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EULER-POINCARÉ PAIRING, DIRAC INDEX AND  
ELLIPTIC PAIRING FOR HARISH-CHANDRA MODULES

BY DAVID RENARD

ABSTRACT. — Let  $G$  be a connected real reductive group with maximal compact subgroup  $K$  of equal rank, and let  $\mathcal{M}$  be the category of Harish-Chandra modules for  $G$ . We relate three differently defined pairings between two finite length modules  $X$  and  $Y$  in  $\mathcal{M}$ : the Euler-Poincaré pairing, the natural pairing between the Dirac indices of  $X$  and  $Y$ , and the elliptic pairing of [2]. (The Dirac index  $\mathrm{I}_{\mathrm{Dir}}(X)$  is a virtual finite-dimensional representation of  $\tilde{K}$ , the spin double cover of  $K$ .) We construct index functions  $f_X$  for any finite length Harish-Chandra module  $X$ . Each of these functions is very cuspidal in the sense of Labesse, and its orbital integral on elliptic elements coincides with the character of  $X$ . From this we deduce that the Dirac index pairing coincide with the elliptic pairing. Analogy with the case of Hecke algebras studied in [8] and [7] and a formal (but not rigorous) computation led us to conjecture that the first two pairings coincide. We show that they are both computed as the indices of Fredholm pairs (defined here in an algebraic sense) of operators acting on the same spaces. Recently, Huang and Sun have established the equality between the Euler-Poincaré and the elliptic pairing, thereby proving directly the analogue of a result of Schneider and Stuhler for  $p$ -adic groups [25].

RÉSUMÉ (Accouplement d'Euler-Poincaré, indice de Dirac et accouplement elliptique des modules de Harish-Chandra)

Soit  $G$  un groupe réductif réel connexe et soit  $K$  un sous-groupe compact maximal que l'on suppose de même rang. Nous relierons trois accouplements entre modules de Harish-Chandra de longueur finie  $X$  et  $Y$  : l'accouplement d'Euler-Poincaré, l'accouplement naturel entre les indices de Dirac de  $X$  et  $Y$  et l'accouplement elliptique d'Arthur [2] (l'indice de Dirac  $\mathrm{I}_{\mathrm{Dir}}(X)$  est une représentation virtuelle de dimension finie de  $\tilde{K}$ , le revêtement Spin à deux feuillets de  $K$ ). Nous construisons des fonctions indices  $f_X$  pour tout module de Harish-Chandra de longueur finie  $X$ . Chacune de ces fonctions est très cuspidale au sens de Labesse, et son intégrale orbitale coïncide sur les éléments elliptiques avec le caractère de  $X$ . De ceci nous déduisons que l'accouplement naturel des indices de Dirac coïncide avec l'accouplement elliptique. Une analogie avec le cas des algèbres de Hecke considéré dans [8] et [7] et un calcul formel (mais non rigoureux) nous ont amenés à conjecturer que les deux premiers accouplements coïncident eux aussi. Nous montrons qu'ils peuvent tout deux être exprimés comme indices de paires de Fredholm (définis ici dans un sens algébrique) d'opérateurs agissant sur les mêmes espaces. Récemment Huang et Sun ont établi l'égalité entre accouplement d'Euler-Poincaré et accouplement elliptique, démontrant ainsi directement l'analogie d'un résultat de Schneider et Stuhler pour les groupes  $p$ -adiques [25].

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## 1. INTRODUCTION

Let  $G$  be the group of rational points of a connected algebraic reductive group defined over a local nonarchimedean field  $F$  of characteristic 0. Assume that  $G$  has compact center. Let  $\mathcal{M} = \mathcal{M}(G)$  be the category of smooth representations of  $G$ . This is also the category of non-degenerate modules over the Hecke algebra  $\mathcal{H} = \mathcal{H}(G)$  of  $G$ . It is known from the work of J. Bernstein that  $\mathcal{M}$  has finite cohomological dimension. Furthermore, for any finitely generated module  $(\pi, V)$  in  $\mathcal{M}$ , Schneider and Stuhler construct an explicit resolution of  $(\pi, V)$  by finitely generated projective modules. They also establish a general theory of Euler-Poincaré functions for modules of finite length, generalizing results of Kottwitz ([20]). Namely, for any finite length modules  $(\pi, V)$ ,  $(\pi', V')$  in  $\mathcal{M}$ , one can define their Euler-Poincaré pairing:

$$(1.1) \quad \mathbf{EP}(\pi, \pi') = \sum_i (-1)^i \dim \operatorname{Ext}_{\mathcal{M}}^i(\pi, \pi').$$

They construct functions  $f_\pi$  (Euler-Poincaré functions) in  $\mathcal{H}$  such that

$$(1.2) \quad \mathbf{EP}(\pi, \pi') = \Theta_{\pi'}(f_\pi),$$

where  $\Theta_{\pi'}$  is the distribution-character of  $\pi'$  (a linear form on  $\mathcal{H}$ ). Following [9] and [27], let us now give another point of view on these functions.

Let  $\mathcal{K}(G)$  be the Grothendieck group of finitely generated projective modules in  $\mathcal{M}$ . Since  $\mathcal{M}$  has finite cohomological dimension, this is also the Grothendieck group of all finitely generated modules in  $\mathcal{M}$ . Let  $\mathcal{R} = \mathcal{R}(G)$  be the Grothendieck group of finite length modules in  $\mathcal{M}$ . If  $(\pi, V)$  is a finitely generated (resp. finite length) module in  $\mathcal{M}$ , we denote by  $[\pi]$  its image in Grothendieck group  $\mathcal{K}$  (resp.  $\mathcal{R}$ ). Set  $\mathcal{K}_{\mathbb{C}} = \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{C}$  and  $\mathcal{R}_{\mathbb{C}} = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$ . Let  $(\pi, V)$  be a finite length module in  $\mathcal{M}$ , and

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow V \longrightarrow 0$$

be a resolution of  $\pi$  by finitely generated projective modules. Then

$$(1.3) \quad \mathbf{EP} : \mathcal{R} \longrightarrow \mathcal{K}, \quad [\pi] \longmapsto \sum_i (-1)^i [P_i]$$

is a well-defined map. Let  $\overline{\mathcal{H}} = \overline{\mathcal{H}(G)} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$  be the abelianization of  $\mathcal{H}$ . The Hattori rank map

$$(1.4) \quad \text{Rk} : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$$

is defined as follows. Let  $P$  be a finitely generated projective module in  $\mathcal{M}$ . Write  $P$  as a direct factor of some  $\mathcal{H}^n$  and let  $e \in \text{End}_{\mathcal{H}}(\mathcal{H}^n)$  be the projector onto  $P$ . Then the trace of  $e$  is an element of  $\mathcal{H}$ , and its image in  $\overline{\mathcal{H}}$  is well-defined. This defines  $\text{Rk}([P])$ . An alternative description of  $\overline{\mathcal{H}}$  as the ‘‘cocenter’’ of the category  $\mathcal{M}$  gives a natural definition of  $\text{Rk}$  for finitely generated projective modules (see [9, §1.3]).

Let  $\mathcal{D}'(G)$  be the space of distributions on  $G$ , and let  $\mathcal{D}'(G)^G$  be the subspace of invariant distributions. Fix a Haar measure on  $G$ , so that  $\mathcal{H}$  is identified with the convolution algebra of compactly supported smooth functions on  $G$ . The orthogonal of  $\mathcal{D}'(G)^G$  in  $\mathcal{H}$  for the natural pairing between  $\mathcal{D}'(G)$  and  $\mathcal{H}$  is exactly  $[\mathcal{H}, \mathcal{H}]$ , so there is an induced non-degenerate pairing:

$$\mathcal{D}'(G)^G \times \overline{\mathcal{H}} \longrightarrow \mathbb{C}, \quad (T, f) \longmapsto \langle T, f \rangle = T(f).$$

Let  $(\pi', V')$  be a finite length module in  $\mathcal{M}$ . Its distribution character  $\Theta_{\pi'}$  is an element of  $\mathcal{D}'(G)^G$ . This defines a pairing

$$(1.5) \quad \mathcal{R}_{\mathbb{C}} \times \overline{\mathcal{H}} \longrightarrow \mathbb{C}, \quad (\pi', f) \longmapsto \langle \Theta_{\pi'}, f \rangle = \Theta_{\pi'}(f).$$

With the notation above, we have the following identity: for all finite length modules  $(\pi, V), (\pi', V')$  in  $\mathcal{M}$ ,

$$(1.6) \quad \mathbf{EP}(\pi, \pi') = \langle \Theta_{\pi'}, \text{Rk} \circ \mathbf{EP}(\pi) \rangle.$$

Thus, the image in  $\overline{\mathcal{H}}$  of the Euler-Poincaré function  $f_{\pi}$  constructed by Schneider and Stuhler is  $\text{Rk} \circ \mathbf{EP}(\pi)$  ([9, Lem. 3.7]).

There is a third way of seeing the space  $\overline{\mathcal{H}}$ , namely, as the space of orbital integrals on  $G$ . More precisely, recall that for a regular semisimple element  $x$  in  $G$ , one can define the orbital integral of  $f \in \mathcal{H}$  at  $x$  as

$$\Phi(f, x) = \int_{G/T} f(gxg^{-1}) \, d\dot{g},$$

where  $T$  is the unique maximal torus containing  $x$ , and  $d\dot{g}$  is an invariant measure on  $G/T$ . When  $f$  is fixed,  $x \mapsto \Phi(f, x)$  is a smooth invariant function on  $G_{\text{reg}}$ , and we denote by  $\mathcal{I}(G)$  the image of  $\Phi : f \in \mathcal{H} \mapsto \Phi(f, \cdot)$  in the space of smooth invariant functions on  $G_{\text{reg}}$ . This space can be explicitly described (by properties of orbital integrals, see [26]). Furthermore the kernel of  $\Phi$  is exactly  $[\mathcal{H}, \mathcal{H}]$  (this is called the geometric density theorem, i.e., the density of the space generated by the distributions  $f \mapsto \Phi(f, x), x \in G_{\text{reg}}$ , in  $\mathcal{D}'(G)^G$ ). Thus, we have an exact sequence

$$0 \longrightarrow [\mathcal{H}, \mathcal{H}] \longrightarrow \mathcal{H} \longrightarrow \mathcal{I}(G) \longrightarrow 0$$

and  $\overline{\mathcal{H}}$  is identified with the space  $\mathcal{I}(G)$  of orbital integrals.

