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On canonical subgroups of Hilbert–Blumenthal abelian varieties

par Shin HATTORI

Résumé. Soit p un nombre premier. Soit F un corps totalement réel non ramifié en p. Dans cet article, nous développons une théorie de sous-groupes canoniques pour les variétés abéliennes de Hilbert–Blumenthal avec $\mathcal{O}_F$-actions, dans laquelle ceux-ci sont liés à des applications de Hodge–Tate si la β-hauteur de Hodge est plus petite que $(p-1)/p^n$ pour tout plongement $\beta : F \to \bar{\mathbb{Q}}_p$.

Abstract. Let p be a rational prime. Let F be a totally real number field which is unramified over p. In this paper, we develop a theory of canonical subgroups for Hilbert–Blumenthal abelian varieties with $\mathcal{O}_F$-actions, in which they are related with Hodge–Tate maps if the β-Hodge height is less than $(p-1)/p^n$ for every embedding $\beta : F \to \bar{\mathbb{Q}}_p$.

1. Introduction

Let p be a rational prime. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with residue field k and $\bar{K}$ an algebraic closure of K. Let A be an abelian scheme of relative dimension g over the integer ring $\mathcal{O}_K$ of K. Let n be a positive integer. If A has ordinary reduction, then the unit component $A[p^n]^0$ of the p^n-torsion subgroup scheme $A[p^n]$ is finite and flat of rank p^{ng} over $\mathcal{O}_K$ and its reduction $A[p^n]^0 \times \mathrm{Spec}(\mathcal{O}_K/p\mathcal{O}_K)$ is equal to the kernel of the n-th iterated Frobenius map of $A \times \mathrm{Spec}(\mathcal{O}_K/p\mathcal{O}_K)$. The theory of canonical subgroups asserts that, if A is sufficiently close to having ordinary reduction, then a similar subgroup scheme $\mathcal{C}_n$ of A exists even for the case of non-ordinary reduction and that $\mathcal{C}_n$ has distinguished properties compared to other subgroup schemes of $A[p^n]$. The subgroup scheme $\mathcal{C}_n$ is called the canonical subgroup of A of level n. We can make explicit how close A should be to abelian varieties with ordinary reduction in order to have the canonical subgroup, in terms of the Hodge height $\mathrm{Hdg}(A)$. It is an element of $\mathbb{Q} \cap [0,1]$ such that $\mathrm{Hdg}(A) = 0$ if and only if A has ordinary reduction.

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This theory is a powerful tool to study overconvergent p-adic modular forms and the Hecke operator U_p at p acting on them. For example, the theory tells us that, for any non-canonical subgroup $\mathcal{D}$ of A, the abelian scheme $A/\mathcal{D}$ is closer to having ordinary reduction than A. From this we can show that U_p is a compact operator and thus has a rich spectral theory. Furthermore, it is shown that we can control the cokernel of the Hodge–Tate map of the Cartier dual $\mathcal{C}_n^\vee$ which relates $\mathcal{C}_n^\vee(\bar{K})$ with the module $\omega_{\mathcal{C}_n}$ of invariant differentials. This property leads to the construction of sheaves of overconvergent Siegel and Hilbert modular forms and their eigenvarieties, which is due to Andreatta–Iovita–Pilloni [1, 2]. The sheaves of overconvergent modular forms of Andreatta–Iovita–Pilloni are defined on the loci in modular varieties classifying abelian varieties A with additional structures where A has the canonical subgroup of some level.

On the other hand, it has been actively studied how overconvergent modular forms are analytically continued over modular varieties (for example, [4, 5, 16, 17, 18, 26, 27, 28]). Especially, Buzzard–Calegari [4] proved that, for the case of 1-analytic weights where weights are sufficiently close to classical ones, any overconvergent elliptic eigenform of finite slope (namely, eigenform with non-zero U_p-eigenvalue) can be analytically continued to the locus on a modular curve where elliptic curves have Hodge heights no more than $1/(p+1)$, while any non-zero overconvergent elliptic eigenform of infinite slope (namely, eigenform killed by U_p) cannot. This clear distinction between extension of overconvergent eigenforms of finite and infinite slopes enabled them to prove a properness of the Coleman–Mazur eigencurve for some cases.

In order to carry out a similar study of extension properties for overconvergent Hilbert modular forms, what we need is a theory of canonical subgroups for Hilbert–Blumenthal abelian varieties (HBAV's) satisfying the following two requirements: First, we need to relate canonical subgroups with Hodge–Tate maps, in order to make them yield sheaves of overconvergent Hilbert modular forms. Second, we need to have a sufficiently large existence locus for canonical subgroups, in order to investigate extension properties of overconvergent Hilbert modular forms far away from the ordinary locus.

Let F be a totally real number field which is unramified over p, $\mathcal{O}_F$ its ring of integers and $f_\mathfrak{p}$ its residue degree at a prime ideal $\mathfrak{p}$ dividing p. We denote by $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. Let $\mathbb{B}_F$ be the set of embeddings $F \to \bar{K}$ and $\mathbb{B}_\mathfrak{p}$ the subset consisting of embeddings factoring through $F_\mathfrak{p}$. For any HBAV A over $\mathcal{O}_K$ with $\mathcal{O}_F$-action, the Hodge height $\mathrm{Hdg}(A)$ is decomposed as the truncated sum

$$\mathrm{Hdg}(A)=\min\left\{\sum_{\beta\in\mathbb{B}_F}\mathrm{Hdg}_\beta(A),1\right\}$$

of more precise invariants $\mathrm{Hdg}_\beta(A)$, the β-Hodge heights, for each embedding $\beta:F\to\bar{K}$. Goren–Kassaei [9] established the theory of canonical subgroups of any level, using the geometry of Hilbert modular varieties over a field of characteristic p. In their theory, the existence locus of canonical subgroups is described in terms of $\mathrm{Hdg}_\beta(A)$ and much larger than what we can obtain by using $\mathrm{Hdg}(A)$ or

$$\mathrm{Hdg}_\mathfrak{p}(A)=\min\left\{\sum_{\beta\in\mathbb{B}_\mathfrak{p}}\mathrm{Hdg}_\beta(A),1\right\}.$$

However, the relation of their canonical subgroups with Hodge–Tate maps had been known only for the locus of $\mathrm{Hdg}_\mathfrak{p}(A)<1/2$ for all $\mathfrak{p}\mid p$ with $p\neq 2$ by comparing with canonical subgroups of Fargues [8], and thus sheaves of overconvergent Hilbert modular forms of p-adic weights [2] had been defined only on this small locus. This causes a constraint on the study of analytic continuation of overconvergent Hilbert modular forms: for example, it is only for $p>\max\{2f_\mathfrak{p}-1\mid\mathfrak{p}\mid p\}$ that overconvergent Hilbert modular forms of p-adic weights are defined on the locus where $\mathrm{Hdg}_\beta(A)=1/(p+1)$ holds for all β, and a similar study of extension properties to [4] would be possible only for a sufficiently large p so far.

In this paper, we present a theory of canonical subgroups controlled by β-Hodge heights and satisfying the above two requirements, which is used in the author's work on a properness of the Hilbert eigenvariety at integral weights [11]. To state the main theorem, we fix some notation. Let v_p be the additive valuation on K normalized as $v_p(p)=1$. For any non-negative real number i, we put

$$m_K^{\geqslant i}=\{x\in\mathcal{O}_K\mid v_p(x)\geq i\},\quad \mathcal{O}_{K,i}=\mathcal{O}_K/m_K^{\geqslant i},\quad \mathscr{S}_i=\mathrm{Spec}(\mathcal{O}_{K,i}).$$

For any extension L/K of valuation fields, we consider the valuation on L extending v_p and define $m_L^{\geqslant i}$, $\mathcal{O}_{L,i}$ and $\mathscr{S}_{L,i}=\mathrm{Spec}(\mathcal{O}_{L,i})$ similarly. Let $\bar{L}$ be an algebraic closure of L. For any finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_L$ with Cartier dual $\mathcal{G}^\vee$, we denote the Hodge–Tate map for $\mathcal{G}$ by $\mathrm{HT}_\mathcal{G}:\mathcal{G}(\mathcal{O}_{\bar{L}})\to\omega_{\mathcal{G}^\vee}\otimes_{\mathcal{O}_L}\mathcal{O}_{\bar{L}}$ (see §2 for the definition) and define the i-th Hodge–Tate map $\mathrm{HT}_{\mathcal{G},i}:\mathcal{G}(\mathcal{O}_{\bar{L}})\to\omega_{\mathcal{G}^\vee}\otimes_{\mathcal{O}_L}\mathcal{O}_{\bar{L},i}$ as the composite of $\mathrm{HT}_\mathcal{G}$ and the natural reduction map. We denote the i-th lower ramification subgroup of $\mathcal{G}$ by $\mathcal{G}_i$. By definition, it is the scheme-theoretic closure in $\mathcal{G}$ of $\mathrm{Ker}(\mathcal{G}(\mathcal{O}_{\bar{L}})\to\mathcal{G}(\mathcal{O}_{\bar{L},i}))$. Let σ be the p-th power Frobenius map on the residue field of K, which naturally acts on $\mathbb{B}_F$ and $\mathbb{B}_{\mathfrak{p}}$. Then the main theorem of this paper is the following.

Theorem 1.1 (Theorem 8.1)**.** *Let L/K be a finite extension in $\bar{K}$. Let $\mathfrak{c}$ be a non-zero fractional ideal of F. Let A be a HBAV over $\mathcal{O}_L$ with $\mathcal{O}_F$-action and $\mathfrak{c}$-polarization. Put $w_\beta = \mathrm{Hdg}_\beta(A)$ and $w = \max\{w_\beta \mid \beta \in \mathbb{B}_F\}$. Suppose that*

$$w_\beta + p w_{\sigma^{-1}\circ\beta} < p^{2-n}$$

holds for all $\beta \in \mathbb{B}_F$.

If the residue field k is perfect, then there exists a finite flat closed $\mathcal{O}_F$-subgroup scheme $\mathcal{C}_n(A)$ of $A[p^n]$ over $\mathcal{O}_L$ such that the $\mathcal{O}_F/p^n\mathcal{O}_F$-module $\mathcal{C}_n(A)(\mathcal{O}_{\bar{K}})$ is free of rank one and $\mathcal{C}_n(A) \times \mathscr{S}_{L,1-p^{n-1}w}$ is equal to the kernel of the n-th iterated Frobenius map of $A \times \mathscr{S}_{L,1-p^{n-1}w}$. We call $\mathcal{C}_n(A)$ the canonical subgroup of A of level n. Put $b = n - w(p^n-1)/(p-1)$. If $w < (p-1)/p^n$, then $\mathcal{C}_n = \mathcal{C}_n(A)$ also satisfies the following (among others).

- *$\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ coincides with $\mathrm{Ker}(\mathrm{HT}_{A[p^n],i})$ for any rational number i satisfying*
$$n - 1 + \frac{w}{p-1} < i \leq b.$$
- *$\mathcal{C}_n = A[p^n]_i$ for any rational number i satisfying*
$$\frac{1}{p^n(p-1)} \leq i \leq \frac{1}{p^{n-1}(p-1)} - \frac{w}{p-1}.$$
- *For any $i \in v_p(\mathcal{O}_L)$ satisfying $i \leq b$, the natural map*
$$\omega_A \otimes_{\mathcal{O}_L} \mathcal{O}_{L,i} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_L} \mathcal{O}_{L,i}$$
is an isomorphism.
- *The cokernel of the map*
$$\mathrm{HT}_{\mathcal{C}_n^\vee} \otimes 1 : \mathcal{C}_n^\vee(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{K}}$$
is killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$.

Moreover, for the case where k is imperfect, the same statements hold if $w < (p-1)/p^n$.

Weights for overconvergent Hilbert modular forms are p-adic continuous characters of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$, and we say that a weight is n-analytic if its restriction to $1 + p^n(\mathcal{O}_F \otimes \mathbb{Z}_p)$ is analytic (see [2, §2]). Using Theorem 1.1, we can enlarge the domain of definition of sheaves of overconvergent Hilbert modular forms of n-analytic weights to the locus where $\mathrm{Hdg}_\beta(A) < (p-1)/p^n$ holds for all β [11], which in particular enables us, for the 1-analytic case, to study extension of overconvergent Hilbert modular forms to the locus of $\mathrm{Hdg}_\beta(A) = 1/(p+1)$ for all β even if p is small.

Our approach to construct canonical subgroups is (reducing to the case where k is perfect and) using the Breuil–Kisin classification of finite flat group schemes over $\mathcal{O}_K$ [3, 20, 21, 23, 25]. For a HBAV A over $\mathcal{O}_K$, the group scheme $A[p^n]$ is decomposed as $A[p^n] = \bigoplus_{\mathfrak{p}|p} A[\mathfrak{p}^n]$, where the direct sum is taken over the set of prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$ dividing p. Each $A[\mathfrak{p}^n]$ is a truncated Barsotti–Tate group with an action of $\mathbb{Z}_{p^f}$ for $f = f_\mathfrak{p}$ and the $\mathcal{O}_{K,n} \otimes \mathbb{Z}_{p^f}$-module $\omega_{A[\mathfrak{p}^n]}$ is free of rank one. Moreover, $A[\mathfrak{p}^n]$ is also endowed with a structure of alternating self-duality coming from the polarization of A. We refer to such a group scheme as a $\mathbb{Z}_{p^f}$-alternating self-dual truncated Barsotti–Tate $\mathbb{Z}_{p^f}$-group or a $\mathbb{Z}_{p^f}$-ADBT$_n$.

For the p-torsion case, Tian [28] constructed the canonical subgroup of a $\mathbb{Z}_{p^f}$-ADBT$_1$ of level one, following the idea of the author [13] for the Siegel case. We can also construct the canonical subgroup of level n for a $\mathbb{Z}_{p^f}$-ADBT$_n$ by a standard induction as in [8, 9], and obtain the canonical subgroup $\mathcal{C}_n(A)$ of A as the direct sum of canonical subgroups of $A[\mathfrak{p}^n]$'s. The main contribution of this paper is to show that canonical subgroups obtained in this way are related with Hodge–Tate maps and equal to lower ramification subgroups. The coincidence with lower ramification subgroups produces a larger existence locus of a family of canonical subgroups in the Hilbert modular variety via [14, Lemma 5.6], and thus those of sheaves of overconvergent Hilbert modular forms, than the other approaches in the literature. To show this coincidence, the author's description of lower ramification subgroups via the Breuil–Kisin classification [14, Theorem 1.1 and Corollary 3.3] is crucially used. Note that the use of lower ramification subgroups in the theory of canonical subgroups had already appeared in the classical case of elliptic curves [19, Theorem 3.10.7].

As a byproduct of our construction of canonical subgroups, we also study the variation of β-Hodge heights by the U_p-correspondence for $\mathbb{Z}_{p^f}$-ADBT$_n$'s of β-Hodge heights $p/(p+1)$ for the case of $f \leq 2$ (Proposition 6.1), which is one of the key ingredients for generalizing Buzzard–Calegari's extension property mentioned above to overconvergent Hilbert modular forms [11].

This paper is organized as follows. In §2, we recall the Breuil–Kisin classification of finite flat group schemes and its relation with invariants on the side of differentials such as the degree, the Hodge height and Hodge–Tate maps. These invariants and the Breuil–Kisin module associated to a finite flat group scheme with $\mathbb{Z}_{p^f}$-action are decomposed by the action of $\mathbb{Z}_{p^f}$, which are studied in §3. In §4, we prove various properties of Tian's canonical subgroup of level one using the Breuil–Kisin classification. In §5, we generalize Goren–Kassaei's theory of variation of β-Hodge heights and β-degrees of HBAV's by isogenies [9, §5] to the case of $\mathbb{Z}_{p^f}$-ADBT$_n$'s. Using these results, canonical subgroups of $\mathbb{Z}_{p^f}$-ADBT$_n$'s and HBAV's are constructed and studied in §7 and §8, respectively. §6 is devoted to studying the U_p-correspondence for $\mathbb{Z}_{p^f}$-ADBT$_n$'s of β-Hodge heights $p/(p+1)$.

2. Breuil–Kisin modules

Let K be as in §1 and suppose that the residue field k is perfect until §5. Let e be the absolute ramification index of K and $W = W(k)$ the Witt ring of k. Put $W_n = W/p^n W$. We denote by σ both the p-th power Frobenius map on k and its natural lift on W. Let π be a uniformizer of K. Let v_p be the additive valuation on K normalized as $v_p(p) = 1$. For any non-negative real number i and any extension L/K of valuation fields, we put $m_K^{\geqslant i}$, $\mathcal{O}_{K,i}$, $\mathscr{S}_i$, $m_L^{\geqslant i}$, $\mathcal{O}_{L,i}$ and $\mathscr{S}_{L,i}$ as in §1. For any element $x \in \mathcal{O}_{L,1}$, we define the truncated valuation $v_p(x)$ by

$$v_p(x) = \min\{v_p(\hat{x}), 1\}$$

with any lift $\hat{x} \in \mathcal{O}_L$ of x. Let us fix an algebraic closure $\bar{K}$ of K and extend v_p naturally to $\bar{K}$. Put $G_K = \mathrm{Gal}(\bar{K}/K)$. We fix a system $(\pi_n)_{n\geq 0}$ of p-power roots of π in $\bar{K}$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$ for any n. Put $K_\infty = \bigcup_{n\geq 0} K(\pi_n)$ and $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$.

Let $E(u) \in W[u]$ be the monic Eisenstein polynomial for π and set $c_0 = p^{-1}E(0) \in W^\times$. Put $\mathfrak{S} = W[[u]]$ and $\mathfrak{S}_n = \mathfrak{S}/p^n\mathfrak{S}$. The ring $\mathfrak{S}_1 = k[[u]]$ is a complete discrete valuation ring with additive valuation v_u normalized as $v_u(u) = 1$. We also denote by φ the σ-semilinear continuous ring homomorphism $\varphi : \mathfrak{S} \to \mathfrak{S}$ defined by $u \mapsto u^p$.

