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Waring–Goldbach Problem with Piatetski-Shapiro Primes

par Yildirim AKBAL et Ahmet M. GÜLOĞLU

Résumé. Dans cet article nous donnons une formule asymptotique pour le nombre de représentations d'un grand entier comme somme de puissances identiques des nombres premiers de Piatetski-Shapiro, établissant donc une variante du problème de Waring–Goldbach pour des suites clairsemées de nombres premiers.

Abstract. In this paper, we exhibit an asymptotic formula for the number of representations of a large integer as a sum of a fixed power of Piatetski-Shapiro primes, thereby establishing a variant of Waring–Goldbach problem with primes from a sparse sequence.

1. Introduction

We define, for a natural number k, and a prime p, $\theta = \theta(p,k)$ to be the largest natural number such that $p^\theta \mid k$, and define $\gamma(p,k)$ by

$$\gamma = \gamma(p,k) = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } 2 \mid k, \\ \theta + 1 & \text{otherwise.} \end{cases}$$

We then put $K(k) = \prod_{(p-1)|k} p^\gamma$. In this work, we establish an asymptotic formula for the number of representations of a positive integer $\mathcal{N}$ in the form

$$\mathcal{N} = p_1^k + \cdots + p_s^k, \qquad \text{with } p_1, \ldots, p_s \in \mathcal{P}_c \tag{1.1}$$

for $k \geqslant 3$, provided that $\mathcal{N}$ is congruent to s modulo $K(k)$, and $c > 1$ takes values in a small interval depending on s and k. Here, the set of primes

$$\mathcal{P}_c = \{\lfloor m^c \rfloor : \lfloor m^c \rfloor \text{ is prime for some } m \in \mathbb{N}\}$$

is named after I.I. Piatetski-Shapiro, since he was the first to prove an analog of the Prime Number Theorem (cf. [12]) for $c \in (1, 12/11)$.

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Theorem 1.1. *Let $t > 0$ be any integer such that the inequality*

$$\int_0^1 \Bigg| \sum_{1 \leqslant n \leqslant X} e^{2\pi i \alpha n^k} \Bigg|^{2t} \mathrm{d}\alpha < C X^{2t-k} \log^\eta X \tag{1.2}$$

holds for all $X > 2$ with some constants $C = C(k, t)$ and $\eta = \eta(t, k) \geqslant 0$. Then, for any integer $s > 2t$, the number of representations $R^{(c)}_{s,k}(\mathcal{N})$ of a positive integer $\mathcal{N}$ as in (1.1) satisfies

$$R^{(c)}_{s,k}(\mathcal{N}) = \frac{\Gamma\left(1 + 1/(ck)\right)^s}{\Gamma\left(s/(ck)\right)} \mathfrak{S}(\mathcal{N}) \frac{\mathcal{N}^{s/(ck)-1}}{\log^s \mathcal{N}} + o\left(\frac{\mathcal{N}^{s/(ck)-1}}{\log^s \mathcal{N}}\right)$$

where $\mathfrak{S}(\mathcal{N})$ defined in (2.3) is the singular series in the classical Waring–Goldbach problem, provided that c is a fixed number satisfying

$$1 < c < 1 + (s - 2t) \begin{cases} 3 \min\left\{\frac{1}{77s+158t}, \frac{1}{75s+164t}\right\} & k = 3, \\ \dfrac{1}{(\nu - 1)s + 2t\nu} & k > 3, \end{cases} \tag{1.3}$$

where

$$\nu = \begin{cases} k(k+1)^2 & \text{if } 4 \leqslant k \leqslant 11, \\ \dfrac{2\lfloor 3k/2 \rfloor \left(\lfloor 3k/2 \rfloor^2 - 1\right)}{\lfloor 3k/2 \rfloor - k} & \text{if } k \geqslant 12. \end{cases} \tag{1.4}$$

By [7, Lemmas 8.10 and 8.12], when $\mathcal{N} \equiv s \pmod{K(k)}$, the singular series satisfies $\mathfrak{S}(\mathcal{N}) \asymp 1$ for the values of s given in Theorem 1.1. Thus, our theorem implies that all sufficiently large integers $\mathcal{N}$ congruent to s modulo $K(k)$ can be written as in (1.1), thereby establishing a variant of Waring–Goldbach problem with Piatetski-Shapiro primes for $k \geqslant 3$. For $k = 2$, it is shown in [15] that every sufficiently large integer $N \equiv 5 \pmod{24}$ can be written as in (1.1) with $s = 5$, provided that $1 < c < \frac{256}{249}$, while for $k = 1$, it follows from [9] that every sufficiently large *odd* integer can be written as in (1.1) with $s = 3$, provided that $1 < c < \frac{53}{50}$.

Following the proof of the main theorem of [15], the current range of c in Theorem 1.1 for the case $k = 3$ can be improved. We shall leave this to a subsequent paper.

In analogy to Waring–Goldbach Problem, one can define $H_c(k)$ to be the least integer s such that every sufficiently large integer congruent to s modulo $K(k)$ can be expressed as in (1.1). Following the proof of Theorem 1 and using the methods in Hua's book [7, §9], one may conclude that, for large k, $H_c(k)$ is bounded above by $4k \log k(1 + o(1))$, when c lies in a slightly larger range than that of Theorem 1.1. However, coupling our results with the recent improvements of Wooley and Kumchev [10, 11] on Waring–Goldbach problem, we intend to futher improve this bound in an another paper.

The range of c in Theorem 1.1 is determined by three different estimates for exponential sums; van der Corput's estimate in Lemma 2.4 for $k = 3$, Heath Brown's new estimate in Lemma 2.3 for $3 < k < 12$, and finally our estimate in Lemma 2.9 for $k \geqslant 12$.