Let us denote by  $G_{\text{ell}}$  the space of regular semisimple elliptic elements in  $G$  (i.e., elements whose centralizer is a compact maximal torus in  $G$ ), and by  $\mathcal{H}_c$  the subspace of functions  $f \in \mathcal{H}$  such that  $\Phi(f, x) = 0$  when  $x$  is not elliptic. Accordingly,

we write  $\overline{\mathcal{H}}_c$  for the image of this subspace in  $\overline{\mathcal{H}}$ . This latter space is isomorphic to the subspace  $\mathcal{I}(G)_c$  of orbital integrals which vanish outside  $G_{\text{ell}}$ . The Selberg principle for Euler-Poincaré functions ([25, Rem. II.4.11]) asserts that the functions  $f_\pi$  are in  $\mathcal{H}_c$ . In fact, it is a theorem of Brylinski-Blanc [4], (see also [9] and [10]) that the image Hattori rank map  $\text{Rk}$  is exactly  $\overline{\mathcal{H}}_c$ .

Recall that by Harish-Chandra regularity theorem, the character  $\Theta_{\pi'}$  of a finite length module  $(\pi', V') \in \mathcal{M}$  is given by a locally integrable function, denoted by  $\theta_{\pi'}$ , on  $G$ , which is smooth on the open dense subset of regular semisimple elements  $G_{\text{reg}}$ , i.e., for all  $f \in \mathcal{H}$ ,

$$\langle \Theta_{\pi'}, f \rangle = \int_{G_{\text{reg}}} \theta_{\pi'}(x) f(x) dx.$$

(Once again, we have chosen a Haar measure on  $G$ ).

Following Kazhdan [17], let us denote by  $C^{\text{ell}}$  the set of regular semisimple elliptic conjugacy classes in  $G$ . Then any orbital integral  $\Phi(f, \cdot)$  or any character function  $\theta_{\pi'}$  can be viewed as functions on  $C^{\text{ell}}$ . By [17, §3, Lem. 1], there is a unique measure  $dc$  on  $C^{\text{ell}}$  such that for all  $f \in \mathcal{H}$  with support in  $G_{\text{ell}}$ ,

$$\int_G f(x) dx = \int_{C^{\text{ell}}} \Phi(f, c) dc.$$

The elliptic pairing between two finite length modules  $(\pi, V)$  and  $(\pi', V')$  in  $\mathcal{M}$  is then defined by

$$(1.7) \quad \langle \pi, \pi' \rangle_{\text{ell}} = \int_{C^{\text{ell}}} \theta_\pi(c) \theta_{\pi'}(c^{-1}) dc.$$

Let us now relate this elliptic pairing to the Euler-Poincaré pairing (1.1). On a regular semisimple elliptic element  $x$ , the orbital integrals of the Euler-Poincaré functions  $f_\pi$  at  $x$  coincide with the character  $\theta_\pi$  of  $\pi$  at  $x^{-1}$  ([25, Th. III.4.16]):

$$(1.8) \quad (\forall x \in G_{\text{ell}}), \quad \Phi(f_\pi, x) = \theta_\pi(x^{-1}).$$

Therefore, the following formula is established ([25, Th. III.4.21]):

$$(1.9) \quad \begin{aligned} \mathbf{EP}(\pi, \pi') &= \langle \Theta_{\pi'}, f_\pi \rangle = \int_{G_{\text{ell}}} \theta_{\pi'}(x) f_\pi(x) dx \\ &= \int_{C^{\text{ell}}} \theta_{\pi'}(c) \theta_\pi(c^{-1}) dc = \langle \pi, \pi' \rangle_{\text{ell}}. \end{aligned}$$

Let us mention that it is explained in [9] how these results can be formulated and proved when the center of  $G$  is not compact.

Many of the objects and results described above make sense over an Archimedean field as well. So assume now that  $G$  is a real reductive group, connected (as a Lie group). Fix a maximal compact subgroup  $K$  of  $G$  and also assume that  $G$  and  $K$  have same rank, i.e.,  $G$  admits discrete series representations. The category of representations we are now considering is the category of Harish-Chandra modules  $\mathcal{M} = \mathcal{M}(\mathfrak{g}, K)$  ([18]). It is also known that  $\mathcal{M}$  has finite cohomological dimension, and furthermore resolutions of finite length modules by explicit finitely generated projective modules can be constructed ([18, §II.7]). Thus, the Euler-Poincaré pairing of

two modules of finite length is well-defined by (1.1). The elliptic pairing (1.9) is also defined, and the purpose of this paper is to relate these two pairings by constructing Euler-Poincaré functions in that case, i.e., compactly supported smooth functions on  $G$  satisfying (1.2), but also (1.8), so that the formula (1.9) is valid. An interesting aspect is that this relation involves a third pairing defined through Dirac cohomology.

Dirac cohomology of Harish-Chandra modules was introduced by D. Vogan (see [15]). Dirac cohomology of a Harish-Chandra module  $X$  of finite length consists in a finite dimensional representation  $H_{\text{Dir}}(X)$  of the compact group  $\tilde{K}$ , the spin cover of  $K$ . In fact,  $H_{\text{Dir}}(X)$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded, and a slightly more refined invariant, the Dirac index  $l_{\text{Dir}}(X)$  can be defined as the formal difference  $l_{\text{Dir}}(X) = H_{\text{Dir}}(X)^{\bar{0}} - H_{\text{Dir}}(X)^{\bar{1}}$  between the even and odd part of  $H_{\text{Dir}}(X)$ , a virtual finite-dimensional representation of  $\tilde{K}$ . If  $[\sigma, \sigma']_{\tilde{K}}$  is the usual pairing between two virtual finite-dimensional representations  $\sigma, \sigma'$  of  $\tilde{K}$ , then the third pairing between two Harish-Chandra modules  $X$  and  $Y$  of finite length that we introduce is

$$(1.10) \quad \langle X, Y \rangle_{\text{Dir}} = [l_{\text{Dir}}(X), l_{\text{Dir}}(Y)]_{\tilde{K}}.$$

In Section 3, we use Labesse “index” functions ([21]) to construct index functions  $f_X$  for any finite length Harish-Chandra module  $X$ , i.e.,  $f_X$  is a smooth compactly supported function on  $G$  satisfying

$$\langle X, Y \rangle_{\text{Dir}} = \Theta_Y(f_X)$$

for any finite length Harish-Chandra module  $Y$ . These functions are very cuspidal in the sense of Labesse, thus their orbital integrals  $\Phi(f_X, x)$  vanish on non-elliptic elements. In Section 4, we show that when  $x$  is an elliptic element in  $G$ , the formula (1.8) is valid in this context. This easily implies that the elliptic pairing coincide with the Dirac pairing. The proof uses the density of tempered characters in the space of invariant distributions to reduce the problem to the case of limits of discrete series. In the case of discrete series, the relevant results are well-known and their extension to the case of limits of discrete series is relatively easy.

A formal (but not rigorous) computation given in Section 5 of the paper led us to conjecture that

$$(1.11) \quad \mathbf{EP}(X, Y) = \langle X, Y \rangle_{\text{Dir}},$$

for any Harish-Chandra modules  $X, Y$  of finite length. Interestingly enough finding a direct proof of this equality seems much more intricate than it seems at first sight.

Recently, Jing-Song Huang and Binyong Sun have established directly the equality between the elliptic and Euler-Poincaré pairings [16], thereby proving the conjecture. Jing-Song Huang also obtained independently (by a different computation) the equality between the Dirac and elliptic pairings [12]. Nevertheless, we still believe that a direct approach to prove (1.11) could be of some interest. We show that both sides of (1.11) are indices of suitable Fredholm pairs defined on the same spaces. We found the definition of Fredholm pairs (and some properties of their indices) in [1] and we adapted it to our purely algebraic setting. Let us just say here that the index of

a Fredholm pair is a generalization of the index of a Fredholm operator. The relevant material is exposed in an appendix. The first of these Fredholm pair is given by the complex computing the Ext groups and its differential, and its index is the Euler-Poincaré characteristic of the complex, i.e. the left-hand side of (1.11). The second Fredholm pair is given (on the same space) by actions of Dirac operators and its index is the right-hand side of (1.11). The conjecture thus boils down to the equality of these two indices.

The idea that the elliptic pairing for Harish-Chandra modules is related to Dirac index originates in the papers [8] and [7] where the result is established for Hecke algebra. The role of the group  $\tilde{K}$  is played in that context by the spin cover  $\tilde{W}$  of the relevant Weyl group. Since modules for the Hecke algebra are finite-dimensional, the difficulties we met in proving (1.11) do not appear.

We learned from G. Zuckerman that he obtained results on Euler-Poincaré pairing for Harish-Chandra modules in the late 70's, but that these were never published. Pavle Pandžić and Peter Trapa informed us that they were also led to the identity (1.11) in a work in progress with Dan Barbasch, and we thank them for useful conversations. We also thank the referee for some useful comments.

## 2. DIRAC COHOMOLOGY AND DIRAC INDEX OF $(\mathfrak{g}, K)$ -MODULES

2.1. NOTATION AND STRUCTURAL FACTS. — Let  $G$  be a connected real reductive Lie group with Cartan involution  $\theta$  such that  $K = G^\theta$  is a maximal compact subgroup of  $G$ . We assume that  $G$  and  $K$  have same rank, i.e.,  $G$  has discrete series representations. Let us denote by  $\mathfrak{g}_0$  the Lie algebra of  $G$ ,  $\mathfrak{g}$  its complexification, with Cartan involutions also denoted by  $\theta$ . We fix an invariant nondegenerate symmetric bilinear form  $B$  on  $\mathfrak{g}_0$ , extending the Killing form on the semisimple part of  $\mathfrak{g}_0$ . Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus_{\theta} \mathfrak{p}_0, \quad \mathfrak{g} = \mathfrak{k} \oplus_{\theta} \mathfrak{p}$$

be the Cartan decompositions of  $\mathfrak{g}_0$  and  $\mathfrak{g}$  respectively. We assume furthermore that in extending the Killing form, we made sure that  $B|_{\mathfrak{p}_0}$  remains definite positive and  $B|_{\mathfrak{k}_0}$  definite negative. Remark that since  $G$  and  $K$  have equal rank, the dimension of  $\mathfrak{p}_0$  is even.