An $\mathfrak{S}$-module $\mathfrak{M}$ is said to be a Breuil–Kisin module (of E-height ≤ 1) if $\mathfrak{M}$ is a finitely generated $\mathfrak{S}$-module equipped with a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearization

$$1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \longrightarrow \mathfrak{M}$$

is killed by $E(u)$. We refer to $\varphi_{\mathfrak{M}}$ as the Frobenius map of the Breuil–Kisin module $\mathfrak{M}$ and often write as φ abusively. A morphism of Breuil–Kisin modules is defined as an $\mathfrak{S}$-linear map compatible with Frobenius maps. Let

$$\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$$

be the category of Breuil–Kisin modules $\mathfrak{M}$ such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is free of finite rank over $\mathfrak{S}_1$. We denote by

$$\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$$

the category of Breuil–Kisin modules $\mathfrak{M}$ such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is finitely generated, p-power torsion and u-torsion free.

The category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$ admits a natural notion of duality, which is denoted by $\mathfrak{M} \mapsto \mathfrak{M}^\vee$ [24, Proposition 3.1.7]. Here we give its explicit definition for the full subcategory $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$. For any object $\mathfrak{M}$ of $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$, let $e_1,\ldots,e_h$ be its basis. Write as

$$\varphi_{\mathfrak{M}}(e_1,\ldots,e_h) = (e_1,\ldots,e_h)A$$

with some $A \in M_h(\mathfrak{S}_1)$. Consider its dual $\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}_1}(\mathfrak{M},\mathfrak{S}_1)$ with the dual basis $e_1^\vee,\ldots,e_h^\vee$. We give $\mathfrak{M}^\vee$ a structure of a Breuil–Kisin module by

$$\varphi_{\mathfrak{M}^\vee}(e_1^\vee,\ldots,e_h^\vee) = (e_1^\vee,\ldots,e_h^\vee)\left(\frac{E(u)}{c_0}\right){}^t\!A^{-1},$$

which is independent of the choice of a basis.

Consider the inverse limit ring

$$R = \varprojlim_{n\geq 0}(\mathcal{O}_{\bar{K},1} \leftarrow \mathcal{O}_{\bar{K},1} \leftarrow \cdots),$$

where every transition map is the p-th power Frobenius map. The absolute Galois group G_K acts on R via the natural action on each entry. We define an element $\underline{\pi}$ of R by

$$\underline{\pi} = (\pi_0,\pi_1,\ldots).$$

The ring R is a complete valuation ring of characteristic p with algebraically closed fraction field, and we normalize the additive valuation v_R on R by $v_R(\underline{\pi}) = 1/e$. We define $m_R^{\geqslant i}$ and $R_i = R/m_R^{\geqslant i}$ as before, using v_R. We consider the Witt ring $W(R)$ as an $\mathfrak{S}$-algebra by the continuous W-linear map defined by $u \mapsto [\underline{\pi}]$. Then we have the following classification of finite flat group schemes over $\mathcal{O}_K$ [3, 20, 21, 23, 25].

Theorem 2.1.

(1) *There exists an exact anti-equivalence*

$$\mathcal{G} \longmapsto \mathfrak{M}^*(\mathcal{G})$$

from the category of finite flat group schemes over $\mathcal{O}_K$ killed by some p-power to the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$. If $\mathcal{G}$ is a truncated Barsotti–Tate group of level n over $\mathcal{O}_K$, then the $\mathfrak{S}$-module $\mathfrak{M}^*(\mathcal{G})$ is free over $\mathfrak{S}_n$.

(2) *Let n be a positive integer satisfying $p^n\mathcal{G} = 0$. Then there exists a natural isomorphism of G_{K_∞}-modules*

$$\mathcal{G}(\mathcal{O}_{\bar{K}}) \longrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^*(\mathcal{G}), W_n(R)).$$

Moreover, we have $\mathrm{lg}_{\mathbb{Z}_p}(\mathcal{G}(\mathcal{O}_{\bar{K}})) = \mathrm{lg}_{\mathfrak{S}}(\mathfrak{M}^*(\mathcal{G}))$.

(3) *Let $\mathcal{G}^\vee$ be the Cartier dual of $\mathcal{G}$. Then there exists a natural isomorphism $\mathfrak{M}^*(\mathcal{G}^\vee) \to \mathfrak{M}^*(\mathcal{G})^\vee$ which, combined with the natural isomorphism of* (2)*, identifies the pairing of Cartier duality*

$$\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}} : \mathcal{G}(\mathcal{O}_{\bar{K}}) \times \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \longrightarrow \mu_{p^n}(\mathcal{O}_{\bar{K}})$$

with the natural perfect pairing

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^*(\mathcal{G}), W_n(R)) \times \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^*(\mathcal{G})^\vee, W_n(R)) \longrightarrow W_n(R).$$

(4) *For any non-negative rational number $i \leq 1$, there exists an ideal $I_{n,i}$ of $W_n(R)$ such that the isomorphism of* (2) *induces an isomorphism*

$$\mathcal{G}_i(\mathcal{O}_{\bar{K}}) \simeq \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^*(\mathcal{G}), I_{n,i}).$$

Moreover, we have $I_{1,i} = m_R^{\geqslant i}$.

Proof. The assertions (1) and (2) are contained in [20, Corollary 4.3]: the assertion on truncated Barsotti–Tate groups of level n over $\mathcal{O}_K$ follows from the fact that they are p^n-torsion parts of p-divisible groups [15, Théorème 4.4 (e)], and the equality on the length follows from the natural isomorphism of (2). The assertion (3) follows from a similar assertion on p-divisible groups [20, §5.1] and a dévissage argument as in [12, Proposition 4.4]. The assertion (4) is [14, Theorem 1.1 and Corollary 3.3]. □

Next we recall, for any extension L/K of complete valuation fields, the definitions of invariants associated to a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_L$ which is killed by p^n with some positive integer n. For any finitely presented torsion $\mathcal{O}_L$-module M, write as $M \simeq \bigoplus_i \mathcal{O}_L/(a_i)$ with some $a_i \in \mathcal{O}_L$ and put $\deg(M) = \sum_i v_p(a_i)$. Since $\mathcal{G}$ is finitely presented over $\mathcal{O}_L$, the module $\omega_{\mathcal{G}}$ of invariant differentials of $\mathcal{G}$ is a finitely presented $\mathcal{O}_L$-module. We put

$$\deg(\mathcal{G}) = \deg(\omega_{\mathcal{G}})$$

and refer to it as the degree of $\mathcal{G}$.

Let $\bar{L}$ be an algebraic closure of L. Note that any element $x \in \mathcal{G}(\mathcal{O}_{\bar{L}})$ defines a homomorphism

$$x : \mathcal{G}^\vee \times_{\mathcal{O}_L} \mathrm{Spec}(\mathcal{O}_{\bar{L}}) \longrightarrow \mathbb{G}_{\mathrm{m}} \times_{\mathcal{O}_L} \mathrm{Spec}(\mathcal{O}_{\bar{L}})$$

by Cartier duality. We define the Hodge–Tate map by

$$\mathrm{HT}_{\mathcal{G}} : \mathcal{G}(\mathcal{O}_{\bar{L}}) \longrightarrow \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{L}}, \quad x \longmapsto x^* \frac{dT}{T}$$

and, for any positive rational number i, the i-th Hodge–Tate map by the composite

$$\mathrm{HT}_{\mathcal{G},i} : \mathcal{G}(\mathcal{O}_{\bar{L}}) \xrightarrow{\mathrm{HT}_{\mathcal{G}}} \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{L}} \longrightarrow \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{L},i}$$

of $\mathrm{HT}_{\mathcal{G}}$ and the reduction map. We also denote them by HT and HT_i.

Suppose that $\mathcal{G}$ is a truncated Barsotti–Tate group of level n, height h and dimension d over $\mathcal{O}_L$. Consider the p-torsion part $\mathcal{G}[p]$. Note that the Lie algebra $\mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1})$ is a free $\mathcal{O}_{L,1}$-module of rank $h-d$. The Verschiebung of $\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1}$ induces a map on the Lie algebra

$$\mathrm{Lie}(V_{\mathcal{G}^\vee[p]\times\mathscr{S}_{L,1}}) : \mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1})^{(p)} \longrightarrow \mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1}).$$

The truncated valuation for v_p of the determinant of a representing matrix of this map is independent of the choice of a basis of the Lie algebra, which we call the Hodge height of $\mathcal{G}$ and denote by

$$\mathrm{Hdg}(\mathcal{G}).$$

Finally, for any truncated Barsotti–Tate group $\mathcal{G}$ of level one over $\mathcal{O}_K$ and any element i of $e^{-1}\mathbb{Z}_{\geq 0}$, the quotient $\mathfrak{M}^*(\mathcal{G})_i = \mathfrak{M}^*(\mathcal{G})/u^{ei}\mathfrak{M}^*(\mathcal{G})$ has a natural structure of a φ-module induced by $\varphi_{\mathfrak{M}}$. We put

$$\mathrm{Fil}^1\mathfrak{M}^*(\mathcal{G})_i = \mathrm{Im}(1 \otimes \varphi : \varphi^*\mathfrak{M}^*(\mathcal{G})_i \longrightarrow \mathfrak{M}^*(\mathcal{G})_i).$$

It also has a natural structure of a φ-module induced by $\varphi_{\mathfrak{M}}$. By the isomorphisms of k-algebras $\mathfrak{S}_1/(u^e) \to \mathcal{O}_{K,1}$ defined by $u \mapsto \pi$ and $R_i \to \mathcal{O}_{\bar{K},i}$ defined by the zeroth projection pr_0 for $i \leq 1$, we identify both sides. For any $x \in \mathfrak{S}_1/(u^e)$, we define the truncated valuation $v_u(x)$ by $v_u(x) = \min\{v_u(\hat{x}), e\}$ with any lift $\hat{x} \in \mathfrak{S}_1$ of x. Then these invariants of $\mathcal{G}$ on the side of differentials can be read off from the associated Breuil–Kisin module, as follows.

Proposition 2.2.

(1) *For any finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ killed by p, there exists a natural isomorphism*

$$\mathfrak{M}^*(\mathcal{G})/(1 \otimes \varphi_{\mathfrak{M}^*(\mathcal{G})})(\varphi^*\mathfrak{M}^*(\mathcal{G})) \longrightarrow \omega_{\mathcal{G}}$$

and we have

$$\deg(\mathcal{G}) = e^{-1}v_u(\det(\varphi_{\mathfrak{M}^*(\mathcal{G})})).$$

(2) *Suppose that $\mathcal{G}$ is a truncated Barsotti–Tate group of level one. Then we have a natural isomorphism*

$$\mathrm{Lie}(\mathcal{G}^\vee \times \mathscr{S}_1) \longrightarrow \mathrm{Fil}^1\mathfrak{M}^*(\mathcal{G})_1.$$

The $\mathcal{O}_{K,1}$-module $\mathrm{Fil}^1\mathfrak{M}^*(\mathcal{G})_1$ is a direct summand of $\mathfrak{M}^*(\mathcal{G})_1$ of rank $h - d$. Moreover, we have the equality of truncated valuations

$$\mathrm{Hdg}(\mathcal{G}) = e^{-1}v_u(\det(\varphi_{\mathrm{Fil}^1\mathfrak{M}^*(\mathcal{G})_1})).$$

(3) *Suppose that $\mathcal{G}$ is a truncated Barsotti–Tate group of level one. For any positive rational number $i \leq 1$, the i-th Hodge–Tate map $\mathrm{HT}_{\mathcal{G},i}$ coincides with the composite*

$$\begin{aligned}\mathcal{G}(\mathcal{O}_{\bar{K}}) \longrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^*(\mathcal{G}), R) &\longrightarrow \mathrm{Hom}_{\mathfrak{S}}(\mathrm{Fil}^1\mathfrak{M}^*(\mathcal{G})_1, R_i)\\ &\simeq \mathrm{Hom}_{\mathcal{O}_K}(\mathrm{Lie}(\mathcal{G}^\vee \times \mathscr{S}_1), \mathcal{O}_{\bar{K},i}) \simeq \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},i}.\end{aligned}$$

Proof. The first isomorphism is shown in [28, Proposition 3.2] and the others are in [13, §2.3]. Note that though [13] assumes $p > 2$, the same proof remains valid also for $p = 2$ by using [20] instead of [21]. $\square$

3. $\mathbb{Z}_{p^f}$-groups

Let f be a positive integer. We assume that the residue field k of K is perfect and contains the finite field $\mathbb{F}_{p^f}$. Let $\mathbb{B}_f$ be the set of embeddings of $\mathbb{F}_{p^f}$ into k. We denote the unramified extension of $\mathbb{Q}_p$ of degree f by $\mathbb{Q}_{p^f}$ and its ring of integers by $\mathbb{Z}_{p^f}$. Any $\beta \in \mathbb{B}_f$ has the canonical lifts $\mathbb{Z}_{p^f} \to \mathcal{O}_K$ and $\mathbb{Q}_{p^f} \to K$, which we also denote by β. Then any $W \otimes \mathbb{Z}_{p^f}$-module M is decomposed as

$$M = \bigoplus_{\beta \in \mathbb{B}_f} M_\beta$$

according with the decomposition $W \otimes \mathbb{Z}_{p^f} \simeq \prod_{\beta \in \mathbb{B}_f} W$.

Let L/K be any extension of complete valuation fields and $\bar{L}$ an algebraic closure of L. A group scheme $\mathcal{G}$ over $\mathcal{O}_L$ is said to be a $\mathbb{Z}_{p^f}$-group if it is equipped with an action of the ring $\mathbb{Z}_{p^f}$. Then we have the decompositions

$$\omega_{\mathcal{G}} = \bigoplus_{\beta \in \mathbb{B}_f} \omega_{\mathcal{G},\beta}, \quad \mathrm{Lie}(\mathcal{G} \times \mathscr{S}_{L,n}) = \bigoplus_{\beta \in \mathbb{B}_f} \mathrm{Lie}(\mathcal{G} \times \mathscr{S}_{L,n})_\beta.$$

When $\mathcal{G}$ is finite and flat over $\mathcal{O}_L$, we define the β-degree of $\mathcal{G}$ by

$$\deg_\beta(\mathcal{G}) = \deg(\omega_{\mathcal{G},\beta}).$$

We have $\deg(\mathcal{G}) = \sum_{\beta \in \mathbb{B}_f} \deg_\beta(\mathcal{G})$. Moreover, for any exact sequence of finite flat $\mathbb{Z}_{p^f}$-groups over $\mathcal{O}_L$

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0,$$

the equality $\deg_\beta(\mathcal{G}) = \deg_\beta(\mathcal{G}') + \deg_\beta(\mathcal{G}'')$ holds.

Let n be a positive integer. A $\mathbb{Z}_{p^f}$-group $\mathcal{G}$ over $\mathcal{O}_L$ is said to be a truncated Barsotti–Tate $\mathbb{Z}_{p^f}$-group of level n if $\mathcal{G}$ is a truncated Barsotti–Tate group of level n, height $2f$ and dimension f such that $\omega_{\mathcal{G}}$ is a free $\mathcal{O}_{L,n} \otimes \mathbb{Z}_{p^f}$-module of rank one. Note that for such $\mathcal{G}$, we have $\deg_\beta(\mathcal{G}) = n$.

Definition 3.1. We say that a truncated Barsotti–Tate $\mathbb{Z}_{p^f}$-group $\mathcal{G}$ of level n is $\mathbb{Z}_{p^f}$-alternating self-dual if it is equipped with an isomorphism of $\mathbb{Z}_{p^f}$-groups $i : \mathcal{G} \simeq \mathcal{G}^\vee$ over $\mathcal{O}_L$ such that the perfect pairing defined via Cartier duality

$$\mathcal{G}(\mathcal{O}_{\bar{L}}) \times \mathcal{G}(\mathcal{O}_{\bar{L}}) \xrightarrow{1 \times i} \mathcal{G}(\mathcal{O}_{\bar{L}}) \times \mathcal{G}^\vee(\mathcal{O}_{\bar{L}}) \xrightarrow{\langle\cdot,\cdot\rangle_{\mathcal{G}}} \mu_{p^n}(\mathcal{O}_{\bar{L}})$$

satisfies $\langle x, i(ax)\rangle_{\mathcal{G}} = 1$ for any $x \in \mathcal{G}(\mathcal{O}_{\bar{L}})$ and $a \in \mathbb{Z}_{p^f}$. In this case, we also say that the isomorphism i is $\mathbb{Z}_{p^f}$-alternating. We abbreviate $\mathbb{Z}_{p^f}$-alternating self-dual truncated Barsotti–Tate $\mathbb{Z}_{p^f}$-group of level n as

$$\mathbb{Z}_{p^f}\text{-ADBT}_n.$$

Note that the above map i is skew-symmetric: namely, we have the commutative diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "i"] \arrow[d, "\text{can.}"'] & \mathcal{G}^\vee \arrow[d, "-1"] \\ \mathcal{G}^{\vee\vee} \arrow[r, "i^\vee"'] & \mathcal{G}^\vee. \end{tikzcd}$$

For $p \neq 2$, an isomorphism of $\mathbb{Z}_{p^f}$-groups $i : \mathcal{G} \simeq \mathcal{G}^\vee$ is $\mathbb{Z}_{p^f}$-alternating if and only if it is skew-symmetric.

