornstein and Weiss [OW80] yields the following.

COROLLARY 5. — *For every right-angled Artin group G , every group obtained as a graph product over the defining graph of G with countably infinite amenable vertex groups, is measure equivalent to G .*

By choosing the vertex groups to be isomorphic to \mathbb{Z}^n , this gives infinitely many right-angled Artin groups that are all measure equivalent to G , but pairwise non-quasi-isometric. As pointed out by a referee, Corollary 5 also yields a continuum of countable groups that are measure equivalent to a given right-angled Artin group (see Theorem 4.1).

Let us also mention that conversely, there are examples of right-angled Artin groups which are quasi-isometric but not measure equivalent. For instance, the groups $G_n = (F_3 \times F_3) * F_n$ are all quasi-isometric [Why99, Th. 1.5] but pairwise not measure equivalent by comparison of their ℓ^2 -Betti numbers [Gab02a, Cor. 0.3].

Corollary 5 also shows that right-angled Artin groups are never superrigid for measure equivalence in the strongest possible sense. This is in strong contrast to the situation for some non-right-angled Artin groups with finite outer automorphism

groups [HH20], or mapping class groups [Kid10]; this also strongly contrasts with the quasi-isometry superrigidity results obtained by the second named author in [Hua18].

In fact, there is another source of failure of measure equivalence superrigidity of right-angled Artin groups, given by the following theorem.

THEOREM 6. — *For every nonabelian right-angled Artin group G , the automorphism group of the universal cover of the Salvetti complex of G contains an infinitely generated (non-uniform) lattice H (in particular H is measure equivalent to G , but not commensurable to G).*

These lattices are constructed as follows. First we take an isometrically embedded tree in the universal cover and a non-uniform lattice H acting on this tree (see e.g. [BL01]). Then we extend this action to a larger group acting on the ambient space. This method only produces non-uniform lattices which are not finitely generated, and we do not know whether there are finitely generated examples.

A word on the proof of the classification theorem. — In the remainder of this introduction, we would like to explain how we prove our main classification theorem, in the form of Theorem 2 (from which Theorem 1 follows). The general structure of our proof is inspired by Kida’s strategy in the mapping class group setting [Kid10]. The orbit equivalence interpretation reduces our proof to a problem about measured groupoids associated to measure-preserving actions of right-angled Artin groups on standard probability spaces. Given two such actions of G_1 and G_2 on the same standard probability space Y with the same orbit structure – in other words, a measured groupoid over Y coming with two cocycles, one towards G_1 and the other towards G_2 , the goal is to build a canonical map that associates to every point of Y , an isomorphism between Γ_1^e and Γ_2^e , where Γ_i^e denotes the extension graph of G_i . To this end, given a transvection-free right-angled Artin group G , we characterize subgroupoids of $G \times Y$ (the groupoid coming from the G -action on Y – or in fact, more generally, a restriction of this groupoid) that naturally correspond to stabilizers of vertices of Γ^e , or pairs of stabilizers of adjacent vertices of Γ^e , in a purely groupoid-theoretic way.

In order to keep this introduction not too technical, we will not provide any groupoid-theoretic statement, however we will describe some of the ideas behind their group-theoretic analogues. Namely, we will explain how to obtain an algebraic characterization of stabilizers of vertices of Γ^e in G_Γ (see Proposition 3.15 for the version for groupoids), and an algebraic characterization of adjacency of vertices.

So let v be a vertex in Γ^e – corresponding to a cyclic parabolic subgroup Z_v of G_Γ . The stabilizer of v for the G_Γ -action on Γ^e is the centralizer of Z_v . Classical work of Servatius [Ser89] ensures that this centralizer splits as a direct product $Z_v \times Z_v^\perp$, where Z_v^\perp is a parabolic subgroup of G_Γ . More precisely, if Z_v is conjugate to the standard cyclic subgroup associated to a vertex \bar{v} of Γ , then Z_v^\perp is conjugate to the standard parabolic subgroup associated to the link of \bar{v} in Γ . The transvection-freeness assumption ensures that Z_v^\perp is nonabelian. In fact the following holds.

(*) The centralizer $C_{G_\Gamma}(Z_v)$ is a nonamenable subgroup of G_Γ that contains an infinite normal amenable subgroup. Among the subgroups of G_Γ , it is maximal with respect to this property.

This is reminiscent of the characterization of centralizers of Dehn twists – i.e., stabilizers of isotopy classes of essential simple closed curves – for finite-type mapping class groups. Property (*) is phrased on purpose in terms of *amenability* and *normality*, as these notions have groupoid-theoretic analogues – in our proof, amenability is used to obtain invariant probability measures on the (compact) Roller boundary of the universal cover of the Salvetti complex, as will be explained below under the heading ‘Geometric tools’.

However (contrary to the case of the curve graph of a surface), Property (*) is not enough to characterize vertex stabilizers of Γ^e : indeed G_Γ could also contain a maximal parabolic subgroup splitting as a direct product of two nonabelian subgroups $P_1 \times P_2$ with trivial center, and in this case a subgroup of the form $\langle g \rangle \times P_2$ with g generic in P_1 , would also satisfy Property (*). To distinguish these two types of subgroups, we make the following observation: when $H = C_{G_\Gamma}(Z_v)$, the centralizer H' of any infinite amenable subgroup Z' not commensurable to Z_v , will look very different from H , in the sense that one can always find a nonamenable subgroup of H that intersects H' trivially. On the other hand, when H is of the form $\langle g \rangle \times P_2$ as above, by choosing g' to be any other generic element of P_1 , the centralizer H' of g' will be much closer to H , and in particular every nonamenable subgroup of H' will have to intersect P_2 , whence H , nontrivially. See assertion (2b) from Proposition 3.15 for the groupoid-theoretic version.

We also need to characterize adjacency in Γ^e . For that, the key observation is that two vertices $v, w \in \Gamma^e$ are adjacent if and only if there are only finitely many vertices of Γ^e that are fixed by $C_{G_\Gamma}(Z_v) \cap C_{G_\Gamma}(Z_w)$.

Geometric tools. — Our proof of Theorem 2 is an implementation of a groupoid version of the above strategy. This groupoid version involves a certain geometric setting (with features of negative curvature), in order to apply an argument introduced by Adams in [Ada94]. In the work of Kida [Kid10], this came in the form of a natural partition of the (compact) Thurston boundary of the Teichmüller space into arational and nonarational laminations. In our previous work on Artin groups of hyperbolic type [HH20], this came in the form of a partition of the horofunction compactification of a certain CAT(−1) simplicial complex on which the group is acting. In the present paper, we exploit the cubical geometry of right-angled Artin groups by considering the Roller boundary $\partial_R \tilde{S}_\Gamma$ of the universal cover of the Salvetti complex, and its partition, introduced by Fernós in [Fer18], into *regular* points and *non-regular* points.

The three important features we exploit are the following: every non-regular point has a natural associated parabolic subgroup (see Theorem 7 below); there is a *barycenter map* that canonically associates a vertex of \tilde{S}_Γ to every triple of pairwise distinct regular points [FLM18]; the action of G_Γ on the set of regular points (in fact on the whole $\partial_R \tilde{S}_\Gamma$) is Borel amenable [NS13, Duc18].

Let us explain how this geometric setting is used, by explaining why a subgroup $H \subseteq G_\Gamma$ satisfying Property (*) above (in particular, containing an infinite normal amenable subgroup A) preserves a parabolic subgroup. Amenability of A is used to get an invariant probability measure ν on the compact metrizable space $\partial_R \tilde{S}_\Gamma$. Theorem 7 below, saying that every non-regular point has a natural associated parabolic subgroup, is exploited to build a canonical A -invariant (whence H -invariant) parabolic subgroup if ν gives positive measure to the set of non-regular points. If ν is supported on the set of regular points, and if its support has cardinality at least 3, then the aforementioned *barycenter map* is used to contradict the fact that A is infinite. Finally, if ν is supported on at most two points, then the amenability of point stabilizers is exploited to get a contradiction to the fact that H is nonamenable – in the groupoid version, this is replaced by the Borel amenability of the action of G_Γ on $\partial_R \tilde{S}_\Gamma$.

Let us finally state the result we use concerning non-regular points (see Section 2.1 for relevant definitions). In this statement $\partial_R \tilde{S}_\Gamma$ is equipped with its Borel σ -algebra.

THEOREM 7. — *Let Γ be a finite simple graph. Then we can assign to each non-regular point $\xi \in \partial_R \tilde{S}_\Gamma$ a unique standard subcomplex $Y \subseteq \tilde{S}_\Gamma$ such that for every combinatorial geodesic ray r representing ξ , the subset Y is the smallest standard subcomplex of \tilde{S}_Γ that contains a subray of r . This assignment is measurable and G_Γ -equivariant, and Y is contained in a standard subcomplex of \tilde{S}_Γ which splits non-trivially as a product.*

Open questions. — Our work raises several questions regarding the measure equivalence classification and rigidity of right-angled Artin groups.

(1) Given a right-angled Artin group G_Γ with finite outer automorphism group, what can be said of countable groups H that are measure equivalent to G_Γ ? For instance, do they necessarily act on a CAT(0) cube complex with amenable vertex stabilizers?

(2) A sharper notion of L^1 -measure equivalence was introduced by Bader, Furman and Sauer in [BFS13], by imposing an integrability condition on the measure equivalence cocycle. Are there right-angled Artin groups that are superrigid for L^1 -measure equivalence? It turns out that free abelian groups of different ranks are not L^1 -measure equivalent to one another (see [Aus16]), thus suggesting that the obstruction coming from Corollary 5 vanishes under this sharper notion. The question may be subdivided into two parts. First, if a finitely generated group H is L^1 -measure equivalent to G_Γ , is it commensurable to a lattice in $\text{Aut}(\tilde{S}_\Gamma)$? Second, are all finitely generated such lattices cocompact?

(3) Classify right-angled Artin groups up to measure equivalence, beyond the class of those with finite outer automorphism group. For instance, Behrstock and Neumann proved in [BN08] that all right-angled Artin groups whose defining graph is a tree of diameter at least 3 are quasi-isometric. Are they all measure equivalent?

Organization of the paper. — The paper has four parts. The first part is mostly a background section on right-angled Artin groups. In the second part, we prove Theorem 7 concerning the geometry of the Roller boundary of the universal cover of the Salvetti complex. The third part implements the groupoid version of the strategy explained in this introduction, and gives a proof of our main classification theorems (Theorems 1 and 2). In Section 4, we present two sources of failure of measure equivalence super-rigidity of right-angled Artin groups: graph products of countably infinite amenable groups, and non-uniform lattices in the automorphism group of the universal cover of the Salvetti complex.

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1. RIGHT-ANGLED ARTIN GROUPS: BACKGROUND AND COMPLEMENTS

In the present section, we review standard facts about right-angled Artin groups and establish a few statements that we will need in the sequel. More background on right-angled Artin groups can be found in [Cha07]. Section 1.1 reviews basic facts about right-angled Artin groups, their parabolic subgroups, their automorphisms, and Salvetti complexes. Section 1.2 reviews earlier work of the second named author [Hua17a] establishing a rigidity statement for extension graphs of right-angled Artin groups. Sections 1.3 and 1.4 establish two extra statements that are used in Section 3.

We will stick to the setting of right-angled Artin groups in the present paper, but would like to mention that many of the results presented here have also been generalized to broader contexts. As a specific example, several results regarding parabolic subgroups have been generalized to graph products in [AM15, §3].

1.1. REVIEW OF BASIC FACTS ABOUT RIGHT-ANGLED ARTIN GROUPS

Definition. — Given a finite simple graph Γ with vertex set $V\Gamma$, the *right-angled Artin group* with defining graph Γ , denoted by G_Γ , is given by the following presentation:

$$\langle V\Gamma \mid [v, w] = 1 \text{ if } v \text{ and } w \text{ are joined by an edge} \rangle.$$

In this way $V\Gamma$ is identified to a *standard generating set* for G_Γ . Given two finite simple graphs Γ and Λ , a theorem of Droms [Dro87] asserts that G_Γ and G_Λ are isomorphic if and only if Γ and Λ are isomorphic.

Parabolic subgroups. — Let $\Lambda \subseteq \Gamma$ be a *full* subgraph, i.e., two vertices of Λ are adjacent in Λ if and only if they are adjacent in Γ . Then there is an injective homomorphism $G_\Lambda \hookrightarrow G_\Gamma$, whose image is called a *standard subgroup* of G_Γ with *type* Λ . A conjugate of such a subgroup is called a *parabolic subgroup of G_Γ with type Λ* . The type of a parabolic subgroup is well-defined: if $\Lambda_1, \Lambda_2 \subseteq \Gamma$ are two full subgraphs, then G_{Λ_1} and G_{Λ_2} are not conjugate unless $\Lambda_1 = \Lambda_2$, as follows from [CCV07, Prop. 2.2]. Note that the definitions of parabolic subgroups and standard subgroups depend on the choice of a standard generating set for G_Γ .

The *star* of a vertex v in Γ , denoted by $\text{st}(v)$, is the full subgraph spanned by v and all the vertices that are adjacent to v . Its *link* $\text{lk}(v)$ is defined to be the full subgraph spanned by all the vertices that are adjacent to v .

Given two graphs Γ_1 and Γ_2 , we denote by $\Gamma_1 \circ \Gamma_2$ the join of Γ_1 and Γ_2 . Every finite simple graph Γ has a canonical join decomposition $\Gamma = \Gamma_0 \circ \Gamma_1 \circ \cdots \circ \Gamma_k$, where Γ_0 is the maximal clique factor of Γ , and for every $i \in \{1, \dots, k\}$, the graph Γ_i is *irreducible*, i.e., it does not admit any nontrivial join decomposition. We call this the *de Rham decomposition* of Γ . There is also an induced *de Rham decomposition* of G_Γ , namely $G_\Gamma = \mathbb{Z}^n \times G_{\Gamma_1} \times \cdots \times G_{\Gamma_k}$.

For a full subgraph $\Lambda \subseteq \Gamma$, we define Λ^\perp to be the full subgraph of Γ spanned by the collection of all vertices of $\Gamma \setminus \Lambda$ which are adjacent to every vertex of Λ . For example, with this terminology, we have $\text{lk}(v) = \{v\}^\perp$. The following was proved by Charney, Crisp and Vogtmann in [CCV07, Prop. 2.2], building on work of Godelle [God03].

PROPOSITION 1.1. — *Let Γ be a finite simple graph, and let $\Lambda \subseteq \Gamma$ be a full subgraph. Then the normalizer of G_Λ in G_Γ is $G_{\Lambda \circ \Lambda^\perp}$.*

Let now $P = gG_\Lambda g^{-1}$ be a parabolic subgroup. We define $P^\perp = gG_{\Lambda^\perp} g^{-1}$. This is well-defined: if we can write the parabolic subgroup P in two different ways $gG_\Lambda g^{-1}$ and $hG_{\Lambda'} h^{-1}$, then $\Lambda = \Lambda'$ and Proposition 1.1 ensures that $gG_{\Lambda^\perp} g^{-1} = hG_{\Lambda^\perp} h^{-1}$. Moreover, the normalizer of P in G_Γ is $P \times P^\perp$.

In the following lemma, we record basic facts about parabolic subgroups of right-angled Artin groups, which follow from the works of Duncan, Kazachkov and Remeslennikov in [DKR07, §2.2]. Note that (5) and (6) generalize Proposition 1.1.

LEMMA 1.2 (Duncan–Kazachkov–Remeslennikov). — *Let Γ be a finite simple graph.*

- (1) *The intersection of two parabolic subgroups of G_Γ is a parabolic subgroup. Moreover, for full subgraphs Γ_1 and Γ_2 of Γ , one has $G_{\Gamma_1} \cap G_{\Gamma_2} = G_{\Gamma_1 \cap \Gamma_2}$.*
- (2) *There is a finite integer $n > 0$ such that every chain $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_k$ of parabolic subgroups of G_Γ has length $k \leq n$.*
- (3) *Every element of G_Γ is contained in a unique smallest parabolic subgroup.*
- (4) *If P_1 and P_2 are two parabolic subgroups of G_Γ with $P_1 \subseteq P_2$, then the type of P_1 is contained in the type of P_2 .*
- (5) *For every parabolic subgroup P of G_Γ and every $g \in G$, if $P \subseteq g(P \times P^\perp)g^{-1}$, then $g \in P \times P^\perp$.*
- (6) *Given any two induced subgraphs Γ_1, Γ_2 of Γ , and any $g \in G_\Gamma$, if $gG_{\Gamma_1}g^{-1} \subseteq G_{\Gamma_2}$, then there exists $h \in G_{\Gamma_2}$ such that $gG_{\Gamma_1}g^{-1} = hG_{\Gamma_1}h^{-1}$ (in particular $G_{\Gamma_1} \subseteq G_{\Gamma_2}$).*

Proof. — The first assertion is [DKR07, Prop. 2.6 & Lem. 2.7]. Assertion (2) follows from [DKR07, Prop. 2.6], and assertion (3) follows from [DKR07, Prop. 2.8]. Assertions (4), (5) and (6) follow from [DKR07, Cor. 2.5] or [CCV07, Prop. 2.2]. \square

REMARK 1.3. — Every parabolic subgroup H of G_Γ is itself a right-angled Artin group, with a choice of generating set induced from the generating set of G_Γ . By Proposition 1.1, this choice is well-defined up to conjugation by elements inside H . Thus it makes sense to talk about parabolic subgroups of H with respect to this choice of generating set of H . Every parabolic subgroup of H is then naturally a parabolic subgroup of G_Γ . Conversely, by Lemma 1.2 (4) and (6), every parabolic subgroup H' of G_Γ with $H' \subseteq H$ is in fact a parabolic subgroup of H . Thus from now on, we will just refer to H' as a parabolic subgroup, without specifying its ambient group.

The smallest parabolic subgroup that contains an element $g \in G_\Gamma$ is called the *support* of g . The *type* of g is then defined as the type of its support.

LEMMA 1.4. — *Let Γ be a finite simple graph, and let \mathbb{P} be the collection of parabolic subgroups of G_Γ , equipped with the conjugation action of G_Γ . Then for every finite set $\mathcal{F} \subset \mathbb{P}$, the pointwise stabilizer of \mathcal{F} in G_Γ coincides with the setwise stabilizer of \mathcal{F} .*

Proof. — Let P be a parabolic subgroup in the set \mathcal{F} . Take $g \in G_\Gamma$ with $g\mathcal{F} = \mathcal{F}$. Then there exists $k \neq 0$ such that g^k normalizes P , so $g^k \in P \times P^\perp$. Hence the subgroup $\langle g \rangle$ has a finite-index subgroup contained in the parabolic subgroup $P \times P^\perp$. By [Min12, Lem. 6.4], we have $\langle g \rangle \subseteq P \times P^\perp$. Hence g fixes P and the lemma follows. \square

Automorphisms. — Let Γ be a finite simple graph. Laurence [Lau95] and Servatius [Ser89] showed that the outer automorphism group $\text{Out}(G_\Gamma)$ is generated by the outer classes of four types of automorphisms, namely *inversions*, *graph automorphisms*, *transvections* and *partial conjugations*. We say that Γ (or G_Γ) is *transvection-free* if $\text{Out}(G_\Gamma)$ does not contain any transvection; equivalently, there do not exist distinct vertices $v, w \in \Gamma$ such that $\text{lk}(w) \subseteq \text{st}(v)$. When Γ is transvection-free, the set of parabolic subgroups of G_Γ does not depend on the choice of standard generating set of G_Γ . It also follows from the work of Laurence and Servatius that $\text{Out}(G_\Gamma)$ is finite if and only if Γ is transvection-free and does not contain any separating star.