Let  $\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{p}; B)$  be the Clifford algebra of  $\mathfrak{p}$  with respect to  $B$ . We follow here the sign convention of [15]. We refer to [15], [19] or [22] for material on Clifford algebras and spinor modules. Let

$$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}^{\bar{0}}(\mathfrak{p}) \oplus \mathrm{Cl}^{\bar{1}}(\mathfrak{p})$$

be the decomposition of  $\mathrm{Cl}(\mathfrak{p})$  into even and odd parts. Recall also that the Clifford algebra  $\mathrm{Cl}(\mathfrak{p})$  is isomorphic as a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space to the exterior algebra  $\Lambda\mathfrak{p}$  by the Chevalley isomorphism. It is convenient to identify  $\mathrm{Cl}(\mathfrak{p})$  to  $\Lambda\mathfrak{p}$ , and to see the latter as a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space endowed with two different algebra products, the graded commutative wedge product (denoted  $x \wedge y$ ) and the Clifford product (denoted  $xy$ ). Let us denote by  $\mathbf{SO}(\mathfrak{p}_0)$  (resp.  $\mathbf{SO}(\mathfrak{p})$ ) the special orthogonal group

of  $(\mathfrak{p}_0, B)$  (resp.  $(\mathfrak{p}, B)$ ). The subspace  $\wedge^2 \mathfrak{p}$  is stable under the Clifford Lie bracket  $[x, y]_{\text{Cl}} = xy - yx$  on  $\wedge \mathfrak{p} = \text{Cl}(\mathfrak{p})$  and is isomorphic as a Lie algebra to the Lie algebra  $\mathfrak{so}(\mathfrak{p})$  of  $\mathbf{SO}(\mathfrak{p})$ .

We denote by  $\tilde{K}$  the spin double cover of  $K$ , i.e., the pull-back of the covering map  $\mathbf{Spin}(\mathfrak{p}_0) \rightarrow \mathbf{SO}(\mathfrak{p}_0)$  by the adjoint action map  $\text{Ad}_{|\mathfrak{p}_0} : K \rightarrow \mathbf{SO}(\mathfrak{p}_0)$ . The compact groups  $\mathbf{Spin}(\mathfrak{p}_0)$  and  $\mathbf{SO}(\mathfrak{p}_0)$  embed in their complexification  $\mathbf{Spin}(\mathfrak{p})$  and  $\mathbf{SO}(\mathfrak{p})$ , so we get the following diagram:

$$\begin{array}{ccccccc}
 \tilde{K} & \longrightarrow & \mathbf{Spin}(\mathfrak{p}_0) & \hookrightarrow & \mathbf{Spin}(\mathfrak{p}) & \hookrightarrow & \text{Cl}^{\bar{0}}(\mathfrak{p})^\times \\
 \downarrow & & \downarrow & & \downarrow & & \\
 K & \xrightarrow{\text{Ad}_{|\mathfrak{p}_0}} & \mathbf{SO}(\mathfrak{p}_0) & \hookrightarrow & \mathbf{SO}(\mathfrak{p}) & & 
 \end{array}$$

The complexification of the differential at the identity of the Lie group morphism  $\text{Ad}_{|\mathfrak{p}_0} : K \rightarrow \mathbf{SO}(\mathfrak{p}_0)$  is the Lie algebra morphism

$$\text{ad}_{|\mathfrak{p}} : \mathfrak{k} \longrightarrow \mathfrak{so}(\mathfrak{p}), \quad X \longmapsto \text{ad}(X)_{|\mathfrak{p}}.$$

Let us denote by  $\alpha$  the composition of this map with the identification between  $\mathfrak{so}(\mathfrak{p})$  and  $\wedge^2 \mathfrak{p}$  and the inclusion of  $\wedge^2 \mathfrak{p}$  in  $\wedge \mathfrak{p} = \text{Cl}(\mathfrak{p})$ :  $\alpha : \mathfrak{k} \longrightarrow \text{Cl}(\mathfrak{p})$ .

A key role is played in the theory of Dirac cohomology of Harish-Chandra modules by the associative  $\mathbb{Z}/2\mathbb{Z}$ -graded superalgebra  $\mathcal{A} = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$ . The  $\mathbb{Z}/2\mathbb{Z}$ -grading comes from the  $\mathbb{Z}/2\mathbb{Z}$ -grading on  $\text{Cl}(\mathfrak{p})$ , i.e., elements in  $U(\mathfrak{g}) \otimes 1$  are even. The super Lie algebra structure on  $\mathcal{A}$  is given by the (super)commutator bracket  $[\cdot, \cdot]_{\mathcal{A}}$ . The group  $K$  acts on  $U(\mathfrak{g})$  through  $K \subset G$  by the adjoint action, and on  $\text{Cl}(\mathfrak{p})$  through the map  $\tilde{K} \rightarrow \text{Cl}^{\bar{0}}(\mathfrak{p})^\times$  in the first row of the diagram above and conjugation in  $\text{Cl}(\mathfrak{p})$  (this action of  $\tilde{K}$  on  $\text{Cl}(\mathfrak{p})$  factors through  $K$ ). Thus we get a linear action of  $K$  on  $\mathcal{A}$ . Differentiating this action at the identity, and taking the complexification, we get a Lie algebra representation of  $\mathfrak{k}$  in  $\mathcal{A}$ . This representation can be described as follows. The map  $\alpha$  above is used to define a map

$$\Delta : \mathfrak{k} \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}), \quad \Delta(X) = X \otimes 1 + 1 \otimes \alpha(X),$$

which is a morphism of Lie algebra (it takes values in the even part of the super Lie algebra  $\mathcal{A}$ ). Thus it extends to an algebra morphism

$$(2.1) \quad \Delta : U(\mathfrak{k}) \longrightarrow \mathcal{A} = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}).$$

The action of an element  $X \in \mathfrak{k}$  on  $\mathcal{A}$  is then given by the adjoint action of  $\Delta(X)$ , i.e.,  $a \in \mathcal{A} \mapsto [\Delta(X), a]_{\mathcal{A}}$ . We denote by  $\mathcal{A}^K$  (resp.  $\mathcal{A}^{\mathfrak{k}}$ ) the subalgebra of  $K$ -invariants (resp.  $\mathfrak{k}$ -invariants) in  $\mathcal{A}$ . Since  $K$  is assumed to be connected,  $\mathcal{A}^K = \mathcal{A}^{\mathfrak{k}}$ .

Recall that  $n = \dim(\mathfrak{p})$  is even. Thus there are:

- two isomorphism classes of irreducible  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\text{Cl}(\mathfrak{p})$ -modules,
- one isomorphism class of irreducible ungraded  $\text{Cl}(\mathfrak{p})$ -modules,
- two isomorphism classes of irreducible  $\text{Cl}^{\bar{0}}(\mathfrak{p})$ -modules.

An irreducible ungraded  $\text{Cl}(\mathfrak{p})$ -module  $S$  can be realized as follows: choose a decomposition  $\mathfrak{p} = U \oplus U^*$  into dual isotropic subspaces. As the notation indicates,  $U^*$  is

identified to the dual of  $U$  by the bilinear form  $B$ . Set  $S = \wedge U$ . Let  $U$  act on  $S$  by wedging and  $U^*$  by contracting. The decomposition  $\wedge U = \wedge^{\bar{0}}U \oplus \wedge^{\bar{1}}U$  give a decomposition  $S = S^+ \oplus S^-$  into the two non-isomorphic simple  $\text{Cl}^{\bar{0}}(\mathfrak{p})$ -modules  $S^+$  and  $S^-$ . The dual of  $S$  is identified with  $\wedge U^*$ . The modules  $S, S^+$  and  $S^-$  are finite-dimensional representations of  $\tilde{K}$  and so are their duals and also  $\text{Cl}(\mathfrak{p}) = \wedge \mathfrak{p}, \text{Cl}(\mathfrak{p})^{\bar{0}}$  and  $\text{Cl}(\mathfrak{p})^{\bar{1}}$ .

PROPOSITION 2.1. — *As virtual  $\tilde{K}$  representations,*

$$(2.2) \quad \wedge^{\bar{0}}\mathfrak{p} := \wedge^{\bar{1}}\mathfrak{p} - \wedge^{\bar{0}}\mathfrak{p} \simeq (S^+ - S^-)^* \otimes (S^+ - S^-).$$

*Proof.* — As  $\tilde{K}$ -modules:  $\wedge \mathfrak{p} \simeq \text{Cl}(\mathfrak{p}) \simeq \text{End}(S) \simeq S^* \otimes S = (S^+ \oplus S^-)^* \otimes (S^+ \oplus S^-)$ . This can be obtained also using:  $\wedge \mathfrak{p} = \wedge(U^* \oplus U) \simeq (\wedge U^*) \otimes (\wedge U) \simeq S^* \otimes S$ . Writing  $(S^+ \oplus S^-)^* \otimes (S^+ \oplus S^-) = ((S^+)^* \otimes S^+) \oplus ((S^-)^* \otimes S^+) \oplus ((S^+)^* \otimes S^-) \oplus ((S^-)^* \otimes S^-)$  and identifying the even and odd parts of these  $\mathbb{Z}/2\mathbb{Z}$ -graded modules, we get

$$\wedge^{\bar{0}}\mathfrak{p} \simeq ((S^+)^* \otimes S^+) \oplus ((S^-)^* \otimes S^-), \quad \wedge^{\bar{1}}\mathfrak{p} \simeq ((S^+)^* \otimes S^-) \oplus ((S^-)^* \otimes S^+).$$

So, as virtual  $\tilde{K}$  representations,  $\wedge^{\bar{0}}\mathfrak{p} = (S^+ - S^-)^* \otimes (S^+ - S^-)$ .  $\square$

2.2. DIRAC COHOMOLOGY OF HARISH-CHANDRA MODULES. — Let us now introduce the Dirac operator  $D$ :

DEFINITION 2.2. — If  $(Y_i)_i$  is a basis of  $\mathfrak{p}$  and  $(Z_i)_i$  is the dual basis with respect to  $B$ , then

$$D = D(\mathfrak{g}, K) = \sum_i Y_i \otimes Z_i \in U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$$

is independent of the choice of basis  $(Y_i)_i$  and  $K$ -invariant for the adjoint action on both factors. The Dirac operator  $D$  (for the pair  $(\mathfrak{g}, K)$ ) is an element of  $\mathcal{A}^K$  (see [15]).