For a $\mathbb{Z}_{p^f}$-ADBT$_n$ $\mathcal{G}$ over $\mathcal{O}_L$, the $\mathcal{O}_{L,n} \otimes \mathbb{Z}_{p^f}$-modules

$$\omega_{\mathcal{G}}, \quad \mathrm{Lie}(\mathcal{G} \times \mathscr{S}_{L,n}), \quad \omega_{\mathcal{G}^\vee}, \quad \mathrm{Lie}(\mathcal{G}^\vee \times \mathscr{S}_{L,n})$$

are all free of rank one. Moreover, the action of the Verschiebung on $\mathrm{Lie}(V_{\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1}})$ can be written as the direct sum of σ-semilinear maps

$$\mathrm{Lie}(V_{\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1}})_\beta : \mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1})_{\sigma^{-1} \circ \beta} \longrightarrow \mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathscr{S}_{L,1})_\beta.$$

Note that both sides are free $\mathcal{O}_{L,1}$-modules of rank one, and by choosing their bases, this map is identified with the multiplication by an element $a_\beta \in \mathcal{O}_{L,1}$. We define the β-Hodge height $\mathrm{Hdg}_\beta(\mathcal{G})$ of $\mathcal{G}$ as the truncated valuation of a_β, namely

$$\mathrm{Hdg}_\beta(\mathcal{G}) = v_p(a_\beta),$$

which is independent of the choice of bases. From the diagram in the proof of [8, Proposition 2] and [6, Lemma 2.3.7], we obtain the equality

$$\mathrm{Hdg}_\beta(\mathcal{G}) = \mathrm{Hdg}_\beta(\mathcal{G}^\vee).$$

A Breuil–Kisin module $\mathfrak{M}$ is called a $\mathbb{Z}_{p^f}$-Breuil–Kisin module if $\mathfrak{M}$ is equipped with an $\mathfrak{S}$-linear action of the ring $\mathbb{Z}_{p^f}$ commuting with $\varphi_{\mathfrak{M}}$. A morphism of $\mathbb{Z}_{p^f}$-Breuil–Kisin modules is that of Breuil–Kisin modules compatible with $\mathbb{Z}_{p^f}$-action. The $\mathbb{Z}_{p^f}$-Breuil–Kisin modules whose underlying $\mathfrak{S}$-modules are free of finite rank over $\mathfrak{S}_1$ (resp. finitely generated, p-power torsion and u-torsion free) form a category, which we denote by

$$\mathbb{Z}_{p^f}\text{-}\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1} \quad (\text{resp. } \mathbb{Z}_{p^f}\text{-}\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}).$$

Note that $\mathfrak{M} \mapsto \mathfrak{M}^\vee$ defines a notion of duality also for these categories. The anti-equivalence $\mathfrak{M}^*(-)$ of the Breuil–Kisin classification induces an anti-equivalence from the category of finite flat $\mathbb{Z}_{p^f}$-groups over $\mathcal{O}_K$ killed by some p-power to $\mathbb{Z}_{p^f}\text{-}\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$.

To give an object $\mathfrak{M}$ of $\mathbb{Z}_{p^f}\text{-}\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$ (resp. $\mathbb{Z}_{p^f}\text{-}\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$) is the same as to give a free $\mathfrak{S}_1$-module $\mathfrak{M}$ of finite rank (resp. a finitely generated

$\mathfrak{S}$-module $\mathfrak{M}$ which is p-power torsion and u-torsion free) equipped with a decomposition into $\mathfrak{S}$-submodules

$$\mathfrak{M} = \bigoplus_{\beta \in \mathbb{B}_f} \mathfrak{M}_\beta$$

and a φ-semilinear map

$$\varphi_{\mathfrak{M},\beta} : \mathfrak{M}_{\sigma^{-1}\circ\beta} \longrightarrow \mathfrak{M}_\beta,$$

which we often write as φ_β, for each $\beta \in \mathbb{B}_f$ such that the cokernel of the linearization $1 \otimes \varphi_\beta : \varphi^*\mathfrak{M}_{\sigma^{-1}\circ\beta} \to \mathfrak{M}_\beta$ is killed by $E(u)$. Since $1 \otimes \varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective, the map $1 \otimes \varphi_\beta$ is also injective. Hence we see that if $\mathfrak{M} \neq 0$, then $\mathfrak{M}_\beta \neq 0$ for any $\beta \in \mathbb{B}_f$. Since $E(u)\mathfrak{M} \subseteq (1 \otimes \varphi)(\varphi^*\mathfrak{M})$, we have $E(u)\mathfrak{M}_\beta \subseteq (1 \otimes \varphi_\beta)(\varphi^*\mathfrak{M}_{\sigma^{-1}\circ\beta})$.

Let $\mathfrak{M}$ be any object of $\mathbb{Z}_{p^f}\text{-Mod}^{1,\varphi}_{/\mathfrak{S}_1}$. The last inclusion implies that the free $\mathfrak{S}_1$-modules $\mathfrak{M}_\beta$ have the same rank for any $\beta \in \mathbb{B}_f$, which is equal to

$$f^{-1}\mathrm{rank}_{\mathfrak{S}_1}(\mathfrak{M}) = \dim_{\mathbb{F}_{p^f}}(\mathrm{Hom}_{\mathfrak{S}_1,\varphi}(\mathfrak{M}, R)).$$

Moreover, Proposition 2.2 (1) implies that, if $\mathcal{G}$ is the finite flat group scheme over $\mathcal{O}_K$ corresponding to $\mathfrak{M}$, then we have

$$\deg_\beta(\mathcal{G}) = e^{-1}\,\mathrm{lg}_{\mathfrak{S}_1}(\mathrm{Coker}(1 \otimes \varphi_\beta : \varphi^*\mathfrak{M}_{\sigma^{-1}\circ\beta} \longrightarrow \mathfrak{M}_\beta)). \tag{3.1}$$

Lemma 3.2. *Let $\mathcal{G}$ be a finite flat $\mathbb{Z}_{p^f}$-group over $\mathcal{O}_K$. Then we have*

$$\deg_\beta(\mathcal{G}) + \deg_\beta(\mathcal{G}^\vee) = \mathrm{lg}_{\mathfrak{S}}(\mathfrak{M}^*(\mathcal{G})_\beta).$$

Proof. Let $\mathcal{H}$ be the scheme-theoretic closure in $\mathcal{G}$ of $\mathcal{G}(\mathcal{O}_{\bar{K}})[p]$. It is a finite flat closed $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}$ killed by p. Since both sides are additive with respect to exact sequences of finite flat $\mathbb{Z}_{p^f}$-groups over $\mathcal{O}_K$, by an induction we may assume that $\mathcal{G}$ is killed by p.

Put $\mathfrak{M} = \mathfrak{M}^*(\mathcal{G})$. Let A_β be the representing matrix of the map $\varphi_{\mathfrak{M},\beta} : \mathfrak{M}_{\sigma^{-1}\circ\beta} \to \mathfrak{M}_\beta$ with some bases. From the definition of the dual, we see that the representing matrix of the map $\varphi_{\mathfrak{M}^\vee,\beta}$ with the dual bases is $c_0^{-1}E(u)^tA_\beta^{-1}$. Then the equality (3.1) implies

$$\begin{aligned}\deg_\beta(\mathcal{G}) + \deg_\beta(\mathcal{G}^\vee) &= e^{-1}v_u(\det(A_\beta)\det(c_0^{-1}E(u)^tA_\beta^{-1}))\\ &= \mathrm{rank}_{\mathfrak{S}_1}(\mathfrak{M}_\beta).\end{aligned}$$

This concludes the proof. □

For any $\mathfrak{M} \in \mathbb{Z}_{p^f}\text{-Mod}^{1,\varphi}_{/\mathfrak{S}_1}$ and $i \in e^{-1}\mathbb{Z} \cap [0,1]$, we put

$$\mathfrak{M}_{\beta,i} = \mathfrak{M}_\beta / u^{ei}\mathfrak{M}_\beta.$$

Then the map φ_β induces an $\mathfrak{S}_1$-semilinear map $\mathfrak{M}_{\sigma^{-1}\circ\beta,i} \to \mathfrak{M}_{\beta,i}$, which we denote also by φ_β. We define

$$\mathrm{Fil}^1\mathfrak{M}_{\beta,i} = \mathrm{Im}(1 \otimes \varphi_\beta : \varphi^*\mathfrak{M}_{\sigma^{-1}\circ\beta,i} \longrightarrow \mathfrak{M}_{\beta,i}).$$

4. Tian's construction

We continue to use the notation in §3. Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_1$ over $\mathcal{O}_K$ of β-Hodge height w_β. Put $\mathfrak{M} = \mathfrak{M}^*(\mathcal{G})$ and $\mathfrak{M}_1 = \mathfrak{M}/u^e\mathfrak{M}$. Then each $\mathfrak{M}_\beta$ is a free $\mathfrak{S}_1$-module of rank two. By Proposition 2.2 (1) and (2), we have an exact sequence of φ-modules over $\mathcal{O}_{K,1} \otimes \mathbb{Z}_{p^f}$

$$0 \longrightarrow \mathrm{Fil}^1\mathfrak{M}_1 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}_1/\mathrm{Fil}^1\mathfrak{M}_1 \longrightarrow 0,$$

where the $\mathcal{O}_{K,1} \otimes \mathbb{Z}_{p^f}$-modules $\mathrm{Fil}^1\mathfrak{M}_1$ and $\mathfrak{M}_1/\mathrm{Fil}^1\mathfrak{M}_1$ are free of rank one. In particular, this splits as a sequence of $\mathcal{O}_{K,1} \otimes \mathbb{Z}_{p^f}$-modules. Hence we also have a split exact sequence of $\mathcal{O}_{K,1}$-modules

$$0 \longrightarrow \mathrm{Fil}^1\mathfrak{M}_{\beta,1} \longrightarrow \mathfrak{M}_{\beta,1} \longrightarrow \mathfrak{M}_{\beta,1}/\mathrm{Fil}^1\mathfrak{M}_{\beta,1} \longrightarrow 0,$$

where the modules on the left-hand side and the right-hand side are free of rank one. As in the proof of [13, Theorem 3.1], we can choose a basis $\{e_\beta, e'_\beta\}$ of $\mathfrak{M}_\beta$ satisfying $e_\beta \in (1\otimes\varphi)(\varphi^*\mathfrak{M}_{\sigma^{-1}\circ\beta})$ such that the image of e_β in $\mathfrak{M}_{\beta,1}$ is a basis of $\mathrm{Fil}^1\mathfrak{M}_{\beta,1}$ and the image of e'_β in $\mathfrak{M}_{\beta,1}/\mathrm{Fil}^1\mathfrak{M}_{\beta,1}$ gives its basis. Then we can write as

$$\varphi(e_{\sigma^{-1}\circ\beta}, e'_{\sigma^{-1}\circ\beta}) = (e_\beta, e'_\beta)\begin{pmatrix} a_{\beta,1} & a_{\beta,2} \\ u^e a_{\beta,3} & u^e a_{\beta,4}\end{pmatrix} \tag{4.1}$$

with some invertible matrix

$$\begin{pmatrix} a_{\beta,1} & a_{\beta,2} \\ a_{\beta,3} & a_{\beta,4}\end{pmatrix} \in GL_2(\mathfrak{S}_1).$$

For any $\mathbb{Z}_{p^f}$-group $\mathcal{G}$ over $\mathcal{O}_K$ killed by p, a finite flat closed $\mathbb{Z}_{p^f}$-subgroup $\mathcal{H}$ of $\mathcal{G}$ over $\mathcal{O}_K$ is said to be cyclic if the $\mathbb{F}_{p^f}$-vector space $\mathcal{H}(\mathcal{O}_{\bar{K}})$ is of rank one. Note that, for such $\mathcal{H}$, the free $\mathfrak{S}_1$-module $\mathfrak{M}^*(\mathcal{H})_\beta$ is of rank one for any $\beta \in \mathbb{B}_f$. If $\mathcal{G}$ is a $\mathbb{Z}_{p^f}$-ADBT$_1$, then the proof of [9, Lemma 2.1.1] shows that the $\mathbb{F}_{p^f}$-subspace $\mathcal{H}(\mathcal{O}_{\bar{K}})$ is automatically isotropic with respect to the $\mathbb{Z}_{p^f}$-alternating perfect pairing on $\mathcal{G}(\mathcal{O}_{\bar{K}})$. Moreover, since finite flat closed subgroup schemes of $\mathcal{G}$ over $\mathcal{O}_K$ are determined by their generic fibers, this implies that the $\mathbb{Z}_{p^f}$-alternating isomorphism $i : \mathcal{G} \simeq \mathcal{G}^\vee$ induces an isomorphism $\mathcal{H} \simeq (\mathcal{G}/\mathcal{H})^\vee$.

Now the existence theorem of the canonical subgroup of level one for a $\mathbb{Z}_{p^f}$-ADBT$_1$ over $\mathcal{O}_K$ is as follows.

Theorem 4.1. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_1$ over $\mathcal{O}_K$ with β-Hodge height w_β. Put $w = \max\{w_\beta \mid \beta \in \mathbb{B}_f\}$. Suppose that the inequality*

$$w_\beta + pw_{\sigma^{-1}\circ\beta} < p$$

holds for all $\beta \in \mathbb{B}_f$. Then there exists a finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}$ of $\mathcal{G}$ over $\mathcal{O}_K$ satisfying

$$\deg_\beta(\mathcal{G}/\mathcal{C}) = w_\beta$$

for all $\beta \in \mathbb{B}_f$. Moreover, the group scheme $\mathcal{C}$ is the unique finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}$ over $\mathcal{O}_K$ satisfying

$$\deg_\beta(\mathcal{C}) + p\deg_{\sigma^{-1}\circ\beta}(\mathcal{C}) > 1$$

for all $\beta \in \mathbb{B}_f$. We refer to $\mathcal{C}$ as the canonical subgroup of $\mathcal{G}$. It has the following properties:

(1) *Let $\mathcal{G}'$ be a $\mathbb{Z}_{p^f}$-ADBT_1 over $\mathcal{O}_K$ satisfying the same condition on the β-Hodge heights as above and $\mathcal{C}'$ the canonical subgroup of $\mathcal{G}'$. Then any isomorphism of $\mathbb{Z}_{p^f}$-groups $j : \mathcal{G} \to \mathcal{G}'$ over $\mathcal{O}_K$ induces an isomorphism $\mathcal{C} \simeq \mathcal{C}'$.*
(2) *$\mathcal{C}$ is compatible with finite base extension of complete discrete valuation rings.*
(3) *$\mathcal{C}$ is compatible with Cartier duality. Namely, $(\mathcal{G}/\mathcal{C})^\vee$ is the canonical subgroup of $\mathcal{G}^\vee$.*
(4) *The kernel of the Frobenius map of $\mathcal{G} \times \mathscr{S}_{1-w}$ coincides with $\mathcal{C} \times \mathscr{S}_{1-w}$.*
(5) *If $w < p/(p+1)$, then $\mathcal{C} = \mathcal{G}_{(1-w)/(p-1)}$.*
(6) *If $w < (p-1)/p$, then $\mathcal{C}(\mathcal{O}_{\bar{K}})$ coincides with $\mathrm{Ker}(\mathrm{HT}_{\mathcal{G},i})$ for any rational number i satisfying $w/(p-1) < i \leq 1-w$.*
(7) *If $w < (p-1)/p$, then $\mathcal{C} = \mathcal{G}_i$ for any rational number i satisfying $1/(p(p-1)) \leq i \leq (1-w)/(p-1)$.*

Proof. Note that, since we have $w < 1$ by assumption, Proposition 2.2 (2) implies

$$w_\beta = e^{-1} v_u(a_{\beta,1}).$$

The existence and the uniqueness in the theorem are due to Tian [28, Theorem 3.10]: the $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}$ is defined as the finite flat closed $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}$ over $\mathcal{O}_K$ corresponding to the quotient $\mathfrak{N} = \mathfrak{M}/\mathfrak{L}$ via the Breuil–Kisin classification, where $\mathfrak{L} = \bigoplus_{\beta \in \mathbb{B}_f} \mathfrak{L}_\beta$ is the unique $\mathbb{Z}_{p^f}$-Breuil–Kisin submodule of $\mathfrak{M}$ satisfying

$$\mathfrak{L}_{\beta,1-w_\beta} = \mathrm{Fil}^1 \mathfrak{M}_{\beta,1-w_\beta}$$

for all $\beta \in \mathbb{B}_f$. In particular, the $\mathfrak{S}_1$-module $\mathfrak{L}_\beta$ is generated by

$$\delta_\beta = e_\beta + u^{e(1-w_\beta)} y_\beta e'_\beta$$

with some $y_\beta \in \mathfrak{S}_1$. The assertions (1) and (2) follow from the uniqueness.

Let us prove the assertion (3). Note that, since $\mathrm{Hdg}_\beta(\mathcal{G}) = \mathrm{Hdg}_\beta(\mathcal{G}^\vee)$, the $\mathbb{Z}_{p^f}$-ADBT_1 $\mathcal{G}^\vee$ over $\mathcal{O}_K$ also has the canonical subgroup $\mathcal{C}'$. By Lemma 3.2, we have

$$\deg_\beta((\mathcal{G}/\mathcal{C})^\vee) = 1 - \deg_\beta(\mathcal{G}/\mathcal{C}) = 1 - w_\beta$$

and the uniqueness assertion of the theorem and the assumption on w_β imply $\mathcal{C}' = (\mathcal{G}/\mathcal{C})^\vee$.

The assertion (4) is also due to Tian [28, Remark 3.11]. Here we give a short proof for the convenience of the reader. Since $1-w \leq 1-w_\beta$, the construction of $\mathfrak{L}$ implies

$$\mathfrak{L}_{1-w} = \mathrm{Fil}^1\mathfrak{M}_{1-w}. \tag{4.2}$$

Then Proposition 2.2 (1) shows that the natural map

$$\omega_{\mathcal{G}/\mathcal{C}} \otimes \mathcal{O}_{K,1-w} \longrightarrow \omega_{\mathcal{G}} \otimes \mathcal{O}_{K,1-w}$$

is zero. By [8, Proposition 1], the closed subgroup scheme $(\mathcal{G}/\mathcal{C})^\vee \times \mathscr{S}_{1-w}$ of $\mathcal{G}^\vee \times \mathscr{S}_{1-w}$ is killed by the Frobenius. Comparing the rank, the former coincides with the kernel of the Frobenius of the latter. Since $\mathcal{G}^\vee \times \mathscr{S}_{1-w}$ is a truncated Barsotti–Tate group of level one, we see by duality and [15, Remark 1.3 (b)] that $\mathcal{C} \times \mathscr{S}_{1-w}$ also coincides with the kernel of the Frobenius of $\mathcal{G} \times \mathscr{S}_{1-w}$.