Remark 1.2. Using Wooley's result [14, Theorem 4.1] in the light of recent developments on Vinogradov's Mean Value Theorem by Bourgain, Demeter and Guth in [4, Theorem 1.1], it follows that the smallest exponent satisfying (1.2) is

k	3	4	5	6	7	8	9	10	11	12
2t	8	16	24	34	48	62	78	98	118	142

while for $k > 12$, it follows from [3, Theorem 11] that $2t$ can be chosen as the smallest even integer no smaller than

$$k^2 + 1 - \max_{s \leqslant k} \left\lceil s \frac{k - s - 1}{k - s + 1} \right\rceil ,$$

and for large k, $2t$ can be taken as large as $k^2 - k + O(\sqrt{k})$.

2. Preliminaries and Notation

2.1. Notation. Throughout the paper, k, m and n are natural numbers with $k \geqslant 3$, and p always denotes a prime number. We write $n \sim N$ to mean that $N < n \leqslant 2N$. Furthermore, $c > 1$ is a fixed real number and we put $\delta = 1/c$.

Given a real number x, we write $e(x) = e^{2\pi i x}$, $\{x\}$ for the fractional part of x, $\lfloor x \rfloor$ for the greatest integer not exceeding x. We write $\mathcal{L} = \log \mathcal{N}^{1/k}$.

For any function f, we put

$$\Delta f(x) = f\left(-(x+1)^{\delta}\right) - f(-x^{\delta}), \qquad (x > 0).$$

We recall that for functions F and real nonnegative G the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. If $F \geqslant 0$ also, then $F \gg G$ is equivalent to $G \ll F$. We also write $F \asymp G$ to indicate that $F \ll G$ and $G \ll F$. In what follows, any implied constants in the symbols $\ll$ and O may depend on the parameters c, ε, k, s, t, but are absolute otherwise. We shall frequently use ε with a slight abuse of notation to mean a small positive number, possibly a different one each time.

Finally we put

$$S_{c,k}(\alpha, X) = \sum_{\substack{p \leqslant X \\ p \in \mathcal{P}_c}} e(\alpha p^k), \qquad T_{c,k}(\alpha, X) = \sum_{p \leqslant X} \delta p^{\delta - 1} e(\alpha p^k).$$

2.2. Preliminaries.

2.2.1. ***Results related to Piatetski-Shapiro sequences.*** The characteristic function of the set $\mathcal{A}_c = \{\lfloor m^c \rfloor : m \in \mathbb{N}\}$ is given by

$$\left\lfloor -n^{\delta} \right\rfloor - \left\lfloor -(n+1)^{\delta} \right\rfloor = \begin{cases} 1 & \text{if } n \in \mathcal{A}_c, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Putting $\psi(x) = x - \lfloor x \rfloor - 1/2$ we obtain

$$\begin{aligned} \left\lfloor -n^{\delta} \right\rfloor - \left\lfloor -(n+1)^{\delta} \right\rfloor &= (n+1)^{\delta} - n^{\delta} + \Delta\psi(n) \\ &= \delta n^{\delta-1} + O(n^{\delta-2}) + \Delta\psi(n). \end{aligned} \tag{2.2}$$

The following result due to Vaaler gives an approximation to $\psi(x)$.

Lemma 2.1 ([5, Appendix])**.** *There exists a trigonometric polynomial*

$$\psi^*(x) = \sum_{1 \leqslant |h| \leqslant H} a_h e(hx), \qquad (a_h \ll |h|^{-1})$$

such that for any real x,

$$|\psi(x) - \psi^*(x)| \leqslant \sum_{|h| < H} b_h e(hx), \qquad (b_h \ll H^{-1}).$$

2.2.2. ***Definitions related to Waring–Goldbach Problem.*** Put

$$\begin{aligned} S(a,q) &= \sum_{\substack{1 \leqslant x \leqslant q \\ (x,q)=1}} e(ax^k/q), \\ S_m(q) &= \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \left(\varphi(q)^{-1} S(a,q)\right)^s e(-ma/q), \qquad (s \in \mathbb{N}, m \in \mathbb{Z}) \\ \mathfrak{S}(m) &= \sum_{q \geqslant 1} S_m(q), \\ \mathcal{J}(z) &= \int_2^{N^{1/k}} \frac{\delta x^{\delta-1} e(zx^k)}{\log x} \, \mathrm{d}x, \quad \mathcal{I}(z) = \int_0^{N^{1/k}} \delta x^{\delta-1} e(zx^k) \, \mathrm{d}x, \\ v(z) &= \varphi(q)^{-1} S(a,q) \mathcal{J}(z). \end{aligned} \tag{2.3}$$

By [7, Lemma 8.5] the estimate

$$S(a,q) \ll q^{1/2+\varepsilon} \tag{2.4}$$

holds for $\gcd(a,q) = 1$. By the substitution $y = zx^k$ and the trivial estimate, it easily follows that

$$\mathcal{I}(z) \ll \min\left(N^{\delta/k}, |z|^{-\delta/k}\right). \tag{2.5}$$

Definition 2.2 (Major and Minor Arcs)**.** For fixed $\kappa > 0$, define

$$\mathfrak{M}_\kappa(a,q) = \{\alpha \in \mathbb{R} : |q\alpha - a| \leqslant \mathcal{L}^\kappa X^{-k}\}.$$

Let $\mathfrak{M}_\kappa$ be the union of all $\mathfrak{M}_\kappa(a,q)$ where a, q are coprime integers such that $1 \leqslant a \leqslant q \leqslant \mathcal{L}^\kappa$. Note that the sets $\mathfrak{M}_\kappa(a,q)$ are pairwise disjoint and are contained in the unit interval $\mathcal{U}_\kappa = (\mathcal{L}^\kappa X^{-k}, 1 + \mathcal{L}^\kappa X^{-k}]$. Put $\mathfrak{m}_\kappa = \mathcal{U}_\kappa \setminus \mathfrak{M}_\kappa$.