The most important property of  $D$  is the formula

$$(2.3) \quad D^2 = -\text{Cas}_{\mathfrak{g}} \otimes 1 + \Delta(\text{Cas}_{\mathfrak{t}}) + (\|\rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)1 \otimes 1$$

due to Parthasarathy [23] (see also [15]). Here  $\text{Cas}_{\mathfrak{g}}$  (respectively  $\text{Cas}_{\mathfrak{t}}$ ) denotes the Casimir element of  $U(\mathfrak{g})$ . The constant  $(\|\rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)$  is explained below. This formula has several important consequences for Harish-Chandra modules. To state them, we need more notation. Let us fix a maximal torus  $T$  in  $K$ , with Lie algebra  $\mathfrak{t}_0$ . Since  $G$  and  $K$  are connected and have same rank,  $T$  is also a Cartan subgroup of  $G$ .

Let  $R = R(\mathfrak{g}, \mathfrak{t})$  denotes the root system of  $\mathfrak{t}$  in  $\mathfrak{g}$ ,  $W = W(\mathfrak{g}, \mathfrak{t})$  its Weyl group. Let us also choose a positive root system  $R^+$  in  $R$ . As usual,  $\rho$  denotes the half-sum of positive roots, an element in  $\mathfrak{t}^*$ . Similarly, we introduce the root system  $R_{\mathfrak{t}} = R(\mathfrak{t}, \mathfrak{t})$ , its Weyl group  $W_{\mathfrak{t}}$ , a positive root system  $R_{\mathfrak{t}}^+$ , compatible with  $R^+$ , and the half-sum of positive roots  $\rho_{\mathfrak{t}}$ . The bilinear form  $B$  on  $\mathfrak{g}$  restricts to a non degenerate symmetric bilinear form on  $\mathfrak{t}$ , which is definite positive on the real form  $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$ . We denote by  $\langle ., . \rangle$  the induced form on  $i\mathfrak{t}_0^*$  and in the same way its extension to  $\mathfrak{t}^*$ . The norm appearing in (2.3) is defined for any  $\lambda \in \mathfrak{t}^*$  by  $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$ .

Recall the Harish-Chandra algebra isomorphism

$$(2.4) \quad \gamma_{\mathfrak{g}} : \mathfrak{Z}(\mathfrak{g}) \simeq S(\mathfrak{t})^W$$

between the center  $\mathfrak{Z}(\mathfrak{g})$  of the enveloping algebra  $U(\mathfrak{g})$  and the  $W$ -invariants in the symmetric algebra  $S(\mathfrak{t})$  on  $\mathfrak{t}$ . Accordingly, a character  $\chi$  of  $\mathfrak{Z}(\mathfrak{g})$  is given by an element of  $\mathfrak{t}^*$  (or rather its Weyl group orbit). If  $\lambda \in \mathfrak{t}^*$ , we denote by  $\chi_\lambda$  the corresponding character of  $\mathfrak{Z}(\mathfrak{g})$ . Let  $X$  be a Harish-Chandra module. We say that  $X$  has infinitesimal character  $\lambda$  if any  $z \in \mathfrak{Z}(\mathfrak{g})$  acts on  $X$  by the scalar operator  $\chi_\lambda(z)\text{Id}_X$ .

Let  $\mathcal{M} = \mathcal{M}(\mathfrak{g}, K)$  be the category of Harish-Chandra modules for the pair  $(\mathfrak{g}, K)$  (see [18] for details). If  $X \in \mathcal{M}$ , then  $\mathcal{A} = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$  acts on  $X \otimes S$ . Then  $X$  decomposes as the direct sum of its  $K$ -isotypic components, these being finite-dimensional if  $X$  is admissible. Accordingly,  $X \otimes S$  decomposes as the direct sum of its (finite-dimensional if  $X$  is admissible)  $\tilde{K}$ -isotypic components. Let us denote by  $F_\tau$  the irreducible representation of  $\tilde{K}$  with highest weight  $\tau \in i\mathfrak{t}_0^*$ . We denote the corresponding  $\tilde{K}$ -isotypic component of  $X \otimes S$  by  $(X \otimes S)(\tau)$ . Assume  $X$  is admissible and has infinitesimal character  $\Lambda \in \mathfrak{t}^*$ . Then  $D^2$  acts on  $(X \otimes S)(\tau)$  by the scalar  $-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2$ . In particular, we see that in that case  $D^2$  acts semi-simply on  $X \otimes S$ , and that the kernel of  $D^2$  on  $X \otimes S$  is a (finite) direct sum of full  $\tilde{K}$ -isotypic components of  $X \otimes S$ : these are exactly those  $(X \otimes S)(\tau)$  for which

$$(2.5) \quad \|\tau + \rho_{\mathfrak{k}}\|^2 = \|\Lambda\|^2.$$

Another important fact is that the action of  $D$  preserves  $\tilde{K}$ -isotypic components of  $X \otimes S$ . If  $X$  is unitary (resp. finite-dimensional), one can put a positive definite hermitian form on  $X \otimes S$  and one sees that  $D$  is hermitian (resp. skew-hermitian). Let us now review Vogan’s definition of Dirac cohomology.

DEFINITION 2.3. — Let  $X \in \mathcal{M}$ . The Dirac operator  $D$  acts on  $X \otimes S$  with kernel  $\ker D$  and image  $\text{Im } D$ . Vogan’s Dirac cohomology of  $X$  is the quotient

$$\mathbf{H}_{\text{Dir}}(X) = \ker D / (\ker D \cap \text{Im } D).$$

Since  $D \in \mathcal{A}^K$ ,  $\tilde{K}$  acts on  $\ker D$ ,  $\text{Im } D$  and  $\mathbf{H}_{\text{Dir}}(X)$ . Also, assume that  $X$  is admissible and has infinitesimal character  $\Lambda \in \mathfrak{t}^*$ . Then, since  $\ker D \subset \ker D^2$  and since we have seen that the latter is the sum the full  $\tilde{K}$ -isotypic components of  $X \otimes S$  satisfying (2.5) (these are obviously in finite number), we see that  $\mathbf{H}_{\text{Dir}}(X)$  is a finite-dimensional representation of  $\tilde{K}$ . This is particularly helpful if  $X$  is unitary, admissible and has infinitesimal character  $\Lambda \in \mathfrak{t}^*$ . Then it follows that  $D$  acts semisimply on  $X \otimes S$  and so

$$(2.6) \quad \ker D^2 = \ker D = \mathbf{H}_{\text{Dir}}(X).$$

In this case, the Dirac cohomology of  $X$  is a sum the full isotypic components  $(X \otimes S)(\tau)$  such that (2.5) holds. For general  $X$ , (2.6) does not hold, but note that  $D$  is always a differential on  $\ker D^2$ , and  $\mathbf{H}_{\text{Dir}}(X)$  is the usual cohomology of this differential.

Let us state the main result of [14], which gives a strong condition on the infinitesimal character of an admissible Harish-Chandra module  $X$  with nonzero Dirac cohomology.

**PROPOSITION 2.4.** — *Let  $X \in \mathcal{M}$  be an admissible Harish-Chandra module with infinitesimal character  $\Lambda \in \mathfrak{h}^*$ . Assume that  $F_\tau$  is an irreducible representation of  $\tilde{K}$  with highest weight  $\tau \in i\mathfrak{t}_0^*$  such that  $(X \otimes S)(\tau)$  contributes to  $\mathbf{H}_{\text{Dir}}(X)$ . Then*

$$(2.7) \quad \Lambda = \tau + \rho_{\mathfrak{k}} \quad \text{up to conjugacy by the Weyl group } W.$$

Thus for unitary  $X$ , (2.5) is equivalent to the stronger condition (2.7), provided that  $F_\tau$  appears in  $X \otimes S$ .

**2.3. DIRAC INDEX.** — The Dirac index of Harish-Chandra modules is a refinement of Dirac cohomology. It uses the decomposition  $S = S^+ \oplus S^-$  of the spinor module as a  $\text{Cl}(\mathfrak{p})^{\bar{0}}$ -module (and thus also as a representation of  $\tilde{K}$ ). Since  $D$  is an odd element in  $U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p})$  its action on  $X \otimes S$ , for any Harish-Chandra module  $X$  exchanges  $X \otimes S^+$  and  $X \otimes S^-$ :  $D : X \otimes S^+ \longleftrightarrow X \otimes S^-$ . Accordingly, the Dirac cohomology of  $X$  decomposes as

$$\mathbf{H}_{\text{Dir}}(X) = \mathbf{H}_{\text{Dir}}(X)^+ \oplus \mathbf{H}_{\text{Dir}}(X)^-.$$

The *index of the Dirac operator* acting on  $X \otimes S$  is the virtual representation

$$l_{\text{Dir}}(X) = \mathbf{H}_{\text{Dir}}(X)^+ - \mathbf{H}_{\text{Dir}}(X)^-$$

of  $\tilde{K}$ . The following proposition is interpreted as an Euler-Poincaré principle.

**PROPOSITION 2.5.** — *Let  $X$  be an admissible Harish-Chandra module with infinitesimal character. Then*

$$l_{\text{Dir}}(X) = X \otimes S^+ - X \otimes S^-$$

as virtual  $\tilde{K}$ -representations.