Next we consider the assertion (5). It can be shown similarly to [13, Theorem 3.1 (c)]. For any $\mathfrak{S}_1$-algebra A, we define an abelian group $\mathcal{H}(\mathfrak{M})(A)$ by

$$\mathcal{H}(\mathfrak{M})(A) = \mathrm{Hom}_{\mathfrak{S}_1,\varphi}(\mathfrak{M}, A),$$

where we consider A as a φ-module with the p-th power Frobenius map. If we take the basis $\{e_\beta, e'_\beta\}_{\beta\in\mathbb{B}_f}$ of $\mathfrak{M}$ as above, it is identified with the set of f-tuples of elements $(x_\beta, x'_\beta) \in A^2$ satisfying

$$(x^p_{\sigma^{-1}\circ\beta}, (x'_{\sigma^{-1}\circ\beta})^p) = (x_\beta, x'_\beta)\begin{pmatrix} a_{\beta,1} & a_{\beta,2} \\ u^e a_{\beta,3} & u^e a_{\beta,4}\end{pmatrix}. \tag{4.3}$$

We define the subgroup $\mathcal{H}(\mathfrak{M})_i(R)$ of $\mathcal{H}(\mathfrak{M})(R)$ by

$$\mathcal{H}(\mathfrak{M})_i(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \longrightarrow \mathcal{H}(\mathfrak{M})(R_i)).$$

We also have the subgroup $\mathcal{H}(\mathfrak{N})(R)$ of $\mathcal{H}(\mathfrak{M})(R)$. Note that we have an exact sequence

$$0 \longrightarrow \mathcal{H}(\mathfrak{N})(R) \longrightarrow \mathcal{H}(\mathfrak{M})(R) \longrightarrow \mathcal{H}(\mathfrak{L})(R) \longrightarrow 0$$

which can be identified with the exact sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(\mathcal{O}_{\bar{K}}) \longrightarrow \mathcal{G}(\mathcal{O}_{\bar{K}}) \longrightarrow (\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}}) \longrightarrow 0.$$

Since $\deg_\beta(\mathcal{G}/\mathcal{C}) = w_\beta$, the basis δ_β of $\mathfrak{L}_\beta$ satisfies

$$\varphi_\beta(\delta_{\sigma^{-1}\circ\beta}) = \lambda_\beta\delta_\beta \text{ with } v_R(\lambda_\beta) = w_\beta.$$

Thus any element of $\mathcal{H}(\mathfrak{L})(R)$ can be identified with an f-tuple $(z_\beta)_{\beta\in\mathbb{B}_f}$ in R satisfying

$$z^p_{\sigma^{-1}\circ\beta} = \lambda_\beta z_\beta \tag{4.4}$$

for all $\beta \in \mathbb{B}_f$.

Lemma 4.2. *For any element* $(z_\beta)_{\beta\in\mathbb{B}_f}\neq 0$ *of* $\mathcal{H}(\mathfrak{L})(R)$*, we have*

$$v_R(z_\beta)\leq \frac{w}{p-1} \text{ for any } \beta\in\mathbb{B}_f.$$

In particular, $\mathcal{H}(\mathfrak{L})_i(R)=0$ *for any* $i>w/(p-1)$*.*

Proof. From the equation (4.4), we see that $(z_\beta)_{\beta\in\mathbb{B}_f}\neq 0$ if and only if $z_\beta\neq 0$ for all $\beta\in\mathbb{B}_f$. This equation also implies

$$pv_R(z_{\sigma^{-1}\circ\beta})=v_R(z_\beta)+w_\beta$$

for any $\beta\in\mathbb{B}_f$ and thus

$$v_R(z_{\sigma^{-1}\circ\beta})=\frac{1}{p^f-1}\sum_{l=0}^{f-1}p^{f-1-l}w_{\sigma^l\circ\beta}\leq\frac{w}{p-1},$$

which concludes the proof. □

We claim that $\mathcal{H}(\mathfrak{N})(R)=\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$. Indeed, take any element $(x_\beta,x'_\beta)_{\beta\in\mathbb{B}_f}$ of the left-hand side. Take the element $y_\beta\in\mathfrak{S}_1$ such that

$$\delta_\beta=e_\beta+u^{e(1-w_\beta)}y_\beta e'_\beta$$

is a basis of the $\mathfrak{S}_1$-module $\mathfrak{L}_\beta$. Then $(x_\beta,x'_\beta)_{\beta\in\mathbb{B}_f}\in\mathcal{H}(\mathfrak{N})(R)$ if and only if $x_\beta+u^{e(1-w_\beta)}y_\beta x'_\beta=0$ for all $\beta\in\mathbb{B}_f$. The equation (4.3) implies

$$(x'_{\sigma^{-1}\circ\beta})^p=x'_\beta(-a_{\beta,2}u^{e(1-w_\beta)}y_\beta+u^e a_{\beta,4})$$

and thus

$$pv_R(x'_{\sigma^{-1}\circ\beta})\geq v_R(x'_\beta)+1-w_\beta.$$

Hence we have

$$v_R(x'_{\sigma^{-1}\circ\beta})\geq\frac{1}{p^f-1}\sum_{l=0}^{f-1}p^{f-1-l}(1-w_{\sigma^l\circ\beta})\geq\frac{1-w}{p-1}$$

for any $\beta\in\mathbb{B}_f$ and we obtain $(x_\beta,x'_\beta)_{\beta\in\mathbb{B}_f}\in\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$.

Conversely, let $(x_\beta,x'_\beta)_{\beta\in\mathbb{B}_f}$ be an element of $\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$. By the equation (4.3), we have

$$(x_\beta,u^e x'_\beta)=(x^p_{\sigma^{-1}\circ\beta},(x'_{\sigma^{-1}\circ\beta})^p)\begin{pmatrix}a_{\beta,1}&a_{\beta,2}\\a_{\beta,3}&a_{\beta,4}\end{pmatrix}^{-1}\tag{4.5}$$

for any $\beta\in\mathbb{B}_f$. Recall that the matrix on the right-hand side is an element of $GL_2(\mathfrak{S}_1)$. Hence we obtain

$$v_R(x_\beta)\geq\frac{p(1-w)}{p-1}$$

and the element $z_\beta = x_\beta + u^{e(1-w_\beta)} y_\beta x'_\beta$ satisfies

$$v_R(z_\beta) \geq \frac{p(1-w)}{p-1}$$

for all $\beta \in \mathbb{B}_f$. By the assumption $w < p/(p+1)$, we have

$$\frac{w}{p-1} < \frac{p(1-w)}{p-1}$$

and Lemma 4.2 implies $z_\beta = 0$ for any $\beta \in \mathbb{B}_f$. Therefore we obtain $(x_\beta, x'_\beta)_{\beta \in \mathbb{B}_f} \in \mathcal{H}(\mathfrak{N})(R)$, from which the claim follows. Now Theorem 2.1 (4) shows the assertion (5).

Let us show the assertion (6). This is shown similarly to [13, Theorem 3.1 (2)]. Since $i \leq 1-w$, Proposition 2.2 (3) and (4.2) show that the kernel of the map $\mathrm{HT}_{\mathcal{G},i}$ is equal to the kernel of the natural map

$$\mathcal{G}(\mathcal{O}_{\bar{K}}) \simeq \mathcal{H}(\mathfrak{M})(R) \longrightarrow \mathcal{H}(\mathfrak{L})(R) \longrightarrow \mathcal{H}(\mathfrak{L})(R_i).$$

Since $i > w/(p-1)$, Lemma 4.2 implies $\mathcal{H}(\mathfrak{L})_i(R) = 0$ and the right arrow in the above map is injective. Thus $\mathrm{Ker}(\mathrm{HT}_{\mathcal{G},i})$ coincides with the inverse image of

$$\mathcal{H}(\mathfrak{N})(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \longrightarrow \mathcal{H}(\mathfrak{L})(R))$$

by the isomorphism $\mathcal{G}(\mathcal{O}_{\bar{K}}) \simeq \mathcal{H}(\mathfrak{M})(R)$, which is $\mathcal{C}(\mathcal{O}_{\bar{K}})$. The assertion (7) follows from the lemma below. □

Lemma 4.3. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT_1 over $\mathcal{O}_K$ with β-Hodge height w_β. Put $w = \max\{w_\beta \mid \beta \in \mathbb{B}_f\}$ and*

$$i_n = \frac{1}{p^{n-1}(p-1)} - \frac{w}{p-1}, \quad i'_n = \frac{1}{p^n(p-1)}.$$

Suppose $w < (p-1)/p^n$ for some positive integer n. Let $\mathcal{C}$ be the canonical subgroup of $\mathcal{G}$, which exists by Theorem 4.1. Then we have

$$\mathcal{C} = \mathcal{G}_{i_m} = \mathcal{G}_{i'_m}$$

for any $1 \leq m \leq n$.

Proof. This can be shown in the same way as [14, Lemma 5.2]. We follow the notation in the proof of Theorem 4.1. By Theorem 2.1 (4) and Theorem 4.1 (5), it is enough to show

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, m_R^{\geqslant i'_n}) \subseteq \mathcal{H}(\mathfrak{N})(R).$$

We identify an element x of the left-hand side with a solution $(x_\beta, x'_\beta)_{\beta \in \mathbb{B}_f}$ of the equation (4.3) in R satisfying $v_R(x_\beta), v_R(x'_\beta) \geq i'_n$ for all $\beta \in \mathbb{B}_f$. From the equality (4.5), we have $v_R(x_\beta) \geq p i'_n > w/(p-1)$. Since we have $1 - w_\beta \geq 1 - w > w/(p-1)$, the element $z_\beta = x_\beta + u^{e(1-w_\beta)} y_\beta x'_\beta$ satisfies

$$v_R(z_\beta) > w/(p-1).$$

Thus Lemma 4.2 implies $z_\beta = 0$ for any $\beta \in \mathbb{B}_f$ and $x \in \mathcal{H}(\mathfrak{N})(R)$. □

The description of the Hodge–Tate map via the Breuil–Kisin classification also yields a torsion property of the Hodge–Tate cokernel, as follows.

Lemma 4.4. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_1$ over $\mathcal{O}_K$ with β-Hodge height w_β. Put $w = \max\{w_\beta \mid \beta \in \mathbb{B}_f\}$. Suppose $w < (p-1)/p$. Then the cokernel of the linearization of the Hodge–Tate map*

$$\mathrm{HT} \otimes 1 : \mathcal{G}(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$$

is killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$.

Proof. For this, we first show the following lemma.

Lemma 4.5. *Let M be a finitely generated $\mathcal{O}_{\bar{K}}$-module. Let N be an $\mathcal{O}_{\bar{K}}$-submodule of M. Suppose that there exist positive rational numbers $r > s$ satisfying $m_{\bar{K}}^{\geqslant s} M \subseteq N + m_{\bar{K}}^{\geqslant r} M$. Then we have $m_{\bar{K}}^{\geqslant s} M \subseteq N$.*

Proof. Put $Q = m_{\bar{K}}^{\geqslant s}(M/N)$. Since the assumption implies $m_{\bar{K}}^{\geqslant r-s} Q = Q$, Nakayama's lemma shows $Q = 0$ and $m_{\bar{K}}^{\geqslant s} M \subseteq N$. □

Put $M = \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$ and

$$N = \mathrm{Im}(\mathrm{HT} \otimes 1 : \mathcal{G}(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}).$$

We claim that

$$m_{\bar{K}}^{\geqslant w/(p-1)} \mathrm{Coker}(\mathrm{HT}_{1-w} \otimes 1 : \mathcal{G}(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{G}^\vee} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},1-w}) = 0.$$

This is equivalent to the inclusion

$$m_{\bar{K}}^{\geqslant w/(p-1)} M \subseteq N + m_{\bar{K}}^{\geqslant 1-w} M.$$

The assumption $w < (p-1)/p$ implies $w/(p-1) < 1-w$ and thus Lemma 4.4 follows from the claim and Lemma 4.5.

Now let us prove the claim. Consider the basis δ_β of $\mathfrak{L}_\beta$ as in the proof of Theorem 4.1. Using this, we identify each element of $\mathcal{H}(\mathfrak{L})(R)$ with an f-tuple $(z_\beta)_{\beta \in \mathbb{B}_f}$ in R satisfying the equation (4.4). By Proposition 2.2 (3) and (4.2), the cokernel of the claim is identified with the cokernel of the natural map

$$\begin{aligned} \mathcal{H}(\mathfrak{L})(R) \otimes R &\longrightarrow \mathrm{Hom}_{\mathfrak{S}_1}(\mathfrak{L}, R_{1-w}) = \mathfrak{L}^\vee \otimes R_{1-w}, \\ (z_\beta)_{\beta \in \mathbb{B}_f} \otimes 1 &\longmapsto \sum_{\beta \in \mathbb{B}_f} \delta_\beta^\vee \otimes z_\beta. \end{aligned}$$

Note that the abelian group $\mathcal{H}(\mathfrak{L})(R)$ has a natural action of the ring $\mathbb{F}_{p^f}$ defined by

$$\alpha (z_\beta)_{\beta \in \mathbb{B}_f} = (\beta(\alpha) z_\beta)_{\beta \in \mathbb{B}_f} \text{ for any } \alpha \in \mathbb{F}_{p^f}.$$

Take a generator α_0 of the extension $\mathbb{F}_{p^f}/\mathbb{F}_p$ and also a non-zero element $(z_\beta)_{\beta\in\mathbb{B}_f}$ of $\mathcal{H}(\mathfrak{L})(R)$. Then the subset $\{\alpha_0^l(z_\beta)_{\beta\in\mathbb{B}_f}\}_{l=0,1,\dots,f-1}$ forms a basis of the $\mathbb{F}_p$-vector space $\mathcal{H}(\mathfrak{L})(R)$. Hence the image of the natural map above is generated by the entries of the f-tuple

$$(\delta_\beta^\vee\otimes 1)_{\beta\in\mathbb{B}_f}(\beta(\alpha_0^l)z_\beta)_{\beta,l}=(\delta_\beta^\vee\otimes 1)_{\beta\in\mathbb{B}_f}\mathrm{diag}(z_\beta)_{\beta\in\mathbb{B}_f}(\beta(\alpha_0^l))_{\beta,l}.$$

Since the matrix $(\beta(\alpha_0^l))_{\beta,l}$ is invertible in $M_f(R)$, the cokernel is isomorphic as an R-module to

$$\bigoplus_{\beta\in\mathbb{B}_f} R_{1-w}/(z_\beta).$$

Thus the claim follows from Lemma 4.2. □

Corollary 4.6. *Let L be a complete discrete valuation field of mixed characteristic $(0,p)$ and $\mathcal{G}$ a $\mathbb{Z}_{p^f}$-ADBT_1 over $\mathcal{O}_L$ with β-Hodge height w_β. Put $w=\max\{w_\beta\mid\beta\in\mathbb{B}_f\}$. If $w<p/(p+1)$, then there exists a finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}$ of $\mathcal{G}$ over $\mathcal{O}_L$ satisfying*

$$\deg_\beta(\mathcal{G}/\mathcal{C})=w_\beta$$

for all $\beta\in\mathbb{B}_f$, which is characterized by the inequality

$$\deg_\beta(\mathcal{C})+p\deg_{\sigma^{-1}\circ\beta}(\mathcal{C})>1$$

for all $\beta\in\mathbb{B}_f$. The $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}$ has the properties (1)–(7) in Theorem 4.1. We also refer to $\mathcal{C}$ as the canonical subgroup of $\mathcal{G}$.

Proof. From [10, Théorème (19.8.6)] we see that $\mathcal{O}_L$ is a finite totally ramified extension of a Cohen ring, and as in the proof of [13, Theorem 3.1] we can choose an extension of complete discrete valuation fields L'/L with relative ramification index one such that the residue field of L' is perfect. Put $\mathcal{C}=\mathcal{G}_{(1-w)/(p-1)}$. Since the map $\mathcal{O}_L\to\mathcal{O}_{L'}$ is flat and lower ramification subgroups are compatible with base extension of complete discrete valuation fields, Theorem 4.1 applied to $\mathcal{G}\otimes_{\mathcal{O}_L}\mathcal{O}_{L'}$ implies the corollary. □

5. Goren–Kassaei's theory

Here we analyze the variation of β-Hodge heights by taking quotients with cyclic $\mathbb{Z}_{p^f}$-subgroups. For the case of abelian varieties, it was obtained by Goren–Kassaei [9, Lemma 5.3.4 and Lemma 5.3.6]. We are still assuming that the residue field $k\supseteq\mathbb{F}_{p^f}$ of K is perfect.

Lemma 5.1. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT_1 over $\mathcal{O}_K$ with β-Hodge height w_β. Let $\mathcal{H}$ be a finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}$ over $\mathcal{O}_K$. Put $v_\beta=\deg_\beta(\mathcal{G}/\mathcal{H})$.*

(1) *If we have*

$$v_\beta+pv_{\sigma^{-1}\circ\beta}<p \textit{ for all } \beta\in\mathbb{B}_f,$$

then $w_\beta=v_\beta$ and $\mathcal{G}$ has a canonical subgroup, which is equal to $\mathcal{H}$.