The Salvetti complex. — Every right-angled Artin group G_Γ is the fundamental group of a locally CAT(0) cube complex S_Γ , called the *Salvetti complex*, obtained from a bouquet of $|V\Gamma|$ circles by gluing a k -cube per k -clique of Γ (recall that a k -clique is a complete subgraph on k vertices). We refer to [Cha07, §3.6] for more details. The 2-skeleton of S_Γ is the presentation complex of G_Γ . For each full subgraph $\Lambda \subseteq \Gamma$, there is an isometric embedding $S_\Lambda \hookrightarrow S_\Gamma$. Let \tilde{S}_Γ be the universal cover of S_Γ , which is a CAT(0) cube complex. A *standard subcomplex* of \tilde{S}_Γ of type Λ is a connected component of the inverse image of $S_\Lambda \subseteq S_\Gamma$ with respect to the covering map $\tilde{S}_\Gamma \rightarrow S_\Gamma$. In particular, a standard subcomplex whose type is the empty subgraph is the same as a vertex of \tilde{S}_Γ . We collect several standard facts on \tilde{S}_Γ .

(1) Two standard subcomplexes of the same type are either disjoint or equal. If a standard subcomplex of type Λ_1 and a standard subcomplex of type Λ_2 have nonempty intersection, then their intersection is a standard subcomplex of type $\Lambda_1 \cap \Lambda_2$.

(2) The stabilizer of a standard subcomplex is a parabolic subgroup. Each parabolic subgroup can be realized as the stabilizer of some (non-unique) standard subcomplex.

1.2. RIGIDITY OF THE EXTENSION GRAPH. — Let Γ be a connected finite simple graph. The *extension graph* of Γ , denoted Γ^e , was defined by Kim and Koberda [KK13] to be the graph whose vertices are the parabolic subgroups of G_Γ isomorphic to \mathbb{Z} , where two vertices are adjacent if the corresponding parabolic subgroups commute. As follows from Proposition 1.1 and Lemma 1.2(6), two distinct cyclic parabolic subgroups $P_1 = g_1\langle v_1 \rangle g_1^{-1}$ and $P_2 = g_2\langle v_2 \rangle g_2^{-1}$ commute if and only if v_1 and v_2 are adjacent in Γ and there exists $g \in G_\Gamma$ such that for every $i \in \{1, 2\}$, one has $P_i = g\langle v_i \rangle g^{-1}$. The conjugation action of G_Γ on itself induces an action $G_\Gamma \curvearrowright \Gamma^e$. The above observation about adjacency can be rephrased by saying that Γ is a fundamental domain for the G_Γ -action on Γ^e .

Now we recall the following rigidity result on extension graphs, which is a combination of [Hua17a, Cor. 4.16 and Lemma 4.17].

THEOREM 1.5 ([Hua17a]). — *Let Γ_1 and Γ_2 be two finite simple graphs, and assume that for every $i \in \{1, 2\}$, the group $\text{Out}(G_{\Gamma_i})$ is finite.*

Then Γ_1^e and Γ_2^e are isomorphic if and only if Γ_1 and Γ_2 are isomorphic.

In particular, Theorem 1 from the introduction follows from Theorem 2.

1.3. A LEMMA ABOUT TRANSVECTION-FREE RIGHT-ANGLED ARTIN GROUPS. — In the sequel of the paper, one way in which the transvection-freeness assumption will be used is through the following easy lemma.

LEMMA 1.6. — *Let $G = G_\Gamma$ be a transvection-free right-angled Artin group. Let Z be a cyclic parabolic subgroup of G , and let P be a nontrivial parabolic subgroup of G such that $Z \times Z^\perp \subseteq P \times P^\perp$.*

- (1) *If P is cyclic, then $Z = P$.*
- (2) *If P is noncyclic, then $P \cap Z^\perp$ is nonabelian.*

Proof. — Up to conjugation, we can assume P is standard, hence $P \times P^\perp$ is also standard. As $Z \times Z^\perp$ can be viewed as a parabolic subgroup of $P \times P^\perp$ (see Remark 1.3), up to conjugating $Z \times Z^\perp$ by an element in $P \times P^\perp$, we can assume without loss of generality that both Z and $Z \times Z^\perp$ are standard as such conjugation brings P to itself.

Let $\Gamma_P \subseteq \Gamma$ be the type of P . Then the type of P^\perp is $(\Gamma_P)^\perp$. Let v_Z be the vertex of Γ which is the type of Z . As $Z \times Z^\perp \subseteq P \times P^\perp$, we have $\text{st}(v_Z) \subseteq \Gamma_P \circ (\Gamma_P)^\perp$.

Let $\text{lk}(v_Z) = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_k$ be the de Rham decomposition of $\text{lk}(v_Z)$, which induces $Z^\perp = Q_1 \times \cdots \times Q_k$. The transvection-free condition implies that no Γ_i is a clique (in particular no Γ_i is reduced to one vertex), as otherwise the link of v_Z would be contained in the star of every vertex of Γ_i . Thus, each Q_i is nonabelian. As each Γ_i is irreducible, either $\Gamma_i \subseteq \Gamma_P$ or $\Gamma_i \subseteq (\Gamma_P)^\perp$.

If P is cyclic, then Γ_P is reduced to a vertex v_P , no Γ_i can be contained in Γ_P , and we deduce that $\text{lk}(v_Z) \subseteq (\Gamma_P)^\perp$. So $\text{lk}(v_Z)$ is contained in the star of v_P , and the transvection-free condition thus implies that $\{v_Z\} = \Gamma_P$. It follows that $Z = P$ as they are both standard, showing that the first assertion of the lemma holds.

If P is noncyclic, then $\text{lk}(v_Z) \not\subseteq (\Gamma_P)^\perp$, otherwise we have $\text{lk}(v_Z) \subseteq \text{st}(w)$ for every vertex $w \in \Gamma_P$. Thus at least one Γ_i is contained in Γ_P . Therefore $\Gamma_P \cap \text{lk}(v_Z)$ contains an irreducible graph with at least two vertices, and the second assertion of the lemma follows. \square

1.4. FULL SUPPORT SUBGROUPS

PROPOSITION 1.7. — *Let Γ be a finite simple graph whose de Rham decomposition has no clique factor. Then G_Γ contains a nonabelian free subgroup F such that no nontrivial element of F is contained in a proper parabolic subgroup of G_Γ .*

Proof. — We claim that if $\Gamma = \Gamma_1 \circ \Gamma_2$, then each parabolic subgroup P of G_Γ splits as a product $P_1 \times P_2$ where $P_i = P \cap G_{\Gamma_i}$ is a parabolic subgroup of G_{Γ_i} . This is clearly true if P is standard, otherwise there is $g = g_1g_2 \in G_\Gamma$ ($g_i \in G_{\Gamma_i}$) such that gPg^{-1} is standard. On the other hand, $gG_{\Gamma_i}g^{-1} = g_iG_{\Gamma_i}g_i^{-1} = G_{\Gamma_i}$. Thus the claim follows. This claim implies that it suffices to prove the lemma when Γ is not a join.

In what follows, we are equipping \tilde{S}_Γ with the standard CAT(0) metric. We will use the following simple observation: if a nontrivial element g belongs to a parabolic subgroup of type Γ' , then it has an axis (with respect to the action $G_\Gamma \curvearrowright \tilde{S}_\Gamma$) contained in a standard subcomplex of type Γ' , hence every axis of g is contained in a finite neighborhood of this standard subcomplex (see [BH99, Chap. II.6] for basic properties of axes).

Let X be the wedge of two circles C_1 and C_2 , with the wedge point denoted by x_0 . We will choose two words W_1 and W_2 in the standard generating set $V\Gamma$ such that

- (1) each W_i uses all the generators of G_Γ ;
- (2) the map $\varphi : X \rightarrow S_\Gamma$ defined by mapping x_0 to the base vertex of S_Γ and mapping each C_i to the edge path in the 1-skeleton of S_Γ corresponding to the word W_i is a local isometry (C_i is metricized so that its length is equal to the word length of W_i).

Then $\varphi_*(\pi_1 X)$ is the free subgroup satisfying our requirement, as (2) implies that any lift $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{S}_\Gamma$ to the universal covers maps an axis of a nontrivial element $g \in \pi_1 X$ to an axis ℓ of $\varphi_*(g)$, and (1) implies that ℓ is not contained in any finite neighborhood of any proper standard subcomplex.

Let Γ^c be the complement graph of Γ (i.e., these two graphs have the same vertex set, two vertices in Γ^c are adjacent if they are not adjacent in Γ). As Γ is not a join, Γ^c is connected. Let u_0, v_0 be two adjacent vertices in Γ^c . For each vertex $v \in \Gamma^c$, let p_v an edge path traveling from v_0 to v then back to v_0 in Γ^c . Listing consecutive vertices in p_v leads to a word W_v starting with v_0 and ending with v_0 . Let W be the product (in any order) of all W_v with v ranging over all vertices of Γ^c . Then the words $W_1 = v_0 W u_0$ and $W_2 = v_0^{-1} u_0 W u_0^{-1}$ satisfy the above two conditions. \square

REMARK 1.8. — Proposition 1.7 can also be proved by showing that, when Γ does not decompose nontrivially as a join, the group G_Γ acts nonelementarily on the graph of parabolic subgroups of G_Γ – having one vertex per proper parabolic subgroup

of G_Γ , where two such subgroups are joined by an edge whenever they have nontrivial intersection – which turns out to be hyperbolic. We decided to provide an elementary proof that does not rely on a hyperbolicity statement.

2. THE ROLLER BOUNDARY OF THE SALVETTI COMPLEX

The goal of the present section is to prove Theorem 7 from the introduction (Theorem 2.1 below). We start with a short review on Roller boundaries of CAT(0) cube complexes.

2.1. BACKGROUND ON THE ROLLER BOUNDARY

General background. — The Roller boundary, implicit in the work of Roller [Rol98] and explicitly introduced in [BCG⁺09], gives a way of compactifying any CAT(0) cube complex. We refer to [Sag12] for background on CAT(0) cube complexes, including a discussion on hyperplanes and halfspaces. We now review the definition and a few facts about the Roller boundary.

Let X be a CAT(0) cube complex. Given a subset $Z \subseteq X$, we let $\mathcal{H}(Z)$ be the set of all hyperplanes of X that intersect Z nontrivially. Let \mathfrak{H} be the set of all halfspaces of X (the boundary hyperplane of a halfspace h will be denoted ∂h). There is an embedding of the vertex set $V(X)$ into $\{0, 1\}^{\mathfrak{H}}$ (equipped with the product topology), sending a vertex v to the map $\mathfrak{H} \rightarrow \{0, 1\}$ that sends a halfspace h to 1 if and only if $v \in h$. The closure of the image of this embedding yields a compactification of $V(X)$. The *Roller boundary* of X , which we denote by $\partial_R X$, is the complement of $V(X)$ in this compactification: it is compact whenever X is locally compact. A point $\xi \in \partial_R X$ is thus a map $\mathfrak{H} \rightarrow \{0, 1\}$, and we let $U_\xi \subseteq \mathfrak{H}$ be the set of all halfspaces sent to 1 under this map. Let $Y \subseteq X$ be a convex subcomplex. Then there is a continuous embedding $\partial_R Y \rightarrow \partial_R X$ such that a point $\xi \in \partial_R X$ lies in the image of this embedding if and only if it corresponds to a map $\mathfrak{H} \rightarrow \{0, 1\}$ that sends every halfspace containing Y to 1. Thus we will identify $\partial_R Y$ with a closed subset of $\partial_R X$.

When X has countably many hyperplanes (e.g. when $X = \tilde{S}_\Gamma$ is the universal cover of the Salvetti complex associated to a finite simple graph Γ), the space $\{0, 1\}^{\mathfrak{H}}$ is metrizable, so $\partial_R X$ is metrizable.

Combinatorial geodesic rays and the Roller boundary. — We denote by $X^{(1)}$ the 1-skeleton of X , equipped with the path metric. Geodesic rays in $X^{(1)}$ are called *combinatorial geodesic rays*. Let $x \in X$ be a vertex, and let r be a combinatorial geodesic ray in $X^{(1)}$ originating at x . For every hyperplane \mathfrak{h} , the ray r selects exactly one of the two halfspaces complementary to \mathfrak{h} , by considering the halfspace which virtually contains r (i.e., contains r up to a finite segment). Thus r defines a point in $\partial_R X$. Two combinatorial geodesic rays are *equivalent* if they cross the same set of hyperplanes. There is a 1-1 correspondence between equivalence classes of combinatorial geodesic rays based at a given point $x \in X$, and points in $\partial_R X$ (see e.g. [Gen20, §A.2]).

Now suppose that two combinatorial geodesic rays r and r' (potentially with different origins x and x') represent the same point of $\partial_R X$. By definition of $\partial_R X$,

a halfspace of X virtually contains r if and only if it virtually contains r' . Also, every hyperplane \mathfrak{h} contained in the symmetric difference $\mathcal{H}(r)\Delta\mathcal{H}(r')$ separates x from x' . In particular $\mathcal{H}(r)\Delta\mathcal{H}(r')$ is finite.

Regular points in the Roller boundary: — Two hyperplanes \mathfrak{h}_1 and \mathfrak{h}_2 of X are *strongly separated* [BC12, Def. 2.1] if there does not exist any hyperplane \mathfrak{h} such that for every $i \in \{1, 2\}$, one has $\mathfrak{h} \cap \mathfrak{h}_i \neq \emptyset$. The notion of a regular point of the Roller boundary, introduced by Fernós in [Fer18, Def. 7.3], is defined as follows: a point $\xi \in \partial_R X$ is *regular* if given any two halfspaces $h_1, h_2 \in U_\xi$, there exists a halfspace $k \in U_\xi$ such that $k \subseteq h_1 \cap h_2$ and for every $i \in \{1, 2\}$, the hyperplanes ∂k and ∂h_i are strongly separated. We denote by $\partial_{\text{reg}} X$ the subspace of $\partial_R X$ made of regular points.

2.2. NON-REGULAR POINTS IN THE ROLLER BOUNDARY OF THE SALVETTI COMPLEX

Let Γ be a finite simple graph. We denote by \mathbb{S}_J the set of all standard subcomplexes of \tilde{S}_Γ that are contained in some join standard subcomplex of \tilde{S}_Γ , i.e., a standard subcomplex whose type admits a nontrivial join decomposition. The goal of the present section is to prove the following theorem.

THEOREM 2.1. — *Let Γ be a finite simple graph. Then $\partial_{\text{reg}} \tilde{S}_\Gamma$ is a Borel subset of $\partial_R \tilde{S}_\Gamma$, and there exists a unique Borel G_Γ -equivariant map*

$$\Phi : \partial_R \tilde{S}_\Gamma \setminus \partial_{\text{reg}} \tilde{S}_\Gamma \longrightarrow \mathbb{S}_J$$

such that for every nonregular point $\xi \in \partial_R \tilde{S}_\Gamma$ and every combinatorial geodesic ray r representing ξ , the subset $\Phi(\xi)$ is the smallest standard subcomplex of \tilde{S}_Γ that virtually contains r .

Theorem 2.1 is a consequence of Lemmas 2.2, 2.4 and 2.5 below.

LEMMA 2.2. — *Let Γ be a finite simple graph. For every $\xi \in \partial_R \tilde{S}_\Gamma$, there exists a unique standard subcomplex $Y \subseteq \tilde{S}_\Gamma$ such that for every combinatorial geodesic ray r representing ξ , the subcomplex Y is the smallest standard subcomplex that virtually contains r .*

Proof. — Recall that the intersection of two standard subcomplexes is again a standard subcomplex and that there is a uniform bound on the length of a strictly descending chain of standard subcomplexes. Therefore, for every combinatorial geodesic ray r , there is a unique minimal standard subcomplex $Y_r \subseteq \tilde{S}_\Gamma$ which virtually contains r .

We are thus left with showing that if r and r' are combinatorial geodesic rays representing the same point of $\partial_R \tilde{S}_\Gamma$ (possibly with different base points), then $Y_r = Y_{r'}$. In this case, as observed in Section 2.1,

- (1) a halfspace of \tilde{S}_Γ virtually contains r if and only if it virtually contains r' ;
- (2) the symmetric difference of $\mathcal{H}(r)$ and $\mathcal{H}(r')$ is finite.

Let Γ_r be the type of Y_r . As the 1-skeleton of \tilde{S}_Γ is the Cayley graph of G_Γ , we label edges of \tilde{S}_Γ by vertices of Γ . Then there exists a subray r_1 of r such that the collection $L(r_2)$ of labels of edges of any further subray r_2 of r_1 satisfies $L(r_2) = V\Gamma_r$.

We define r'_1 and $\Gamma_{r'}$ similarly. By (2), for any subray r'_3 of r'_1 , there is a subray r_3 of r_1 with $L(r_3) \subseteq L(r'_3)$. Thus $\Gamma_r \subseteq \Gamma_{r'}$. Similarly $\Gamma_{r'} \subseteq \Gamma_r$. Therefore $\Gamma_r = \Gamma_{r'}$, and in fact $Y_r = Y_{r'}$: indeed (1) implies that $Y_r \cap Y_{r'} \neq \emptyset$, as otherwise Y_r and $Y_{r'}$ would be separated by a hyperplane (see [HW08, Cor. 13.10]). This concludes our proof. \square

Lemma 2.2 yields a well-defined map $\Phi : \partial_R \tilde{S}_\Gamma \rightarrow \mathbb{S}$, where \mathbb{S} is the set of all standard subcomplexes of \tilde{S}_Γ (including \tilde{S}_Γ itself). Moreover, the map Φ is equivariant with respect to the natural actions of G_Γ .

REMARK 2.3. — The map Φ can be reinterpreted as follows: Lemma 2.2 shows that given any two standard subcomplexes $Y_1, Y_2 \subseteq \tilde{S}_\Gamma$, one has $\partial_R Y_1 \cap \partial_R Y_2 = \partial_R(Y_1 \cap Y_2)$ (viewed as subsets of $\partial_R \tilde{S}_\Gamma$), and the map Φ then sends a point $\xi \in \partial_R \tilde{S}_\Gamma$ to the smallest standard subcomplex Y such that $\xi \in \partial_R Y$.

We equip the countable set \mathbb{S} with the discrete topology.

LEMMA 2.4. — *The map Φ is Borel.*

Proof. — Let $Y \in \mathbb{S}$ be a standard subcomplex, and let $x \in \tilde{S}_\Gamma$. Then a point $\xi \in \partial_R \tilde{S}_\Gamma$ belongs to $\Phi^{-1}(Y)$ if and only if it is represented by a combinatorial geodesic ray based at x which is virtually contained in Y , but not virtually contained in any proper standard subcomplex of Y . Thus $\Phi^{-1}(Y) = \partial_R Y \setminus (\bigcup_{Z \in \mathbb{S}, Z \subsetneq Y} \partial_R Z)$. The lemma follows as the subspaces $\partial_R Z$ and $\partial_R Y$ are closed subsets of $\partial_R \tilde{S}_\Gamma$. \square

Now we recall some standard facts to prepare for the next lemma. As $\tilde{S}_\Gamma^{(1)}$ is the Cayley graph of G_Γ with respect to its standard generating set, we label each edge of \tilde{S}_Γ by a vertex of Γ . Recall that an edge is *dual* to a hyperplane if this hyperplane intersects the edge in its midpoint. Note that if two edges are dual to the same hyperplane, then they have the same label. Hence each hyperplane has a well-defined label. Note that two hyperplanes having the same label are either equal or disjoint. Let h be a hyperplane in \tilde{S}_Γ . Then the smallest subcomplex of \tilde{S}_Γ containing h splits as a product $h \times [0, 1]$. Note that both $h \times \{0\}$ and $h \times \{1\}$ are standard subcomplexes of type $\text{lk}(v)$, where $v \in \Gamma$ is the label of h . Thus it is natural to consider *standard subcomplexes* of h and their types, which correspond to standard subcomplexes of $h \times \{0\}$ (or $h \times \{1\}$).