2.3. Standard Lemmas.

Lemma 2.3 ([6, Theorem 1])**.** *Let $k \geqslant 3$ be an integer, and suppose that $f : [0,N] \to \mathbb{R}$ has continuous derivatives of order up to k on $(0,N)$. Suppose further that $0 < \lambda_k \leqslant f^{(k)}(x) \leqslant A\lambda_k$ for $x \in (0,N)$. Then,*

$$\sum_{n \leqslant N} e(f(n))$$
$$\ll_{A,k,\varepsilon} N^{1+\varepsilon}\left(\lambda_k{}^{1/k(k-1)} + N^{-1/k(k-1)} + N^{-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}\right).$$

Lemma 2.4 ([5, Theorem 2.8])**.** *Let q be a positive integer. Suppose that f is a real valued function with $q+2$ continuous derivatives on some interval I. Suppose also that for some $\lambda > 0$ and for some $\alpha > 1$,*

$$\lambda \leqslant |f^{(q+2)}(x)| \leqslant \alpha\lambda$$

on I. Let $Q = 2^q$. Then,

$$\sum_{n \in I} e(f(n)) \ll$$
$$|I|(\alpha^2\lambda)^{1/(4Q-2)} + |I|^{1-1/(2Q)}\alpha^{1/(2Q)} + |I|^{1-2/Q+1/Q^2}\lambda^{-1/(2Q)}$$

where the implied constant is absolute.

Lemma 2.5. *Assume I_1 is a subinterval of an interval I with $|I_1| > 1$, and $g(x)$ is defined on I. Then,*

$$\sum_{n \in I_1} e\left(g(n)\right) \ll \log(1 + |I|) \sup_{\gamma \in [0,1]} \left| \sum_{n \in I} e\left(g(n) + \gamma n\right) \right|.$$

Proof. The result follows upon taking the supremum over all $\gamma \in [0,1]$ in

$$\sum_{n \in I_1} e\left(g(n)\right) = \int_0^1 \sum_{n \in I} e\left(g(n) + \gamma n\right) \sum_{m \in I_1} e(-\gamma m)\, \mathrm{d}\gamma,$$

and using the fundamental inequality

$$\int_0^1 \left| \sum_{m \in I_1} e(-\gamma m) \right| \mathrm{d}\gamma \ll \int_0^1 \min\left\{|I_1|, \frac{1}{\|\gamma\|}\right\} \mathrm{d}\gamma \ll \log(1 + |I|)$$

where $\|\gamma\| = \min_{n \in \mathbb{Z}} |n - \gamma|$. □

Lemma 2.6 ([3, Theorem 5])**.** *Let $k \geqslant 3$ be an integer, and let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that there exists a natural number j with $2 \leqslant j \leqslant k$ such that, for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, one has $|\alpha_j - a/q| \leqslant q^{-2}$. Then,*

$$\sum_{1 \leqslant n \leqslant N} e\left(\alpha_1 n + \cdots + \alpha_k n^k\right) \ll N^{1+\varepsilon}\left(q^{-1} + N^{-1} + qN^{-j}\right)^{1/k(k-1)}.$$

The following result can be deduced from [8, Proposition 13.4].

Lemma 2.7 (Vaughan's Identity)**.** *Let $u, v \geqslant 1$ be real numbers. If $n > v$ then,*

$$\Lambda(n) = \sum_{\substack{ab=n \\ a \leqslant u}} \mu(a) \log b - \sum_{\substack{ab=n \\ a>v, b>u}} \Lambda(a) \sum_{\substack{d \mid b \\ d \leqslant u}} \mu(d) - \sum_{\substack{abc=n \\ b \leqslant u, a \leqslant v}} \mu(b)\Lambda(a).$$

Lemma 2.8. *For any nonzero $\beta \in \mathbb{R}$,*

$$\mathcal{J}(\beta) - \mathcal{L}^{-1}\mathcal{I}(\beta) \ll \frac{\mathcal{N}^{\delta/k}}{\mathcal{L}^{\kappa+2}} + \min\{\mathcal{N}^{\delta/k}, |\beta|^{-\delta/k}\} \frac{\log \log \mathcal{N}}{\mathcal{L}^2}.$$

Proof. Using trivial estimate

$$\int_2^J \delta x^{\delta-1} e(\beta x^k) \left((\log x)^{-1} - \mathcal{L}^{-1}\right) \mathrm{d}x \ll \frac{J^{\delta/k}}{\log J},$$

for any $2 < J < \mathcal{N}^{1/k}$. By partial integration

$$\int_J^{N^{1/k}} \delta x^{\delta-1} e(\beta x^k) \left((\log x)^{-1} - \mathcal{L}^{-1}\right) \mathrm{d}x \ll \frac{\log\left(\mathcal{N}^{1/k}/J\right)}{\mathcal{L} \log J} \sup_{J < t \leqslant N^{1/k}} |\Phi(\beta, t)|$$

where

$$\Phi(\beta, t) = \int_2^t \delta x^{\delta-1} e(\beta x^k) \, \mathrm{d}x \ll \min\{t^\delta, |\beta|^{-\delta/k}\}$$

uniformly for $2 < t \leqslant N^{1/k}$. Choosing $J = \mathcal{N}^{1/k}(\log \mathcal{N})^{-c(\kappa+1)}$ and combining the above estimates completes the proof. □

2.4. Exponential sum estimates. This part constitutes the backbone of the entire paper and is to be used in the proof of Theorem 1.1.