(2) *If we have*

$$v_\beta + p v_{\sigma^{-1}\circ\beta} > p \text{ for all } \beta \in \mathbb{B}_f,$$

then $w_\beta = p(1 - v_{\sigma^{-1}\circ\beta})$ and $\mathcal{G}$ has a canonical subgroup, which is not equal to $\mathcal{H}$. We refer to any $\mathcal{H}$ satisfying this inequality as an anti-canonical subgroup of $\mathcal{G}$.

(3) *If both of the inequalities in* (1) *and* (2) *are not satisfied, then*

$$w_\beta + p w_{\sigma^{-1}\circ\beta} \geq p \text{ for some } \beta \in \mathbb{B}_f.$$

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be the Breuil–Kisin modules corresponding to $\mathcal{G}/\mathcal{H}$ and $\mathcal{H}$, respectively. We have an exact sequence of $\mathfrak{S}_1$-modules

$$0 \longrightarrow \mathfrak{P}_\beta \longrightarrow \mathfrak{M}_\beta \longrightarrow \mathfrak{Q}_\beta \longrightarrow 0$$

for any $\beta \in \mathbb{B}_f$. Note that $\mathfrak{S}_1$-modules $\mathfrak{P}_\beta$ and $\mathfrak{Q}_\beta$ are free of rank one. Let $\{f_\beta, f'_\beta\}$ be a basis of the free $\mathfrak{S}_1$-module $\mathfrak{M}_\beta$ such that f_β is a basis of $\mathfrak{P}_\beta$ and the image of f'_β is a basis of $\mathfrak{Q}_\beta$. We can write as

$$\varphi_\beta(f_{\sigma^{-1}\circ\beta}, f'_{\sigma^{-1}\circ\beta}) = (f_\beta, f'_\beta)\begin{pmatrix} a_\beta & b_\beta \\ 0 & c_\beta \end{pmatrix}$$

with some $a_\beta, b_\beta, c_\beta \in \mathfrak{S}_1$ such that $v_R(a_\beta) = v_\beta$ and $v_R(c_\beta) = \deg_\beta(\mathcal{H}) = 1 - v_\beta$. Thus we obtain

$$\mathrm{Fil}^1\mathfrak{M}_{\beta,1} = \langle a_\beta f_\beta, b_\beta f_\beta + c_\beta f'_\beta \rangle.$$

Since it is a direct summand of $\mathfrak{M}_{\beta,1}$ of rank one over $\mathcal{O}_{K,1}$, we have

$$\begin{cases} v_R(a_\beta) = 0 & (v_\beta = 0), \\ v_R(b_\beta) = 0 & (0 < v_\beta < 1), \\ v_R(c_\beta) = 0 & (v_\beta = 1) \end{cases}$$

and

$$\mathrm{Fil}^1\mathfrak{M}_{\beta,1} = \begin{cases} \langle f_\beta \rangle & (v_\beta = 0), \\ \langle b_\beta f_\beta + c_\beta f'_\beta \rangle & (v_\beta > 0). \end{cases}$$

Moreover, in $\mathfrak{M}_{\sigma\circ\beta,1}$ we have

$$\begin{cases} \varphi_{\sigma\circ\beta}(f_\beta) = a_{\sigma\circ\beta} f_{\sigma\circ\beta}, \\ \varphi_{\sigma\circ\beta}(b_\beta f_\beta + c_\beta f'_\beta) = (b_\beta^p a_{\sigma\circ\beta} + c_\beta^p b_{\sigma\circ\beta}) f_{\sigma\circ\beta} + c_\beta^p c_{\sigma\circ\beta} f'_{\sigma\circ\beta}. \end{cases} \tag{5.1}$$

Now let us consider the assertion (1). The assumption implies $v_{\sigma\circ\beta} < 1$ for any $\beta \in \mathbb{B}_f$. Hence $w_{\sigma\circ\beta}$ is equal to the truncated valuation of the coefficient of $f_{\sigma\circ\beta}$ of the right-hand side of the equality (5.1) in both cases.

- If $v_\beta = 0$, then $w_{\sigma\circ\beta} = v_R(a_{\sigma\circ\beta}) = v_{\sigma\circ\beta}$.

- If $0 < v_\beta < 1$, then $v_R(b_\beta) = 0$ and
$$w_{\sigma\circ\beta} = \min\{v_R(b_\beta^p a_{\sigma\circ\beta} + c_\beta^p b_{\sigma\circ\beta}), 1\}.$$
The assumption also yields $v_{\sigma\circ\beta} < p(1 - v_\beta)$ and thus we have $w_{\sigma\circ\beta} = v_{\sigma\circ\beta}$.

Thus $\mathcal{G}$ satisfies the assumption of Theorem 4.1 and has the canonical subgroup $\mathcal{C}$. Since $\deg_\beta(\mathcal{H}) = 1 - v_\beta$, the uniqueness of the theorem implies $\mathcal{H} = \mathcal{C}$.

Next we treat the assertion (2). The assumption implies $v_{\sigma\circ\beta} > 0$ for any $\beta \in \mathbb{B}_f$.

- If $0 < v_{\sigma\circ\beta} < 1$, then $v_R(b_{\sigma\circ\beta}) = 0$ and
$$w_{\sigma\circ\beta} = \min\{v_R(b_\beta^p a_{\sigma\circ\beta} + c_\beta^p b_{\sigma\circ\beta}), 1\}.$$
By assumption we have $v_{\sigma\circ\beta} > p(1-v_\beta)$ and thus we obtain $w_{\sigma\circ\beta} = p(1 - v_\beta)$.
- If $v_{\sigma\circ\beta} = 1$, then $v_R(c_{\sigma\circ\beta}) = 0$. This implies that $w_{\sigma\circ\beta}$ is equal to the truncated valuation of the coefficient of $f'_{\sigma\circ\beta}$ of the equality (5.1), namely
$$w_{\sigma\circ\beta} = \min\{v_R(c_\beta^p c_{\sigma\circ\beta}), 1\} = p(1 - v_\beta).$$

From this we see that $\mathcal{G}$ has the canonical subgroup $\mathcal{C}$. If $\mathcal{C} = \mathcal{H}$, then we have
$$v_\beta = \deg_\beta(\mathcal{G}/\mathcal{H}) = w_\beta = p(1 - v_{\sigma^{-1}\circ\beta})$$
for any $\beta \in \mathbb{B}_f$, which contradicts the assumption.

The assertion (3) can be shown as in [9, Corollary 5.3.7]: Take $\beta' \in \mathbb{B}_f$ such that $v_{\beta'} + pv_{\sigma^{-1}\circ\beta'} \leq p$. Let $i \geq 1$ be the minimal integer satisfying $v_{\sigma^i\circ\beta'} + pv_{\sigma^{i-1}\circ\beta'} \geq p$. The minimality shows that $\beta = \sigma^{i-1} \circ \beta'$ satisfies

$$v_\beta + pv_{\sigma^{-1}\circ\beta} \leq p, \quad v_{\sigma\circ\beta} + pv_\beta \geq p. \tag{5.2}$$

We claim that
$$w_\beta \geq v_\beta, \quad w_{\sigma\circ\beta} \geq p(1 - v_\beta).$$
Indeed, if $v_\beta = 0$ then the first inequality is trivial. If $0 < v_\beta < 1$, then (5.1) implies
$$w_\beta = \begin{cases} v_R(a_\beta) & (v_{\sigma^{-1}\circ\beta} = 0), \\ \min\{v_R(b_{\sigma^{-1}\circ\beta}^p a_\beta + c_{\sigma^{-1}\circ\beta}^p b_\beta), 1\} & (v_{\sigma^{-1}\circ\beta} > 0). \end{cases}$$
From this and (5.2), we obtain $w_\beta \geq v_\beta$. If $v_\beta = 1$, then (5.1) gives
$$w_\beta = \begin{cases} 1 & (v_{\sigma^{-1}\circ\beta} = 0), \\ \min\{v_R(c_{\sigma^{-1}\circ\beta}^p c_\beta), 1\} = \min\{p(1 - v_{\sigma^{-1}\circ\beta}), 1\} & (v_{\sigma^{-1}\circ\beta} > 0) \end{cases}$$
and the inequality $w_\beta \geq v_\beta$ follows from (5.2).

Let us consider the second inequality. If $v_{\sigma\circ\beta} = 0$, then (5.2) implies $v_\beta = 1$ and the inequality is trivial. If $0 < v_{\sigma\circ\beta} < 1$, then (5.1) implies

$$w_{\sigma\circ\beta} = \begin{cases} v_R(a_{\sigma\circ\beta}) & (v_\beta = 0), \\ \min\{v_R(b_\beta^p a_{\sigma\circ\beta} + c_\beta^p b_{\sigma\circ\beta}), 1\} & (v_\beta > 0) \end{cases}$$

and from (5.2) we obtain $w_{\sigma\circ\beta} \geq p(1 - v_\beta)$ for both cases. If $v_{\sigma\circ\beta} = 1$, then we have $v_\beta \geq (p-1)/p > 0$ and

$$w_{\sigma\circ\beta} = \min\{v_R(c_\beta^p c_{\sigma\circ\beta}), 1\} = p(1 - v_\beta).$$

This concludes the proof of the claim. Now we have

$$w_{\sigma\circ\beta} + p w_\beta \geq p(1 - v_\beta) + p v_\beta = p$$

and the assertion (3) follows. □

Lemma 5.2. *Let n be a positive integer, L/K any extension of complete discrete valuation fields and $\bar{L}$ an algebraic closure of L. Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_{n+1}$ over $\mathcal{O}_L$ with $\mathbb{Z}_{p^f}$-alternating isomorphism $i : \mathcal{G} \simeq \mathcal{G}^\vee$. Let $\mathcal{H}$ be a finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}[p]$ over $\mathcal{O}_L$. Then $p^{-n}\mathcal{H}/\mathcal{H}$ is a $\mathbb{Z}_{p^f}$-ADBT$_n$ over $\mathcal{O}_L$, with its $\mathbb{Z}_{p^f}$-alternating isomorphism of self-duality induced by i.*

Proof. Note that $p^{-n}\mathcal{H}/\mathcal{H}$ is a truncated Barsotti–Tate group of level n over $\mathcal{O}_L$. Indeed, by taking a base change we may assume that the residue field of L is perfect, and in this case it follows from the fact that $\mathcal{G}$ is the p^{n+1}-torsion part of a p-divisible group [15, Théorème 4.4 (e)]. Put $\mathcal{H}' = (\mathcal{G}[p]/\mathcal{H})^\vee$. Since $\mathcal{H}(\mathcal{O}_{\bar{L}})$ is isotropic, the map i induces an isomorphism $\mathcal{H} \simeq \mathcal{H}'$. On the other hand, Cartier duality gives a natural isomorphism $j : p^{-n}\mathcal{H}'/\mathcal{H}' \simeq (p^{-n}\mathcal{H}/\mathcal{H})^\vee$ satisfying

$$\langle \bar{x}, j(\bar{y}) \rangle_{p^{-n}\mathcal{H}/\mathcal{H}} = \langle x, y \rangle_{\mathcal{G}}$$

for any $x \in p^{-n}\mathcal{H}(\mathcal{O}_{\bar{L}})$ and $y \in p^{-n}\mathcal{H}'(\mathcal{O}_{\bar{L}})$, which can be shown as in [13, §4, Proof of Theorem 1.1 (b)]. Thus these maps induce a $\mathbb{Z}_{p^f}$-alternating isomorphism

$$p^{-n}\mathcal{H}/\mathcal{H} \xrightarrow{i} p^{-n}\mathcal{H}'/\mathcal{H}' \overset{j}{\simeq} (p^{-n}\mathcal{H}/\mathcal{H})^\vee.$$

It remains to prove that the $\mathcal{O}_{L,n} \otimes \mathbb{Z}_{p^f}$-module $\omega_{p^{-n}\mathcal{H}/\mathcal{H}}$ is free of rank one. Consider the decomposition

$$\omega_{p^{-n}\mathcal{H}/\mathcal{H}} = \bigoplus_{\beta \in \mathbb{B}_f} \omega_{p^{-n}\mathcal{H}/\mathcal{H},\beta}.$$

Since we know that the left-hand side is free of rank f as an $\mathcal{O}_{L,n}$-module, each $\omega_{p^{-n}\mathcal{H}/\mathcal{H},\beta}$ is a free $\mathcal{O}_{L,n}$-module of rank f_β with some non-negative integer f_β. For $n=1$, we have exact sequences

$$0 \longrightarrow \omega_{\mathcal{H},\beta} \xrightarrow{\times p} \omega_{p^{-1}\mathcal{H},\beta} \longrightarrow \omega_{\mathcal{G}[p],\beta} \longrightarrow 0,$$

$$0 \longrightarrow \omega_{p^{-1}\mathcal{H}/\mathcal{H},\beta} \longrightarrow \omega_{p^{-1}\mathcal{H},\beta} \longrightarrow \omega_{\mathcal{H},\beta} \longrightarrow 0$$

and thus $\lg_{\mathcal{O}_L}(\omega_{p^{-1}\mathcal{H}/\mathcal{H},\beta}) = \lg_{\mathcal{O}_L}(\omega_{\mathcal{G}[p],\beta})$. Since the $\mathcal{O}_{L,1}$-module $\omega_{\mathcal{G}[p],\beta}$ is free of rank one, we obtain $f_\beta = 1$ and the lemma follows. □

Corollary 5.3. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_2$ over $\mathcal{O}_K$ with β-Hodge height w_β. Suppose that the inequality*

$$w_\beta + p w_{\sigma^{-1}\circ\beta} < p$$

holds for all $\beta \in \mathbb{B}_f$. Theorem 4.1 ensures that the canonical subgroup $\mathcal{C}$ of $\mathcal{G}[p]$ exists.

(1) *For any finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup $\mathcal{H} \neq \mathcal{C}$ of $\mathcal{G}[p]$ over $\mathcal{O}_K$, we have*

$$\mathrm{Hdg}_\beta(p^{-1}\mathcal{H}/\mathcal{H}) = p^{-1} w_{\sigma\circ\beta} \text{ for any } \beta \in \mathbb{B}_f.$$

Moreover, $p^{-1}\mathcal{H}/\mathcal{H}$ has the canonical subgroup $\mathcal{G}[p]/\mathcal{H}$.

(2) *Suppose that the inequality*

$$w_\beta + p w_{\sigma^{-1}\circ\beta} < 1$$

holds for all $\beta \in \mathbb{B}_f$. Consider the $\mathbb{Z}_{p^f}$-ADBT$_1$ $p^{-1}\mathcal{C}/\mathcal{C}$ over $\mathcal{O}_K$. Then we have

$$\mathrm{Hdg}_\beta(p^{-1}\mathcal{C}/\mathcal{C}) = p w_{\sigma^{-1}\circ\beta} \text{ for any } \beta \in \mathbb{B}_f.$$

Moreover, $\mathcal{G}[p]/\mathcal{C}$ is an anti-canonical subgroup of $p^{-1}\mathcal{C}/\mathcal{C}$.

Proof. For the assertion (1), Lemma 5.1 (3) implies that $\mathcal{H}$ is an anti-canonical subgroup and

$$\deg_\beta(\mathcal{G}[p]/\mathcal{H}) + p\deg_{\sigma^{-1}\circ\beta}(\mathcal{G}[p]/\mathcal{H}) > p,$$
$$\mathrm{Hdg}_\beta(\mathcal{G}[p]) = p(1-\deg_{\sigma^{-1}\circ\beta}(\mathcal{G}[p]/\mathcal{H})) = p\deg_{\sigma^{-1}\circ\beta}(\mathcal{H}).$$

Hence we have

$$\deg_\beta((p^{-1}\mathcal{H}/\mathcal{H})/(\mathcal{G}[p]/\mathcal{H})) + p\deg_{\sigma^{-1}\circ\beta}((p^{-1}\mathcal{H}/\mathcal{H})/(\mathcal{G}[p]/\mathcal{H})) < 1.$$

Lemma 5.1 (1) shows that $p^{-1}\mathcal{H}/\mathcal{H}$ has the canonical subgroup $\mathcal{G}[p]/\mathcal{H}$ and

$$\mathrm{Hdg}_\beta(p^{-1}\mathcal{H}/\mathcal{H}) = \deg_\beta(\mathcal{H}) = p^{-1}\mathrm{Hdg}_{\sigma\circ\beta}(\mathcal{G}).$$

Let us consider the assertion (2). Since $\deg_\beta(\mathcal{G}[p]/\mathcal{C}) = w_\beta$, we have

$$\deg_\beta((p^{-1}\mathcal{C}/\mathcal{C})/(\mathcal{G}[p]/\mathcal{C})) = \deg_\beta(\mathcal{C}) = 1 - w_\beta.$$

The assumption implies

$$\deg_\beta((p^{-1}\mathcal{C}/\mathcal{C})/(\mathcal{G}[p]/\mathcal{C})) + p\deg_{\sigma^{-1}\circ\beta}((p^{-1}\mathcal{C}/\mathcal{C})/(\mathcal{G}[p]/\mathcal{C})) > p$$

and Lemma 5.1 (2) yields the assertion. □

6. The case of $\mathrm{Hdg}_\beta(\mathcal{G}) = p/(p+1)$

In this subsection, we investigate the behavior of the U_p-correspondence at the locus where all the β-Hodge heights are $p/(p+1)$, under the assumption of $f \leq 2$. The results of this section is one of the key ingredients of [11], while it will not be used in the sequel. Let K be as in §1 and we assume $k \supseteq \mathbb{F}_{p^f}$.

Proposition 6.1. *Suppose $f \leq 2$. Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_2$ over $\mathcal{O}_K$ with $\mathrm{Hdg}_\beta(\mathcal{G}) = w_\beta$. Suppose $w_\beta = p/(p+1)$ for all $\beta \in \mathbb{B}_f$. Then, for any finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup $\mathcal{H}$ of $\mathcal{G}[p]$ over $\mathcal{O}_K$, we have*

$$\deg_\beta(\mathcal{G}[p]/\mathcal{H}) = \frac{p}{p+1}, \quad \mathrm{Hdg}_\beta(p^{-1}\mathcal{H}/\mathcal{H}) = \frac{1}{p+1}$$

and $p^{-1}\mathcal{H}/\mathcal{H}$ has the canonical subgroup $\mathcal{G}[p]/\mathcal{H}$.