Let X_1 and X_2 be standard subcomplexes of \tilde{S}_Γ . It is a standard fact that one can find a standard subcomplex $B \subseteq X_1$ such that $\mathcal{H}(B) = \mathcal{H}(X_1) \cap \mathcal{H}(X_2)$. Actually, we can take B to be the subset of X_1 made of all points whose distance to X_2 is equal to $d(X_1, X_2)$. Then B is a standard subcomplex of X_1 (see [Hua17a, Lem. 3.1]) and $\mathcal{H}(B) = \mathcal{H}(X_1) \cap \mathcal{H}(X_2)$ (see e.g. [Hua17b, Lem. 2.14]). By the discussion in the previous paragraph, the fact still holds if we require X_1 and X_2 to be hyperplanes.

LEMMA 2.5. — *Let Γ be a finite simple graph. Let r be a combinatorial geodesic ray in \tilde{S}_Γ . Then r represents a point in $\partial_{\text{reg}} \tilde{S}_\Gamma$ if and only if r is not virtually contained in a join standard subcomplex of \tilde{S}_Γ .*

Proof. — First we show that if r is virtually contained in a join standard subcomplex X representing $\xi \in \partial_R \tilde{S}_\Gamma$, then ξ is not regular. To see this, let X_1 and X_2 be two unbounded standard subcomplexes of \tilde{S}_Γ such that X splits as $X = X_1 \times X_2$. We assume without loss of generality that $r \subset X$, and the image of r under the natural projection $X \rightarrow X_1$ is unbounded. Take halfspaces $h_1, h_2 \in U_\xi$ such that $\partial h_i \in \mathcal{H}(X_1) \cap \mathcal{H}(r)$. By definition of a regular point, it suffices to show that if k is a halfspace with $k \in U_\xi$ and $k \subset h_1 \cap h_2$, then ∂k and ∂h_i are not strongly separated for any $i \in \{1, 2\}$. Note that $\partial k \cap X \neq \emptyset$, as otherwise we will have $r \subset k$, and therefore $r \subset h_1$, contradicting that $\partial h_1 \in \mathcal{H}(r)$. As $\mathcal{H}(X) = \mathcal{H}(X_1) \sqcup \mathcal{H}(X_2)$, we have $\partial k \in \mathcal{H}(X_1)$ or $\partial k \in \mathcal{H}(X_2)$. The latter is not possible as it will imply $\partial k \cap \partial h_1 \neq \emptyset$, contradicting $k \subset h_1$. Now $\partial k \in \mathcal{H}(X_1)$ implies that ∂k and ∂h_1 cannot be strongly separated, because any hyperplane in $\mathcal{H}(X_2)$ will intersect both ∂k and ∂h_1 .

Now we suppose that r does not represent a regular point. Let $\{\mathfrak{h}_i\}_{i \in \mathbb{N}}$ be an infinite collection of hyperplanes crossed by r such that for every $i \in \mathbb{N}$, the hyperplane \mathfrak{h}_i separates \mathfrak{h}_{i-1} from \mathfrak{h}_{i+1} (such a collection can be found by looking at hyperplanes with the same label, though this is a general fact for geodesic rays in finite dimensional CAT(0) cube complexes). By [Fer18, Prop. 7.5], up to passing to a subcollection, we can assume that for all $i, j \in \mathbb{N}$, the hyperplanes \mathfrak{h}_i and \mathfrak{h}_j are not strongly separated.

For every $i \in \mathbb{N}$, let $B_i \subseteq \mathfrak{h}_1$ be a standard subcomplex of \mathfrak{h}_1 with $\mathcal{H}(B_i) = \mathcal{H}(\mathfrak{h}_1) \cap \mathcal{H}(\mathfrak{h}_i)$. For every $i_1 < i_2 < i_3$, every hyperplane that intersects both \mathfrak{h}_{i_1} and \mathfrak{h}_{i_3} also intersects \mathfrak{h}_{i_2} . This implies that for every $j \leq i$, one has $\mathcal{H}(B_i) \subseteq \mathcal{H}(B_j)$. Thus the type Γ_i of B_i is contained in Γ_j , with $\Gamma_i = \Gamma_j$ if and only if $\mathcal{H}(B_i) = \mathcal{H}(B_j)$. Thus there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, one has $\Gamma_i = \Gamma_{i_0}$. We define $A_1 = B_{i_0}$. Then $\mathcal{H}(A_1) = \bigcap_{j=1}^\infty \mathcal{H}(\mathfrak{h}_j)$. Moreover, as \mathfrak{h}_1 and \mathfrak{h}_{i_0} are not strongly separated, we have $\Gamma_{i_0} \neq \emptyset$. Similarly, for each $i \geq 1$ we define a standard subcomplex $A_i \subseteq \mathfrak{h}_i$ such that $\mathcal{H}(A_i) = \bigcap_{k=i}^\infty \mathcal{H}(\mathfrak{h}_k)$. Thus $\mathcal{H}(A_j) \subseteq \mathcal{H}(A_i)$ whenever $j \leq i$. By a similar argument as before (the type of every A_i is a subgraph of Γ), up to passing to an infinite subset of $\{\mathfrak{h}_i\}_{i \geq 1}$, we can (and will) assume that $\mathcal{H}(A_i) = \mathcal{H}(A_j)$ for every $i \neq j$. Let Λ be the type of A_1 , and let Y be the standard subcomplex of \tilde{S}_Γ with type $\Lambda \circ \Lambda^\perp$ that contains A_1 . Then for every $i \geq 1$, we have $A_i \subseteq Y$. We already know $\Lambda \neq \emptyset$ from the previous discussion, and $\Lambda^\perp \neq \emptyset$ as the label of any edge dual to the hyperplane containing A_1 belongs to Λ^\perp . We pass to a further infinite subset of $\{\mathfrak{h}_i\}_{i=1}^\infty$ so that $\mathcal{H}(A_1) = \mathcal{H}(\mathfrak{h}_1) \cap \mathcal{H}(\mathfrak{h}_2)$.

Let r' be a combinatorial subray of r with $r' \cap \mathfrak{h}_2 = \emptyset$. We claim that $\mathcal{H}(r') \subseteq \mathcal{H}(Y)$. This claim will imply that r' is contained in a standard subcomplex of the same type as Y , thus concluding the proof of the lemma.

We are thus left proving the above claim. Suppose towards a contradiction that there exists $\mathfrak{h}' \in \mathcal{H}(r') \setminus \mathcal{H}(Y)$. If \mathfrak{h}' intersects infinitely many members of $\{\mathfrak{h}_i\}_{i=1}^\infty$, then there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{h}' \in \bigcap_{j=j_0}^\infty \mathcal{H}(\mathfrak{h}_j) = \mathcal{H}(A_{j_0}) = \mathcal{H}(A_1) \subseteq \mathcal{H}(Y)$, a contradiction. Thus \mathfrak{h}' intersects at most finitely many members from $\{\mathfrak{h}_i\}_{i=1}^\infty$. Let e' (resp. e_i) be the edge of r dual to \mathfrak{h}' (resp. \mathfrak{h}_i). Then there exists $\ell_0 \in \mathbb{N}$ such that e_{ℓ_0} occurs after e' in r' and $\mathfrak{h}_{\ell_0} \cap \mathfrak{h}' = \emptyset$. We will now argue separately depending on whether \mathfrak{h}' intersects \mathfrak{h}_1 or not, and reach a contradiction in both cases.

We first assume that $\mathfrak{h}' \cap \mathfrak{h}_1 \neq \emptyset$. As r' intersects \mathfrak{h}' but not \mathfrak{h}_2 , it follows that \mathfrak{h}' also intersects \mathfrak{h}_2 . Therefore \mathfrak{h}' has nontrivial intersection with both \mathfrak{h}_1 and \mathfrak{h}_2 , which implies that $\mathfrak{h}' \in \mathcal{H}(\mathfrak{h}_1) \cap \mathcal{H}(\mathfrak{h}_2) = \mathcal{H}(A_1) \subseteq \mathcal{H}(Y)$. This is a contradiction.

We now assume that $\mathfrak{h}' \cap \mathfrak{h}_1 = \emptyset$. As e_1 occurs before e' in r , we know that \mathfrak{h}' separates \mathfrak{h}_1 from \mathfrak{h}_{ℓ_0} . As \mathfrak{h}_1 and \mathfrak{h}_{ℓ_0} belong to $\mathcal{H}(Y)$, we have $\mathfrak{h}' \in \mathcal{H}(Y)$ by convexity of Y , a contradiction. This concludes our proof. \square

We denote by \mathbb{P}_J the set of all parabolic subgroups of G_Γ which are contained in a parabolic subgroup whose type admits a nontrivial join decomposition. Combining the map Φ given by Theorem 2.1 with the G_Γ -equivariant map $\mathbb{S}_J \rightarrow \mathbb{P}_J$ sending a standard subcomplex to its stabilizer, we reach the following corollary, which is the form in which Theorem 2.1 will be used in the sequel.

COROLLARY 2.6. — *Let Γ be a finite simple graph. There exists a Borel G_Γ -equivariant map $\partial_R \tilde{\mathcal{S}}_\Gamma \setminus \partial_{\text{reg}} \tilde{\mathcal{S}}_\Gamma \rightarrow \mathbb{P}_J$.* \square

REMARK 2.7. — It is natural to ask how much of the discussion in this section applies to more general CAT(0) cube complexes. As a starting point, we consider the following geometric question. Let X be a CAT(0) cube complex, possibly with a geometric group action. Is it possible to split $\partial_R X$ into a regular part K_1 , corresponding to “hyperbolic directions” in X , and a non-regular part $K_2 = \partial_R X \setminus K_1$, corresponding to “product regions” of X in the sense that each point in K_2 is contained in the Roller boundary of a subcomplex which splits as a product of two unbounded cube complexes?

Note that in general we cannot take K_1 to be $\partial_{\text{reg}} X$. Indeed, consider $X = [0, 1] \times \mathbb{R}$ with the usual cubical structure. Then every point of the Roller boundary is non-regular, but no subcomplex in X splits as a product of two unbounded cube complexes. To remedy this, we define a point $\xi \in \partial_R X$ to be *weakly regular* if there exist $N \geq 0$ and an infinite descending chain of halfspaces $\{h_i\}_{i=1}^\infty \subset U_\xi$ such that for any $i \neq j$, there are at most N hyperplanes of X intersecting both ∂h_i and ∂h_j . Note that the definition of a regular point corresponds to the case where $N = 0$ (see [Fer18, Prop. 7.5]). In the above example $X = [0, 1] \times \mathbb{R}$, every point of the Roller boundary is now weakly regular. And we can take K_1 to be the collection of weakly regular points.

A second issue is that a non weakly regular point ξ might be contained in the Roller boundary of a cubical staircase inside X (see e.g. [HS20, Fig. 1]), but if this staircase is not contained in a quarter plane, then ξ is not contained in the Roller boundary of a product subcomplex. Work of Hagen and Susse [HS20] gives conditions on when one can complete the staircases in X to quarter planes, i.e., [HS20, Th. A] implies the existence of a *factor system* as defined in [HS20, §1.2], and factor systems lead to completions of staircases as discussed in [HS20, §7]. In particular [HS20, Th. A] applies to universal covers of Salvetti complexes and many other cube complexes.

Given the above connection, one can prove Lemma 2.5 using results from [HS20]. However, we chose to write out a self-contained proof for the convenience of the reader, as in the setting of Lemma 2.5 the proof is a short argument.

3. MAIN MEASURE EQUIVALENCE CLASSIFICATION THEOREM

In this section, we prove our main theorems concerning the measure equivalence classification of right-angled Artin groups with finite outer automorphism groups (Theorems 1 and 2).

3.1. MEASURED GROUPOIDS

General definitions. — General background about measured groupoids can be found in [AD13, §2.1] or [Kid09], for instance. By definition, a Borel groupoid \mathcal{G} is a standard Borel space which comes equipped with a base space Y (which is also a standard Borel space, and every element $g \in \mathcal{G}$ can be thought of as an arrow with a source $s(g)$ and a range $r(g)$ in Y , the maps s and r being Borel). A Borel groupoid \mathcal{G} also comes equipped by definition with a composition law, an inverse map (all Borel) and a neutral element e_y for every $y \in Y$, satisfying the axioms of groupoids (see e.g. [Kid09, Def. 2.10]). All Borel groupoids considered in the present paper are *discrete*, i.e., there are at most countably many elements with a given source or range.

As in [Kid09, Def. 2.13], we define a *measured groupoid* as a Borel groupoid \mathcal{G} for which the base space Y has a quasi-invariant σ -finite measure – which will always be a probability measure in the present paper. There is a natural notion of measured subgroupoid of \mathcal{G} , as well as a notion of restriction $\mathcal{G}|_U$ of a measured groupoid \mathcal{G} to a Borel subset U of the base space Y , by only considering elements of \mathcal{G} whose source and range both belong to U .

Stably trivial groupoids, groupoids of infinite type. — A measured groupoid \mathcal{G} over a base space Y is *trivial* if $\mathcal{G} = \{e_y | y \in Y\}$. It is *stably trivial* if there exist a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, the groupoid $\mathcal{G}|_{Y_i}$ is trivial. It is *of infinite type* if for every Borel subset $U \subseteq Y$ of positive measure, and a.e. $y \in U$, there are infinitely many elements of $\mathcal{G}|_U$ with source y . Notice that any restriction of a subgroupoid of infinite type is again of infinite type.

Cocycles. — Given a measured groupoid \mathcal{G} over a base space Y and a countable group G , a *strict cocycle* $\rho : \mathcal{G} \rightarrow G$ is a Borel map such that for all $g_1, g_2 \in \mathcal{G}$ with $s(g_1) = r(g_2)$ (where s and r denote the source and range maps, respectively), one has $\rho(g_1 g_2) = \rho(g_1) \rho(g_2)$. Here the word *strict* refers to the fact that this relation is assumed to hold everywhere, not just on a conull Borel subset. The *kernel* of a strict cocycle ρ is the subgroupoid of \mathcal{G} made of all elements g such that $\rho(g) = e$.

Let \mathcal{G} be a measured groupoid over a base space Y , and let G be a countable group. We say that a strict cocycle $\mathcal{G} \rightarrow G$ is *action-type* if it has trivial kernel and for every infinite subgroup $H \subseteq G$, the subgroupoid $\rho^{-1}(H)$ is of infinite type. An important example, which motivates the terminology, is that when \mathcal{G} is the measured groupoid naturally associated to a measure-preserving action of a countable group G on a finite measure space Y , the cocycle ρ given by the action is action-type (see [Kid09, Prop. 2.26] for details, which relies on work of Adams [Ada94]). We emphasize that

it is crucial in this example that the G -action on Y preserves a finite measure, see Remark 3.3 below.

Invariant maps and stabilizers. — When G is acting on a standard Borel space Δ by Borel automorphisms, and $\rho : \mathcal{G} \rightarrow G$ is a strict cocycle, we say that a Borel map $\phi : Y \rightarrow \Delta$ is (\mathcal{G}, ρ) -invariant if there exists a conull Borel subset $Y^* \subseteq Y$ such that for all $g \in \mathcal{G}_{Y^*}$, one has $\phi(r(g)) = \rho(g)\phi(s(g))$. Also, for every $\delta \in \Delta$, we say that δ is (\mathcal{G}, ρ) -invariant if the constant map with value δ is (\mathcal{G}, ρ) -invariant. In other words, there exists a conull Borel subset $Y^* \subseteq Y$ such that $\rho(\mathcal{G}_{Y^*}) \subseteq \text{Stab}_G(\delta)$. The (\mathcal{G}, ρ) -stabilizer of δ is defined as the subgroupoid of \mathcal{G} made of all $g \in \mathcal{G}$ such that $\rho(g) \in \text{Stab}_G(\delta)$.

Stable containment and stable equivalence. — Let \mathcal{G} be a measured groupoid over a base space Y , and let $\mathcal{H}, \mathcal{H}'$ be two measured subgroupoids of \mathcal{G} . We say that \mathcal{H} is *stably contained* in \mathcal{H}' if there exists a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets, such that for every $i \in I$, one has $\mathcal{H}|_{Y_i} \subseteq \mathcal{H}'|_{Y_i}$. We say that \mathcal{H} and \mathcal{H}' are *stably equivalent* if each of them is stably contained in the other – in other words, there exists a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, one has $\mathcal{H}|_{Y_i} = \mathcal{H}'|_{Y_i}$.

Normalization. — A notion of normality of a subgroupoid was proposed by Kida in [Kid08, Chap. 4, §6.1], building on work of Feldman, Sutherland and Zimmer [FSZ89], defined as follows. Given a measured groupoid \mathcal{G} over a base space Y , a measured subgroupoid $\mathcal{H} \subseteq \mathcal{G}$, and a Borel subset $B \subseteq \mathcal{G}$, one says that \mathcal{H} is *B-invariant* if there exists a conull Borel subset $Y^* \subseteq Y$ such that for every $g_1, g_2 \in B \cap \mathcal{G}_{Y^*}$ and every $h \in \mathcal{G}_{Y^*}$ satisfying $s(h) = s(g_1)$ and $r(h) = s(g_2)$, one has $h \in \mathcal{H}$ if and only if $g_2 h g_1^{-1} \in \mathcal{H}$. Given two measured subgroupoids $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{G}$, one says that \mathcal{H} is *normalized* by \mathcal{H}' if there exists a covering of \mathcal{H}' by countably many Borel subsets $B_n \subseteq \mathcal{G}$ so that for every $n \in \mathbb{N}$, the groupoid \mathcal{H} is B_n -invariant. One says that \mathcal{H} is *stably normalized* by \mathcal{H}' if there exists a partition $Y = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, the groupoid $\mathcal{H}|_{Y_i}$ is normalized by $\mathcal{H}'|_{Y_i}$.

An important example is that if \mathcal{G} is equipped with a strict cocycle ρ towards a countable group G , and if $H_1, H_2 \subseteq G$ are two subgroups such that H_1 is normalized by H_2 , then $\rho^{-1}(H_1)$ is normalized by $\rho^{-1}(H_2)$.

Using a theorem of Lusin and Novikov (see [Kec95, Th. 18.10]), the subsets B_n that arise in the definition of \mathcal{H} being normalized by \mathcal{H}' can always be chosen so that the restrictions of the source and range maps to each B_n are Borel isomorphisms to Borel subsets of Y .

Amenability. — We will adopt the same definition of amenability of a measured groupoid as in [Kid09]. The crucial way in which amenability is used is the following:

if \mathcal{G} is a measured groupoid over a base space Y with a cocycle ρ towards a countable group G , and if G acts by homeomorphisms on a compact metrizable space K , then there exists a (\mathcal{G}, ρ) -invariant Borel map $Y \rightarrow \text{Prob}(K)$, where $\text{Prob}(K)$ denotes the space of Borel probability measures on K equipped with the weak-* topology (coming from the duality with the space of real-valued continuous functions on K), see [Kid09, Prop. 4.14]. We say that a groupoid \mathcal{G} over a base space Y is *everywhere nonamenable* if for every Borel subset $U \subseteq Y$ of positive measure, the groupoid $\mathcal{G}|_U$ is nonamenable. We will need a few invariance properties: every subgroupoid of an amenable groupoid is amenable [Kid09, Th. 4.16] (and every restriction of an amenable groupoid is amenable); also, if $Y^* \subseteq Y$ is a conull Borel subset of the base space and $Y^* = \bigsqcup_{i \in I} Y_i$ is a partition into at most countably many Borel subsets, and if for every $i \in I$, the groupoid $\mathcal{G}|_{Y_i}$ is amenable, then \mathcal{G} is amenable.