Lemma 2.9. *Assume $k \geqslant 3$, $D > 0$, and $g(x) \in \mathbb{R}[x]$ is a polynomial of degree not exceeding k. Let $\ell \geqslant k+1$ be an integer. Then, the estimate*

$$\sum_{n \in I} e\left(g(n) + Dn^\delta\right) \ll N^{1+\varepsilon}\left((DN^{\delta-k-1})^\sigma + (DN^\delta)^{\frac{\sigma}{\ell+1}} N^{-\sigma} + (DN^\delta)^{-\frac{\ell-k}{\ell+1}\sigma}\right)$$

holds with $\sigma^{-1} = \ell(\ell-1)$ or with $\sigma^{-1} = 2^k$ whenever $\ell = k+1$, for any subinterval I of $(N, 2N]$, where the implied constant depends only on ε, k and ℓ.

Proof. We shall first bound the sum

$$\sum_{n\sim N} e(g(n) + Dn^{\delta})$$

for an arbitrary $g(x) \in \mathbb{R}[x]$ with $\deg g \leqslant k$, and the result will follow by Lemma 2.5.

We can assume that $2^{\ell+1} < DN^{\delta} < N^{k+1}$, since otherwise the claimed estimate holds trivially. Put $f(x) = g(x) + Dx^{\delta}$. For $m \in \mathbb{Z}$ with $1 \leqslant m \leqslant M < N/2$,

$$\sum_{n\sim N} e(f(n)) = \sum_{n\sim N} e\left(f(n+m)\right) + O(m).$$

Thus, summing over $m \in [1, M]$,

$$\sum_{n\sim N} e(f(n)) \ll \frac{1}{M} \sum_{n\sim N} \left| \sum_{1\leqslant m\leqslant M} e\left(f(n+m)\right) \right| + M.$$

Let $R_j(x) = (1+x)^{\delta} - F_j(x)$, where $F_j(x) = \sum_{0\leqslant i\leqslant j} \binom{\delta}{i} x^i$ is the jth Taylor polynomial of $(1+x)^{\delta}$. Then, taking $x = m/n$,

$$\begin{aligned} f(n+m) &= g(n+m) + Dn^{\delta}\left(F_{\ell}(m/n) + R_{\ell}(m/n)\right) \\ &= P_{\ell}(m) + Dn^{\delta} R_{\ell}(m/n) \end{aligned}$$

where $P_{\ell}(x) = \sum_{i=0}^{\ell} c_i x^i \in \mathbb{R}[x]$, $c_{k+1} = CDn^{\delta-k-1}$, and $0 < |C| < 1$. Noting that $R'_{\ell}(x) \ll |x|^{\ell}$ uniformly for $|x| \leqslant M/N < 1/2$, we derive by partial integration and Lemma 2.5 that

$$\begin{aligned} &\sum_{m\leqslant M} e\left(f(n+m)\right) \\ &\qquad \ll \left(1 + DN^{\delta}(M/N)^{\ell+1}\right) \sup_{\gamma\in[0,1)} \left| \sum_{1\leqslant m\leqslant M} e\left(P_{\ell}(m) + \gamma m\right) \right| \log M. \end{aligned}$$

Note that $|c_{k+1} \pm 1/q| \leqslant q^{-2}$, where $q = \lfloor |c_{k+1}|^{-1} \rfloor \geqslant 1$ since $|c_{k+1}| < 1$. Then, taking $\ell = k+1$ and applying Weyl's inequality (cf. [13, Lemma 2.4]) yields for *any* $\gamma \in \mathbb{R}$ that

$$\sum_{1\leqslant m\leqslant M} e\left(P_{k+1}(m) + \gamma m\right) \ll M^{1+\varepsilon}\left(q^{-1} + M^{-1} + qM^{-k-1}\right)^{2^{-k}},$$

while it follows from Lemma 2.6 that for arbitrary $\ell \geqslant k+1$,

$$\sum_{1\leqslant m\leqslant M} e\left(P_{\ell}(m) + \gamma m\right) \ll M^{1+\varepsilon}\left(q^{-1} + M^{-1} + qM^{-k-1}\right)^{1/\ell(\ell-1)}.$$

In either case, we choose $M = N(DN^\delta)^{-\frac{1}{\ell+1}}$ so that we have $1 < M < N/2$, and thus we obtain

$$\sum_{n\sim N} e(f(n)) \ll N^{1+\varepsilon}\left(q^{-1} + M^{-1} + qM^{-k-1}\right)^\sigma + M.$$

Using the definitions of M and q, and the fact that $\sigma < (\ell-k)^{-1}$, we see that the contribution of $N^{1+\varepsilon}(qM^{-k-1})^\sigma$ is already larger than M, thus M can be eliminated, and the result follows. □

Lemma 2.10. *Uniformly for any complex numbers a_n, b_m with $|a_n|, |b_m| \leqslant 1$, and $g(t) \in \mathbb{R}[t]$ of degree not exceeding k,*

$$S(x,y) = \sum_{\substack{m\sim x\\ n\sim y\\ mn\sim N}} a_n b_m e(hn^\delta m^\delta + g(mn)) \ll N^{1+\varepsilon}\min\{S_1, S_2, S_3\},$$

where

$$\begin{aligned} S_1 &= (|h|N^{\delta-1}x^{-\ell})^{\frac{\sigma/2}{\sigma+\ell+1}} + (|h|N^{\delta-1}x^{-k})^{\frac{\sigma/2}{1+\sigma}} \\ &\qquad + (|h|N^\delta)^{-\frac{\ell-k}{\ell+1}\sigma/2} + (x/N)^{1/2} + x^{-\frac{\ell-k}{\ell-k+1}\sigma/2}, \\ S_2 &= (x/N)^{1/2} + (|h|N^\delta)^{-1/k(k+1)^2} + x^{-1/2k(k+1)} \\ &\qquad + (|h|N^{\delta-1}x^{-k})^{1/2(k^2+k+1)}, \\ S_3 &= x^{-2^{-k-1}} + (x/N)^{1/2} + x^{2^{1-2k}-2^{1-k}}(|h|N^\delta x^{-1-k})^{-2^{-k-1}} \\ &\qquad + (|h|N^{\delta-1}x^{-k})^{\frac{1}{2^{k+2}-2}}. \end{aligned} \tag{2.6}$$