*Proof.* — A virtual  $\tilde{K}$ -representation is by definition an element of the Grothendieck group  $\mathcal{R}(\tilde{K})$  of the category of finite-dimensional representations of  $\tilde{K}$ . This is the free  $\mathbb{Z}$ -module generated by equivalence classes of irreducible representations, i.e., one can write

$$\mathcal{R}(\tilde{K}) = \bigoplus_{\gamma \in (\tilde{K})^\sim} \mathbb{Z}.$$

The right-hand side of the equation in the proposition cannot a priori be interpreted as an element of  $\mathcal{R}(\tilde{K})$ , but only of the larger group  $\prod_{\gamma \in (\tilde{K})^\sim} \mathbb{Z}$ .

Let us now prove the equality. We have seen that  $D^2$  acts semisimply on  $X \otimes S$ . Furthermore, each eigenspace of  $D^2$  in  $X \otimes S$  is a sum of full  $\tilde{K}$ -isotypic components and that these are preserved by the action of  $D$ . Each of these  $\tilde{K}$ -isotypic components  $(X \otimes S)(\tau)$  decomposes also as  $(X \otimes S)(\tau) = (X \otimes S)(\tau)^+ \oplus (X \otimes S)(\tau)^-$  where

$$(X \otimes S)(\tau)^\pm := (X \otimes S)(\tau) \cap (X \otimes S^\pm).$$

For  $\tilde{K}$ -isotypic components corresponding to a non-zero eigenvalue of  $D^2$ , we thus get that  $D$  is a bijective intertwining operator (for the  $\tilde{K}$ -action) between  $(X \otimes S)(\tau)^+$  and  $(X \otimes S)(\tau)^-$ . Thus the contribution of these  $\tilde{K}$ -isotypic components to

$$X \otimes S^+ - X \otimes S^-$$

is zero. So only  $\ker D^2$  will contribute, i.e.,

$$X \otimes S^+ - X \otimes S^- = (\ker D^2 \cap (X \otimes S^+)) - (\ker D^2 \cap (X \otimes S^-)).$$

Let us write

$$\ker D^2 \cap (X \otimes S^\pm) = \ker D \cap (X \otimes S^\pm) \oplus W^\pm$$

for some  $\tilde{K}$ -invariant complementary subspaces  $W^\pm$ . Then, as above,  $D$  is bijective intertwining operator for the  $\tilde{K}$ -action between  $W^\pm$  and  $D(W^\pm) \subset \ker D^2 \cap (X \otimes S^\mp)$ . So these contributions also cancel, and what remains is exactly the virtual  $\tilde{K}$ -representation  $H_{\text{Dir}}(X)^+ - H_{\text{Dir}}(X)^-$ .  $\square$

2.4. DIRAC INDEX PAIRING. — If  $\gamma, \sigma$  are virtual finite-dimensional representations of  $\tilde{K}$ , and  $\chi_\gamma, \chi_\sigma$  are their characters, we denote by

$$[\gamma, \sigma]_{\tilde{K}} = \int_{\tilde{K}} \overline{\chi_\gamma(k)} \chi_\sigma(k) dk$$

the usual (hermitian) pairing between these virtual representations ( $dk$  is the normalized Haar measure on  $\tilde{K}$ ). Since virtual finite-dimensional representations of  $K$  are also virtual representations of  $\tilde{K}$ , we use also the notation for their pairing. Notice that when  $\gamma, \sigma$  are actual finite-dimensional representations, then  $[\gamma, \sigma]_{\tilde{K}} = \dim \text{Hom}_{\tilde{K}}(\gamma, \sigma)$ .

DEFINITION 2.6. — Assume that  $X$  and  $Y$  are admissible Harish-Chandra modules with infinitesimal character<sup>(1)</sup>. Set  $\langle X, Y \rangle_{\text{Dir}} = [l_{\text{Dir}}(X), l_{\text{Dir}}(Y)]_{\tilde{K}}$ , the Dirac index pairing of  $X$  and  $Y$ .

2.5. DIRAC INDEX OF LIMITS OF DISCRETE SERIES. — Harish-Chandra modules of limits of discrete series of  $G$  are obtained as cohomologically induced  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ -modules (see [18]), where  $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$  is a Borel subalgebra containing  $\mathfrak{t}$  with nilpotent radical  $\mathfrak{u}$ , and  $\mathbb{C}_\lambda$  is the one-dimensional representation of  $T$  with weight  $\lambda \in i\mathfrak{t}_0^*$ . Some positivity conditions on  $\lambda$  are required, that we now describe. The Borel subalgebra  $\mathfrak{b}$  determines a set of positive roots  $R_{\mathfrak{b}}^+$  of  $R = R(\mathfrak{g}, \mathfrak{t})$  (the roots of  $\mathfrak{t}$  in  $\mathfrak{u}$ ). Let us denote by  $\rho(\mathfrak{b})$ , (resp.  $\rho_c(\mathfrak{b})$ , resp.  $\rho_n(\mathfrak{b})$ ) the half-sum of (resp. compact, resp. non-compact) roots in  $R_{\mathfrak{b}}^+$ . The positivity condition on  $\lambda$  is that

$$(2.8) \quad \langle \lambda + \rho(\mathfrak{b}), \alpha \rangle \geq 0, \quad (\forall \alpha \in R_{\mathfrak{b}}^+).$$

<sup>(1)</sup>A technical remark is in order here: recall that a Harish-Chandra module with infinitesimal character is of finite length if and only if it is admissible.

Then,  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  is a discrete series modules if the inequalities in (2.8) are strict, and otherwise a limit of discrete series ( $\chi = \lambda + \rho(\mathfrak{b})$  is the infinitesimal character of  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ ), or 0. Assume furthermore that  $\chi$  is not singular for  $K$ , i.e.,  $\langle \chi, \alpha \rangle \neq 0$  for all  $\alpha \in R(\mathfrak{k}, \mathfrak{t})$ . Set

$$(2.9) \quad \Lambda = \lambda + 2\rho_n(\mathfrak{b}) = \chi + \rho_n(\mathfrak{b}) - \rho_c(\mathfrak{b}) = \chi + \rho(\mathfrak{b}) - 2\rho_c(\mathfrak{b}).$$

Then, for all simple root  $\alpha \in R^+(\mathfrak{k}, \mathfrak{t})$ , we have  $\langle \chi, \alpha \rangle \geq 1$ ,  $\langle \rho(\mathfrak{b}), \alpha \rangle \geq 1$ ,  $\langle \rho_c(\mathfrak{b}), \alpha \rangle = 1$  and thus  $\Lambda$  is dominant for  $\mathfrak{k}$ . The lowest  $K$ -type of  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  has multiplicity one and highest weight  $\Lambda$ , and all other  $K$ -types have highest weights of the form  $\Lambda + \sum_{\beta \in R(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})} n_\beta \beta$ , with  $n_\beta$  nonnegative integers.

Let us compute the Dirac cohomology of  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ , following [13, §5]. There it is assumed that  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  is a discrete series, and for these it goes back more or less explicitly to [3]. Let  $\mu$  be the highest weight of a  $\tilde{K}$ -type in  $A_{\mathfrak{b}}(\mathbb{C}_\lambda) \otimes S$ . Weights of  $S$  are of the form  $-\rho_n(\mathfrak{b}) + \Phi$  where  $\Phi$  is a sum of positive non-compact roots and thus  $\gamma$  must be of the form

$$(2.10) \quad \mu = \Lambda + \Psi - \rho_n(\mathfrak{b}) + \Phi = \chi - \rho_c(\mathfrak{b}) + \Psi + \Phi,$$

where  $\Psi$  is an integral linear combination of positive non-compact roots with non-negative coefficients. By Proposition 2.4,  $\gamma$  contributes to the Dirac cohomology of  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  if and only if there exists  $w \in W$  such that  $w \cdot \chi = \mu + \rho_c(\mathfrak{b})$  and in that case, (2.10) gives

$$\chi - w \cdot \chi + \Psi + \Phi = 0.$$

Since  $\chi$  is dominant,  $\chi - w \cdot \chi$  is an integral linear combination of positive roots with nonnegative coefficients ([13, Lem. 2.3]). Hence  $\Psi = \Phi = 0$ ,  $w \cdot \chi = \chi$  and

$$(2.11) \quad \mu = \Lambda - \rho_n(\mathfrak{b}) = \chi - \rho_c(\mathfrak{b}) = \lambda + \rho_n(\mathfrak{b}).$$

The  $\tilde{K}$ -type  $F_\mu$  occurs with multiplicity one in  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$ , thus the Dirac index of  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  is  $\pm F_\mu$ . To determine the sign, recall that the spinor module  $S$ , as a  $\tilde{K}$ -representation, doesn't depend on any choice, nor does the set  $\{S^+, S^-\}$ , in particular not on the way we realized this module, but the distinction between  $S^+$  and  $S^-$  does (resulting on a sign change in the Dirac index). So suppose we have fixed once for all a Borel subalgebra  $\mathfrak{b}_1 = \mathfrak{k} \oplus \mathfrak{u}_1$  and choose  $\mathfrak{u}_1 \cap \mathfrak{p}$  as the isotropic subspace  $U$  of  $\mathfrak{p}$  used to construct the spinor modules in Section 2.1. Then  $S^+$  is the  $\tilde{K}$ -representation containing the weight  $-\rho_n(\mathfrak{b}_1)$  and  $S^-$  is the one containing the weight  $\rho_n(\mathfrak{b}_1)$ . With this choice, it is easy to see that  $\text{I}_{\text{Dir}}(A_{\mathfrak{b}}(\mathbb{C}_\lambda)) = \text{sgn}(w)F_\mu$ , where  $w \in W$  is the Weyl group element sending  $\mathfrak{b}_1$  to  $\mathfrak{b}$  and  $\text{sgn}$  is the sign character.