REMARK 3.1. — Given a separable metrizable topological space X , the set $\text{Prob}(X)$ of Borel probability measures on X is endowed with the topology generated by the maps $\mu \mapsto \int f d\mu$, where f varies over the set of all real-valued continuous bounded functions on X . When X is compact, the Riesz–Markov–Kakutani theorem identifies $\text{Prob}(X)$ with a subspace of the unit ball of the dual of $C(X; \mathbb{R})$ (the space of real-valued continuous functions on X), and under this identification the above topology coincides with the weak-* topology. When X is countable, this is the same as the topology of pointwise convergence. In our applications, the space X will always be either a subspace of a compact metrizable space or a countable set with discrete topology. By [Kec95, Th. 17.24], the Borel σ -algebra on $\text{Prob}(X)$ is the σ -algebra generated by the maps $\mu \mapsto \mu(A)$, where A varies over the Borel subsets of X . As a consequence, every Borel map $f : X \rightarrow Y$ induces a Borel map $\text{Prob}(X) \rightarrow \text{Prob}(Y)$.

3.2. MEASURED GROUPOIDS WITH COCYCLES TOWARDS FREE GROUPS. — Throughout the paper, we will extensively use the following observation of Kida.

LEMMA 3.2 (Kida [Kid10, Lem. 3.20]). — *Let G be a countable group, and let \mathcal{G} be a measured groupoid over a standard finite measure space Y .*

If there exists a strict action-type cocycle $\mathcal{G} \rightarrow G$, and if G contains a nonabelian free subgroup, then \mathcal{G} is everywhere nonamenable.

REMARK 3.3. — In [Kid10, Lem. 3.20], Kida assumes that the groupoid \mathcal{G} preserves a probability measure. In his proof, this assumption is used to ensure that for every infinite cyclic subgroup $A \subseteq G$ and every Borel subset $U \subseteq Y$ of positive measure, the groupoid $\rho^{-1}(A)|_U$ is of infinite type. For us, this is automatically ensured by our assumption that the cocycle $\mathcal{G} \rightarrow G$ is action-type.

We warn the reader about the following example. Let $\partial_\infty F_N \rtimes F_N$ be the measured groupoid coming from the action of a non-amenable free group on its boundary (which quasi-preserves a probability measure, but does not preserve any). Then $\partial_\infty F_N \rtimes F_N$ is amenable: this follows from the universal amenability of the action of a free group on its boundary. But the natural cocycle $\partial_\infty F_N \rtimes F_N \rightarrow F_N$ is not of action type.

We mention that on the other hand, if there exists a strict cocycle with trivial kernel from \mathcal{G} to an amenable group, then \mathcal{G} is amenable (this follows from [Kid08, Prop. 4.33], for instance). We will need the following lemma, whose proof relies on techniques introduced by Adams in [Ada94].

LEMMA 3.4. — *Let A_1, A_2, A_3 be finitely generated free groups, and let \mathcal{G} be a measured groupoid over a base space Y , equipped with a strict cocycle $\rho : \mathcal{G} \rightarrow A_1 * A_2 * A_3$ with trivial kernel. Let \mathcal{A} be an amenable measured subgroupoid of \mathcal{G} .*

Then there exist $i \in \{1, 2, 3\}$ and a Borel subset $U \subseteq Y$ of positive measure such that $(\mathcal{A} \cap \rho^{-1}(A_i))|_U$ is trivial.

In the following proof, we will use Remark 3.1 implicitly when we check Borel measurability of maps.

Proof. — For every $i \in \{1, 2, 3\}$, we let $\mathcal{A}_i = \mathcal{A} \cap \rho^{-1}(A_i)$. Given $i \neq j$, we let $\mathcal{A}_{ij} = \mathcal{A} \cap \rho^{-1}(A_i * A_j)$. Assume towards a contradiction that for every Borel subset $U \subseteq Y$ of positive measure, none of the subgroupoids $(\mathcal{A}_1)|_U, (\mathcal{A}_2)|_U, (\mathcal{A}_3)|_U$ is trivial (in particular, neither are the restrictions of \mathcal{A} and of the \mathcal{A}_{ij} to U). Let $F = A_1 * A_2 * A_3$. For every $i \in \{1, 2, 3\}$, let B_i be a free basis of A_i , and let T be the Cayley graph of F with respect to the free basis $B_1 \cup B_2 \cup B_3$, on which F naturally acts by isometries. As $\partial_\infty T$ is compact and metrizable and \mathcal{A} is amenable, there exists an (\mathcal{A}, ρ) -invariant Borel map $\theta : Y \rightarrow \text{Prob}(\partial_\infty T)$.

We claim that for every such map θ , and almost every $y \in Y$, the support of $\theta(y)$ has cardinality at most 2. Indeed, otherwise, there exists a Borel subset $U \subseteq Y$ of positive measure such that for every $y \in U$, the support of the probability measure $\theta(y)$ has cardinality at least 3. Thus, the probability measure $\theta(y) \otimes \theta(y) \otimes \theta(y)$ on $(\partial_\infty T)^3$ gives positive measure to the subset $(\partial_\infty T)^{(3)}$ made of pairwise distinct triples. By restricting and renormalizing, we thus derive an $(\mathcal{A}|_U, \rho)$ -invariant Borel map $U \rightarrow \text{Prob}((\partial_\infty T)^{(3)})$. Denoting by $V(T)$ the vertex set of T , there is a natural F -equivariant barycenter map $(\partial_\infty T)^{(3)} \rightarrow V(T)$. This map is Borel (in fact continuous) with respect to the natural topology on $(\partial_\infty T)^{(3)}$. This yields an $(\mathcal{A}|_U, \rho)$ -invariant Borel map $U \rightarrow \text{Prob}(V(T))$. Let $\mathcal{F}(V(T))$ denote the countable set of all nonempty finite subsets of $V(T)$. As $V(T)$ is countable, there is an F -equivariant map $\text{Prob}(V(T)) \rightarrow \mathcal{F}(V(T))$, sending a probability measure ν to the finite set made of all elements of maximal ν -measure. One readily checks that this map is Borel. We deduce that there is an $(\mathcal{A}|_U, \rho)$ -invariant Borel map $U \rightarrow \mathcal{F}(V(T))$. Restricting to a positive measure Borel subset $U' \subseteq U$ where this map takes a constant value, we deduce (up to replacing U' by a conull Borel subset) that $\rho(\mathcal{A}|_{U'}) = \{1\}$. As ρ has trivial kernel, it follows that $\mathcal{A}|_{U'}$ is trivial, a contradiction.

Let $\mathcal{P}_{\leq 2}(\partial_\infty T)$ (resp. $\mathcal{P}_{=2}(\partial_\infty T)$) denote the set of all nonempty subsets of $\partial_\infty T$ of cardinality at most 2 (resp. equal to 2). The above argument shows that there exists an (\mathcal{A}, ρ) -invariant Borel map $\theta : Y \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$ (obtained after mapping a probability measure to its support). In particular, this map is (\mathcal{A}_i, ρ) -invariant for

every $i \in \{1, 2, 3\}$. Notice also that given two such maps, the above argument also ensures that their union should still take its values (essentially) in $\mathcal{P}_{\leq 2}(\partial_\infty T)$.

For every $i \in \{1, 2, 3\}$, we let T_i be the Cayley tree of A_i with respect to the free basis B_i . For $i \neq j$, we let T_{ij} be the Cayley tree of $A_i * A_j$ with respect to the free basis $B_i \cup B_j$. There are natural embeddings $T_i \hookrightarrow T_{ij} \hookrightarrow T$ and $\partial_\infty T_i \hookrightarrow \partial_\infty T_{ij} \hookrightarrow \partial_\infty T$.

Given distinct $i, j \in \{1, 2, 3\}$, the following three assertions follow from the same argument as above, using the fact that all subgroupoids \mathcal{A}_i and \mathcal{A}_{ij} are amenable and have nontrivial restrictions to any Borel subset of positive measure:

- (1) there exists an (\mathcal{A}_i, ρ) -invariant Borel map $\alpha_i : Y \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T_i)$;
- (2) there exists an (\mathcal{A}_{ij}, ρ) -invariant Borel map $\beta_{ij} : Y \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T_{ij})$ (in particular β_{ij} is both (\mathcal{A}_i, ρ) -invariant and (\mathcal{A}_j, ρ) -invariant);
- (3) given any Borel subset $U \subseteq Y$ of positive measure, the union of any two $((\mathcal{A}_i)_{|U}, \rho)$ -invariant Borel maps $U \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$ again takes its values (essentially) in $\mathcal{P}_{\leq 2}(\partial_\infty T)$.

The third point ensures that $\alpha_i \cup \beta_{ij}$ takes (essentially) its values in $\mathcal{P}_{\leq 2}(\partial_\infty T_{ij})$. As $\partial_\infty T_i \cap \partial_\infty T_j = \emptyset$ whenever $i \neq j$ (viewed as subsets of $\partial_\infty T_{ij}$), we can find a Borel partition $Y = Y_i \cup Y_j$ such that

- (1) for a.e. $y \in Y_i$, the set $\alpha_i(y) \cup \beta_{ij}(y)$ has cardinality exactly 2;
- (2) for a.e. $y \in Y_j$, the set $\alpha_j(y) \cup \beta_{ij}(y)$ has cardinality exactly 2.

In particular, we can find a Borel subset $U \subseteq Y$ of positive measure, and $i_{12} \in \{1, 2\}$, such that there exists an $((\mathcal{A}_{i_{12}})_{|U}, \rho)$ -invariant Borel map $U \rightarrow \mathcal{P}_{=2}(\partial_\infty T_{12})$. Using the third assertion above, this implies that any $((\mathcal{A}_{i_{12}})_{|U}, \rho)$ -invariant Borel map $U \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$ must take its values in $\mathcal{P}_{\leq 2}(\partial_\infty T_{12})$. Likewise, up to restricting U to a further positive measure Borel subset, we find $i_{13} \in \{1, 3\}$ and $i_{23} \in \{2, 3\}$ such that every $((\mathcal{A}_{i_{13}})_{|U}, \rho)$ -invariant Borel map $U \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$ must take its values in $\mathcal{P}_{\leq 2}(\partial_\infty T_{13})$, and every $((\mathcal{A}_{i_{23}})_{|U}, \rho)$ -invariant Borel map $U \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$ must take its values in $\mathcal{P}_{\leq 2}(\partial_\infty T_{23})$. As $\partial_\infty T_{12} \cap \partial_\infty T_{13} \cap \partial_\infty T_{23} = \emptyset$, we have reached a contradiction to the existence of an $(\mathcal{A}_{|U}, \rho)$ -invariant Borel map $U \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty T)$. □

COROLLARY 3.5. — *Let A_1, A_2, A_3 be finitely generated free groups, and let \mathcal{G} be a measured groupoid over a standard finite measure space Y , equipped with a strict cocycle $\rho : \mathcal{G} \rightarrow A_1 * A_2 * A_3$ with trivial kernel. Let \mathcal{A} be an amenable measured subgroupoid of \mathcal{G} .*

Then there exists a Borel partition $Y = Y_1 \sqcup Y_2 \sqcup Y_3$ such that for every $i \in \{1, 2, 3\}$, the groupoid $(\mathcal{A} \cap \rho^{-1}(A_i))_{|Y_i}$ is stably trivial.

Proof. — For every $i \in \{1, 2, 3\}$, let $\mathcal{A}_i = \mathcal{A} \cap \rho^{-1}(A_i)$, and let Y_i be a Borel subset of maximal measure such that $(\mathcal{A}_i)_{|Y_i}$ is stably trivial (this exists because if $(Y_{i,n})$ is a measure-maximizing sequence of such sets, then \mathcal{A}_i is still stably trivial in restriction to their countable union). It is enough to show that $Y = Y_1 \cup Y_2 \cup Y_3$ (up to null sets). Otherwise, there would exist a Borel subset $U \subseteq Y$ of positive measure such that for every Borel subset $V \subseteq U$ of positive measure, none of the three subgroupoids

$(\mathcal{A}_1)|_V, (\mathcal{A}_2)|_V, (\mathcal{A}_3)|_V$ is trivial. This contradicts Lemma 3.4 applied to the ambient groupoid $\mathcal{G}|_U$. \square

3.3. SUPPORT OF A GROUPOID WITH A COCYCLE TO A RIGHT-ANGLED ARTIN GROUP

In the present section, we consider the following general situation: G is a countable group, \mathcal{G} is a measured groupoid over a finite measure space Y , coming with a cocycle $\rho : \mathcal{G} \rightarrow G$, and \mathbb{P} is an *admissible* collection of subgroups of G , i.e., \mathbb{P} is countable, conjugation-invariant, stable under intersections, $G \in \mathbb{P}$, and there does not exist any infinite descending chain $P_1 \supsetneq P_2 \supsetneq \dots$ in \mathbb{P} . The group G acts on \mathbb{P} by conjugation. We aim at defining a notion of support for the pair (\mathcal{G}, ρ) , which can be viewed as a canonical assignment of a subgroup in \mathbb{P} to (almost) every point $y \in Y$. In the sequel of the paper, this will always be applied with $G = G_\Gamma$ a right-angled Artin group, and \mathbb{P} the collection of parabolic subgroups with respect to a fixed standard generating set of G_Γ , see Remark 3.9 below.

DEFINITION 3.6. — Let G be a countable group, let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict cocycle. Let \mathbb{P} be an admissible collection of subgroups of G , and let $P \in \mathbb{P}$. We say that (\mathcal{G}, ρ) is *tightly P -supported with respect to \mathbb{P}* if the following two conditions hold:

- (1) there exists a conull Borel subset $Y^* \subseteq Y$ such that $\rho(\mathcal{G}|_{Y^*}) \subseteq P$;
- (2) there does not exist a Borel subset $U \subseteq Y$ of positive measure and a subgroup $Q \in \mathbb{P}$ with $Q \subsetneq P$ such that $\rho(\mathcal{G}|_U) \subseteq Q$.

A subgroup $P \in \mathbb{P}$ such that (\mathcal{G}, ρ) is tightly P -supported with respect to \mathbb{P} , if it exists, is unique: this follows from the fact that \mathbb{P} is assumed to be stable under intersections. The existence of P is ensured up to a countable partition of the base space Y by the following lemma.

LEMMA 3.7. — Let G be a countable group, let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict cocycle. Let \mathbb{P} be an admissible collection of subgroups of G .

Then there exists a partition $Y = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, there exists $P_i \in \mathbb{P}$ such that $(\mathcal{G}|_{Y_i}, \rho)$ is tightly P_i -supported with respect to \mathbb{P} .

Proof. — For every $P \in \mathbb{P}$, we claim that there exists a Borel subset $U_P \subseteq Y$ such that

- (1) there exists a conull Borel subset $U_P^* \subseteq U_P$ and a partition $U_P^* = \bigsqcup_{j \in J} U_j$ into at most countably many Borel subsets, such that for every $j \in J$, one has $\rho(\mathcal{G}|_{U_j}) \subseteq P$;
- (2) the subset U_P is maximal up to measure 0 with respect to the above property (i.e., if V_P is another Borel subset of Y that satisfies this property, then $V_P \setminus U_P$ is a null set).

Indeed, this is shown by taking U_P of maximal (finite) measure satisfying Property 1 (this is possible because if $(U_{P,k})_{k \in \mathbb{N}}$ is a maximizing sequence for the measure, then the countable union of all sets $U_{P,k}$ again satisfies the property).

Now, for every $P \in \mathbb{P}$, we let $Y_P = U_P \setminus \bigcup_{Q \subsetneq P} U_Q$, where the union is taken over all subgroups $Q \in \mathbb{P}$ that are properly contained in P : this is a Borel subset of Y because \mathbb{P} is countable. By construction, there exists a partition $Y_P = \bigsqcup_{j \in J} Y_{P,j}$ into at most countably many Borel subsets such that $(\mathcal{G}_{|Y_{P,j}}, \rho)$ is tightly P -supported for every $j \in J$. We finally observe that the sets Y_P form a countable partition of Y (up to null sets). This is shown by observing that the set $\{P \in \mathbb{P} \mid y \in U_P\}$ depends measurably on y , is almost everywhere nonempty (as $Y = U_G$ up to null sets), and using the fact that \mathbb{P} does not contain any infinite descending chain, we see that this set almost surely has a minimum P_y . Then for almost every $y \in Y$, we have $y \in Y_{P_y}$. Finally, the pairwise intersections of the sets Y_P are null sets because given $P_1, P_2 \in \mathbb{P}$, one has $U_{P_1} \cap U_{P_2} = U_{P_1 \cap P_2}$ (up to null sets). \square

We finally establish a canonicity statement for the support of a groupoid, coming in the form of its invariance by normalizers.

LEMMA 3.8. — *Let G be a countable group, let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict cocycle. Let \mathbb{P} be an admissible collection of subgroups of G . Let \mathcal{H} and \mathcal{H}' be two measured subgroupoids of \mathcal{G} . Assume that there exists $P \in \mathbb{P}$ such that (\mathcal{H}, ρ) is tightly P -supported with respect to \mathbb{P} . Assume also that \mathcal{H} is normalized by \mathcal{H}' .*

Then P is (\mathcal{H}', ρ) -invariant: in other words, denoting by $N_G(P)$ the normalizer of P in G , there exists a conull Borel subset $Y^ \subseteq Y$ such that $\rho(\mathcal{H}'_{|Y^*}) \subseteq N_G(P)$.*

Proof. — As \mathcal{H} is normalized by \mathcal{H}' , there exists a covering of \mathcal{H}' by countably many Borel subsets B_k that all leave \mathcal{H} invariant, with $s(B_k)$ Borel. Up to passing to a further covering, we can assume that for every $k \in \mathbb{N}$, the ρ -image of B_k is constant, with value an element $g_k \in G$. Up to replacing Y by a conull Borel subset, we will also assume that $s(B_k)$ has positive measure for every $k \in \mathbb{N}$.

As a consequence of normality, there exists a conull Borel subset $Y^* \subseteq Y$ such that for every $k \in \mathbb{N}$, one has $\rho(\mathcal{H}'_{|s(B_k) \cap Y^*}) \subseteq P \cap g_k^{-1} P g_k$. As (\mathcal{H}, ρ) is tightly P -supported and $P \cap g_k^{-1} P g_k$ belongs to \mathbb{P} , it follows that $P \subseteq g_k^{-1} P g_k$. Likewise, up to replacing Y^* by a further conull Borel subset, for every $k \in \mathbb{N}$, one has $\rho(\mathcal{H}'_{|r(B_k) \cap Y^*}) \subseteq P \cap g_k P g_k^{-1}$. It follows that $P \subseteq g_k P g_k^{-1}$. Combining the above two inclusions, we derive that $P = g_k P g_k^{-1}$, i.e., $g_k \in N_G(P)$. As this is true for every k and the subsets B_k cover \mathcal{H}' , we deduce that $\rho(\mathcal{H}'_{|Y^*}) \subseteq N_G(P)$, which concludes our proof. \square

REMARK 3.9. — From now on, we will let $G = G_\Gamma$ be a right-angled Artin group with a standard generating set $S = V\Gamma$, and let \mathbb{P} be the set of all parabolic subgroups of G (which is admissible by Lemma 1.2). In what follows, parabolic subgroups always refer to parabolic subgroups with respect to S . In fact, when Γ is transvection-free (as we will ultimately assume in this section), the collection of parabolic subgroups

does not depend on the choice of S . When \mathcal{G} is a measured groupoid coming with a cocycle $\rho : \mathcal{G} \rightarrow G$, and $P \in \mathbb{P}$, we will simply say that (\mathcal{G}, ρ) is *tightly P -supported* if it is tightly P -supported with respect to the collection \mathbb{P} of all parabolic subgroups.