Proof. We may assume that $|h|N^{\delta-1}x^{-k} < 1$; otherwise, the assertion is trivial. Applying Weyl-van der Corput inequality (cf. [5, Lemma 2.5]) we see that

$$S^2(x,y) \ll \frac{(xy)^2}{Q} + \frac{xy^2}{Q}\sum_{1\leqslant|q|\leqslant Q}\max_{y<n,n+q\leqslant 2y}|\Gamma(q,n,x)|$$

where $1 \leqslant Q \leqslant y$ is to be chosen optimally, and

$$\Gamma(q,n,x) = \sum_{m\in I} e\left(h((n+q)^\delta - n^\delta)m^\delta + g\left((n+q)m\right) - g(nm)\right), \tag{2.7}$$

where $I \subseteq (x, 2x]$ is an interval determined by the conditions $m \sim x$, $nm \sim N$, and $(n+q)m \sim N$.

If we apply Lemma 2.9 with $D = |h\left((n+q)^\delta - n^\delta\right)|$, noting that $D \asymp |hq|y^{\delta-1}$, we obtain

$$\begin{aligned} \Gamma(q,n,x) \ll x^{1+\varepsilon}(&(|hq|N^{\delta-1}x^{-k})^\sigma \\ &+ (|hq|N^{\delta-1}x^{-\ell})^{\frac{\sigma}{\ell+1}} + (|hq|N^{\delta-1}x)^{-\frac{\ell-k}{\ell+1}\sigma}). \end{aligned}$$

Inserting this estimate above and summing over q yields

$$S^2(x,y)(xy)^{-2-\varepsilon} \ll Q^{-1} + (Q|h|N^{\delta-1}x^{-k})^{\sigma} + (Q|h|N^{\delta-1}x^{-\ell})^{\frac{\sigma}{\ell+1}} + (Q|h|N^{\delta-1}x)^{-\frac{\ell-k}{\ell+1}\sigma}.$$

Using [5, Lemma 2.4] to choose $1 \leqslant Q \leqslant y$ optimally, we conclude that

$$\begin{aligned} S^2(x,y)(xy)^{-2-\varepsilon} \ll{}& (|h|N^{\delta-1}x^{-\ell})^{\frac{\sigma}{\sigma+\ell+1}} + (|h|N^{\delta-1}x^{-\ell})^{\frac{\sigma}{\ell+1}} + xN^{-1} \\ &+ (|h|N^{\delta-1}x^{-k})^{\sigma} + (|h|N^{\delta-1}x^{-k})^{\frac{\sigma}{1+\sigma}} \\ &+ (|h|N^{\delta})^{-\frac{\ell-k}{\ell+1}\sigma} + x^{-\frac{\ell-k}{\ell-k+1}\sigma} + (x^{-k-1})^{\frac{\ell-k}{2\ell+1-k}\sigma}. \end{aligned}$$

Since $|h|N^{\delta-1}x^{-k} < 1$, we can eliminate the second and the fourth terms, and the last term is smaller than the penultimate one.

If we instead apply Lemma 2.3 (with $k+1$ in place of k) to (2.7), we obtain

$$\Gamma(q,n,x) \ll x^{1+\varepsilon}\Big((|hq|N^{\delta-1}x^{-k})^{1/k(k+1)} + x^{-1/k(k+1)} + (|hq|N^{\delta-1}x)^{-2/k(k+1)^2}\Big),$$

which yields

$$S^2(x,y)(xy)^{-2-\varepsilon} \ll Q^{-1} + (Q|h|N^{\delta-1}x^{-k})^{1/k(k+1)} + x^{-1/k(k+1)} + (Q|h|N^{\delta-1}x)^{-2/k(k+1)^2}.$$

Using [5, Lemma 2.4] once again, we conclude that

$$\begin{aligned} S^2(x,y)(xy)^{-2-\varepsilon} \ll{}& x/N + (|h|N^{\delta-1}x^{-k})^{1/k(k+1)} + x^{-1/k(k+1)} \\ &+ (|h|N^{\delta})^{-2/k(k+1)^2} + x^{-2/k(k+3)} + (|h|N^{\delta-1}x^{-k})^{1/(k^2+k+1)}. \end{aligned}$$

Since $|h|N^{\delta-1}x^{-k} < 1$, we eliminate the second term.

Finally, if we apply van der Corput's result, Lemma 2.4, to (2.7) and carry on similar calculations as above, we derive the desired estimate. □

Lemma 2.11. *For any $\varepsilon > 0$, and $c \in (1,2)$,*

$$S_{c,3}(\alpha,X) = T_{c,3}(\alpha,X) + O\Big(X^{\varepsilon}\max\Big\{X^{\frac{76\delta+77}{156}}, X^{\frac{79\delta+75}{157}}\Big\}\Big)$$

$$S_{c,k}(\alpha,X) = T_{c,k}(\alpha,X) + O\left(X^{(1+\delta)\frac{\nu-1}{2\nu-1}+\varepsilon}\right), \quad k \geqslant 4$$

holds uniformly for $\alpha \in \mathbb{R}$, where ν is given by (1.4).