Let us now determine the Dirac index of the virtual modules  $X$  which are the linear combinations of limits of discrete series with the same infinitesimal character, whose characters are the supertempered distributions constructed by Harish-Chandra (see [11] and [5, §7]). Namely take an integral but non necessarily regular weight  $\chi$  in  $i\mathfrak{t}_0^*$ , but assume that  $\chi$  is regular for  $\mathfrak{k}$ , and consider the limits of discrete series  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  as above with  $\chi = \lambda + \rho(\mathfrak{b})$  satisfying (2.8). Notice that  $\chi$  being fixed,  $\mathfrak{b}$  determines  $A_{\mathfrak{b}}(\mathbb{C}_\lambda)$  and that Borel subalgebras  $\mathfrak{b}$  which occurs are the one such that the

corresponding Weyl chamber  $C_{\mathfrak{b}}$  has  $\chi$  in its closure. So we can forget the  $\lambda$  in the notation, and the set of limits of discrete series we are considering is  $\mathcal{L} = \{A_{\mathfrak{b}} \mid \chi \in \overline{C_{\mathfrak{b}}}\}$ . Choose one of them as a base point, say  $A_{\mathfrak{b}_2}$ . Then the linear combination introduced by Harish-Chandra is

$$(2.12) \quad X_{\chi, \mathfrak{b}_2} = \frac{1}{|W_{\chi}|} \sum_{A_{\mathfrak{b}} \in \mathcal{L}} \varepsilon(\mathfrak{b}) A_{\mathfrak{b}}$$

where  $\varepsilon(\mathfrak{b}) = \text{sgn}(w)$ , and  $w \in W$  is the Weyl group element sending  $\mathfrak{b}_2$  to  $\mathfrak{b}$  and  $W_{\chi} = \{w \in W \mid w \cdot \chi = \chi\}$ . This construction is made because all the  $\varepsilon(\mathfrak{b})A_{\mathfrak{b}}$  have the same character formula on  $T$  and this is also the formula for the character of  $X_{\chi, \mathfrak{b}_2}$ . Notice that a different choice of  $\mathfrak{b}_2$  as base point would result in at most a sign change.

The fact that  $\chi$  is regular for  $\mathfrak{k}$  implies that the corresponding Weyl chambers  $C_{\mathfrak{b}}$  are all included in a single Weyl chamber for  $R(\mathfrak{k}, \mathfrak{t})$ . In particular, the various  $\rho_c(\mathfrak{b})$  are all equal (let say to  $\rho_c$ ). This shows that the Dirac cohomology of all the  $A_{\mathfrak{b}} \in \mathcal{L}$  is the same, namely the multiplicity-one  $\tilde{K}$ -type  $F_{\mu}$  with highest weight  $\mu = \chi - \rho_c$ . Taking signs into account, we see that all the  $\varepsilon(\mathfrak{b})A_{\mathfrak{b}}$  have same Dirac index  $\text{sgn}(w)F_{\mu}$  where  $w$  is the Weyl group element sending  $\mathfrak{b}_1$  to  $\mathfrak{b}_2$ . This is thus also the Dirac index of  $X_{\chi, \mathfrak{b}_2}$ .

For discrete series,  $\chi$  is regular, so  $\mathcal{L}$  contains only one element  $A_{\mathfrak{b}}(\lambda)$  with  $\lambda = \chi - \rho(\mathfrak{b})$ .

REMARK 2.7. — If  $\chi$  is singular for  $\mathfrak{k}$ , the virtual representation (2.12) is still defined, but it is 0.

PROPOSITION 2.8. — *Let the notation be as above, with  $\chi$  regular for  $\mathfrak{k}$ . If  $\mathfrak{l}_{\text{Dir}}(X_{\chi, \mathfrak{b}}) = \pm \mathfrak{l}_{\text{Dir}}(X_{\chi', \mathfrak{b}'})$ , then  $\chi$  and  $\chi'$  are conjugate by a element of  $W(\mathfrak{k}, \mathfrak{t})$ , and thus by an element in  $K$ . Therefore  $X_{\chi, \mathfrak{b}} = X_{\chi', \mathfrak{b}'}$ .*

Proof. — Say that  $\mathfrak{l}_{\text{Dir}}(X_{\chi, \mathfrak{b}}) = \pm F_{\mu} = \pm I(X_{\chi', \mathfrak{b}'})$ . For any  $g \in G$  normalizing  $T$ ,  $X_{g \cdot \chi, g \cdot \mathfrak{b}} = X_{\chi, \mathfrak{b}}$ , so we can assume that  $C_{\mathfrak{b}}$  and  $C_{\mathfrak{b}'}$  are in the same Weyl chamber for  $R(\mathfrak{k}, \mathfrak{t})$ . We have then  $\mu = \chi - \rho_c(\mathfrak{b}) = \chi' - \rho_c(\mathfrak{b}')$  and  $\rho_c(\mathfrak{b}) = \rho_c(\mathfrak{b}')$  so that  $\chi = \chi'$ . □

### 3. LABESSE INDEX FUNCTIONS AND EULER-POINCARÉ FUNCTIONS

Let us compute the Dirac index pairing for two admissible Harish-Chandra modules  $X$  and  $Y$  with infinitesimal character. We have

$$\langle X, Y \rangle_{\text{Dir}} = [\mathfrak{l}_{\text{Dir}}(X), \mathfrak{l}_{\text{Dir}}(Y)]_{\tilde{K}} = \sum_{\gamma \in (\tilde{K})^{\wedge}} [\gamma, \mathfrak{l}_{\text{Dir}}(X)]_{\tilde{K}} \times [\gamma, \mathfrak{l}_{\text{Dir}}(Y)]_{\tilde{K}}.$$

Let us now introduce some material from [21]. If  $\gamma$  is a genuine virtual finite-dimensional representation of  $\tilde{K}$ , let us denote by  $I_K(\gamma)$  the virtual finite-dimensional representation

$$\gamma \otimes (S^+ - S^-)^*$$

of  $K$  and by  $\tilde{\chi}_\gamma$  of the character of  $I_K(\gamma)$  (a conjugation invariant function on  $K$ ). If  $(\pi, X)$  is an admissible Harish-Chandra module, with  $\pi$  denoting the action of  $K$  on  $X$ , the operator

$$\pi(\overline{\tilde{\chi}_\gamma}) = \int_K \pi(k) \overline{\tilde{\chi}_\gamma(k)} dk$$

is of finite rank, and so is a trace operator. Let us denote its trace by  $I(X, \gamma)$ . If furthermore  $X$  has infinitesimal character, we have

$$I(X, \gamma) = [\gamma \otimes (S^+ - S^-)^*, X]_{\tilde{K}} = [\gamma, X \otimes (S^+ - S^-)]_{\tilde{K}} = [\gamma, \mathbf{l}_{\text{Dir}}(X)]_{\tilde{K}}.$$

Labesse main result is the existence for all  $\gamma \in (\tilde{K})^\wedge$  of a smooth, compactly supported and bi- $K$ -finite function  $f_\gamma$  which satisfies, for all finite length Harish-Chandra module  $X$ , with  $\Theta_X$  denoting the distribution-character of  $X$ ,

$$\Theta_X(f_\gamma) = I(X, \gamma).$$

The main ingredient in Labesse's construction is the Paley-Wiener theorem of Arthur [2]. From this, we get

$$\langle X, Y \rangle_{\text{Dir}} = \sum_{\gamma \in (\tilde{K})^\wedge} I(X, \gamma) \times I(Y, \gamma) = \sum_{\gamma \in (\tilde{K})^\wedge} \Theta_X(f_\gamma) \Theta_Y(f_\gamma).$$

Let us call

$$f_Y = \sum_{\gamma \in (\tilde{K})^\wedge} \Theta_Y(f_\gamma) f_\gamma$$

an index function for  $Y$ . Note that the sum above has finite support, so that  $f_Y$  is smooth, compactly supported and bi- $K$ -finite. Furthermore, Labesse shows ([21, Prop. 7]) that the functions  $f_\gamma$  are "very cuspidal", i.e., that their constant terms for all proper parabolic subgroups of  $G$  vanish. Thus the same property holds for  $f_Y$ . This implies the vanishing of the orbital integrals  $\Phi(f_Y, x)$  on regular non-elliptic element  $x$  in  $G$ . We have obtained:

**THEOREM 3.1.** — *For any admissible Harish-Chandra module  $Y$  with infinitesimal character, there exists a smooth, compactly supported, bi- $K$ -finite and very cuspidal function  $f_Y$  on  $G$  such that for any admissible Harish-Chandra module  $X$  with infinitesimal character,*

$$\langle X, Y \rangle_{\text{Dir}} = \Theta_X(f_Y).$$

#### 4. ORBITAL INTEGRAL OF $f_Y$ AS THE CHARACTER OF $Y$ ON ELLIPTIC ELEMENTS

The goal of this section is to show that the value of the orbital integral  $\Phi(f_Y, x)$  of  $f_Y$  at an elliptic regular element  $x$  coincides with the value of the character  $\theta_Y$  of  $Y$  at  $x^{-1}$ . The character  $\Theta_Y$  of  $Y$  is a distribution on  $G$ , but recall that according to Harish-Chandra regularity theorem, there is an analytic, conjugation invariant function that we will denote by  $\theta_Y$  on the set  $G_{\text{reg}}$  such that for all  $f \in \mathcal{C}_c^\infty(G)$ ,

$$\Theta_Y(f) = \int_G \theta_Y(x) f(x) dx.$$

Using Weyl integration formula, this could be written as

$$\Theta_Y(f) = \sum_{[H]} \frac{1}{|W(G, H)|} \int_H |D_G(h)| \theta_Y(h) \Phi(f, h) dh,$$

where the first sum is on a system of representative of conjugacy classes of Cartan subgroups  $H$  of  $G$ ,  $W(G, H) = N_G(H)/H$  is the real Weyl group of  $H$  and  $|D_G|$  is the usual Jacobian. The assertion is thus that:

$$(4.1) \quad \theta_Y(x^{-1}) = \Phi(f_Y, x), \quad (x \in G_{\text{ell}}).$$

The characterization of orbital integrals due to A. Bouaziz [5] shows that there indeed exists a function  $\psi_Y$  in  $C^\infty(G)$  such that

$$\Phi(\psi_Y, x) = \begin{cases} \theta_Y(x^{-1}) & \text{if } x \in G_{\text{ell}}, \\ 0 & \text{if } x \in G_{\text{reg}} \setminus G_{\text{ell}}. \end{cases}$$

Let  $\mathcal{F}$  be a family of elements  $X$  in  $\mathcal{R}_{\mathbb{C}}$  such that the space generated by the  $\Theta_X$  is dense in the space of invariant distributions  $\mathcal{D}'(G)^G$  on  $G$ . For instance,  $\mathcal{F}$  could be the set of characters of all irreducible tempered representations, but we will rather take  $\mathcal{F}$  to be the family of virtual representations with characters  $\Theta_{h^*}$  defined in [5, §7]. The density of this family of invariant distributions is a consequence of the inversion formula of orbital integrals [6]. Elements in  $\mathcal{F}$  are generically irreducible tempered representations, but in general, they are linear combinations of some of these with same infinitesimal character. The distributions  $\Theta_{h^*}$  are supertempered in the sense of Harish-Chandra [11]. In any case to prove (4.1), it is enough by density to show that for all  $X \in \mathcal{F}$ , we have:

$$(4.2) \quad \Theta_X(f_Y) = \Theta_X(\psi_Y).$$

If  $X$  is a linear combination of parabolically induced representations, then both side are 0 since  $\Theta_X$  vanishes on elliptic elements. Thus, it is sufficient to prove (4.2) for  $X$  corresponding to the  $\Theta_{h^*}$  of [5] attached to the compact Cartan subgroup  $T$ . Then  $X$  is either a discrete series or a linear combination of limits of discrete series (with same infinitesimal character) described in Section 2.5.