3.4. TAKING ADVANTAGE OF AMENABLE NORMALIZED SUBGROUPOIDS. — In this section, we set up arguments towards the groupoid-theoretic version of Property (*) from the introduction. Following an argument of Adams [Ada94], we will exploit the Borel amenability of the action of a right-angled Artin group on the Roller boundary of the universal cover of its Salvetti complex ([Duc18, Th. 5.11], building on [BCG⁺09]) in a crucial way. We start by reviewing facts about Borel amenability of group actions.

3.4.1. Review on Borel amenability of group actions. — Let G be a countable group, and let X be a standard Borel space equipped with a G -action by Borel automorphisms. The G -action on X is *Borel amenable* if there exists a sequence of Borel maps $\nu_n : X \rightarrow \text{Prob}(G)$ (where $\text{Prob}(G)$ is equipped with the topology of pointwise convergence) such that for every $x \in X$ and every $g \in G$, one has $\|\nu_n(gx) - g\nu_n(x)\|_1 \rightarrow 0$ as n goes to $+\infty$. We will need the following facts about Borel amenability of group actions.

(1) A countable group G is amenable if and only if the trivial G -action on a point is Borel amenable.

(2) If the G -action on X is Borel amenable, then so is its restriction to every G -invariant Borel subset of X .

(3) Let G be a countable group, and let X be a Borel space equipped with a G -action by Borel automorphisms. Let $\mathcal{P}_{\leq 2}(X)$ be the set of all nonempty subsets of X of cardinality at most 2, equipped with its natural structure of standard Borel space coming from that of X . Then the G -action on $\mathcal{P}_{\leq 2}(X)$ is Borel amenable. This is an easy consequence of the definition, see e.g. [HH20, Lem. 6.14].

(4) Let G_1, \dots, G_k be countable groups, and let X_1, \dots, X_k be standard Borel spaces such that for every $i \in \{1, \dots, k\}$, the space X_i comes equipped with a Borel amenable G_i -action by Borel automorphisms. Then the product action of $G = G_1 \times \dots \times G_k$ on $X = X_1 \times \dots \times X_k$ is Borel amenable. Indeed, an elementary computation shows that if $\nu_n^i : X_i \rightarrow \text{Prob}(G_i)$ is a sequence of probability measures that witnesses the Borel amenability of the G_i -action on X_i , then the probability measures ν_n defined by letting $\nu_n(x_1, \dots, x_k)(g_1, \dots, g_k) = \nu_n^1(x_1)(g_1) \times \dots \times \nu_n^k(x_k)(g_k)$ witness the Borel amenability of the G -action on X .

We also refer the reader to [GHL20, §2] for a discussion on the relation to other notions of amenability of group actions – in particular, Borel amenability of a group action implies its *universal amenability* (i.e., Zimmer amenability with respect to every quasi-invariant probability measure), another notion that is commonly used in the literature (in particular this is the notion used by Kida in [Kid08]).

3.4.2. Using amenable normalized subgroupoids

LEMMA 3.10. — *Let G be a right-angled Artin group. Let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let \mathcal{A} and \mathcal{H} be measured subgroupoids*

of \mathcal{G} . Let $\rho : \mathcal{G} \rightarrow G$ be a strict cocycle, such that $\rho|_{\mathcal{H}}$ has trivial kernel. Let $P \in \mathbb{P}$ be a parabolic subgroup, and assume that (\mathcal{A}, ρ) is tightly P -supported. Assume that \mathcal{A} is amenable and normalized by \mathcal{H} .

Then $\mathcal{H} \cap \rho^{-1}(P)$ is amenable. If additionally P^\perp is amenable, then \mathcal{H} is amenable.

REMARK 3.11. — If $P = \{1\}$, the conclusion is clear. Indeed, as $\rho|_{\mathcal{H}}$ has trivial kernel, $\mathcal{H} \cap \rho^{-1}(P)$ is then trivial (whence amenable). In addition, amenability of P^\perp is then equivalent to the amenability of G , and if G is amenable, then \mathcal{H} is also amenable since $\rho|_{\mathcal{H}}$ has trivial kernel.

We may therefore as well assume that $P \neq \{1\}$. Note that in this case \mathcal{A} is not trivial, even after restricting it to a positive measure Borel subset: indeed, if $\mathcal{A}|_U$ were trivial for some Borel subset $U \subseteq Y$ of positive measure, then (\mathcal{A}, ρ) would not be tightly P -supported by definition.

Proof. — The proof relies on an argument that dates back to work of Adams [Ada94]. Up to replacing Y by a conull Borel subset, we can assume that $\rho(\mathcal{A}) \subseteq P$. As \mathcal{A} is normalized by \mathcal{H} , using Lemma 3.8 together with the fact that the normalizer of P is $P \times P^\perp$, we can also assume that $\rho(\mathcal{H}) \subseteq P \times P^\perp$.

Let $P = P_1 \times \dots \times P_k \times \mathbb{Z}^m$ be the de Rham decomposition of P . For every $i \in \{1, \dots, k\}$, we let $\rho_i : \mathcal{A} \rightarrow P_i$ be the cocycle obtained by postcomposing ρ with the i^{th} projection. For every $i \in \{1, \dots, k\}$, let $\partial_R \tilde{S}_i$ be the Roller boundary of the universal cover \tilde{S}_i of the Salvetti complex of P_i , and let $\partial_{\text{reg}} \tilde{S}_i$ be the subspace of $\partial_R \tilde{S}_i$ made of regular points.

Let $i \in \{1, \dots, k\}$. As \mathcal{A} is amenable and $\partial_R \tilde{S}_i$ is compact and metrizable, by [Kid09, Prop. 4.14], there exists an (\mathcal{A}, ρ_i) -invariant Borel map $\theta : Y \rightarrow \text{Prob}(\partial_R \tilde{S}_i)$ (recall that $\text{Prob}(\partial_R \tilde{S}_i)$ is equipped with the weak- $*$ topology, coming from the duality with the space of continuous functions on $\partial_R \tilde{S}_i$).

We claim that for a.e. $y \in Y$, the probability measure $\theta(y)$ gives full measure to the Borel subset $\partial_{\text{reg}} \tilde{S}_i$. Indeed, otherwise, there exists a Borel subset $V \subseteq Y$ of positive measure for which we can find an $(\mathcal{A}|_V, \rho_i)$ -invariant Borel map $V \rightarrow \text{Prob}(\partial_R \tilde{S}_i \setminus \partial_{\text{reg}} \tilde{S}_i)$. Let $\mathbb{P}_J(P_i)$ be the set of all nontrivial parabolic subgroups of P_i that are contained in some parabolic subgroup of P_i whose type decomposes nontrivially as a join. By Corollary 2.6, there is a P_i -equivariant Borel map $\partial_R \tilde{S}_i \setminus \partial_{\text{reg}} \tilde{S}_i \rightarrow \mathbb{P}_J(P_i)$. Therefore, we get an $(\mathcal{A}|_V, \rho_i)$ -invariant Borel map $V \rightarrow \text{Prob}(\mathbb{P}_J(P_i))$. Let $\mathcal{F}(\mathbb{P}_J(P_i))$ be the set of all nonempty finite subsets of $\mathbb{P}_J(P_i)$. As $\mathbb{P}_J(P_i)$ is countable, there is also a P_i -equivariant Borel map $\text{Prob}(\mathbb{P}_J(P_i)) \rightarrow \mathcal{F}(\mathbb{P}_J(P_i))$, sending a probability measure η to the finite set of all elements of $\mathbb{P}_J(P_i)$ of maximal η -measure. In summary, we obtain an $(\mathcal{A}|_V, \rho_i)$ -invariant Borel map $V \rightarrow \mathcal{F}(\mathbb{P}_J(P_i))$. By Lemma 1.4, the setwise P_i -stabilizer of every finite subset of $\mathbb{P}_J(P_i)$ coincides with its pointwise stabilizer. By restricting to a positive measure Borel subset $W \subseteq V$ where the above map is constant, we thus get an $(\mathcal{A}|_W, \rho_i)$ -invariant parabolic subgroup $Q'_i \in \mathbb{P}_J(P_i)$, i.e., $\rho_i(\mathcal{A}|_W) \subseteq Q'_i \times (Q'_i)^\perp$ (where \perp_i denotes the orthogonality relation taken with respect to the ambient group P_i).

The fact that $Q'_i \in \mathbb{P}_J(P_i)$ means that there exists a proper parabolic subgroup $Q_i \subset P_i$ such that $Q'_i \subseteq Q_i \times Q_i^{\perp_i}$, with both Q_i and $Q_i^{\perp_i}$ nontrivial. We now claim that $Q'_i \times (Q'_i)^{\perp_i} \subsetneq P_i$. Indeed, if $(Q'_i)^{\perp_i}$ is trivial, then Q'_i is contained in the join subgroup $Q_i \times Q_i^{\perp_i}$ which is properly contained in P_i as the type of P_i does not split as a nontrivial join. If $(Q'_i)^{\perp_i}$ is nontrivial, then $Q'_i \times (Q'_i)^{\perp_i}$ is proper for the same reason. We then have $\rho(\mathcal{A}|_W) \subseteq (Q'_i \times (Q'_i)^{\perp_i}) \times \prod_{j \neq i} P_j \times \mathbb{Z}^m$, which is a proper parabolic subgroup of P . This contradicts the fact that (\mathcal{A}, ρ) is tightly P -supported.

We next claim that for a.e. $y \in Y$, the support of the probability measure $\theta(y)$ has cardinality at most 2. Indeed, otherwise, there exists a Borel subset $U \subseteq Y$ of positive measure such that for all $y \in U$, the probability measure $\theta(y) \otimes \theta(y) \otimes \theta(y)$ gives positive measure to the space $(\partial_{\text{reg}} \tilde{S}_i)^{(3)}$ made of pairwise distinct triples. Thus, we get an $(\mathcal{A}|_U, \rho_i)$ -invariant Borel map $U \rightarrow \text{Prob}((\partial_{\text{reg}} \tilde{S}_i)^{(3)})$. By work of Fernós, Lécureux and Mathéus [FLM18, Lem. 5.14 & 6.21], there exists a P_i -equivariant Borel map $(\partial_{\text{reg}} \tilde{S}_i)^{(3)} \rightarrow V(\tilde{S}_i)$ – where $V(\tilde{S}_i)$ denotes the vertex set of \tilde{S}_i . This yields an $(\mathcal{A}|_U, \rho_i)$ -invariant Borel map $U \rightarrow \text{Prob}(V(\tilde{S}_i))$. Using the fact that $V(\tilde{S}_i)$ is countable, we deduce as above an $(\mathcal{A}|_U, \rho_i)$ -invariant Borel map $\varphi : U \rightarrow \mathcal{F}(V(\tilde{S}_i))$. Then there exists a Borel subset $W \subseteq U$ of positive measure (in restriction to which φ is a constant map), such that $\rho_i(\mathcal{A}|_W) = \{1\}$, which is a contradiction to the fact that (\mathcal{A}, ρ) is tightly P -supported, as above.

In fact, the above argument shows that for every (\mathcal{A}, ρ_i) -invariant Borel map $\eta : Y \rightarrow \text{Prob}(\partial_{\text{reg}} \tilde{S}_i)$ and a.e. $y \in Y$, the support of the probability measure $\eta(y)$ has cardinality at most 2 (this observation will be important when applying Adams’ argument right below; it corresponds to Adams’ *bipolarity* assumption [Ada94, Def. 3.1]).

Using an argument due to Adams [Ada94, Lem. 3.2 & 3.3], we get a Borel map $Y \rightarrow \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_i)$ which is both (\mathcal{A}, ρ_i) -invariant and (\mathcal{H}, ρ_i) -invariant. By combining these maps, we derive an (\mathcal{H}, ρ) -invariant Borel map

$$Y \longrightarrow \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_k).$$

By [Duc18, Th. 5.11], for every $i \in \{1, \dots, k\}$, the action of P_i on $\partial_R \tilde{S}_i$ is Borel amenable. Therefore, so are the P_i -actions on $\partial_{\text{reg}} \tilde{S}_i$ and on $\mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_i)$ – we could alternatively have applied a theorem of Nevo and Sageev [NS13, Th. 7.2] here. We have a product action of $P = P_1 \times \cdots \times P_k \times \mathbb{Z}^m$ on $\mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_k) \times \{*\}$, where the \mathbb{Z}^m factor is acting trivially on the point $*$, and also trivially on the other factors. Combining the fact that the trivial action of \mathbb{Z}^m on a point is Borel amenable, with Fact 4 from Section 3.4.1, we deduce that the action of P on $\mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_k)$ is Borel amenable. As $\rho : \mathcal{H} \cap \rho^{-1}(P) \rightarrow P_1 \times \cdots \times P_k \times \mathbb{Z}^m$ has trivial kernel, it follows from [Kid08, Prop. 4.33] (see also [GH21, Prop. 3.38]) that $\mathcal{H} \cap \rho^{-1}(P)$ is amenable. This shows that the first conclusion of the lemma holds. We deduce the second conclusion in a similar way from the amenability of the action of $P \times P^\perp$ on $\mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_{\text{reg}} \tilde{S}_k) \times \{*\}$ whenever P^\perp is amenable. \square

We now add a maximality condition.

LEMMA 3.12. — *Let G be a transvection-free right-angled Artin group. Let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a*

strict action-type cocycle. Let \mathcal{A} and \mathcal{H} be measured subgroupoids of \mathcal{G} . Let $P \in \mathbb{P}$ be a parabolic subgroup. Assume that

- (1) \mathcal{A} is amenable, and (\mathcal{A}, ρ) is tightly P -supported;
- (2) \mathcal{H} is everywhere nonamenable;
- (3) \mathcal{A} is normalized by \mathcal{H} ;
- (4) if \mathcal{H}' is a measured subgroupoid of \mathcal{G} which is everywhere nonamenable and normalizes an amenable subgroupoid of \mathcal{G} of infinite type, and if $\mathcal{H} \subseteq \mathcal{H}'$, then \mathcal{H} is stably equivalent to \mathcal{H}' .

Then P either has trivial center or is isomorphic to \mathbb{Z} .

REMARK 3.13. — In the sequel, this will be applied in the weaker form that P is either isomorphic to \mathbb{Z} or nonamenable.

Proof. — Assume that P has nontrivial center. By the centralizer theorem in [Ser89], we can then find a cyclic parabolic subgroup $Z \subseteq P$ contained in the center of P , and we aim to prove that $P = Z$.

As P is (\mathcal{H}, ρ) -invariant (Lemma 3.8), up to replacing Y by a conull Borel subset, we can assume that $\rho(\mathcal{H})$ is contained in the normalizer of P , which is equal to $P \times P^\perp$ (Proposition 1.1). Moreover $P \times P^\perp$ is nonamenable: otherwise, as ρ has trivial kernel, the groupoid $\rho^{-1}(P \times P^\perp)$ would be amenable, and \mathcal{H} , as a subgroupoid of $\rho^{-1}(P \times P^\perp)$, would also be amenable.

Let P' be the centralizer of Z (equal to $Z \times Z^\perp$). Then P' is a parabolic subgroup of G that contains $P \times P^\perp$; in particular it is nonamenable. Let $\mathcal{H}' = \rho^{-1}(P')$. Then $\mathcal{H} \subseteq \mathcal{H}'$ and these subgroupoids are everywhere nonamenable. In addition \mathcal{H}' normalizes $\rho^{-1}(Z)$, an amenable subgroupoid of \mathcal{G} of infinite type. The maximality assumption thus implies that \mathcal{H} is stably equivalent to \mathcal{H}' . Using that ρ is action-type, we deduce in particular that each element in P' has a non-zero power which is contained in $P \times P^\perp$, hence $P' \subseteq P \times P^\perp$ by Lemma 1.4. Thus $P' = P \times P^\perp$.

We now prove that P is abelian. Otherwise, it contains a nonabelian free group, so Lemma 3.2 implies that $\rho^{-1}(P)$ is everywhere nonamenable. On the other hand, since ρ is action-type, it has trivial kernel, and Lemma 3.10 thus implies that $\mathcal{H} \cap \rho^{-1}(P)$ is amenable. But $\mathcal{H} \cap \rho^{-1}(P)$ is stably equivalent to $\mathcal{H}' \cap \rho^{-1}(P) = \rho^{-1}(P)$, so we get a contradiction.

As P is abelian and $P \times P^\perp$ is equal to the centralizer of Z , the transvection-free assumption implies that $P = Z$: otherwise, there would exist a transvection that multiplies a generator of Z by a nontrivial element of P . This concludes our proof. \square

3.5. A UNIQUENESS STATEMENT

LEMMA 3.14. — Let G be a transvection-free right-angled Artin group. Let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict action-type cocycle. Let $Z, Z' \in \mathbb{P}$ be two cyclic parabolic subgroups of G .

If there exists a Borel subset $U \subseteq Y$ of positive measure such that the $(\mathcal{G}|_U, \rho)$ -stabilizer of Z is contained in the $(\mathcal{G}|_U, \rho)$ -stabilizer of Z' , then $Z = Z'$.

Proof. — We will prove the contrapositive statement. Assume that $Z \neq Z'$, and let $U \subseteq Y$ be a Borel subset of positive measure. As G is transvection-free, Lemma 1.6 ensures that there exists an infinite order element g contained in $Z \times Z^\perp$ but not in $Z' \times (Z')^\perp$. As ρ is action-type, the groupoid $(\rho^{-1}(\langle g \rangle))|_U$ is of infinite type, and it is contained in the $(\mathcal{G}|_U, \rho)$ -stabilizer of Z but not in the $(\mathcal{G}|_U, \rho)$ -stabilizer of Z' . \square

3.6. CHARACTERIZATION OF VERTEX STABILIZERS OF THE EXTENSION GRAPH. — Given a measured groupoid \mathcal{G} over a base space Y , we say that a measured subgroupoid \mathcal{H} is *stably maximal* (among subgroupoids of \mathcal{G}) with respect to a property (P) if for every measured subgroupoid \mathcal{H}' of \mathcal{G} satisfying (P), if \mathcal{H} is stably contained in \mathcal{H}' , then \mathcal{H} is stably equivalent to \mathcal{H}' .

The following key proposition gives a groupoid-theoretic characterization of vertex stabilizers of the extension graph; Property (2a) should be viewed as the groupoid-theoretic analogue of Property (*) from the introduction. Some intuition on Property (2b) coming from the group-theoretic setting is also provided in the introduction, under the heading *A word on the proof of the classification theorem*.