Proof. By (2.1), (2.2) and Merten's Theorem (see [8, (2.15)])

$$S_{c,k}(\alpha,X) = T_{c,k}(\alpha,X) + \sum_{p\leqslant X} e(\alpha p^k)\Delta\psi(p) + O(\log\log X). \tag{2.8}$$

In order to bound the middle term on the right, we divide the range of summation $[2, X]$ into dyadic intervals of the form $(N, 2N]$. Applying Lemma 2.1 on each such interval, we see that

$$\sum_{p\sim N} e(\alpha p^k)\Delta(\psi-\psi^*)(p) \ll H_N^{-1} \sum_{|h|<H_N} \Big|\sum_{n\sim N} e(hn^\delta)\Big|.$$

Using the exponent pair $(1/2, 1/2)$ (cf. [5, Chapter 3]) we obtain the estimate

$$\sum_{n\sim N} e(hn^\delta) \ll |h|^{1/2}N^{\delta/2} + |h|^{-1}N^{1-\delta} \qquad (h\neq 0)$$

so that

$$\sum_{p\sim N} e(\alpha p^k)\Delta(\psi-\psi^*)(p) \ll NH_N^{-1} + B^{1/2} + H_N^{-1}\log H_N N^{1-\delta}, \tag{2.9}$$

where $B = H_N N^\delta$.

Next, we turn to the sum involving ψ^*. First using partial summation and then introducing von Mangoldt function we obtain

$$\sum_{p\sim N} e(\alpha p^k)\Delta\psi^*(p) \ll \frac{1}{\log N}\max_{N'\leqslant 2N}\Big|\sum_{N<n\leqslant N'} \Delta\psi^*(n)e(\alpha n^k)\Lambda(n)\Big| + \sqrt{N}.$$

Recalling the definition of ψ^* it is not too hard (see [5, 4.6]) to derive that

$$\begin{aligned} &\sum_{p\sim N} e(\alpha p^k)\Delta\psi^*(p) \ll \Theta(N) + \sqrt{N}, \\ &\Theta(N) = \frac{N^{\delta-1}}{\log N}\sum_{1\leqslant|h|\leqslant H_N}\max_{N'\leqslant 2N}\Big|\sum_{N<n\leqslant N'} e(\alpha n^k + hn^\delta)\Lambda(n)\Big|. \end{aligned} \tag{2.10}$$

Assume that $u, v\geqslant 1$ are real numbers with $uv\leqslant N$. Using Lemma 2.7 we write the inner sum on the right as $E_1 - E_2 - E_3$ where

$$\begin{aligned} E_1 = &\sum_{n\leqslant u}\mu(n)\sum_{N/n<m\leqslant N'/n} e\left(\alpha(nm)^k + h(nm)^\delta\right)\log m \\ &- \sum_{m\leqslant u}\Bigg(\sum_{\substack{ab=m\\ b\leqslant u, a\leqslant v}}\mu(b)\Lambda(a)\Bigg)\sum_{N/m<n\leqslant N'/m} e\left(\alpha(nm)^k + h(nm)^\delta\right), \end{aligned}$$

$$E_2 = \sum_{\substack{N<nm\leqslant N'\\ n>v, m>u}} \Lambda(n)\Bigg(\sum_{\substack{d|m\\ d\leqslant u}}\mu(d)\Bigg)e\left(\alpha(nm)^k + h(nm)^\delta\right),$$

and

$$E_3 = \sum_{\substack{N<nm\leqslant N'\\ u<m\leqslant uv}} \Bigg(\sum_{\substack{ab=m\\ b\leqslant u, a\leqslant v}} \mu(b)\Lambda(a) \Bigg) e\left(\alpha(nm)^k + h(nm)^\delta\right).$$

Note that

$$E_1 \ll \log N \sum_{1\leqslant n\leqslant u} \max_{N<N'\leqslant 2N} \Bigg| \sum_{N/n<m\leqslant N'/n} e\left(\alpha(nm)^k + h(nm)^\delta\right) \Bigg|. \tag{2.11}$$

Thus, applying Lemma 2.9 with $D = |h|n^\delta$ to the inner sum above and summing over $n \leqslant u$ we obtain

$$E_1 \ll N^{1+\varepsilon}\Big((|h|N^\delta)^\sigma (u/N)^{(k+1)\sigma-\varepsilon} + (|h|N^\delta)^{\frac{\sigma}{\ell+1}} (u/N)^{\sigma-\varepsilon} + (|h|N^\delta)^{-\frac{\ell-k}{\ell+1}\sigma}\Big).$$

Hence, the contribution to (2.10) from E_1 is

$$\ll B^{1+\varepsilon}\Big(B^\sigma (u/N)^{(k+1)\sigma-\varepsilon} + B^{\frac{\sigma}{\ell+1}} (u/N)^{\sigma-\varepsilon} + B^{-\frac{\ell-k}{\ell+1}\sigma}\Big). \tag{2.12}$$

Next, we estimate the bilinear sums E_2 and E_3. We first note that $E_2 \ll N^\varepsilon \sum_{x,y} |S(x,y)|$, where

$$S(x,y) = \sum_{m\sim x} b_m \sum_{\substack{n\sim y\\ N<nm\leqslant N'}} a_n e\left(\alpha(nm)^k + h(nm)^\delta\right),$$

with $y > v, x > u$, $xy \asymp N$ and $|a_n|, |b_m| \leqslant 1$. Also,

$$E_3 \ll \log N \sum_{x,y} |S(x,y)|$$

with a similar bilinear sum $S(x,y)$, where $u < x \leqslant uv$, $xy \asymp N$ and $|a_n|, |b_m| \leqslant 1$. Applying the bound S_1 in Lemma 2.10 we obtain

$$E_2 + E_3 \ll N^{1+2\varepsilon}\Big(v^{-1/2} + (uv/N)^{1/2} + u^{-\frac{\ell-k}{\ell-k+1}\sigma/2} + (|h|N^\delta)^{-\frac{\ell-k}{\ell+1}\sigma/2} + (|h|N^{\delta-1}u^{-\ell})^{\frac{\sigma/2}{\sigma+\ell+1}} + (|h|N^{\delta-1}u^{-k})^{\frac{\sigma/2}{1+\sigma}}\Big).$$