Proof. By a base change argument as before, we may assume that the residue field k is perfect. Put $\mathfrak{M} = \mathfrak{M}^*(\mathcal{G}[p])$ and $\mathfrak{P} = \mathfrak{M}^*(\mathcal{G}[p]/\mathcal{H})$. We take a basis $\{e_\beta, e'_\beta\}$ of the $\mathfrak{S}_1$-module $\mathfrak{M}_\beta$ as in §4 and consider the equation (4.1). Take $x_\beta, y_\beta \in \mathfrak{S}_1$ such that the element $f_\beta = x_\beta e_\beta + y_\beta e'_\beta$ is a basis of the free $\mathfrak{S}_1$-module $\mathfrak{P}_\beta$ of rank one. Then there exists an f-tuple $(\lambda_\beta)_{\beta\in\mathbb{B}_f}$ in $\mathfrak{S}_1$ satisfying

$$\begin{pmatrix} a_{\beta,1} & a_{\beta,2} \\ u^e a_{\beta,3} & u^e a_{\beta,4} \end{pmatrix} \begin{pmatrix} x^p_{\sigma^{-1}\circ\beta} \\ y^p_{\sigma^{-1}\circ\beta} \end{pmatrix} = \lambda_\beta \begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix} \tag{6.1}$$

for all $\beta \in \mathbb{B}_f$. Note that $v_R(a_{\beta,1}) = p/(p+1)$. Since the matrix

$$\begin{pmatrix} a_{\beta,1} & a_{\beta,2} \\ a_{\beta,3} & a_{\beta,4} \end{pmatrix}$$

is an element of $GL_2(\mathfrak{S}_1)$, we have $v_R(a_{\beta,2}) = v_R(a_{\beta,3}) = 0$.

We claim that the inequalities $0 < w_\beta < 1$ for all $\beta \in \mathbb{B}_f$ imply $v_R(y_{\sigma^{-1}\circ\beta}) > 0$. Indeed, if $v_R(y_{\sigma^{-1}\circ\beta}) = 0$, then $v_R(\lambda_\beta) = v_R(x_\beta) = 0$ and $v_R(y_\beta) \geq 1$. This is a contradiction if $f = 1$. Using (6.1) for $\sigma\circ\beta$ yields $v_R(\lambda_{\sigma\circ\beta}) \leq w_{\sigma\circ\beta}$ and thus $v_R(y_{\sigma\circ\beta}) \geq 1 - w_{\sigma\circ\beta}$. This is a contradiction if $f = 2$ and the claim follows.

Since $x_\beta e_\beta + y_\beta e'_\beta$ generates the direct summand $\mathfrak{P}_\beta$ of the $\mathfrak{S}_1$-module $\mathfrak{M}_\beta$, the claim implies $v_R(x_\beta) = 0$. Replacing f_β by $x_\beta^{-1} f_\beta$, we may assume $x_\beta = 1$ for all $\beta \in \mathbb{B}_f$. Then $(y_\beta)_{\beta\in\mathbb{B}_f}$ satisfies the equation

$$\begin{cases} a_{\beta,1} + a_{\beta,2} y^p_{\sigma^{-1}\circ\beta} = \lambda_\beta, \\ u^e(a_{\beta,3} + a_{\beta,4} y^p_{\sigma^{-1}\circ\beta}) = \lambda_\beta y_\beta. \end{cases} \tag{6.2}$$

Next we claim that every solution $(y_\beta)_{\beta\in\mathbb{B}_f}$ of this equation satisfies $v_R(\lambda_\beta)=p/(p+1)$ for any $\beta\in\mathbb{B}_f$. For $f=1$, we see that $y=y_\beta$ satisfies the equation

$$y^{p+1}-u^e a_{\beta,2}^{-1}a_{\beta,4}y^p+a_{\beta,2}^{-1}a_{\beta,1}y-u^e a_{\beta,2}^{-1}a_{\beta,3}=0.$$

An inspection of its Newton polygon shows $v_R(y)=1/(p+1)$. Then the second equation of (6.2) implies $v_R(\lambda_\beta)=p/(p+1)$.

Let us consider the case $f=2$. Take any $\beta\in\mathbb{B}_f$. Put

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\begin{pmatrix} a_{\sigma\circ\beta,1} & a_{\sigma\circ\beta,2} \\ u^e a_{\sigma\circ\beta,3} & u^e a_{\sigma\circ\beta,4}\end{pmatrix}\begin{pmatrix} a_{\beta,1}^p & a_{\beta,2}^p \\ u^{pe}a_{\beta,3}^p & u^{pe}a_{\beta,4}^p\end{pmatrix}.$$

Note that

$$v_R(A)\geq p,\quad v_R(B)=\frac{p}{p+1},\quad v_R(C)=1+\frac{p^2}{p+1},\quad v_R(D)=1.$$

We have

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} 1 \\ y_{\sigma^{-1}\circ\beta}^{p^2}\end{pmatrix}=\lambda_{\sigma\circ\beta}\lambda_\beta^p\begin{pmatrix} 1 \\ y_{\sigma^{-1}\circ\beta}\end{pmatrix}$$

and thus $y=y_{\sigma^{-1}\circ\beta}$ satisfies the equation

$$y^{p^2+1}-B^{-1}Dy^{p^2}+B^{-1}Ay-B^{-1}C=0,$$

where the coefficients are all integral. An inspection of its Newton polygon shows $v_R(y_{\sigma^{-1}\circ\beta})=1/(p+1)$. Then the second equation of (6.2) yields $v_R(\lambda_{\sigma^{-1}\circ\beta})=p/(p+1)$. Since $\beta\in\mathbb{B}_f$ is arbitrary, we obtain

$$v_R(y_\beta)=\frac{1}{p+1},\quad v_R(\lambda_\beta)=\frac{p}{p+1}$$

for any $\beta\in\mathbb{B}_f$.

Now the claim shows

$$\begin{aligned}\deg_\beta((p^{-1}\mathcal{H}/\mathcal{H})/(\mathcal{G}[p]/\mathcal{H}))=\deg_\beta(\mathcal{H})=1-v_R(\lambda_\beta)=1/(p+1),\\ \deg_\beta(\mathcal{G}[p]/\mathcal{H})=\deg_\beta(p^{-1}\mathcal{H}/\mathcal{H})-1/(p+1)=p/(p+1)\end{aligned}$$

for any $\beta\in\mathbb{B}_f$. Then Lemma 5.1 (1) implies that

$$\mathrm{Hdg}_\beta(p^{-1}\mathcal{H}/\mathcal{H})=1/(p+1)$$

and that $\mathcal{G}[p]/\mathcal{H}$ is the canonical subgroup of $p^{-1}\mathcal{H}/\mathcal{H}$. $\square$

Remark 6.2. A naive generalization of Proposition 6.1 has a counterexample for $f=3$, if $p\neq 2$. Suppose $k=\bar{k}$ and $p+1\mid e$. Replacing the uniformizer π by a scalar multiple, we may assume that $c_0=p^{-1}E(0)$ satisfies $c_0\equiv 1 \bmod p$. Let r be a positive integer. Fix $\beta'\in\mathbb{B}_3$ and consider the following elements of $M_2(\mathfrak{S})$.

$$\hat{A}_{\beta'} = \begin{pmatrix} u^{\frac{pe}{p+1}} & 1 \\ c_0^{-1}E(u) & c_0^{-1}E(u) \end{pmatrix}, \quad \hat{A}_{\sigma\circ\beta'} = \begin{pmatrix} u^{\frac{pe}{p+1}} & -1 \\ c_0^{-1}E(u) & c_0^{-1}E(u) \end{pmatrix},$$
$$\hat{A}_{\sigma^2\circ\beta'} = \begin{pmatrix} u^{\frac{pe}{p+1}} & 1 \\ c_0^{-1}E(u) & c_0^{-1}u^r E(u) \end{pmatrix}$$

We define the $\mathbb{Z}_{p^3}$-Breuil–Kisin module $\hat{\mathfrak{M}} = \bigoplus_{\beta\in\mathbb{B}_3} \hat{\mathfrak{M}}_\beta$ by

$$\hat{\mathfrak{M}}_\beta = \mathfrak{S}\hat{e}_\beta \oplus \mathfrak{S}\hat{e}'_\beta, \quad \varphi_\beta(\hat{e}_{\sigma^{-1}\circ\beta}, \hat{e}'_{\sigma^{-1}\circ\beta}) = (\hat{e}_\beta, \hat{e}'_\beta)\hat{A}_\beta.$$

Take $\hat{\alpha}_\beta \in \mathfrak{S}^\times$ for each $\beta \in \mathbb{B}_3$ satisfying

$$\varphi(\hat{\alpha}_{\sigma^{-1}\circ\beta}) = c_0 E(u)^{-1} \det(\hat{A}_\beta)\hat{\alpha}_\beta \text{ for all } \beta \in \mathbb{B}_3.$$

Then the map

$$(\hat{e}_\beta, \hat{e}'_\beta) \longmapsto (\hat{e}_\beta^\vee, (\hat{e}'_\beta)^\vee)\hat{\alpha}_\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

gives a skew-symmetric isomorphism $\hat{\mathfrak{M}} \to \hat{\mathfrak{M}}^\vee$. Since $\hat{\mathfrak{M}}$ corresponds to a Barsotti–Tate group Γ over $\mathcal{O}_K$, we see that $\hat{\mathfrak{M}}/p^2\hat{\mathfrak{M}}$ corresponds to a $\mathbb{Z}_{p^3}$-ADBT$_2$ $\mathcal{G} = \Gamma[p^2]$ over $\mathcal{O}_K$. Let $\mathcal{H}$ be any cyclic $\mathbb{Z}_{p^3}$-subgroup of $\mathcal{G}[p]$ over $\mathcal{O}_K$. We write the images of $\hat{e}_\beta, \hat{e}'_\beta$ in $\hat{\mathfrak{M}}/p\hat{\mathfrak{M}}_\beta$ as e_β, e'_β and a basis of $\mathfrak{M}^*(\mathcal{G}/\mathcal{H})_\beta$ as $x_\beta e_\beta + y_\beta e'_\beta$.

Suppose $v_R(y_\beta) > 0$ for all $\beta \in \mathbb{B}_3$. Then we may assume $x_\beta = 1$, and we see that $y = y_{\sigma^2\circ\beta'}$ is a root of the equation

$$y^{p^3+1} - B^{-1}Dy^{p^3} + B^{-1}Ay - B^{-1}C = 0,$$

where we put

$$\begin{aligned} A &= u^{\frac{e(p^3+p^2+p)}{p+1}} + u^{e(p^2+p)}, \\ B &= 2u^{ep} - u^{\frac{e(p^3+p^2+p)}{p+1}} + u^{e(p^2+p)}, \\ C &= u^{\frac{e(p^3+p^2+2p+1)}{p+1}+r} + u^{e(p^2+p+1)+r}, \\ D &= u^{\frac{e(p^2+p+1)}{p+1}} - u^{e(p^2+1)} + u^{e(p+1)+r} + u^{e(p^2+p+1)+r}. \end{aligned}$$

An inspection of its Newton polygon and derivation shows that this equation has exactly p^3 roots satisfying $v_R(y) = 1/(p+1)$ and one root satisfying $v_R(y) = 1 + e^{-1}r$.

The latter case does not occur, since it contradicts the second equation of (6.2). In the former case, put $y = u^{e/(p+1)}\eta$. Then η satisfies a monic polynomial of degree p^3+1 whose reduction modulo u is $X(X^{p^3} - 2^{-1}X^{p^3-1} + 2^{-1})$. Hensel's lemma and the assumption on k imply $y \in \mathfrak{S}_1$. Thus $\mathcal{G}[p]$

has exactly p^3 cyclic $\mathbb{Z}_{p^3}$-subgroups over $\mathcal{O}_K$ such that, for any $\beta \in \mathbb{B}_3$, we have $v_R(y_\beta) > 0$.

By the assumption $k = \bar{k}$, there exist exactly p^3-1 characters $G_K \to \mathbb{F}_{p^3}^\times$. Hence, among these p^3 cyclic $\mathbb{Z}_{p^3}$-subgroups, two define the same character on the generic fiber. This means that G_K acts on $\mathcal{G}[p](\mathcal{O}_{\bar{K}})$ via this character. In particular, any $\mathbb{F}_{p^3}$-subspace of $\mathcal{G}[p](\mathcal{O}_{\bar{K}})$ is G_K-stable. Taking the scheme-theoretic closure, we see that $\mathcal{G}[p]$ has one cyclic $\mathbb{Z}_{p^3}$-subgroup $\mathcal{H}$ over $\mathcal{O}_K$ satisfying $v_R(y_{\sigma^{-1}\circ\beta_0}) = 0$ for some $\beta_0 \in \mathbb{B}_3$.

For this $\mathcal{H}$, the equation (6.1) gives $v_R(\lambda_{\beta_0}) = v_R(x_{\beta_0}) = 0$ and $v_R(y_{\beta_0}) \geq 1$. This in turn gives $v_R(\lambda_{\sigma\circ\beta_0}) \leq p/(p+1)$ and $v_R(y_{\sigma\circ\beta_0}) \geq 1/(p+1)$. Since $x_{\sigma\circ\beta_0}e_{\sigma\circ\beta_0} + y_{\sigma\circ\beta_0}e'_{\sigma\circ\beta_0}$ generates a direct summand, we have $v_R(x_{\sigma\circ\beta_0}) = 0$ and this implies $v_R(\lambda_{\sigma\circ\beta_0}) = p/(p+1)$. Thus we obtain

$$\deg_{\sigma\circ\beta_0}(\mathcal{H}) + p\deg_{\beta_0}(\mathcal{H}) = p + \frac{1}{p+1}$$

and $\mathcal{G}[p]/\mathcal{H}$ is not the canonical subgroup of $p^{-1}\mathcal{H}/\mathcal{H}$.

7. Canonical subgroups of higher levels

Let K be as in §1, with residue field $k \supseteq \mathbb{F}_{p^f}$ which may be imperfect. In this section, we derive from Theorem 4.1 the existence of the canonical subgroup of level n for a $\mathbb{Z}_{p^f}$-ADBT$_n$, by following an argument of Fargues–Tian [8, §7] as in [13, §4]. A similar result was also obtained by Goren–Kassaei [9, Proposition 5.4.5] except the compatibility with Hodge–Tate maps and lower ramification subgroups. This compatibility shown here will be used in [11] to enlarge the locus where the sheaf of overconvergent Hilbert modular forms is defined from that of [2].

Theorem 7.1. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-*ADBT$_n$ *over $\mathcal{O}_K$ with β-Hodge height w_β. Put $w = \max\{w_\beta \mid \beta \in \mathbb{B}_f\}$. Suppose that we have*

$$w_\beta + pw_{\sigma^{-1}\circ\beta} < p^{2-n}$$

for all $\beta \in \mathbb{B}_f$.

If k is perfect, then there exists a finite flat closed $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}_n$ of $\mathcal{G}$ of rank p^{nf} over $\mathcal{O}_K$ satisfying

$$\deg_\beta(\mathcal{G}/\mathcal{C}_n) = \sum_{l=0}^{n-1} p^l w_{\sigma^{-l}\circ\beta}.$$

We refer to $\mathcal{C}_n$ as the canonical subgroup of level n of $\mathcal{G}$. It has the following properties:

(1) *Let $\mathcal{G}'$ be a $\mathbb{Z}_{p^f}$-*ADBT$_n$ *over $\mathcal{O}_K$ satisfying the same condition on the β-Hodge heights as above and $\mathcal{C}'_n$ the canonical subgroup of level n of $\mathcal{G}'$. Then any isomorphism of $\mathbb{Z}_{p^f}$-groups $j : \mathcal{G} \to \mathcal{G}'$ over $\mathcal{O}_K$ induces an isomorphism $\mathcal{C}_n \simeq \mathcal{C}'_n$.*

(2) $\mathcal{C}_n$ *is compatible with finite base extension of complete discrete valuation rings.*
(3) $\mathcal{C}_n$ *is compatible with Cartier duality. Namely,* $(\mathcal{G}/\mathcal{C}_n)^\vee$ *is the canonical subgroup of level* n *of* $\mathcal{G}^\vee$*.*
(4) *The kernel of the* n*-th iterated Frobenius map of* $\mathcal{G}\times\mathscr{S}_{1-p^{n-1}w}$ *coincides with* $\mathcal{C}_n\times\mathscr{S}_{1-p^{n-1}w}$*.*
(5) *The* $\mathbb{Z}_{p^f}/p^n\mathbb{Z}_{p^f}$*-module* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ *is free of rank one.*
(6) *The scheme-theoretic closure of* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ *in* $\mathcal{C}_n$ *is the canonical subgroup* $\mathcal{C}_i$ *of level* i *of* $\mathcal{G}[p^i]$ *for any* $0\le i\le n-1$*.*
(7) *If* $w<(p-1)/p^n$*, then* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ *coincides with* $\mathrm{Ker}(\mathrm{HT}_{\mathcal{G},i})$ *for any rational number* i *satisfying*

$$n-1+\frac{w}{p-1}<i\le n-\frac{w(p^n-1)}{p-1}.$$

(8) *If* $w<(p-1)/p^n$*, then* $\mathcal{C}_n=\mathcal{G}_i$ *for any rational number* i *satisfying*

$$i'_n=\frac{1}{p^n(p-1)}\le i\le i_n=\frac{1}{p^{n-1}(p-1)}-\frac{w}{p-1}.$$

Moreover, for the case where k *is imperfect, the same statements hold if* $w<(p-1)/p^n$*.*

Proof. The last assertion on the case where k is imperfect follows from the assertion (8) in the case of a perfect residue field and a similar base change argument as in the proof of Corollary 4.6. Thus we may assume that k is perfect.