PROPOSITION 3.15. — *Let G be a non-cyclic transvection-free right-angled Artin group. Let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict action-type cocycle. Let $\mathcal{H} \subseteq \mathcal{G}$ be a measured subgroupoid. Then the following conditions are equivalent.*

(1) *There exist a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, there exists a cyclic parabolic subgroup Z_i of G such that $\mathcal{H}|_{Y_i}$ is equal to the $(\mathcal{G}|_{Y_i}, \rho)$ -stabilizer of Z_i , i.e., $\mathcal{H}|_{Y_i} = (\rho^{-1}(Z_i \times Z_i^\perp))|_{Y_i}$.*

(2) *The following two conditions hold:*

(a) *the subgroupoid \mathcal{H} is everywhere nonamenable and stably normalizes an amenable subgroupoid of \mathcal{G} of infinite type, and \mathcal{H} is stably maximal (among subgroupoids of \mathcal{G}) with respect to these two properties;*

(b) *for every Borel subset $U \subseteq Y$ of positive measure, whenever $\mathcal{H}' \subseteq \mathcal{G}|_U$ is an everywhere nonamenable measured subgroupoid which normalizes an amenable subgroupoid \mathcal{A}' of $\mathcal{G}|_U$ of infinite type with $\mathcal{A}' \cap \mathcal{H}'$ stably trivial, there exists an everywhere nonamenable measured subgroupoid $\mathcal{H}'' \subseteq \mathcal{H}'|_U$ such that $\mathcal{H}' \cap \mathcal{H}''$ is stably trivial.*

In the sequel, a subgroupoid \mathcal{H} of \mathcal{G} satisfying one of the equivalent conclusions of Proposition 3.15 will be called a (\mathcal{G}, ρ) -*vertex subgroupoid*: this terminology suggests that it plays the role of a vertex stabilizer of the extension graph Γ^e – where Γ is the underlying graph of G .

Notice that Proposition 3.15 gives a purely groupoid-theoretic characterization of (\mathcal{G}, ρ) -vertex subgroupoids, with no reference to the cocycle ρ ; therefore, being a (\mathcal{G}, ρ) -vertex subgroupoid is a notion that does not depend on the choice of a strict action-type cocycle $\rho : \mathcal{G} \rightarrow G$. Notice also that a (\mathcal{G}, ρ) -vertex subgroupoid \mathcal{H}

naturally comes with a Borel map $\theta : Y \rightarrow V\Gamma^e$, by letting $\theta(y) = Z_i$ whenever $y \in Y_i$. Up to measure 0, this map does not depend on the choice of a Borel partition as in the first conclusion of Proposition 3.15, in view of the uniqueness statement given in Lemma 3.14. We call it the *parabolic map* of (\mathcal{H}, ρ) – we insist here that this map *does* depend on ρ .

Proof. – We first prove that $1 \Rightarrow 2$. Assume that \mathcal{H} satisfies assertion 1. Without loss of generality, we can assume that all subsets Y_i in the given partition of Y^* have positive measure.

We first prove that \mathcal{H} satisfies assertion (2a). As G is transvection-free and not isomorphic to \mathbb{Z} , for every $i \in I$, the group $Z_i \times Z_i^\perp$ contains a nonabelian free subgroup. As ρ is action-type, Lemma 3.2 shows that $\mathcal{H}|_{Y_i}$ is everywhere nonamenable. Thus \mathcal{H} is everywhere nonamenable. Let now \mathcal{A} be a measured subgroupoid of \mathcal{G} such that for every $i \in I$, one has $\mathcal{A}|_{Y_i} = (\rho^{-1}(Z_i))|_{Y_i}$. Then \mathcal{A} is stably normalized by \mathcal{H} , and as Z_i is cyclic and ρ has trivial kernel the groupoid \mathcal{A} is amenable – and it is of infinite type because ρ is action-type.

Let us now prove the maximality condition from assertion (2a). For this, let $\widehat{\mathcal{H}}$ be a subgroupoid of \mathcal{G} which is everywhere nonamenable, stably normalizes an amenable subgroupoid $\widehat{\mathcal{A}}$ of \mathcal{G} of infinite type, and such that \mathcal{H} is stably contained in $\widehat{\mathcal{H}}$. By Lemma 3.7, for every $i \in I$, we can find a partition $Y_i = \bigsqcup_{j \in J_i} V_{i,j}$ into at most countably many Borel subsets of positive measure such that for every $j \in J_i$, there exists a parabolic subgroup $P_{i,j}$ such that $(\widehat{\mathcal{A}}|_{V_{i,j}}, \rho)$ is tightly $P_{i,j}$ -supported, and $P_{i,j}$ is nontrivial since $\widehat{\mathcal{A}}$ is of infinite type and ρ has trivial kernel. As $\widehat{\mathcal{A}}$ is stably normalized by $\widehat{\mathcal{H}}$, up to refining the above partition, we can further assume that for every $j \in J_i$, the groupoid $\widehat{\mathcal{A}}|_{V_{i,j}}$ is normalized by $\mathcal{H}|_{V_{i,j}}$. Lemma 3.8 then implies that up to replacing Y_i by a conull Borel subset, for every $j \in J_i$, one has $\rho(\widehat{\mathcal{H}}|_{V_{i,j}}) \subseteq P_{i,j} \times P_{i,j}^\perp$. As $\widehat{\mathcal{H}}$ is everywhere nonamenable, Lemma 3.10 implies that all groups $P_{i,j}^\perp$ are nonamenable, and $(\widehat{\mathcal{H}} \cap \rho^{-1}(P_{i,j}))|_{V_{i,j}}$ is amenable.

We now fix $i \in I$ and $j \in J_i$. As \mathcal{H} is stably contained in $\widehat{\mathcal{H}}$ and ρ is action-type, we know that for every $g \in Z_i \times Z_i^\perp$, there exists an integer $k \neq 0$ such that $g^k \in P_{i,j} \times P_{i,j}^\perp$. Lemma 1.4 then shows that $Z_i \times Z_i^\perp \subseteq P_{i,j} \times P_{i,j}^\perp$. Since G is transvection-free and $P_{i,j}$ is nontrivial, Lemma 1.6 ensures that either $P_{i,j} = Z_i$, or else $P_{i,j} \cap Z_i^\perp$ is nonamenable. But if the latter holds, then $(\mathcal{H} \cap \rho^{-1}(P_{i,j} \cap Z_i^\perp))|_{V_{i,j}}$ (which is everywhere nonamenable) is not stably contained in $(\widehat{\mathcal{H}} \cap \rho^{-1}(P_{i,j} \cap Z_i^\perp))|_{V_{i,j}}$ (which is amenable), a contradiction. This shows that $P_{i,j} = Z_i$.

Since the above is true for every $i \in I$ and every $j \in J_i$, we deduce that $\widehat{\mathcal{H}}$ is stably contained in \mathcal{H} , so they are stably equivalent. This finishes the proof of assertion (2a).

We now turn to proving assertion (2b). Let $U \subseteq Y$ be a Borel subset of positive measure, and \mathcal{A}' and \mathcal{H}' be measured subgroupoids of $\mathcal{G}|_U$ as in the statement. For every $i \in I$, we let $U_i = U \cap Y_i$. By Lemma 3.7, there exists a partition $U_i = \bigsqcup_{n \in N} U_{i,n}$ into at most countably many Borel subsets such that for every $n \in N$, there exists a parabolic subgroup $P_{i,n}$ such that $(\mathcal{A}'|_{U_{i,n}}, \rho)$ is tightly $P_{i,n}$ -supported. Up to replacing U_i by a conull Borel subset and refining the above partition, Lemma 3.8 allows

us to assume that for every $n \in N$, we have $\rho(\mathcal{H}'_{|U_{i,n}}) \subseteq P_{i,n} \times P_{i,n}^\perp$ (and we can also assume that $\rho(\mathcal{A}'_{|U_{i,n}}) \subseteq P_{i,n}$).

We aim to construct an everywhere nonamenable subgroupoid \mathcal{H}'' of $\mathcal{H}|_U$ such that $\mathcal{H}' \cap \mathcal{H}''$ is stably trivial. For this, we will only define \mathcal{H}'' on each $U_{i,n}$ – and \mathcal{H}'' will have no elements with source and range in different subsets $U_{i,n}$.

We will first define $\mathcal{H}''_{|U_{i,n}}$ for values of i, n such that Z_i^\perp is not contained in $P_{i,n} \times P_{i,n}^\perp$. As G is transvection-free, the group Z_i^\perp has no abelian factor in its de Rham decomposition. By Proposition 1.7, we can (and shall) thus choose elements $a_1, a_2 \in Z_i^\perp$ that freely generate a rank 2 free subgroup, so that the support of every nontrivial element of $\langle a_1, a_2 \rangle$ is equal to Z_i^\perp . Let $\mathcal{H}''_{|U_{i,n}} = (\rho^{-1}(\langle a_1, a_2 \rangle))_{|U_{i,n}}$. Clearly $\mathcal{H}''_{|U_{i,n}} \subseteq \mathcal{H}_{|U_{i,n}}$. Since a_1 and a_2 generate a nonabelian free subgroup and ρ is action-type, Lemma 3.2 implies that $\mathcal{H}''_{|U_{i,n}}$ is everywhere nonamenable. In addition, as Z_i^\perp is not contained in $P_{i,n} \times P_{i,n}^\perp$, it follows that $\rho(\mathcal{H}''_{|U_{i,n}} \cap \mathcal{H}'_{|U_{i,n}})$ is the trivial subgroup. As ρ has trivial kernel, $\mathcal{H}''_{|U_{i,n}} \cap \mathcal{H}'_{|U_{i,n}}$ is trivial.

We will now define $\mathcal{H}''_{|U_{i,n}}$ for values of i, n such that $Z_i^\perp \subseteq P_{i,n} \times P_{i,n}^\perp$. Similar to the proof of Proposition 1.7, we have $Z_i^\perp = M_1 \times M_2$ where $M_1 = Z_i^\perp \cap P_{i,n}$ and $M_2 = Z_i^\perp \cap P_{i,n}^\perp$ are parabolic subgroups. We claim that M_1 is nontrivial. Otherwise we have $Z_i^\perp \subseteq P_{i,n}^\perp$. Thus $P_{i,n} \subseteq (Z_i^\perp)^\perp = Z_i$, with the last equality following from transvection-freeness. Thus $\mathcal{A}'_{|U_{i,n}} \subseteq \mathcal{H}_{|U_{i,n}}$. As \mathcal{A}' is of infinite type, this contradicts our assumption that $\mathcal{A}' \cap \mathcal{H}$ is stably trivial, thereby proving our claim. As G is transvection-free, Z_i^\perp does not have any abelian de Rham factor, and neither does M_1 , in particular M_1 is nonabelian. We can thus find three finitely generated nonabelian free subgroups $A_1, A_2, A_3 \subseteq M_1$ such that $\langle A_1, A_2, A_3 \rangle = A_1 * A_2 * A_3$. Notice that

$$(\mathcal{H}' \cap \rho^{-1}(\langle A_1, A_2, A_3 \rangle))_{|U_{i,n}} \subseteq (\mathcal{H}' \cap \rho^{-1}(M_1))_{|U_{i,n}} \subseteq (\mathcal{H}' \cap \rho^{-1}(P_{i,n}))_{|U_{i,n}}.$$

By Lemma 3.10, the groupoid $(\mathcal{H}' \cap \rho^{-1}(P_{i,n}))_{|U_{i,n}}$ is amenable. Therefore

$$(\mathcal{H}' \cap \rho^{-1}(\langle A_1, A_2, A_3 \rangle))_{|U_{i,n}}$$

is also amenable. By Corollary 3.5, we can thus find a Borel partition $U_{i,n} = U_{i,n,1} \sqcup U_{i,n,2} \sqcup U_{i,n,3}$ such that for every $\ell \in \{1, 2, 3\}$, the groupoid $(\mathcal{H}' \cap \rho^{-1}(A_\ell))_{|U_{i,n,\ell}}$ is stably trivial. We then let $\mathcal{H}''_{|U_{i,n}}$ be a measured subgroupoid of $\mathcal{H}_{|U_{i,n}}$ such that for every $\ell \in \{1, 2, 3\}$, one has $\mathcal{H}''_{|U_{i,n,\ell}} = (\rho^{-1}(A_\ell))_{|U_{i,n,\ell}}$. Then $\mathcal{H}'_{|U_{i,n}} \cap \mathcal{H}''_{|U_{i,n}}$ is stably trivial. In addition, as every A_ℓ is a nonabelian free group, Lemma 3.2 implies that $\mathcal{H}''_{|U_{i,n}}$ is everywhere nonamenable. We have thus constructed a subgroupoid \mathcal{H}'' satisfying the conclusion of assertion (2b). This concludes our proof of (1) \Rightarrow (2).

We now prove that (2) \Rightarrow (1). Assume that \mathcal{H} satisfies assertions (2a) and (2b), and let \mathcal{A} be an amenable subgroupoid of \mathcal{G} of infinite type which is stably normalized by \mathcal{H} . Lemma 3.7 gives a partition $Y = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets of positive measure such that for every $i \in I$, there exists a parabolic subgroup P_i such that $(\mathcal{A}_{|Y_i}, \rho)$ is tightly P_i -supported, and P_i is nontrivial since \mathcal{A} is of infinite type and ρ has trivial kernel. Up to refining this partition if needed, we can further assume that for every $i \in I$, the groupoid $\mathcal{A}_{|Y_i}$ is normalized by $\mathcal{H}_{|Y_i}$.

We will prove that for every $i \in I$, the group P_i is cyclic: by Lemma 3.8, the groupoid $\mathcal{H}_{|Y_i}$ will then be contained in the $(\mathcal{G}_{|Y_i}, \rho)$ -stabilizer of P_i (up to replacing Y_i by a conull Borel subset), and the maximality assumption from assertion (2a) together with (1) \Rightarrow (2a) will complete the proof.

So let $i \in I$, and assume towards a contradiction that P_i is noncyclic. Lemma 3.10 ensures that P_i^\perp is nonamenable, and Lemma 3.12 together with assertion (2a) ensure that P_i is nonamenable (notice that if we replace each groupoid appearing in this assertion by its restriction on Y_i , then the assertion still holds). Up to replacing Y_i by a conull Borel subset, we can assume that $\rho(\mathcal{A}_{|Y_i}) \subseteq P_i$, and Lemma 3.8 enables us to further assume that $\rho(\mathcal{H}_{|Y_i}) \subseteq P_i \times P_i^\perp$. Notice also that Lemma 3.10 implies that $(\mathcal{H} \cap \rho^{-1}(P_i))_{|Y_i}$ is amenable.

Let $\mathcal{H}' = \rho^{-1}(P_i^\perp)_{|Y_i}$. As P_i^\perp contains a nonabelian free subgroup and ρ is action-type, Lemma 3.2 ensures that \mathcal{H}' is everywhere nonamenable. As P_i is nonamenable, we can find three finitely generated nonabelian free subgroups $A_1, A_2, A_3 \subseteq P_i$ such that $\langle A_1, A_2, A_3 \rangle = P_i$. By Corollary 3.5 (applied to the amenable groupoid $(\mathcal{H} \cap \rho^{-1}(P_i))_{|Y_i}$), we can find a Borel partition $Y_i = Y_{i,1} \sqcup Y_{i,2} \sqcup Y_{i,3}$ such that for every $\ell \in \{1, 2, 3\}$, the groupoid $(\mathcal{H} \cap \rho^{-1}(A_\ell))_{|Y_{i,\ell}}$ is stably trivial. For every $\ell \in \{1, 2, 3\}$, let $a_\ell \in A_\ell$ be a nontrivial element. Let then \mathcal{A}' be a measured subgroupoid of $\mathcal{G}_{|Y_i}$ such that for every $\ell \in \{1, 2, 3\}$, one has $\mathcal{A}'_{|Y_{i,\ell}} = \rho^{-1}(\langle a_\ell \rangle)_{|Y_{i,\ell}}$. Then \mathcal{A}' is amenable and of infinite type (as ρ is action-type), it is stably normalized by \mathcal{H}' , and $\mathcal{A}' \cap \mathcal{H}_{|Y_i}$ is stably trivial.

We now aim to contradict assertion (2b) with $U = Y_i$. For that, we let \mathcal{H}'' be a measured subgroupoid of $\mathcal{H}_{|Y_i}$ such that $\mathcal{H}' \cap \mathcal{H}''$ is stably trivial, and we aim to prove that there exists a Borel subset $V \subseteq Y_i$ of positive measure such that $\mathcal{H}''_{|V}$ is amenable.

As $\mathcal{H}'' \subseteq \mathcal{H}_{|Y_i}$, one has $\rho(\mathcal{H}'') \subseteq P_i \times P_i^\perp$. We post-compose $\rho : \mathcal{H}'' \rightarrow P_i \times P_i^\perp$ with the projection $P_i \times P_i^\perp \rightarrow P_i$ to obtain a strict cocycle $\rho' : \mathcal{H}'' \rightarrow P_i$. The kernel of ρ' is the set of all elements $g \in \mathcal{H}''$ such that $\rho(g) \in P_i^\perp$, a subgroupoid of \mathcal{H}' . Being a subgroupoid of $\mathcal{H}'' \cap \mathcal{H}'$, it is stably trivial. We can therefore find a Borel subset $V \subseteq Y_i$ of positive measure such that $\rho' : \mathcal{H}''_{|V} \rightarrow P_i$ has trivial kernel. As $\mathcal{H}''_{|V} \subseteq \mathcal{H}_{|V}$, it normalizes $\mathcal{A}_{|V}$. We can therefore apply Lemma 3.10 to the subgroupoid of $\mathcal{G}_{|V}$ generated by $\mathcal{H}''_{|V}$ and $\mathcal{A}_{|V}$, and to its natural cocycle towards P_i , to derive that $\mathcal{H}''_{|V}$ is amenable, as desired. \square

3.7. CHARACTERIZATION OF ADJACENCY. — In Section 3.6, given a measured groupoid \mathcal{G} with a strict action-type cocycle towards a transvection-free right-angled Artin group G not isomorphic to \mathbb{Z} , we characterized subgroupoids of \mathcal{G} that stabilize rank one parabolic subgroups of G (up to a countable Borel partition of the base space). In the present section, given two such subgroupoids, we will characterize when the parabolic subgroups they stabilize commute – i.e., define adjacent vertices of the extension graph. This is the contents of Lemma 3.17 below; we start with a group-theoretic version of this statement.

LEMMA 3.16. — *Let Γ be a finite simple graph, let $G = G_\Gamma$, and let Γ^e be the extension graph of Γ , equipped with its natural G -action. Let $v, w \in V\Gamma^e$ be two vertices.*

Then v and w are adjacent or equal in Γ^e if and only if there are finitely many vertices $u \in V\Gamma^e$ such that every element of $\text{Stab}_G(v) \cap \text{Stab}_G(w)$ has a nontrivial power that fixes u . In this case there are in fact at most $|V\Gamma|$ such vertices.

Proof. — Let \mathbb{Z}_v be the cyclic parabolic subgroup of G associated with a vertex $v \in V\Gamma^e$. Let $g_v \in G$ be a generator of \mathbb{Z}_v . Let $\bar{v} \in \Gamma$ be the type of \mathbb{Z}_v . Note that $\text{Stab}_G(v)$ is the centralizer of \mathbb{Z}_v , a parabolic subgroup of type $\text{st}(\bar{v})$ (Proposition 1.1).

We first assume that $d_{\Gamma^e}(v, w) \geq 2$, and aim to show that $H = \text{Stab}_G(v) \cap \text{Stab}_G(w)$ fixes infinitely many vertices of Γ^e . The group H centralizes both \mathbb{Z}_v and \mathbb{Z}_w , so H centralizes $\langle \mathbb{Z}_v, \mathbb{Z}_w \rangle$. By the proof of [KK13, Th. 1.3], the group $\langle \mathbb{Z}_v, \mathbb{Z}_w \rangle$ contains a nonabelian free subgroup F , generated by high enough powers of g_v and g_w : in fact, Kim and Koberda prove that there exists an injective homomorphism from G_Γ into the mapping class group of a finite-type surface, such that g_v and g_w are mapped to powers of Dehn twists about essential simple closed curves that intersect essentially. Therefore H fixes infinitely many vertices of Γ^e arising from conjugates of \mathbb{Z}_v by elements in F .