Assume that (4.1) is established for all such  $X$ . The left-hand side of (4.2) then also equals

$$\begin{aligned} \Theta_Y(f_X) &= \frac{1}{|W(G, T)|} \int_T |D_G(t)| \theta_Y(t) \Phi(f_X, t) dt \\ &= \frac{1}{|W(G, T)|} \int_T |D_G(t)| \theta_Y(t) \theta_X(t^{-1}) dt \end{aligned}$$

and equals the right-hand side, by using the definition of  $\psi_Y$  and the Weyl integration formula again. By definition of the measure  $dc$  on the set  $C^{\text{ell}}$  of regular semisimple elliptic conjugacy classes in  $G$  and of the elliptic pairing recalled in the introduction, we have also

$$\Theta_Y(f_X) = \int_{C^{\text{ell}}} \theta_Y(c) \theta_X(c^{-1}) dc = \langle X, Y \rangle_{\text{ell}}.$$

Thus we have reduced the proof of (4.1) for all  $Y$  to the case when  $Y$  is either a discrete series or a linear combination of limits of discrete series as described above. In turns, it is enough to show (4.2) when both  $X$  and  $Y$  are of this kind. In case  $Y$  corresponds to a parameter  $h^*$  of [6],  $\psi_Y$  is exactly the function denoted  $\psi_{h^*}$  there, and in particular:

$$\Theta_X(\psi_Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq \pm Y. \end{cases}$$

With the notation of Section 2.5, we can take  $X = X_{\chi, \mathfrak{b}}$  and  $Y = X_{\chi', \mathfrak{b}'}$ . Then  $f_X = \sum_{\gamma \in (\tilde{K})^\times} \Theta_X(f_\gamma) f_\gamma = \sum_{\gamma \in (\tilde{K})^\times} [\gamma, I(X, \gamma)]_{\tilde{K}} f_\gamma$  and by the results in Section 2.5, we get  $f_X = \varepsilon(\mathfrak{b}) f_{X-\rho_c(\mathfrak{b})}$ . Similarly we get  $f_Y = \varepsilon(\mathfrak{b}') f_{X'-\rho_c(\mathfrak{b}'})$  and

$$\Theta_X(f_Y) = [\mathbf{l}_{\text{Dir}}(X), \mathbf{l}_{\text{Dir}}(Y)]_K = [\varepsilon(\mathfrak{b}) F_{X-\rho(\mathfrak{b})}, \varepsilon(\mathfrak{b}') F_{X'-\rho(\mathfrak{b}')}]_K.$$

By Proposition 2.8, we see that this is 0 if  $X \neq \pm Y$  and 1 if  $X = Y$ . This finishes the proof of (4.2) in the case under consideration. We have proved:

**THEOREM 4.1.** — *Let  $X$  and  $Y$  be finite length Harish-Chandra modules with infinitesimal character in  $\mathcal{M}$  and let  $f_X, f_Y$  be the Euler-Poincaré functions for  $X$  and  $Y$  respectively, constructed in Section 3. Then, the orbital integral  $\Phi(f_X, x)$  at a regular element  $x$  of  $G$  is 0 if  $x$  is not elliptic, and equals  $\theta_X(x^{-1})$  if  $x$  is elliptic. Furthermore:*

$$\langle X, Y \rangle_{\text{Dir}} = \Theta_X(f_Y) = \Theta_Y(f_X) = \int_{C^{\text{ell}}} \theta_Y(c) \theta_X(c^{-1}) dc = \langle X, Y \rangle_{\text{ell}}.$$

### 5. EULER-POINCARÉ PAIRING AND DIRAC PAIRING

For two finite length Harish-Chandra modules  $X$  and  $Y$ , one can define their Euler-Poincaré pairing as the alternating sum of dimensions of Ext functors as in (1.1). Recall that  $\mathcal{M}$  has finite cohomological dimension, so this sum has finite support. More precisely, an explicit projective resolution of  $X$  in  $\mathcal{M}(\mathfrak{g}, K)$  is given by ([18, II.7])

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with  $P_i = (U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \wedge^i \mathfrak{p}) \otimes X$ . Set  $C^i = \text{Hom}_{\mathcal{M}}(P_i, Y) \simeq \text{Hom}_K(\wedge^i \mathfrak{p} \otimes X, Y)$ . Thus  $\text{Ext}^i(X, Y)$  is given by the  $i$ -th cohomology group of the complex  $C = (C^i)_i$ , with differential  $d^i$  given explicitly in loc. cit.

**THEOREM 5.1.** — *Let  $X$  and  $Y$  be two admissible Harish-Chandra modules with infinitesimal character. Then*

$$\mathbf{EP}(X, Y) = [\mathbf{l}_{\text{Dir}}(X), \mathbf{l}_{\text{Dir}}(Y)]_{\tilde{K}} = \langle X, Y \rangle_{\text{Dir}}.$$

*Proof.* — We have seen in the previous section that  $\langle X, Y \rangle_{\text{Dir}} = \langle X, Y \rangle_{\text{ell}}$ . Huang and Sun [16] have proved that  $\langle X, Y \rangle_{\text{ell}} = \mathbf{EP}(X, Y)$ . □

The main content of this statement is that the **EP** pairing of two finite length Harish-Chandra modules with infinitesimal character factors through their Dirac indices, and thus through their Dirac cohomology. In particular, the results on Dirac

cohomology and Dirac index recalled in Section 2.2 put severe conditions on modules  $X$  and  $Y$  for their Dirac pairing to be non-zero. For many interesting modules, the Dirac index is explicitly known, and then so is the Dirac pairing between these modules.

REMARK 5.2. — In the case where  $X = F$  is a finite-dimensional irreducible Harish-Chandra module,  $\text{Ext}^i(F, Y)$  is also the  $i$ -th  $(\mathfrak{g}, K)$ -cohomology group of  $Y \otimes F^*$ . Thus, in that case,

$$\sum_i (-1)^i \dim H^i(\mathfrak{g}, K; Y \otimes F^*) = [\mathbf{l}_{\text{Dir}}(F), \mathbf{l}_{\text{Dir}}(Y)]_{\tilde{K}}.$$

In the case where  $Y$  is unitary, we have much stronger results: the differential  $d$  on  $\text{Hom}_K(\wedge^i \mathfrak{p} \otimes F, Y)$  is 0, so

$$\begin{aligned} \bigoplus_i H^i(\mathfrak{g}, K; Y \otimes F^*) &= \bigoplus_i \text{Hom}_K(\wedge^i \mathfrak{p} \otimes F, Y) \\ &= \text{Hom}_K(\wedge \mathfrak{p} \otimes F, Y) \simeq \text{Hom}_{\tilde{K}}(F \otimes S, Y \otimes S) \end{aligned}$$

and the only common  $\tilde{K}$ -types between  $F \otimes S$  and  $Y \otimes S$  have their isotypic components in  $\ker(D|_{F \otimes S}) = \ker(D^2|_{F \otimes S}) = \mathbf{H}_{\text{Dir}}(F)$  and  $\ker(D|_{X \otimes S}) = \ker(D^2|_{X \otimes S}) = \mathbf{H}_{\text{Dir}}(X)$  respectively. See [28, §9.4] and [15, §8.3.4].

We would like a direct proof of the equality in the theorem. Consider the following computation:

$$\begin{aligned} \mathbf{EP}(X, Y) &= \sum_i (-1)^i \dim \text{Ext}^i(X, Y) = \sum_i (-1)^i \dim \text{Hom}_K(\wedge^i \mathfrak{p} \otimes X, Y) \\ &= \sum_i (-1)^i [\wedge^i \mathfrak{p} \otimes X, Y]_{\tilde{K}} = \left[ \sum_i (-1)^i \wedge^i \mathfrak{p} \otimes X, Y \right]_{\tilde{K}} = [\wedge^{\bullet} \mathfrak{p} \otimes X, Y]_{\tilde{K}} \\ &= [(S^+ - S^-)^* \otimes (S^+ - S^-) \otimes X, Y]_{\tilde{K}} = [(S^+ - S^-) \otimes X, (S^+ - S^-) \otimes Y]_{\tilde{K}} \\ &= [\mathbf{H}_{\text{Dir}}(X)^+ - \mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^+ - \mathbf{H}_{\text{Dir}}(Y)^-]_{\tilde{K}} = [\mathbf{l}_{\text{Dir}}(X), \mathbf{l}_{\text{Dir}}(Y)]_{\tilde{K}}. \end{aligned}$$

The second equality is the Euler-Poincaré principle. We have also used (2.2) and Proposition 2.5. The attentive reader probably noticed a small problem with the application of the Euler-Poincaré principle since the terms  $C^i = \text{Hom}_K(\wedge^i \mathfrak{p} \otimes X, Y)$  could be infinite dimensional. We were not able to find a direct proof of the equality of the two extreme terms so far, but let us try to give some ideas of how this could be done. More details on this can be found in [24]. We may assume that the infinitesimal characters of  $X$  and  $Y$  are the same, since otherwise, both side of the identity we aim to prove are 0.