We proceed by induction on n. The case $n=1$ is Theorem 4.1. Suppose that $n\ge 2$ and the assertions hold for $n-1$. Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT$_n$ satisfying the assumption. Then we have the canonical subgroup $\mathcal{C}_1$ of the $\mathbb{Z}_{p^f}$-ADBT$_1$ $\mathcal{G}[p]$ and Lemma 5.2 implies that $p^{1-n}\mathcal{C}_1/\mathcal{C}_1$ is also a $\mathbb{Z}_{p^f}$-ADBT$_{n-1}$. By Corollary 5.3 (2), we have

$$\mathrm{Hdg}_\beta(p^{1-n}\mathcal{C}_1/\mathcal{C}_1)=pw_{\sigma^{-1}\circ\beta}$$

and by the induction hypothesis, $p^{1-n}\mathcal{C}_1/\mathcal{C}_1$ has the canonical subgroup of level $n-1$, which we write as $\mathcal{C}_n/\mathcal{C}_1$ with some $\mathbb{Z}_{p^f}$-subgroup $\mathcal{C}_n$ of $\mathcal{G}$. Then we have

$$\begin{aligned}\deg_\beta(\mathcal{G}/\mathcal{C}_n)&=\deg_\beta(\mathcal{G}/p^{1-n}\mathcal{C}_1)+\deg_\beta((p^{1-n}\mathcal{C}_1/\mathcal{C}_1)/(\mathcal{C}_n/\mathcal{C}_1))\\&=\deg_\beta(\mathcal{G}[p]/\mathcal{C}_1)+\sum_{l=0}^{n-2}p^l(pw_{\sigma^{-l-1}\circ\beta})=\sum_{l=0}^{n-1}p^lw_{\sigma^{-l}\circ\beta}.\end{aligned}$$

The assertions (1) and (2) follow from the construction and the induction hypothesis. The assertions (3) and (4) can be shown exactly in the same way as [13, Theorem 1.1 (b) and (1)], using the assertion (1).

Let us show the assertion (5). By an induction, we can show $\mathcal{C}_{n-1} \subseteq \mathcal{C}_n$. By the induction hypothesis, it suffices to show $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \cap \mathcal{G}[p](\mathcal{O}_{\bar{K}}) = \mathcal{C}_1(\mathcal{O}_{\bar{K}})$ for any $n \geq 2$. From the assertion (6) for $p^{1-n}\mathcal{C}_1/\mathcal{C}_1$, we see that $(\mathcal{C}_n/\mathcal{C}_1)(\mathcal{O}_{\bar{K}})[p]$ is the generic fiber of the canonical subgroup of $p^{-1}\mathcal{C}_1/\mathcal{C}_1$. On the other hand, Corollary 5.3 (2) implies that $\mathcal{G}[p]/\mathcal{C}_1$ is not the canonical subgroup of $p^{-1}\mathcal{C}_1/\mathcal{C}_1$. Then we have $(\mathcal{C}_n/\mathcal{C}_1)(\mathcal{O}_{\bar{K}}) \cap (\mathcal{G}[p]/\mathcal{C}_1)(\mathcal{O}_{\bar{K}}) = 0$ and thus $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \cap \mathcal{G}[p](\mathcal{O}_{\bar{K}}) \subseteq \mathcal{C}_1(\mathcal{O}_{\bar{K}})$, from which the assertion (5) follows. The assertion (6) follows from $\mathcal{C}_{n-1} \subseteq \mathcal{C}_n$ and the assertion (5).

Next we show the assertion (7). Let i be as in the assertion. Put $\epsilon = n-i$. Since we have

$$w/(p-1) < 1-\epsilon \leq 1-w,$$

by using Theorem 4.1 (6) we can show $\sharp\mathrm{Ker}(\mathrm{HT}_i) \leq p^{nf}$ as in the proof of [8, Proposition 13]. On the other hand, since $\deg_\beta(\mathcal{C}_1^\vee) = w_\beta$, the $\mathcal{O}_{\bar{K}}$-module $\omega_{\mathcal{C}_1^\vee} \otimes \mathcal{O}_{\bar{K}}$ is killed by $m_{\bar{K}}^{\geqslant w}$. Take any element $x \in \mathcal{C}_n(\mathcal{O}_{\bar{K}})$ and denote its image in $(\mathcal{G}/\mathcal{C}_1)(\mathcal{O}_{\bar{K}})$ by $\bar{x}$. By the induction hypothesis, we have $\mathrm{HT}_j(\bar{x}) = 0$ for any j satisfying

$$n-2+pw/(p-1) < j \leq n-1-w(p^n-p)/(p-1).$$

Thus we obtain

$$m_{\bar{K}}^{\geqslant n-1-j}\mathrm{HT}(\bar{x}) = 0, \quad m_{\bar{K}}^{\geqslant n-1-j+w}\mathrm{HT}(x) = 0$$

and $\mathrm{HT}_{1-w+j}(x) = 0$, which yields $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \subseteq \mathrm{Ker}(\mathrm{HT}_i)$. Then the assertion (7) follows from $\sharp\mathcal{C}_n(\mathcal{O}_{\bar{K}}) = p^{nf}$.

Finally, we show the assertion (8) following the proof of [14, Theorem 1.2]. Using Lemma 4.3 and Theorem 4.1 (4), the same argument as in the proof of [14, Lemma 5.4] shows $\mathcal{G}_{i'_n} \subseteq \mathcal{C}_n$. For the reverse inclusion, we need the following variant of [14, Proposition 5.5].

Lemma 7.2. *The image of the map $\mathcal{G}_{i_n}(\mathcal{O}_{\bar{K}}) \xrightarrow{\times p} \mathcal{G}[p^{n-1}]_{pi_n}(\mathcal{O}_{\bar{K}})$ contains $\mathcal{G}[p^{n-1}]_{i_{n-1}}(\mathcal{O}_{\bar{K}})$.*

Proof. Note that the map in the lemma is well-defined by [14, Lemma 5.3]. Put $\mathfrak{M} = \mathfrak{M}^*(\mathcal{G}[p])$. Consider the basis $\{\delta_\beta, e'_\beta\}$ of the $\mathfrak{S}_1$-module $\mathfrak{M}$ as in the proof of Theorem 4.1. Write as

$$\varphi(\delta_{\sigma^{-1}\circ\beta}, e'_{\sigma^{-1}\circ\beta}) = (\delta_\beta, e'_\beta)\begin{pmatrix}\lambda_\beta & \mu_\beta \\ 0 & \nu_\beta\end{pmatrix}.$$

We have $v_R(\lambda_\beta) = w_\beta$ and $v_R(\nu_\beta) = 1 - w_\beta$. Then, in the same way as in the proof of [14, Proposition 5.5], we reduce ourselves to showing that for any $\xi_\beta, \eta_\beta \in m_R^{\geqslant i_{n-1}}$, there exist $\zeta_\beta, \omega_\beta \in m_R^{\geqslant i_n}$ satisfying

$$(\xi_\beta, \eta_\beta) + (\zeta^p_{\sigma^{-1}\circ\beta}, \omega^p_{\sigma^{-1}\circ\beta}) = (\zeta_\beta, \omega_\beta)\begin{pmatrix}\lambda_\beta & \mu_\beta \\ 0 & \nu_\beta\end{pmatrix}$$

for all $\beta \in \mathbb{B}_f$. We can show by recursion that the equation on ζ_β's has a solution satisfying $v_R(\zeta_\beta) \geq pi_n$ for all $\beta \in \mathbb{B}_f$. Fixing such ζ_β's, we obtain the system of equations on ω_β's

$$\omega^p_{\sigma^{-1}\circ\beta} - \nu_\beta \omega_\beta - \mu_\beta \zeta_\beta + \eta_\beta = 0.$$

Take any $a \in R$ satisfying $v_R(a) = i_n$ and put $\omega_\beta = a\alpha_\beta$. Then $(\alpha_\beta)_{\beta\in\mathbb{B}_f}$ is a solution of the system of equations

$$\alpha^p_{\sigma^{-1}\circ\beta} - \frac{\nu_\beta}{a^{p-1}}\alpha_\beta - \frac{\mu_\beta\zeta_\beta}{a^p} + \frac{\eta_\beta}{a^p} = 0,$$

where all the coefficients are contained in R. This system defines a finite R-algebra which is free of rank p^f. Since $\mathrm{Frac}(R)$ is algebraically closed and R is normal, we can find a solution $(\alpha_\beta)_{\beta\in\mathbb{B}_f}$ in R and the lemma follows. □

By the induction hypothesis, we have $\mathcal{G}[p^{n-1}]_{i_{n-1}} = \mathcal{C}_{n-1}$. By Lemma 4.3, we also have $\mathcal{G}[p]_{i_n} = \mathcal{C}_1$. Then Lemma 7.2 implies $\sharp\mathcal{G}_{i_n}(\mathcal{O}_{\bar{K}}) \geq \sharp\mathcal{C}_n(\mathcal{O}_{\bar{K}})$. Now the assertion (8) follows from the inclusions $\mathcal{G}_{i_n} \subseteq \mathcal{G}_{i'_n} \subseteq \mathcal{C}_n$. This concludes the proof of Theorem 7.1. □

Corollary 7.3. *Let n be a positive integer. Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT_{n+1} over $\mathcal{O}_K$ with β-Hodge height w_β satisfying*

$$w_\beta + pw_{\sigma^{-1}\circ\beta} < p^{2-n}$$

for all $\beta \in \mathbb{B}_f$.

Suppose that the residue field k of K is perfect. Let $\mathcal{C}_{n-1}$ and $\mathcal{C}_1$ be the canonical subgroups of level $n-1$ and level one of $\mathcal{G}[p^{n-1}]$ and $\mathcal{G}[p]$, respectively. Let $\mathcal{H} \neq \mathcal{C}_1$ be a finite flat closed cyclic $\mathbb{Z}_{p^f}$-subgroup of $\mathcal{G}[p]$ over $\mathcal{O}_K$. Then the $\mathbb{Z}_{p^f}$-ADBT_n $p^{-n}\mathcal{H}/\mathcal{H}$ has the canonical subgroup $p^{-1}\mathcal{C}_{n-1}/\mathcal{H}$. Moreover, the natural map $\mathcal{C}_n \to p^{-1}\mathcal{C}_{n-1}/\mathcal{H}$ is an isomorphism over K.

For the case where k is imperfect, the same statements hold if

$$\max\{w_\beta \mid \beta \in \mathbb{B}_f\} < (p-1)/p^n.$$

Proof. By a base change, we may assume that k is perfect. By Corollary 5.3 (1), the $\mathbb{Z}_{p^f}$-ADBT_i $p^{-i}\mathcal{H}/\mathcal{H}$ has the canonical subgroup of level i for any positive integer $i \leq n$, which we denote by $\bar{\mathcal{C}}_i$. Moreover, we have $\bar{\mathcal{C}}_1 = \mathcal{G}[p]/\mathcal{H}$. By the construction of the canonical subgroup in Theorem 7.1, the quotient $\bar{\mathcal{C}}_n/\bar{\mathcal{C}}_1$ is equal to the canonical subgroup of level $n-1$ of the $\mathbb{Z}_{p^f}$-ADBT_{n-1} $p^{1-n}\bar{\mathcal{C}}_1/\bar{\mathcal{C}}_1$. We have the map

$$p^{1-n}\bar{\mathcal{C}}_1/\bar{\mathcal{C}}_1 = p^{1-n}(\mathcal{G}[p]/\mathcal{H})/(\mathcal{G}[p]/\mathcal{H}) = (\mathcal{G}[p^n]/\mathcal{H})/(\mathcal{G}[p]/\mathcal{H}) \xrightarrow{\times p} \mathcal{G}[p^{n-1}],$$

where the last arrow is an isomorphism. By Theorem 7.1 (1), we obtain $\bar{\mathcal{C}}_n = p^{-1}\mathcal{C}_{n-1}/\mathcal{H}$. Moreover, Theorem 7.1 (6) implies $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \cap \mathcal{H}(\mathcal{O}_{\bar{K}}) = 0$ and the map $\mathcal{C}_n \to p^{-1}\mathcal{C}_{n-1}/\mathcal{H}$ is an injection over K. Since both sides have the same rank over $\mathcal{O}_K$, the last assertion follows. □

Finally, we show the following generalization of [1, Proposition 3.2.1] to our setting.

Proposition 7.4. *Let $\mathcal{G}$ be a $\mathbb{Z}_{p^f}$-ADBT_n over $\mathcal{O}_K$ with β-Hodge height w_β. Put $w = \max\{w_\beta \mid \beta \in \mathbb{B}_f\}$. Suppose $w < (p-1)/p^n$. Let $\mathcal{C}_n$ be the canonical subgroup of $\mathcal{G}$ of level n, which exists by Theorem 7.1.*

(1) *For any $i \in e^{-1}\mathbb{Z}_{\geq 0}$ satisfying $i \leq n - w(p^n-1)/(p-1)$, the natural map*
$$\omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,i} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,i}$$
is an isomorphism.

(2) *The cokernel of the linearization of the Hodge–Tate map*
$$\mathrm{HT}_{\mathcal{C}_n^\vee} \otimes 1 : \mathcal{C}_n^\vee(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$$
is killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$.

Proof. By a base change argument as before, we may assume that the residue field k is perfect. Put $b = n - w(p^n-1)/(p-1)$. For the first assertion, consider the exact sequence
$$0 \longrightarrow \omega_{\mathcal{G}/\mathcal{C}_n} \longrightarrow \omega_{\mathcal{G}} \longrightarrow \omega_{\mathcal{C}_n} \longrightarrow 0$$
and the decompositions
$$\omega_{\mathcal{G}/\mathcal{C}_n} = \bigoplus_{\beta \in \mathbb{B}_f} \omega_{\mathcal{G}/\mathcal{C}_n,\beta}, \quad \omega_{\mathcal{G}} = \bigoplus_{\beta \in \mathbb{B}_f} \omega_{\mathcal{G},\beta}.$$
Note that $\omega_{\mathcal{G},\beta} \simeq \mathcal{O}_{K,n}$. Theorem 7.1 implies
$$\deg_\beta(\mathcal{G}/\mathcal{C}_n) = \sum_{l=0}^{n-1} p^l w_{\sigma^{-l}\circ\beta} \leq \frac{w(p^n-1)}{p-1} \leq n - i.$$
Thus the image of the natural map $\omega_{\mathcal{G}/\mathcal{C}_n,\beta} \to \omega_{\mathcal{G},\beta}$ is contained in $m_K^{\geqslant i}\omega_{\mathcal{G},\beta}$ for any $\beta \in \mathbb{B}_f$ and the first assertion follows.

For the second assertion, consider the commutative diagram
$$\begin{tikzcd}
\mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \arrow[r, "\mathrm{HT}_{\mathcal{G}^\vee}"] \arrow[d, "p^{n-1}"'] & \omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \arrow[r] \arrow[d] & \omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},1} \arrow[d] \\
\mathcal{G}^\vee[p](\mathcal{O}_{\bar{K}}) \arrow[r, "\mathrm{HT}_{\mathcal{G}^\vee[p]}"] & \omega_{\mathcal{G}[p]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \arrow[r, equal] & \omega_{\mathcal{G}[p]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},1},
\end{tikzcd}$$
where the horizontal composites are the first Hodge–Tate maps and the left vertical arrow is surjective. Since the right vertical arrow is an isomorphism, the map $\mathrm{HT}_{\mathcal{G}^\vee,1}$ factors through $\mathcal{G}^\vee[p](\mathcal{O}_{\bar{K}})$ and we obtain a natural isomorphism of $\mathcal{O}_{\bar{K}}$-modules
$$\mathrm{Coker}(\mathrm{HT}_{\mathcal{G}^\vee,1} \otimes 1) \simeq \mathrm{Coker}(\mathrm{HT}_{\mathcal{G}^\vee[p],1} \otimes 1).$$

By Lemma 4.4, they are killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$ and thus

$$m_{\bar{K}}^{\geqslant w/(p-1)}(\omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}) \subseteq \mathrm{Im}(\mathrm{HT}_{\mathcal{G}^\vee} \otimes 1) + p(\omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}).$$

Since $w < 1$, Lemma 4.5 implies that the $\mathcal{O}_{\bar{K}}$-module $\mathrm{Coker}(\mathrm{HT}_{\mathcal{G}^\vee} \otimes 1)$ is killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$.