We now assume that v and w are equal or adjacent. Actually it suffices to consider the case where they are adjacent. Up to conjugation, we can assume without loss of generality that $g_v = \bar{v}$ and $g_w = \bar{w}$ (after identifying vertices of Γ with the associated standard generators of G_Γ). Then

$$H = \text{Stab}_G(v) \cap \text{Stab}_G(w) = G_{\text{st}(\bar{v})} \cap G_{\text{st}(\bar{w})} = G_{\bar{e} \circ \bar{e}^\perp},$$

where $\bar{e} \subset \Gamma$ is the edge joining \bar{v} to \bar{w} (the second equality follows from Proposition 1.1, and the last one from Lemma 1.2(1)). By Proposition 1.1, we have $N_G(H) = H$ (where $N_G(H)$ denotes the normalizer of H in G). Let now $u \in V\Gamma^e$ be a vertex such that every element of H has a power that fixes u . Notice that if u is fixed by h^n for some $h \in H$ and $n \in \mathbb{N}$, then h^n centralizes \mathbb{Z}_u , and in fact h centralizes \mathbb{Z}_u (as follows from Lemma 1.4). Therefore H centralizes \mathbb{Z}_u . Hence $\mathbb{Z}_u \subseteq N_G(H) = H$. It follows that \mathbb{Z}_u lies in the center of H , which is equal to G_{Γ_1} , where Γ_1 is the maximal clique factor of $\bar{e} \circ \bar{e}^\perp$ [Ser89]. Thus vertices of Γ^e fixed by H arise from cyclic parabolic subgroups contained in the abelian group G_{Γ_1} , and there are only finitely many of those – in fact at most $|V\Gamma_1|$. \square

We now move on to the groupoid-theoretic version of our statement.

LEMMA 3.17. — *Let G be a non-cyclic transvection-free right-angled Artin group, with underlying graph Γ . Let \mathcal{G} be a measured groupoid over a standard finite measure space Y , and let $\rho : \mathcal{G} \rightarrow G$ be a strict action-type cocycle. Let \mathcal{H} and \mathcal{H}' be two (\mathcal{G}, ρ) -vertex subgroupoids, with parabolic maps $\theta, \theta' : Y \rightarrow V\Gamma^e$, respectively. Then the following two assertions are equivalent.*

- (1) *For a.e. $y \in Y$, the groups $\theta(y)$ and $\theta'(y)$ commute.*

(2) *There exist finitely many (\mathcal{G}, ρ) -vertex subgroupoids $\mathcal{G}_1, \dots, \mathcal{G}_k$, such that for every (\mathcal{G}, ρ) -vertex subgroupoid \mathcal{K} , if $\mathcal{H} \cap \mathcal{H}'$ is stably contained in \mathcal{K} , then there exists a Borel partition $Y = W_1 \sqcup \dots \sqcup W_k$ such that for every $\ell \in \{1, \dots, k\}$, the groupoid $\mathcal{K}|_{W_\ell}$ is stably contained in $(\mathcal{G}_\ell)|_{W_\ell}$.*

Proof. — We first prove that (1) \Rightarrow (2). We can find a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, the maps $\theta|_{Y_i}$ and $\theta'|_{Y_i}$ are constant, with respective values commuting cyclic parabolic subgroups Z_i and Z'_i . Let $k = |\text{V}\Gamma|$. By Lemma 3.16, for every $i \in I$, there exists a finite set $\mathcal{Z}_i = \{Z_{i,1}, \dots, Z_{i,k}\}$ of cyclic parabolic subgroups, of cardinality at most k (written possibly with repetitions), such that for every cyclic parabolic subgroup Z , if every element of $\text{Stab}_G(Z_i) \cap \text{Stab}_G(Z'_i)$ has a power that fixes Z , then $Z \in \mathcal{Z}_i$. For every $\ell \in \{1, \dots, k\}$, we let \mathcal{G}_ℓ be a measured subgroupoid of \mathcal{G} such that for every $i \in I$, the groupoid $(\mathcal{G}_\ell)|_{Y_i}$ is equal to the $(\mathcal{G}|_{Y_i}, \rho)$ -stabilizer of $Z_{i,\ell}$.

Let now $\mathcal{K} \subseteq \mathcal{G}$ be a (\mathcal{G}, ρ) -vertex subgroupoid, and assume that $\mathcal{H} \cap \mathcal{H}'$ is stably contained in \mathcal{K} . Up to replacing Y^* by a conull Borel subset and refining the above Borel partition of Y^* , we can assume that for every $i \in I$, the groupoid $\mathcal{K}|_{Y_i}$ is equal to the $(\mathcal{G}|_{Y_i}, \rho)$ -stabilizer of some rank one parabolic subgroup P_i . We can further assume that every element of $\text{Stab}_G(Z_i) \cap \text{Stab}_G(Z'_i)$ has a power contained in $\text{Stab}_G(P_i)$: indeed, since ρ is action-type, each element of $\text{Stab}_G(Z_i) \cap \text{Stab}_G(Z'_i)$ has a power which is the ρ -image of an element of $(\mathcal{H} \cap \mathcal{H}')|_{Y_i}$. It follows from the above that $P_i \in \mathcal{Z}_i$. Assertion (2) follows.

We now prove that $\neg(1) \Rightarrow \neg(2)$. Assume that there exists a Borel subset $U \subseteq Y$ of positive measure such that the maps $\theta|_U$ and $\theta'|_U$ are constant, with values two cyclic parabolic subgroups Z and Z' that do not commute. Let $\mathcal{G}_1, \dots, \mathcal{G}_k$ be finitely many (\mathcal{G}, ρ) -vertex subgroupoids. Up to restricting to a further Borel subset of U of positive measure, we can assume that for every $\ell \in \{1, \dots, k\}$, the groupoid $(\mathcal{G}_\ell)|_U$ is equal to the $(\mathcal{G}|_U, \rho)$ -stabilizer of some rank one cyclic parabolic subgroup P_ℓ . By Lemma 3.16, there exists a cyclic parabolic subgroup $P \notin \{P_1, \dots, P_k\}$ such that $\text{Stab}_G(Z) \cap \text{Stab}_G(Z') \subseteq \text{Stab}_G(P)$. Let then \mathcal{K} be a measured subgroupoid of \mathcal{G} such that $\mathcal{K}|_U$ is equal to the $(\mathcal{G}|_U, \rho)$ -stabilizer of P , and $\mathcal{K}|_{Y \setminus U} = \mathcal{H}|_{Y \setminus U}$. Then $\mathcal{H} \cap \mathcal{H}'$ is stably contained in \mathcal{K} ; moreover \mathcal{K} is a (\mathcal{G}, ρ) -vertex subgroupoid. On the other hand, Lemma 3.14 implies that for every $\ell \in \{1, \dots, k\}$, there is no Borel subset $V \subseteq U$ of positive measure such that $\mathcal{K}|_V$ is contained in $(\mathcal{G}_\ell)|_V$. This shows that assertion (2) fails to hold. \square

3.8. CONCLUSION. — In this section, we complete the proof of our main theorem, which is Theorem 2 from the introduction (from which, as already explained, Theorem 1 follows using Theorem 1.5).

THEOREM 3.18. — *Let G_1 and G_2 be two transvection-free right-angled Artin groups with respective defining graphs Γ_1 and Γ_2 . If G_1 and G_2 are measure equivalent, then the extension graphs Γ_1^e and Γ_2^e are isomorphic.*

Notice that the only nontrivial transvection-free abelian right-angled Artin group is \mathbb{Z} ; all the other ones are nonamenable, and therefore not measure equivalent to \mathbb{Z} (see [Fur11, Cor. 3.2]). We will therefore assume without loss of generality that G_1 and G_2 are not cyclic. Using [HH20, Prop. 5.11] which builds on the seminal work of Furman [Fur99b], to prove Theorem 3.18 it suffices to show the following statement for measured groupoids.

THEOREM 3.19. — *Let G_1 and G_2 be two non-cyclic transvection-free right-angled Artin groups, with respective defining graphs Γ_1 and Γ_2 . Let \mathcal{G} be a measured groupoid over a standard finite measure space Y (of positive measure), and assume that for every $i \in \{1, 2\}$, there exists a strict action-type cocycle $\mathcal{G} \rightarrow G_i$.*

Then the extension graphs Γ_1^e and Γ_2^e are isomorphic.

Proof. — Let Z_1 be a rank one parabolic subgroup of G_1 (which also corresponds to a vertex of Γ_1^e). The (\mathcal{G}, ρ_1) -stabilizer \mathcal{G}_{Z_1} of Z_1 is a (\mathcal{G}, ρ_1) -vertex subgroupoid. In view of Proposition 3.15 – which characterizes (\mathcal{G}, ρ_1) -vertex subgroupoids in a purely groupoid-theoretic manner, i.e., without reference to the cocycle ρ_1 – it follows that \mathcal{G}_{Z_1} is also a (\mathcal{G}, ρ_2) -vertex subgroupoid. This means that there exist a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets of positive measure such that for every $i \in I$, the subgroupoid $(\mathcal{G}_{Z_1})_{|Y_i}$ is equal to the $(\mathcal{G}_{|Y_i}, \rho_2)$ -stabilizer of some rank one parabolic subgroup $Z_{2,i}$ of G_2 (i.e., a vertex of Γ_2^e), which is unique in view of Lemma 3.14. This allows us to define a Borel map $\theta_{Z_1} : Y^* \rightarrow V\Gamma_2^e$, with $\theta_{Z_1}(y) = Z_{2,i}$ whenever $y \in Y_i$ – in other words θ_{Z_1} is the parabolic map of $(\mathcal{G}_{Z_1}, \rho_2)$.

By varying Z_1 (and up to replacing Y^* by a conull Borel subset), we then get a Borel map $\theta : V\Gamma_1^e \times Y^* \rightarrow V\Gamma_2^e$, defined by letting $\theta(Z_1, y) = \theta_{Z_1}(y)$. We now claim that for a.e. $y \in Y^*$, the map $\theta(\cdot, y)$ actually determines a graph isomorphism between Γ_1^e and Γ_2^e , which will conclude our proof.

We first prove that for a.e. $y \in Y^*$, the map $\theta(\cdot, y)$ is injective. Otherwise, as $V\Gamma_1^e$ is countable, we can find a Borel subset $U \subseteq Y^*$ of positive measure, two distinct rank one parabolic subgroups $Z_1, Z'_1 \in V\Gamma_1^e$, and a rank one parabolic subgroup $Z_2 \in V\Gamma_2^e$, such that for all $y \in U$, one has $\theta_{Z_1}(y) = \theta_{Z'_1}(y) = Z_2$. This implies that the $(\mathcal{G}_{|U}, \rho_1)$ -stabilizers of Z_1 and Z'_1 are stably equivalent – as they are both stably equivalent to the $(\mathcal{G}_{|U}, \rho_2)$ -stabilizer of Z_2 . This contradicts Lemma 3.14.

We now prove that for a.e. $y \in Y$, the map $\theta(\cdot, y)$ is surjective. Let $Z_2 \in V\Gamma_2^e$. Then the (\mathcal{G}, ρ_2) -stabilizer \mathcal{G}_{Z_2} of Z_2 is a (\mathcal{G}, ρ_2) -vertex stabilizer. By Proposition 3.15, it is also a (\mathcal{G}, ρ_1) -vertex subgroupoid. This means that there exists a conull Borel subset Y^{**} of Y^* and a partition $Y^{**} = \bigsqcup_{j \in J} Y_j$ such that for every $j \in J$, the subgroupoid $(\mathcal{G}_{Z_2})_{|Y_j}$ is equal to the $(\mathcal{G}_{|Y_j}, \rho_1)$ -stabilizer of some rank one parabolic subgroup $Z_{1,j} \in V\Gamma_1^e$. For every $j \in J$ and a.e. $y \in Y_j$, we then have $\theta_{Z_{1,j}}(y) = Z_2$. As $V\Gamma_2^e$ is countable, surjectivity holds.

Finally, the fact that for a.e. $y \in Y$, the map $\theta(\cdot, y)$ preserves both adjacency and nonadjacency, follows from Lemma 3.17 combined with Proposition 3.15 – which

gives a purely groupoid-theoretic characterization of adjacency, with no reference to the cocycles. This completes our proof. \square

3.9. APPLICATION TO W^* -RIGIDITY. — We finish this section by establishing Corollary 3 from the introduction.

COROLLARY 3.20. — *Let G_1 and G_2 be two right-angled Artin groups with finite outer automorphism groups. Assume that G_1 and G_2 have free ergodic probability measure-preserving actions on standard probability spaces whose von Neumann algebras have isomorphic amplifications.*

Then G_1 and G_2 are isomorphic.

As mentioned in the introduction, if we assumed that the actions were W^* -equivalent (without passing to amplifications), then the corollary would also follow from [HHL20, Cor. 4.2]. It would also follow from recent work of Chifan and Kunnawalkam Elayavalli [CKE21, Th. 1.3]. In fact, the statement of [CKE21, Th. 1.3] does not mention amplifications, but the work in [CKE21, Th. 3.1] also deals with virtual Cartan subalgebras. For the convenience of the reader, we are including a proof of Corollary 3, explaining how it can be deduced from important work of Popa and Vaes [PV14].

Proof. — We will prove in the next paragraph of the proof that G_1 and G_2 satisfy the property $(\text{HH})^+$ introduced by Ozawa and Popa in [OP10b, Def. 1]. For now, let us explain why this is enough to prove our corollary. Assume that $G_1 \curvearrowright X$ and $G_2 \curvearrowright Y$ are two actions as in the statement, whose associated von Neumann algebras have isomorphic amplifications: $(L^\infty(X) \rtimes G_1)^s \simeq (L^\infty(Y) \rtimes G_2)^t$. As G_1 and G_2 satisfy property $(\text{HH})^+$, a theorem of Popa and Vaes [PV14, Th. 1.2 and Remark 1.3] implies that $L^\infty(X) \rtimes G_1$ and $L^\infty(Y) \rtimes G_2$ have a unique *virtual Cartan subalgebra* up to unitary conjugacy: more precisely, $L^\infty(X)$ is (up to unitary conjugacy) the unique maximal abelian subalgebra of $L^\infty(X) \rtimes G_1$ whose normalizer is a finite index subfactor of $L^\infty(X) \rtimes G_1$ (and likewise for $L^\infty(Y)$ inside $L^\infty(Y) \rtimes G_2$). By [OP10a, Prop. 4.12], the amplifications of these von Neumann algebras also have a unique virtual Cartan subalgebra (in the above sense) up to unitary conjugacy. The fact that $(L^\infty(X) \rtimes G_1)^s$ and $(L^\infty(Y) \rtimes G_2)^t$ are isomorphic means that there exist positive measure Borel subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that the restrictions to X' and Y' of the equivalence relations $\mathcal{R}_1, \mathcal{R}_2$ associated to the actions $G_1 \curvearrowright X, G_2 \curvearrowright Y$ have isomorphic von Neumann algebras: $L((\mathcal{R}_1)_{|X'}) \simeq L((\mathcal{R}_2)_{|Y'})$. The above uniqueness statement implies that this isomorphism can be chosen to send the Cartan subalgebra $L^\infty(X')$ to $L^\infty(Y')$. By a theorem of Feldman and Moore [FM77], this in turn implies that the equivalence relations $(\mathcal{R}_1)_{|X'}$ and $(\mathcal{R}_2)_{|Y'}$ are isomorphic, i.e., the actions $G_1 \curvearrowright X$ and $G_2 \curvearrowright Y$ are stably orbit equivalent. As two countable groups admit stably equivalent free ergodic probability measure-preserving actions if and only if they are measure equivalent (cf. [Fur99b, Th. 3.3] or [Gab02b, Th. 2.3]), we deduce that G_1 and G_2 are measure equivalent, whence isomorphic by Theorem 1.

We now prove that G_1 and G_2 satisfy property $(\text{HH})^+$. As property $(\text{HH})^+$ is stable under direct products [OP10b, Th. 2.3(2)], it is enough to prove that every irreducible nonamenable right-angled Artin group satisfies property $(\text{HH})^+$. By [OP10b, Th. 2.3(5)], it is enough to show that for every hyperplane h of the universal cover of the Salvetti complex of G , the G -stabilizer of h is not co-amenable in G . Since the stabilizer of h is conjugate to $G_{\text{lk}(v)}$ for some vertex $v \in V\Gamma$, and since G_Γ splits as an amalgamated free product $G_\Gamma = G_{\text{st}(v)} *_{G_{\text{lk}(v)}} G_{\Gamma \setminus \{v\}}$ (where the edge group has infinite index in both vertex groups as a consequence of the irreducibility of G_Γ), the conclusion follows from [CKE21, Lem. 2.4]. \square

4. TWO SOURCES OF FAILURE OF SUPERRIGIDITY

So far, our results have concerned measure equivalence classification within the class of right-angled Artin groups. Given a right-angled Artin group G_Γ and a countable group H which is measure equivalent to G_Γ , it is natural to ask what can be said about the structure of H . In particular, motivated by the strong measure equivalence rigidity of mapping class groups [Kid10] and some non-right-angled Artin groups [HH20], one naturally wonders whether H is virtually G_Γ . However, this turns out to always be false.

THEOREM 4.1. — *For every right-angled Artin group G , there exists a continuum of countable groups that are measure equivalent to G but pairwise non commensurable up to finite kernels.*

We are grateful to a referee for pointing out that our constructions actually yield a continuum of examples, improving the statement we had in an earlier version of this work. In this section, we will present two sources of examples leading to this phenomenon: graph products of amenable groups, and non-uniform lattices in the automorphism group of the universal cover of the Salvetti complex when G is non-abelian.

4.1. GRAPH PRODUCTS OF AMENABLE GROUPS. — Given a finite simple graph Γ and an assignment of a group G_v to every vertex v of Γ , we recall that the *graph product* over Γ with vertex groups $\{G_v\}_{v \in V\Gamma}$ is the group obtained from the free product of the groups G_v by adding as only extra relations that every element of G_v commutes with every element of G_w whenever v and w are adjacent in Γ . In particular, a right-angled Artin group is a graph product over its defining graph Γ , with all vertex groups isomorphic to \mathbb{Z} .

PROPOSITION 4.2. — *Let Γ be a finite simple graph. Let G and H be two graph products over Γ , with countable vertex groups $\{G_v\}_{v \in V\Gamma}$ and $\{H_v\}_{v \in V\Gamma}$, respectively. Assume that for every $v \in V\Gamma$, the groups G_v and H_v admit orbit equivalent free measure-preserving actions on standard probability spaces.*

Then G and H admit orbit equivalent free measure-preserving actions on standard probability spaces. In particular G and H are measure equivalent.

Notice that the last sentence of the lemma follows from the previous one by [Gab02b, Th. 2.3].

Proof. — The proof is an adaptation of an argument of Gaboriau for free products [Gab05, P_{ME}6], specializing ideas that he introduced for graphings in an earlier work [Gab00, Part IV]. Arguing by induction on the number of vertices of Γ , we can assume that there exists a unique vertex $v \in V\Gamma$ such that G_v is not isomorphic to H_v . Given any vertex $w \neq v$, we fix once and for all an isomorphism between G_w and H_w . We will construct orbit equivalent actions of G and H .