Let  $\mathcal{C} = \text{Hom}_{\tilde{K}}(X \otimes S, Y \otimes S)$ . Then  $\mathcal{C} = \mathcal{C}^{\bar{0}} \oplus \mathcal{C}^{\bar{1}}$ , with

$$\begin{aligned} \mathcal{C}^{\bar{0}} &= \text{Hom}_{\tilde{K}}(X \otimes S^+, Y \otimes S^+) \oplus \text{Hom}_{\tilde{K}}(X \otimes S^-, Y \otimes S^-), \\ \mathcal{C}^{\bar{1}} &= \text{Hom}_{\tilde{K}}(X \otimes S^+, Y \otimes S^-) \oplus \text{Hom}_{\tilde{K}}(X \otimes S^-, Y \otimes S^+). \end{aligned}$$

Let us consider the following various actions of the Dirac operator:

$$\begin{aligned} D_X^{+-} : X \otimes S^+ &\longrightarrow X \otimes S^-, & D_X^{-+} : X \otimes S^- &\longrightarrow X \otimes S^+, \\ D_Y^{+-} : Y \otimes S^+ &\longrightarrow Y \otimes S^-, & D_Y^{-+} : Y \otimes S^- &\longrightarrow Y \otimes S^+, \end{aligned}$$

For  $\phi^{++} \in \text{Hom}_{\tilde{K}}(X \otimes S^+, Y \otimes S^+)$ , and  $\phi^{--} \in \text{Hom}_{\tilde{K}}(X \otimes S^-, Y \otimes S^-)$  set

$$\begin{aligned} S\phi^{++} &= -\phi^{++} \circ D_X^{-+} + D_Y^{+-} \circ \phi^{++}, \\ S\phi^{--} &= -\phi^{--} \circ D_X^{+-} - D_Y^{-+} \circ \phi^{--}. \end{aligned}$$

This defines a linear map  $S : \mathcal{E}^{\bar{0}} \rightarrow \mathcal{E}^{\bar{1}}$ .

For  $\psi^{+-} \in \text{Hom}_{\tilde{K}}(X \otimes S^+, Y \otimes S^-)$ , and  $\psi^{-+} \in \text{Hom}_{\tilde{K}}(X \otimes S^-, Y \otimes S^+)$ , set

$$\begin{aligned} T\psi^{+-} &= -\psi^{+-} \circ D_X^{-+} + D_Y^{-+} \circ \psi^{+-}, \\ T\psi^{-+} &= -\psi^{-+} \circ D_X^{+-} - D_Y^{+-} \circ \psi^{-+}. \end{aligned}$$

This defines a linear map  $T : \mathcal{E}^{\bar{1}} \rightarrow \mathcal{E}^{\bar{0}}$ .

Let us take  $\phi^{++} + \phi^{--}$  in  $\ker S$ . We see easily that  $\phi^{++}$  induces

$$\bar{\phi}^{++} : \mathbf{H}_{\text{Dir}}(X)^+ = \frac{\ker(D_X^{+-})}{\ker(D_X^{+-}) \cap \text{Im}(D_X^{-+})} \longrightarrow \mathbf{H}_{\text{Dir}}(Y)^+ = \frac{\ker(D_Y^{+-})}{\ker(D_Y^{+-}) \cap \text{Im}(D_Y^{-+})},$$

and  $\phi^{--}$  induces

$$\bar{\phi}^{--} : \mathbf{H}_{\text{Dir}}(X)^- = \frac{\ker(D_X^{-+})}{\ker(D_X^{-+}) \cap \text{Im}(D_X^{+-})} \longrightarrow \mathbf{H}_{\text{Dir}}(Y)^- = \frac{\ker(D_Y^{-+})}{\ker(D_Y^{-+}) \cap \text{Im}(D_Y^{+-})}.$$

Furthermore, if  $\phi^{++} + \phi^{--} \in \ker S \cap \text{Im } T$ , then  $(\bar{\phi}^{++}, \bar{\phi}^{--}) = (0, 0)$ . Therefore, there are well-defined morphisms  $\phi^{++} + \phi^{--} \mapsto \bar{\phi}^{++} + \bar{\phi}^{--}$  from  $\ker S / \ker S \cap \text{Im } T$  to

$$\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^+) \oplus \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^-).$$

and  $\psi^{+-} + \psi^{-+} \mapsto \bar{\psi}^{+-} + \bar{\psi}^{-+}$  from  $\ker T / \ker T \cap \text{Im } S$  to

$$\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^-) \oplus \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^+).$$

LEMMA 5.3. — *The two morphisms defined above are isomorphisms:*

$$\ker S / \ker S \cap \text{Im } T \simeq \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^+) \oplus \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^-),$$

$$\ker T / \ker T \cap \text{Im } S \simeq \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^-) \oplus \text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^+).$$

*Proof.* — See [24]. □

Let us introduce now the index  $\text{ind}(S, T)$  of the Fredholm pair  $(S, T)$ . The material about Fredholm pairs and their indices is exposed in the appendix, and by definition

$$\text{ind}(S, T) = \dim\left(\frac{\ker S}{\ker S \cap \text{Im } T}\right) - \dim\left(\frac{\ker T}{\ker T \cap \text{Im } S}\right).$$

COROLLARY 5.4. — *With the notation as above,  $\text{ind}(S, T) = [\mathbf{l}_{\text{Dir}}(X), \mathbf{l}_{\text{Dir}}(Y)]_{\tilde{K}}$ .*

*Proof.* — We have

$$\begin{aligned} \text{ind}(S, T) &= \dim\left(\frac{\ker S}{\ker S \cap \text{Im } T}\right) - \dim\left(\frac{\ker T}{\ker T \cap \text{Im } S}\right) \\ &= \dim(\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^+)) + \dim(\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^-)) \\ &\quad - \dim(\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^+, \mathbf{H}_{\text{Dir}}(Y)^-)) - \dim(\text{Hom}_{\tilde{K}}(\mathbf{H}_{\text{Dir}}(X)^-, \mathbf{H}_{\text{Dir}}(Y)^+)) \\ &= [\mathbf{l}_{\text{Dir}}(X), \mathbf{l}_{\text{Dir}}(Y)]_{\tilde{K}}. \quad \square \end{aligned}$$

Theorem 5.1 is then equivalent to the fact that  $\text{ind}(S, T)$  is equal to the Euler-Poincaré characteristic of the complex computing the Ext groups of  $X$  and  $Y$ , namely  $\mathbf{C} = \sum_{i \in \mathbb{N}} C^i$ , where  $C^i = \text{Hom}_K(\wedge^i \mathfrak{p} \otimes X, Y)$ . By example 6.3, this Euler-Poincaré characteristic is also the index of the Fredholm pair given by the differentials between the even and odd part of the complex. More precisely, with

$$d^+ : \mathbf{C}^{\bar{0}} = \bigoplus_{i \in \mathbb{N}} C^{2i} \longrightarrow \mathbf{C}^{\bar{1}} = \bigoplus_{i \in \mathbb{N}} C^{2i+1},$$

and  $d^- : \mathbf{C}^{\bar{1}} \rightarrow \mathbf{C}^{\bar{0}}$ , we have  $\mathbf{EP}(X, Y) = \text{ind}(d^+, d^-)$ . So we would like to show that  $\text{ind}(S, T) = \text{ind}(d^+, d^-)$ . To facilitate the comparison, first notice that

$$\mathbf{C} = \bigoplus_{i \in \mathbb{N}} C^i = \bigoplus_{i \in \mathbb{N}} \text{Hom}_K(\wedge^i \mathfrak{p} \otimes X, Y) \simeq \text{Hom}_K(\wedge \mathfrak{p} \otimes X, Y) \simeq \text{Hom}_{\tilde{K}}(X \otimes S, Y \otimes S) = \mathcal{C}$$

Then transport  $(S, T)$  to  $(\mathcal{D}^+, \mathcal{D}^-)$  via this isomorphism. Thus

$$\mathcal{D}^+ : \mathbf{C}^{\bar{0}} \longrightarrow \mathbf{C}^{\bar{1}}, \quad \mathcal{D}^- : \mathbf{C}^{\bar{1}} \longrightarrow \mathbf{C}^{\bar{0}},$$

and set  $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ , an operator on  $\mathbf{C}$ . So we would like to establish that

$$(5.1) \quad \text{ind}(d^+, d^-) = \text{ind}(\mathcal{D}^+, \mathcal{D}^-).$$

The point here is that we want to prove the equality of the indices of two Fredholm pairs living on the same spaces.

## 6. APPENDIX: FREDHOLM PAIRS

In this section, we adapt from [1] the definition of the index of a Fredholm pair. We do this in a purely algebraic setting, while the theory is developed for Banach spaces in [1].

**DEFINITION 6.1.** — Let  $X, Y$  be complex vector spaces and let  $S \in \mathcal{L}(X; Y)$ ,  $T \in \mathcal{L}(Y; X)$ . Then  $(S, T)$  is called a Fredholm pair if the following dimensions are finite:

$$a := \dim \ker(S) / \ker(S) \cap \text{Im}(T); \quad b := \dim \ker(T) / \ker(T) \cap \text{Im}(S).$$

In this case, the number  $\text{ind}(S, T) := a - b$  is called the index of  $(S, T)$ .

**EXAMPLE 6.2.** — Take  $T = 0$ . Then  $(S, 0)$  is a Fredholm pair if and only if  $S$  is a Fredholm operator and

$$\text{ind}(S, 0) = \text{ind}(S) = \dim \ker(S) - \dim \text{coker}(S).$$

EXAMPLE 6.3. — Consider a differential complex

$$\dots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \dots$$

Suppose the cohomology groups  $H^i := \ker d^i / \operatorname{Im} d^{i-1}$  of this complex are finite dimensional, and non zero only for a finite number of them. Put  $X = \bigoplus_{i \in \mathbb{Z}} C^{2i}$ ,  $Y = \bigoplus_{i \in \mathbb{Z}} C^{2i+1}$ ,  $S = \bigoplus_{i \in 2\mathbb{Z}} d^{2i}$ ,  $T = \bigoplus_{i \in 2\mathbb{Z}} d^{2i+1}$ . Then  $(S, T)$  is a Fredholm pair and its index is equal to the Euler-Poincaré characteristic of the complex:

$$\operatorname{ind}(S, T) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i.$$

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