On the other hand, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \arrow[r, "\mathrm{HT}_{\mathcal{G}^\vee}"] \arrow[d] & \omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \arrow[r] \arrow[d] & \omega_{\mathcal{G}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},b} \arrow[d] \\
\mathcal{C}_n^\vee(\mathcal{O}_{\bar{K}}) \arrow[r, "\mathrm{HT}_{\mathcal{C}_n^\vee}"] & \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \arrow[r] & \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K},b},
\end{tikzcd}$$

where the left vertical arrow is surjective. By a base change argument using Theorem 7.1 (2), the first assertion implies that the right vertical arrow is an isomorphism. Thus we have a surjection of $\mathcal{O}_{\bar{K}}$-modules

$$\mathrm{Coker}(\mathrm{HT}_{\mathcal{G}^\vee} \otimes 1) \twoheadrightarrow \mathrm{Coker}(\mathrm{HT}_{\mathcal{G}^\vee,b} \otimes 1) \simeq \mathrm{Coker}(\mathrm{HT}_{\mathcal{C}_n^\vee,b} \otimes 1)$$

and $\mathrm{Coker}(\mathrm{HT}_{\mathcal{C}_n^\vee,b} \otimes 1)$ is also killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$. This is equivalent to the inclusion

$$m_{\bar{K}}^{\geqslant w/(p-1)}(\omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}) \subseteq \mathrm{Im}(\mathrm{HT}_{\mathcal{C}_n^\vee} \otimes 1) + m_{\bar{K}}^{\geqslant b}(\omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}).$$

Since $w < (p-1)/p^n$, we have $b > w/(p-1)$ and the proposition follows from Lemma 4.5. $\square$

8. Canonical subgroups for Hilbert–Blumenthal abelian varieties

Let p be a rational prime. Let F be a totally real number field of degree g which is unramified over p. We denote its ring of integers by $\mathcal{O}_F$ and its different by $\mathcal{D}_F$. For any prime ideal $\mathfrak{p} \mid p$ of $\mathcal{O}_F$, let $f_{\mathfrak{p}}$ be the residue degree of $\mathfrak{p}$. Fix a complete discrete valuation field K of mixed characteristic $(0,p)$ with residue field k such that $F \otimes K$ splits completely. We denote by $\mathbb{B}_F$ the set of embeddings $F \to K$ and by $\mathbb{B}_{\mathfrak{p}}$ the subset consisting of embeddings which factor through the completion $F_{\mathfrak{p}}$. Then we can identify $\mathbb{B}_{\mathfrak{p}}$ with $\mathbb{B}_{f_{\mathfrak{p}}}$. The set $\mathbb{B}_F$ is decomposed as

$$\mathbb{B}_F = \coprod_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}}.$$

Let $N \geq 4$ be an integer with $p \nmid N$ and $\mathfrak{c}$ a non-zero fractional ideal of F. Let S be a scheme over $\mathcal{O}_K$. A Hilbert–Blumenthal abelian variety over S, which we abbreviate as HBAV, is a quadruple $(A, \iota, \lambda, \psi)$ such that

- A is an abelian scheme over S of relative dimension g.
- $\iota : \mathcal{O}_F \to \mathrm{End}_S(A)$ is a ring homomorphism.

- λ is a $\mathfrak{c}$-polarization. Namely, $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c} \simeq A^\vee$ is an isomorphism of abelian schemes to the dual abelian scheme $A^\vee$ compatible with $\mathcal{O}_F$-action such that the map

$$\mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee) \simeq \mathrm{Hom}_{\mathcal{O}_F}(A, A \otimes_{\mathcal{O}_F} \mathfrak{c}), \quad f \longmapsto \lambda^{-1} \circ f$$

  induces an isomorphism of $\mathcal{O}_F$-modules with notion of positivity $(\mathcal{P}_A, \mathcal{P}_A^+) \simeq (\mathfrak{c}, \mathfrak{c}^+)$. Here $\mathcal{P}_A$ denotes the $\mathcal{O}_F$-module of symmetric $\mathcal{O}_F$-homomorphisms from A to $A^\vee$, $\mathcal{P}_A^+$ is the subset of $\mathcal{O}_F$-linear polarizations, $\mathfrak{c}^+$ is the subset of totally positive elements of $\mathfrak{c}$ and any element $\gamma \in \mathfrak{c}$ is identified with the element $(x \mapsto x \otimes \gamma)$ of $\mathrm{Hom}_{\mathcal{O}_F}(A, A \otimes_{\mathcal{O}_F} \mathfrak{c})$.
- $\psi : \mathcal{D}_F^{-1} \otimes \mu_N \to A$ is an $\mathcal{O}_F$-linear closed immersion of group schemes, which we call a $\Gamma_{00}(N)$-structure.

Note that for such data, the $\mathcal{O}_S \otimes \mathcal{O}_F$-module $\mathrm{Lie}(A)$ is locally free of rank one [7, Corollaire 2.9].

Let $(A, \iota, \lambda, \psi)$ be a HBAV over S with $\mathfrak{c}$-polarization λ and $\Gamma_{00}(N)$-structure ψ. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_F$. The $\mathfrak{c}$-polarization λ defines a perfect pairing

$$\langle \cdot, \cdot \rangle : A[\mathfrak{a}] \times A[\mathfrak{a}] \otimes_{\mathcal{O}_F} \mathfrak{c}\mathfrak{a} \longrightarrow \mathbb{G}_{\mathrm{m}}$$

satisfying $\langle x, ax \otimes \gamma \rangle = 1$ for any $x \in A[\mathfrak{a}](T)$ with any scheme T over S, $a \in \mathcal{O}_F$ and $\gamma \in \mathfrak{c}\mathfrak{a}$. Let $\mathcal{H}$ be a finite locally free closed subgroup scheme of A over S which is stable under the $\mathcal{O}_F$-action such that $\mathcal{H}$ is isomorphic, etale locally on S, to the constant group scheme $\underline{\mathcal{O}_F/\mathfrak{a}}$. Then $\mathcal{H}$ is isotropic with respect to this pairing. Therefore, if we have $\mathrm{Im}(\psi) \cap \mathcal{H} = 0$ in addition, then we can define on $A/\mathcal{H}$ a natural structure of a HBAV $(A/\mathcal{H}, \bar{\iota}, \bar{\lambda}, \bar{\psi})$ with $\mathfrak{c}\mathfrak{a}$-polarization $\bar{\lambda}$ [22, §1.9].

Let L/K be an extension of complete valuation fields, $\bar{L}$ an algebraic closure of L and $(A, \iota, \lambda, \psi)$ a HBAV over $\mathcal{O}_L$. In this case, we say that a finite flat closed $\mathcal{O}_F$-subgroup scheme $\mathcal{H}$ of A over $\mathcal{O}_L$ is $\mathfrak{a}$-cyclic if the $\mathcal{O}_F$-module $\mathcal{H}(\mathcal{O}_{\bar{L}})$ is isomorphic to $\mathcal{O}_F/\mathfrak{a}$. Since the generic fiber of $\mathcal{H}$ is isotropic and any finite flat closed subgroup scheme of A over $\mathcal{O}_L$ is uniquely determined by its generic fiber, we see that $\mathcal{H}$ is also isotropic. Thus, if $(\mathfrak{a}, N) = 1$, then we can define a structure of a HBAV on $A/\mathcal{H}$ similarly.

For any HBAV $(A, \iota, \lambda, \psi)$ over S, the group scheme $A[p^n]$ is decomposed as

$$A[p^n] = \bigoplus_{\mathfrak{p}|p} A[p^n]_{\mathfrak{p}} = \bigoplus_{\mathfrak{p}|p} A[\mathfrak{p}^n]$$

according with the decomposition

$$\mathcal{O}_F \otimes \mathbb{Z}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{F_\mathfrak{p}}.$$

If $S = \mathrm{Spec}(\mathcal{O}_L)$ with some extension L/K of complete valuation fields, then each $A[\mathfrak{p}^n]$ is a truncated Barsotti–Tate group of level n, height $2f_\mathfrak{p}$ and dimension $f_\mathfrak{p}$. Moreover, for any prime ideal $\mathfrak{p} \mid p$, the $\mathcal{O}_{F_\mathfrak{p}}$-module $\mathfrak{c}\otimes_{\mathcal{O}_F}\mathcal{O}_{F_\mathfrak{p}}$ and the $\mathcal{O}_F/p^n\mathcal{O}_F$-module $\mathfrak{c}/p^n\mathfrak{c}$ are free of rank one. This implies that, for any element x of $\mathfrak{c}$ which generates the $\mathcal{O}_F$-module $\mathfrak{c}/p^n\mathfrak{c}$, the element $p^n x$ and the above pairing define isomorphisms

$$i_n : A[p^n] \longrightarrow A[p^n]^\vee, \quad i_{n,\mathfrak{p}} : A[\mathfrak{p}^n] \longrightarrow A[\mathfrak{p}^n]^\vee$$

such that $i_{n,\mathfrak{p}}$ is $\mathcal{O}_{F_\mathfrak{p}}$-alternating. Hence each $A[\mathfrak{p}^n]$ is an $\mathcal{O}_{F_\mathfrak{p}}$-ADBT$_n$ over $\mathcal{O}_L$. Moreover, i_n is $\mathcal{O}_F$-alternating, namely $\langle x, i_n(ax)\rangle_{A[p^n]} = 1$ for any $x \in A[p^n](\mathcal{O}_{\bar{L}})$ and $a \in \mathcal{O}_F$. For any $\beta \in \mathbb{B}_\mathfrak{p}$, we put

$$\mathrm{Hdg}_\beta(A) = \mathrm{Hdg}_\beta(A[\mathfrak{p}]).$$

On the other hand, for any finite flat group scheme $\mathcal{H}$ over $\mathcal{O}_L$ with an $\mathcal{O}_F$-action, we have the decompositions

$$\mathcal{H} = \bigoplus_{\mathfrak{p}\mid p} \mathcal{H}_\mathfrak{p}, \quad \omega_\mathcal{H} = \bigoplus_{\beta\in\mathbb{B}_F} \omega_{\mathcal{H},\beta}$$

as above such that $\mathcal{H}_\mathfrak{p}$ is a finite flat closed subgroup scheme of $\mathcal{H}$ over $\mathcal{O}_L$ and $\omega_{\mathcal{H},\beta} = \omega_{\mathcal{H}_\mathfrak{p},\beta}$ for any $\beta \in \mathbb{B}_\mathfrak{p}$. Since the i-th Hodge–Tate map $\mathrm{HT}_i : \mathcal{H}(\mathcal{O}_{\bar{L}}) \to \omega_{\mathcal{H}^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{L},i}$ is $\mathcal{O}_F$-linear, it is also decomposed as the direct sum of the maps

$$\mathrm{HT}_i = \bigoplus_{\mathfrak{p}\mid p} \mathrm{HT}_{\mathcal{H}_\mathfrak{p},i}, \quad \mathrm{HT}_{\mathcal{H}_\mathfrak{p},i} : \mathcal{H}_\mathfrak{p}(\mathcal{O}_{\bar{L}}) \longrightarrow \omega_{\mathcal{H}_\mathfrak{p}^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{L},i}.$$

Now we can show the main theorem of this paper.

Theorem 8.1. *Let L/K be a finite extension in $\bar{K}$. Let $\mathfrak{c}$ be a non-zero fractional ideal of F. Let A be a HBAV over $\mathcal{O}_L$ with $\mathfrak{c}$-polarization. Put $w_\beta = \mathrm{Hdg}_\beta(A)$ and $w = \max\{w_\beta \mid \beta \in \mathbb{B}_F\}$. Suppose that*

$$w_\beta + p w_{\sigma^{-1}\circ\beta} < p^{2-n}$$

holds for all $\beta \in \mathbb{B}_F$.

Suppose that k is perfect. For any prime ideal $\mathfrak{p} \mid p$, let $\mathcal{C}_{n,\mathfrak{p}}$ be the canonical subgroup of the $\mathcal{O}_{F_\mathfrak{p}}$-ADBT$_n$ $A[\mathfrak{p}^n]$ of level n, which exists by Theorem 7.1. The finite flat closed subgroup scheme

$$\mathcal{C}_n(A) = \bigoplus_{\mathfrak{p}\mid p} \mathcal{C}_{n,\mathfrak{p}}$$

of $A[p^n]$ over $\mathcal{O}_L$ is stable under the $\mathcal{O}_F$-action. We call $\mathcal{C}_n(A)$ the canonical subgroup of A of level n. It satisfies

$$\deg_\beta(A[p^n]/\mathcal{C}_n(A)) = \sum_{l=0}^{n-1} p^l w_{\sigma^{-l}\circ\beta}$$

for any $\beta \in \mathbb{B}_F$ and the $\mathcal{O}_F/p^n\mathcal{O}_F$-module $\mathcal{C}_n(A)(\mathcal{O}_{\bar{K}})$ is free of rank one. The canonical subgroup $\mathcal{C}_n = \mathcal{C}_n(A)$ also satisfies the following.

(1) *Let A' be a HBAV over $\mathcal{O}_L$ satisfying the same condition on the β-Hodge heights as above. Then any isomorphism of HBAV's $j : A \to A'$ over $\mathcal{O}_L$ induces an isomorphism $\mathcal{C}_n(A) \simeq \mathcal{C}_n(A')$.*

(2) *$\mathcal{C}_n$ is compatible with finite base extension of complete discrete valuation rings.*

(3) *$\mathcal{C}_n$ is isotropic with respect to the $\mathcal{O}_F$-alternating isomorphism $i_n : A[p^n] \simeq A[p^n]^\vee$ defined by the $\mathfrak{c}$-polarization of A and any element $x \in \mathfrak{c}$ generating the $\mathcal{O}_F$-module $\mathfrak{c}/p^n\mathfrak{c}$.*

(4) *The kernel of the n-th iterated Frobenius map of $A[p^n] \times \mathscr{S}_{L,1-p^{n-1}w}$ coincides with $\mathcal{C}_n \times \mathscr{S}_{L,1-p^{n-1}w}$.*

(5) *The scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ in $\mathcal{C}_n$ is the canonical subgroup $\mathcal{C}_i$ of level i of $A[p^i]$ for any $1 \le i \le n-1$.*

Put $b = n - w(p^n-1)/(p-1)$. If $w < (p-1)/p^n$, then $\mathcal{C}_n = \mathcal{C}_n(A)$ also has the following properties:

(6) *$\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ coincides with $\mathrm{Ker}(\mathrm{HT}_{A[p^n],i})$ for any rational number i satisfying*

$$n - 1 + \frac{w}{p-1} < i \le b.$$

(7) *$\mathcal{C}_n = A[p^n]_i$ for any rational number i satisfying*

$$\frac{1}{p^n(p-1)} \le i \le \frac{1}{p^{n-1}(p-1)} - \frac{w}{p-1}.$$

(8) *For any $i \in v_p(\mathcal{O}_L)$ satisfying $i \le b$, the natural map*

$$\omega_A \otimes_{\mathcal{O}_L} \mathcal{O}_{L,i} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_L} \mathcal{O}_{L,i}$$

is an isomorphism.

(9) *The cokernel of the map*

$$\mathrm{HT}_{\mathcal{C}_n^\vee} \otimes 1 : \mathcal{C}_n^\vee(\mathcal{O}_{\bar{K}}) \otimes \mathcal{O}_{\bar{K}} \longrightarrow \omega_{\mathcal{C}_n} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{K}}$$

is killed by $m_{\bar{K}}^{\geqslant w/(p-1)}$.

(10) *For any prime ideal $\mathfrak{p} \mid p$ and any finite flat closed $\mathfrak{p}$-cyclic $\mathcal{O}_F$-subgroup scheme $\mathcal{H} \neq \mathcal{C}_{1,\mathfrak{p}}$ of $A[p]$ over $\mathcal{O}_L$, the HBAV $A/\mathcal{H}$ has the canonical subgroup $\mathcal{C}_n(A/\mathcal{H})$ of level n, which is equal to*

$$\left(\bigoplus_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} \mathcal{C}_{n,\mathfrak{q}} \right) \oplus (p^{-1}\mathcal{C}_{n-1,\mathfrak{p}}/\mathcal{H}).$$

Moreover, the natural map $A \to A/\mathcal{H}$ induces a map $\mathcal{C}_n(A) \to \mathcal{C}_n(A/\mathcal{H})$ which is an isomorphism over L.

For the case where k is imperfect, the same statements hold if $w < (p-1)/p^n$.

Proof. The assertion on $\deg_\beta$ follows from that of Theorem 7.1, since we have

$$\deg_\beta(A[p^n]/\mathcal{C}_n(A)) = \deg_\beta(A[p^n]_\mathfrak{p}/\mathcal{C}_{n,\mathfrak{p}})$$

for $\mathfrak{p} \mid p$ satisfying $\beta \in \mathbb{B}_\mathfrak{p}$. The assertion on the freeness follows from Theorem 7.1 (5). The assertions (1), (2) and (5) also follow from Theorem 7.1.

Let us show the assertion (3). Theorem 7.1 (3) implies that $(A[p^n]/\mathcal{C}_n)^\vee$ can be identified with the canonical subgroup of $A[p^n]^\vee$. By Theorem 7.1 (1), the isomorphism i_n induces an isomorphism $\mathcal{C}_n \simeq (A[p^n]/\mathcal{C}_n)^\vee$. This shows the assertion (3). Put

$$w_\mathfrak{p} = \max\{w_\beta \mid \beta \in \mathbb{B}_\mathfrak{p}\}.$$

Since we have $1 - p^{n-1}w \leq 1 - p^{n-1}w_\mathfrak{p}$, the assertion (4) follows from Theorem 7.1 (4).

Suppose $w < (p-1)/p^n$. Then we have

$$n - 1 + \frac{w_\mathfrak{p}}{p-1} \leq n - 1 + \frac{w}{p-1} < n - \frac{w(p^n-1)}{p-1} \leq n - \frac{w_\mathfrak{p}(p^n-1)}{p-1}$$

for any $\mathfrak{p} \mid p$. Since the map $\mathrm{HT}_{A[p^n],b}$ is the direct sum of the maps

$$\mathrm{HT}_{A[\mathfrak{p}^n],b} : A[\mathfrak{p}^n](\mathcal{O}_{\bar{K}}) \longrightarrow \omega_{A[\mathfrak{p}^n]^\vee} \otimes_{\mathcal{O}_L} \mathcal{O}_{\bar{K},b},$$

Theorem 7.1 (7) implies the assertion (6). Since the formation of lower ramification subgroups commutes with product, the assertion (7) follows from Theorem 7.1 (8). Similarly, the assertions (8) and (9) follow from Proposition 7.4. Since we have the decomposition

$$(A/\mathcal{H})[p^n] = \left(\bigoplus_{\mathfrak{q} \mid p, \mathfrak{q} \neq \mathfrak{p}} A[\mathfrak{q}^n]\right) \oplus p^{-n}\mathcal{H}/\mathcal{H},$$

Corollary 7.3 shows the assertion (10). The last assertion on the case where k is imperfect follows from Theorem 7.1 and Proposition 7.4. This concludes the proof. □

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