Recall that every countable group G has a free measure-preserving ergodic action on a standard probability space, by considering the Bernoulli action $G \curvearrowright \{0, 1\}^G$, which preserves the probability measure $(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes G}$. Let Y_v be a standard probability space equipped with orbit equivalent free measure-preserving actions of G_v and H_v . Up to replacing Y_v by an invariant conull Borel subset, we can assume that for every $x \in Y_v$, one has $G_v \cdot x = H_v \cdot x$. Consider also a free measure-preserving action of G on a standard probability space Z . Let $X = Z \times Y_v$ (equipped with the product probability measure). Let $\psi_v : G \rightarrow G_v$ be the retraction sending all other factors to $\{1\}$. Then we have a G -action on X , where the action on Y_v is through ψ_v . This action is free and measure-preserving. Let $p : X \rightarrow Y_v$ be the projection map, which is G -equivariant.

We now construct an action of H on X , in the following way. We first define the action of H_v , and we will then extend it to the whole group H . Let $x \in X$, and let $h \in H_v$. As the G_v -action on Y_v is free and orbit equivalent to the H_v -action, there is a unique element $g \in G_v$ such that $gp(x) = hp(x)$, and we then let $hx = gx$. We claim that this gives a well-defined action of H_v on X . Indeed, let $x \in X$, and let $h_1, h_2 \in H_v$. Then there exists a unique element $g_2 \in G_v$ such that $h_2p(x) = g_2p(x)$, and a unique element $g_1 \in G_v$ such that $h_1(h_2p(x)) = g_1(g_2p(x))$, and g_1g_2 is then the unique element of G_v such that $(h_1h_2)p(x) = (g_1g_2)p(x)$. This ensures that $(h_1h_2)x = h_1(h_2x)$. Notice also that this action of H_v on X is free (as the action of H_v on Y_v is free).

We now claim that this action of H_v on X commutes with the original action of $H_{\text{lk}(v)} = G_{\text{lk}(v)}$. Indeed, let $x \in X$, let $h \in H_v$ and let $k \in G_{\text{lk}(v)}$. There is a unique element $g \in G_v$ such that $hp(x) = gp(x)$. As the action of $G_{\text{lk}(v)}$ on Y_v is trivial, we also have $hp(kx) = gp(kx)$, and therefore $h k x = g k x = k g x = k h x$, as desired (here the first and third equalities come from the definition of the H_v -action, and the second comes from the fact that g and k commute in G).

The above claim ensures that the H_v -action on X defined above extends to an action of H , which coincides with the original action of $G_w = H_w$ on each factor with $w \neq v$. Also, by construction, this H -action on X is orbit equivalent to the original G -action. In particular it is measure-preserving: given $h \in H$ and a Borel subset $A \subseteq X$, the fact that A and hA have the same measure is proved by decomposing A into countably many mutually disjoint Borel subsets A_i so that on A_i , the action of h coincides with the action of a given element $g_i \in G$.

We finally claim that this H -action on X is free. Indeed, let $h \in H$ and $x \in X$ be such that $hx = x$. Using the graph product structure, we can write $h = h_1 \dots h_k$, where for every $i \in \{1, \dots, k\}$, the element h_i is nontrivial and belongs to one of the vertex groups of the graph product, and if h_i and h_j belong to the same vertex group H_{ij} for some $i \neq j$, then there exists $i < k < j$ such that the associated vertex group H_k does not belong to the star of H_{ij} . By construction, there exists an element $g = g_1 \dots g_k$ satisfying $gx = x$, and such that for every $i \in \{1, \dots, k\}$, the element g_i is nontrivial, and if $h_i \in H_w$, then $g_i \in G_w$ (in fact $h_i = g_i$ whenever $w \neq v$, through our chosen isomorphism between H_w and G_w); here the nontriviality of g_i when $h_i \in H_v$ comes from the freeness of the H_v -action on X . As the G -action on X is free, it follows that $g = \text{id}$. The graph product structure thus ensures that $k = 0$ (this is essentially a consequence of the normal form theorem in [Gre90], see also [HM95, §3]), and therefore $h = \text{id}$. This completes our proof. \square

As a consequence of Proposition 4.2 and a theorem of Ornstein and Weiss [OW80] saying that any two ergodic measure-preserving free actions of countably infinite amenable groups are orbit equivalent, we reach the following corollary.

COROLLARY 4.3. — *For every finite simple graph Γ , any two graph products over Γ with countably infinite amenable vertex groups are measure equivalent.*

In particular, given any right-angled Artin group G , there exist infinitely many pairwise non-commensurable right-angled Artin groups that are measure equivalent to G . \square

In order to prove Theorem 4.1, we are thus left with constructing a continuum of pairwise non-commensurable graph products over Γ with amenable vertex groups. The basic idea is to let one of the vertex groups vary, using a continuum of amenable groups.

Proof of Theorem 4.1. — Let Γ be the defining graph of G . A theorem of Bartholdi and Erschler [BE04] ensures that there exists $\alpha < 1$ such that for every $t \in (\alpha, 1)$, there exists a finitely generated amenable group A_t whose growth function is equivalent to $n \mapsto e^{n^t}$ (for the equivalence relation \sim where $f \sim g$ if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there exists $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$, one has $f(n) \leq g(Cn)$). Moreover we can assume that A_t is torsion-free [BE04, Rem. 1.1]. Let $v \in V\Gamma$ be a vertex, and for every $t \in (\alpha, 1)$, let G_t be the graph product over Γ such that the vertex group G_v is A_t and all other vertex groups are \mathbb{Z} . Then G_t is torsion-free [AM15, Cor. 5.9]. By Corollary 4.3, all groups G_t are measure equivalent, so we are left with proving that they are pairwise not commensurable (hence not commensurable up to finite kernel).

Given $t \in (\alpha, 1)$, we claim that every amenable subgroup $H \subseteq G_t$ is contained in an amenable subgroup K which is either finitely generated and virtually abelian, or equal to the direct product of A_t and a finitely generated virtually abelian subgroup of G_t . We prove this by induction on the number of vertices of Γ . Let $\Gamma = \Gamma_0 \circ \Gamma_1 \circ \dots \circ \Gamma_k$ be

the de Rham decomposition of Γ with Γ_0 being the clique factor. The claim is trivial if $\Gamma = \Gamma_0$. Now we assume $\Gamma \neq \Gamma_0$. If this decomposition is non-trivial, we know the claim is true for each G_{Γ_i} by induction. As $G_\Gamma = \bigoplus_{i=0}^k G_{\Gamma_i}$, the claim follows by considering the projection of H to each G_{Γ_i} . If the de Rham decomposition has only one factor, then [AM15, Th. 4.1] ensures that either H is virtually cyclic (and we are done), or else H is contained in $G_{\Gamma'}$ for some proper subgraph Γ' of Γ , in which case the claim follows by induction.

Now take $t < t'$ and suppose that G_t has a finite index subgroup G_t^0 which is isomorphic to a finite index subgroup $G_{t'}^0$ of $G_{t'}$. Let $A_t^0 = G_t^0 \cap A_t$, whose growth function is also equivalent to $n \mapsto e^{n^{t'}}$. By the previous paragraph, $A_{t'}^0$ can be realized as a subgroup of $H = A_t \times P$ where P is virtually isomorphic to \mathbb{Z}^ℓ for some integer ℓ . As the growth of H is equivalent to $n^\ell e^{n^t} \prec e^{n^{t'}}$ and the map $A_{t'}^0 \rightarrow H$ is injective and Lipschitz with respect to the word metric, this leads to a contradiction. So G_t is not commensurable to $G_{t'}$. □

REMARK 4.4. — Given a right-angled Artin group G , let us explain how the methods used in the present section can be leveraged to build two free, ergodic, measure-preserving actions of G on standard probability spaces which are orbit equivalent but not conjugate.

Let $v \in VT$ be a vertex in the defining graph of G . Let $G \curvearrowright Z$ be a Bernoulli action, and let $G_v \curvearrowright Y_1$ and $G_v \curvearrowright Y_2$ be two Bernoulli actions of the infinite cyclic group G_v whose base spaces have distinct entropies (so these actions are orbit equivalent but not conjugate). Let θ be the product action of G on $Z \times Y_1$ (equipped with the product probability measure), using the retraction $G \rightarrow G_v$ for the action on the second factor. Let θ' be the action obtained from θ by replacing $G_v \curvearrowright Y_1$ by $G_v \curvearrowright Y_2$ as in the proof of Proposition 4.2. More precisely, the orbit equivalence between the G_v -actions on Y_1 and Y_2 yields an isomorphism $\varphi : Y_2 \rightarrow Y_1$ (which extends to $\varphi : Z \times Y_2 \rightarrow Z \times Y_1$, extending by the identity on Z) and a cocycle $c : G_v \times Y_2 \rightarrow G_v$. The action θ' is defined on the standard generators of G as follows: if g_v is the standard generator associated to v , then $\theta'(g_v)(x) = \varphi^{-1}(\theta(c(g_v, \pi(x))(\varphi(x))))$, where $\pi : Z \times Y_2 \rightarrow Y_2$ is the projection; if g_w is a standard generator associated to a vertex $w \neq v$, then $\theta'(g_w)(x) = \varphi^{-1}(\theta(g_w)(\varphi(x)))$. The actions θ and θ' are then free and orbit equivalent as in the proof of Proposition 4.2. As θ is ergodic, we know from the orbit equivalence that θ' is ergodic.

We claim that these two actions are not conjugate. Indeed, assume by contradiction that θ and θ' are conjugate, through a group automorphism $\alpha : G \rightarrow G$ and an isomorphism $f : Z \times Y_2 \rightarrow Z \times Y_1$. Note that if $g \in G$ has trivial projection to G_v (under the natural retraction), then the space of ergodic components for the θ -action $\langle g \rangle \curvearrowright Z \times Y_1$ can be identified with Y_1 . If $g \in G$ has nontrivial projection to G_v , then $\langle g \rangle$ acts ergodically on $Z \times Y_1$ (for the action θ).

Take a vertex $w \neq v$. By construction, the action (under θ') of G_w on $Z \times Y_2$ is not ergodic. As this is conjugate to the action (under θ) of $H = \alpha(G_w)$ on $Z \times Y_1$, the

θ -action of H on $Z \times Y_1$ is not ergodic. Hence H has trivial projection to G_v . By the above description of the spaces of ergodic components of the actions $G_w \curvearrowright Z \times Y_2$ (under θ') and $H \curvearrowright Z \times Y_1$ (under θ), we see that there must exist a measure-preserving map $f_1 : Z \times Y_2 \rightarrow Z$ and a measure space isomorphism $f_2 : Y_2 \rightarrow Y_1$ such that for almost every $(z, y) \in Z \times Y_2$, one has $f(z, y) = (f_1(z, y), f_2(y))$. Let $r_v : G \rightarrow G_v$ be the retraction sending generators which are not v to identity. As $r_v(\alpha(G_w)) = \{1\}$ for every $w \neq v$, and the subgroups $\alpha(G_v)$ together with all other $\alpha(G_w)$ generate G , we deduce that r_v induces an isomorphism between $\alpha(G_v)$ and G_v . Therefore, the map f_2 must conjugate the G_v -action on Y_2 (under θ') to the action of G_v (identified with $r_v(\alpha(G_v))$) on Y_1 . This contradicts the fact that the entropies of the actions $G_v \curvearrowright Y_1$ and $G_v \curvearrowright Y_2$ are distinct.

As a final remark, let us mention that if we further assume that $\text{Out}(G)$ is finite, then the above argument can be extended to construct a continuum of pairwise nonconjugate actions. A subtle point in the above argument comes from the slight asymmetry in the roles of θ and θ' : we have a complete description of the cyclic subgroups of G that act ergodically for θ , but not for θ' . When $\text{Out}(G)$ is finite, the group H appearing in the above argument is automatically conjugate to a cyclic group generated by one of the standard generators of G . We claim that H acts ergodically on $Z \times Y_2$ if and only if it is conjugate to G_v , and otherwise its space of ergodic components is isomorphic to Y_2 . Indeed, for every vertex $u \in V\Gamma$, the θ' -action of G_u on $Z \times Y_2$ is orbit equivalent (by construction) to the θ -action of G_u on $Z \times Y_1$, so one is ergodic if and only if the other is ergodic. And the action of G_v on $Z \times Y_1$ is ergodic (being a product of two mixing actions), while the action of G_w on $Z \times Y_1$ is not if $w \neq v$, which proves our claim. Arguing as in the previous paragraph and letting θ_t be the action built from θ by using a base space with entropy t in place of Y_2 , we can then deduce that θ_t and $\theta_{t'}$ are nonconjugate as soon as $t \neq t'$.

4.2. NON-UNIFORM LATTICES ACTING ON SALVETTI COMPLEXES. — Let X be a locally finite polyhedral complex, and let $\text{Aut}(X)$ be the group of cellular automorphisms of X , equipped with the compact-open topology. Then $\text{Aut}(X)$ is a second countable locally compact topological group, see e.g. [CdH16, Ex. 5.B.11]. Let μ be a left invariant Haar measure on $\text{Aut}(X)$. A discrete subgroup $H \subseteq \text{Aut}(X)$ is a *lattice* if the quotient space $H \backslash \text{Aut}(X)$ (with respect to the left translation action of H on $\text{Aut}(X)$) has finite μ -measure. Note that any two lattices in $\text{Aut}(X)$ are measure equivalent (see e.g. [Fur99a, Ex. 1.2]). We will use the following characterization of lattices due to Serre [Ser71], see also [BL01, Chap. 1.5].

THEOREM 4.5 (Serre). — *Let X be a locally finite polyhedral complex with finitely many $\text{Aut}(X)$ -orbits of cells. Let $H \subseteq \text{Aut}(X)$ be a discrete subgroup. Let \mathcal{O} be the set of all H -orbits of the vertex set of X . For each orbit $x \in \mathcal{O}$, let α_x be the cardinality of the H -stabilizer of an element in the orbit x .*

Then H is a lattice in $\text{Aut}(X)$ if and only if $\sum_{x \in \mathcal{O}} 1/\alpha_x < \infty$.

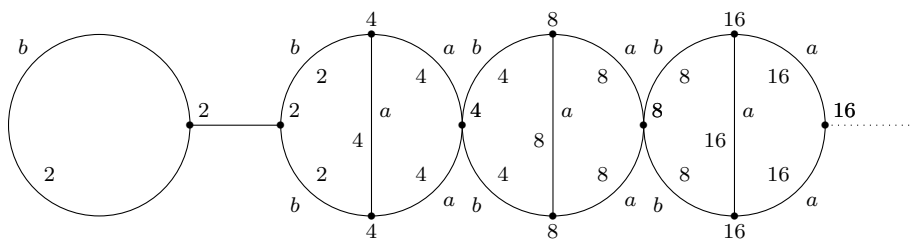


FIGURE 1. A label-preserving lattice in $\text{Aut}(T)$.

We will be interested in the case where $X = \tilde{S}_\Gamma$, in which case the above theorem applies (i.e., X is a locally finite polyhedral complex with finitely many $\text{Aut}(X)$ -orbits of cells).

PROPOSITION 4.6. — *Let Γ be a finite simple graph which is not a complete graph. Then there is a non-uniform lattice H in $\text{Aut}(\tilde{S}_\Gamma)$ which is not finitely generated. In particular H is measure equivalent to G_Γ but not commensurable to G_Γ .*

Proof. — Since Γ is not a complete graph, there are two vertices $a, b \in V\Gamma$ which are not adjacent. Let $T \subset \tilde{S}_\Gamma$ be a standard subcomplex of type $\{a, b\}$. Then T is a 4-valent tree. Recall that edges of \tilde{S}_Γ are labeled by vertices of Γ .

We start with a non-uniform lattice H in $\text{Aut}(T)$ whose action on T preserves the labeling of edges. Such H always exists, and we give a concrete example as follows in terms of the graph of group structure on $H \backslash T$, see Figure 1. Each edge of the graph is labeled by a or b . Each vertex group and edge group is a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$, and we choose injective homomorphisms from the edge groups to the incident vertex groups. The number at each vertex (resp. edge) represents the order of the vertex group (resp. edge group). For example, if a vertex is labeled by 8, that means the vertex group is a direct sum of 3 copies of $\mathbb{Z}/2\mathbb{Z}$. The graph is infinite, though only a finite portion of the graph is drawn, the omitted portion of the graph follows a similar pattern. One readily checks that the Bass–Serre tree of the graph of groups in Figure 1 is a 4-valent tree with the standard edge labeling by a and b . Moreover, according to Theorem 4.5, the fundamental group H of this graph of groups is a lattice in $\text{Aut}(T)$.

The next step is to extend the action $H \curvearrowright T$ to obtain a lattice in $\text{Aut}(\tilde{S}_\Gamma)$. For this purpose, we adjust the argument in [Hua18, Th. 9.5]. We first construct an auxiliary complex X as follows. Consider the homomorphism $h : G_\Gamma \rightarrow F_{a,b}$ to the free group on a and b defined by sending any generator in $V\Gamma \setminus \{a, b\}$ to the identity element (and sending a to a and b to b). Let X be the cover of S_Γ corresponding to $\ker h$. We orient each edge of S_Γ and equip X with the induced edge labeling and orientation from S_Γ . Note that the vertex set of X and the vertex set of T can be identified, and T is a subcomplex of X . The 1-skeleton $X^{(1)}$ is obtained from T by attaching a collection of circles based at each vertex of T – one circle for each vertex in $V\Gamma \setminus \{a, b\}$. The action $H \curvearrowright T$ clearly extends to an action $\rho : H \curvearrowright X^{(1)}$ such that ρ preserves labels of edges and preserves orientation of edges whose labels are in $V\Gamma \setminus \{a, b\}$. As ρ is label-preserving, it extends to an action $H \curvearrowright X$.

Let now $H' \subseteq \text{Aut}(\tilde{S}_\Gamma)$ be the subgroup made of all automorphisms which are lifts of elements of H . Letting $\pi : \tilde{S}_\Gamma \rightarrow X$ be the covering map, and $\text{Aut}(\pi)$ the corresponding group of deck transformations, we have a short exact sequence

$$1 \longrightarrow \text{Aut}(\pi) \longrightarrow H' \longrightarrow H \longrightarrow 1.$$

We thus have $H' \backslash \tilde{S}_\Gamma = H \backslash X$, and in particular there is a 1-1 correspondence between H' -orbits of $\tilde{S}_\Gamma^{(0)}$ and H -orbits of $T^{(0)}$. Moreover, as every element of H fixing a point $x \in X$ has a unique lift fixing a given preimage \tilde{x} of x , the cardinality of vertex stabilizers is preserved under this correspondence. Theorem 4.5 therefore implies that H' is a lattice in $\text{Aut}(\tilde{S}_\Gamma)$. As H is a non-uniform lattice acting on a tree, H is not finitely generated, see [BL01, Cor. 5.16]. Using the above short exact sequence, we deduce that H' is not finitely generated either. \square

REMARK 4.7. — The above proof involves extending the action of a group on a convex subcomplex Y of a CAT(0) cube complex X to an action of a larger group on X . This has been studied in several other contexts, see e.g. [Hag08, HW08].

REMARK 4.8. — If one wants to construct further examples of groups that are measure equivalent to a given right-angled Artin group G , a natural attempt is to embed G as a lattice in some locally compact second countable topological group \hat{G} and find other lattices in \hat{G} . It is therefore natural to try to classify all possible groups \hat{G} in which G embeds as a lattice. Actually, in the case where $\text{Out}(G)$ is finite and G does not split as a product, the compactly generated groups \hat{G} as above admits a more explicit description as follows. First, by work of Bader, Furman and Sauer [BFS20, Th. 1.4 & Th. A], G is cocompact on \hat{G} . Thus \hat{G} has a quasi-action on G . The possible groups \hat{G} are then understood from work of Kleiner and the second named author [KH18]: there is a homomorphism with compact kernel and cocompact image from \hat{G} to the automorphism group of a cube complex which is a small “deformation” of the universal cover of the Salvetti complex.

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