Choosing $v = (N/u)^{1/2}$, we see that the contribution of $E_2 + E_3$ to (2.10) is

$$\ll B^{1+2\varepsilon}\Big((u/N)^{1/4} + u^{-\frac{\ell-k}{\ell-k+1}\sigma/2} + B^{-\frac{\ell-k}{\ell+1}\sigma/2} + (BN^{-1}u^{-\ell})^{\frac{\sigma/2}{\sigma+\ell+1}} + (BN^{-1}u^{-k})^{\frac{\sigma/2}{1+\sigma}}\Big). \tag{2.13}$$

Combining (2.9), (2.12) and (2.13) we conclude that

$$\sum_{p\sim N} e(\alpha p^k)\Delta\psi(p) \ll NH_N^{-1} + B^{1-\frac{\ell-k}{\ell+1}\sigma/2+\varepsilon} + B^{1/2} + B^{1+\varepsilon}\Big(B^{\sigma}(u/N)^{(k+1)\sigma-\varepsilon} + (u/N)^{1/4} + B^{\frac{\sigma}{\ell+1}}(u/N)^{\sigma-\varepsilon} + u^{-\frac{\ell-k}{\ell-k+1}\sigma/2} + (BN^{-1}u^{-\ell})^{\frac{\sigma/2}{\sigma+\ell+1}} + (BN^{-1}u^{-k})^{\frac{\sigma/2}{1+\sigma}}\Big).$$

Note that the second term dominates the third. Since the first two terms are independent of u, we choose

$$H_N = N^{1-(1+\delta)\frac{1-A(\ell)}{2-A(\ell)}}, \qquad A(\ell) = \frac{(\ell-k)\sigma}{2(\ell+1)}$$

so as to balance them first. Note that with this choice, we have $1 < H_N < N$, and

$$NH_N^{-1} = N^{(1+\delta)\frac{1-A(\ell)}{2-A(\ell)}}, \qquad B = H_N N^{\delta} = N^{\frac{1+\delta}{2-A(\ell)}}.$$

In order to minimize NH_N^{-1}, we set

$$A = A_k = \max_{\ell\geqslant k+1} A(\ell).$$

It follows by an easy computation that $\ell = \lfloor 3k/2\rfloor$ maximizes $A(\ell)$, and with this choice of A, we find by setting $u = N^{1/2}$ that all the remaining terms are smaller than NH_N^{-1}, and thus we conclude that

$$\sum_{p\sim N} e(\alpha p^k)\Delta\psi(p) \ll N^{(1+\delta)\frac{1-A}{2-A}+\varepsilon}. \tag{2.14}$$

Next, we estimate the inner sum in (2.11) using Lemma 2.3. This gives

$$E_1 \ll N^{1+\varepsilon}\Big((hN^{\delta})^{1/k(k+1)}(u/N)^{1/k} + (u/N)^{1/k(k+1)} + (hN^{\delta})^{-2/k(k+1)^2}\Big),$$

whose contribution to (2.10) is

$$\ll B^{1+\varepsilon}\Big(B^{1/k(k+1)}(u/N)^{1/k} + (u/N)^{1/k(k+1)} + B^{-2/k(k+1)^2}\Big). \tag{2.15}$$

Using the bound S_2 in (2.6), we see that the contribution of E_2+E_3 to (2.10) is

$$\ll B^{1+2\varepsilon}\Big((u/N)^{1/4} + B^{-1/k(k+1)^2} + u^{-1/2k(k+1)} + (BN^{-1}u^{-k})^{1/2(k^2+k+1)}\Big). \tag{2.16}$$

Combining (2.9), (2.15) and (2.16) shows that (2.10) is bounded by

$$NH_N^{-1} + B^{1-1/k(k+1)^2} + B^{1+\varepsilon}\Big(B^{1/k(k+1)}(u/N)^{1/k} + (u/N)^{1/k(k+1)} + u^{-1/2k(k+1)} + (BN^{-1}u^{-k})^{1/2(k^2+k+1)}\Big).$$

Choosing H_N to balance the first two terms again gives

$$NH_N^{-1} = N^{(1+\delta)\frac{1-C}{2-C}}, \qquad C = \frac{1}{k(k+1)^2}.$$

As before, for $u = N^{1/2}$, all the remaining terms are dominated by NH_N^{-1}. Thus,

$$\sum_{p\sim N} e(\alpha p^k)\Delta\psi(p) \ll N^{(1+\delta)\frac{1-C}{2-C}+\varepsilon}. \tag{2.17}$$

One can easily check that for $3 \leqslant k \leqslant 11$, using Heath-Brown's result (Lemma 2.3) gives a better estimate since $C > A$. For $k \geqslant 12$, however, $A > C$, which explains our choice in (1.4).

Finally, (only) for $k = 3$, one can do slightly better than Heath-Brown's estimate by using van der Corput's estimate; namely, by Lemma 2.4 with $q = 2$, it follows that

$$E_1 \ll N \log N \left(\left(hN^\delta (u/N)^4\right)^{\frac{1}{14}} + (u/N)^{\frac{1}{8}} + (hN^\delta)^{-\frac{1}{8}} N^{\frac{1}{16}}\right)$$

whose contribution to (2.10) is

$$\ll B \log N\Big((B(u/N)^4)^{\frac{1}{14}} + (u/N)^{\frac{1}{8}} + B^{-\frac{1}{8}} N^{\frac{1}{16}}\Big). \tag{2.18}$$

On the other hand, using S_3 in (2.10), we obtain for $k = 3$,

$$E_2 + E_3 \ll N^{1+\varepsilon}\Big(u^{-\frac{1}{16}} + B^{\frac{1}{30}} u^{-\frac{1}{10}} + v^{-\frac{1}{2}} + B^{-\frac{1}{16}} (N/v)^{\frac{1}{32}} + (uv/N)^{\frac{1}{2}} + B^{-\frac{1}{16}} (uv)^{\frac{1}{32}}\Big).$$

Choosing $v = \sqrt{N/u}$ and summing over h, the contribution from $E_2 + E_3$ is

$$\ll B^{1+\varepsilon}\Big(u^{-\frac{1}{16}} + B^{\frac{1}{30}} N^{-\frac{1}{30}} u^{-\frac{1}{10}} + (u/N)^{\frac{1}{4}} + B^{-\frac{1}{16}} (uN)^{\frac{1}{64}}\Big). \tag{2.19}$$

Combining (2.9), (2.18) and (2.19), the total contribution is

$$\ll NH_N^{-1} + B^{1-\frac{1}{8}} N^{\frac{1}{16}} + B^{1+\varepsilon}\Big(u^{-\frac{1}{16}} + B^{\frac{1}{30}} N^{-\frac{1}{30}} u^{-\frac{1}{10}} + B^{-\frac{1}{16}} (uN)^{\frac{1}{64}} + B^{\frac{1}{14}} (u/N)^{\frac{4}{14}} + (u/N)^{\frac{1}{8}}\Big).$$

Choosing u optimally above by using [5, Lemma 2.4] with $1 \leqslant u \leqslant N$, we have that (2.10) is bounded by

$$\ll NH_N^{-1} + B^{1+\varepsilon}\Big(N^{-\frac{1}{16}} + B^{\frac{1}{30}} N^{-\frac{2}{15}} + B^{-\frac{1}{16}} N^{\frac{1}{64}} + B^{\frac{1}{14}} N^{-\frac{4}{14}} + B^{-\frac{1}{20}} N^{\frac{1}{80}} + B^{\frac{1}{78}} N^{-\frac{4}{78}} + N^{-\frac{1}{24}} + B^{-\frac{11}{222}} N^{\frac{1}{111}} + B^{\frac{7}{162}} N^{-\frac{8}{81}} + B^{\frac{1}{54}} N^{-\frac{2}{27}}\Big),$$

Here, only the first term has H_N with a negative exponent. Balancing terms with an appropriate $1 \leqslant H_N \leqslant N$ using [5, Lemma 2.4] again, (2.10) is bounded by

$$\ll N^{\varepsilon}\Big(1+N^{\frac{14\delta+1}{16}}+N^{\frac{31\delta-4}{30}}+N^{\frac{60\delta+1}{64}}+N^{\frac{15\delta-4}{14}}+N^{\frac{76\delta+1}{80}}+N^{\frac{79\delta-4}{78}}+N^{\frac{24\delta-1}{24}} +N^{\frac{211\delta+2}{222}}+N^{\frac{169\delta-16}{162}}+N^{\frac{55\delta-4}{54}}+N^{\frac{7\delta}{15}+\frac{1}{2}}+N^{\frac{31\delta+27}{61}}+N^{\frac{60\delta+61}{124}}+N^{\frac{15\delta+11}{29}} +N^{\frac{76\delta+77}{156}}+N^{\frac{79\delta+75}{157}}+N^{\frac{\delta}{2}+\frac{23}{48}}+N^{\frac{211\delta+213}{433}}+N^{\frac{169\delta+153}{331}}+N^{\frac{55\delta+51}{109}}\Big).$$

Comparing all the terms under the assumption that $\delta \in (1/2, 1)$, we end up with

$$\sum_{p\sim N} e(\alpha p^k)\Delta\psi(p) \ll N^{\varepsilon}\max\left\{N^{\frac{76\delta+77}{156}}, N^{\frac{79\delta+75}{157}}\right\}. \tag{2.20}$$

The result follows by inserting (2.14), (2.17) and (2.20) back to (2.8). □

Lemma 2.12. *If* $1 < c < 12/11$*, then for any* $\alpha \in \mathfrak{M}_\kappa(a,q)$ *with* $\gcd(a,q) = 1$*,* $1 \leqslant a \leqslant q \leqslant \mathcal{L}^\kappa$ *and sufficiently large* X*, we have*

$$S_{c,k}(\alpha, X) - v(\alpha - a/q) \ll X^{\delta}\exp(-C\sqrt{\log X}),$$

where $C > 0$ *is an absolute constant and the implied constant depends only on* κ *and* k*.*

Proof. Combining (2.8) with equations (2.9) and (2.10), in which we take $H_N = N^{1-\delta+\varepsilon}$, we see that

$$S_{c,k}(\alpha, X) - T_{c,k}(\alpha, X) \ll \sum_{N=2^l\leqslant X}\Big(N^{\delta-\varepsilon} + \Theta(N)\Big).$$

The inner sum in the definition of $\Theta(N)$ can be written as

$$\sum_{\substack{1\leqslant m\leqslant q\\ (m,q)=1}} e\left(\frac{am^k}{q}\right)\sum_{\substack{N<n\leqslant N'\\ n\equiv m \mod q}} e(\beta n^k + hn^{\delta})\Lambda(n) + O(q\log N).$$

Removing $\mathbf{e}(\beta n^k)$ by partial summation this double sum is bounded by

$$\sum_{\substack{1\leqslant m\leqslant q\\ (m,q)=1}}\left(1 + N^k\mathcal{L}^\kappa X^{-k}q^{-1}\right)\max_{N<N'\leqslant 2N}\Bigg|\sum_{\substack{N<n\leqslant N'\\ n\equiv m \mod q}} e(hn^{\delta})\Lambda(n)\Bigg|.$$

Applying the estimate given as the first equation on page 323 of [1], which is uniform both in m and q, we derive that

$$\begin{aligned}\Theta(N) &\ll N^{\delta-1}\mathcal{L}^{\kappa-1}\sum_{\substack{1\leqslant m\leqslant q\\ (m,q)=1}} q^{-1}\sum_{1\leqslant|h|\leqslant H_N}\max_{N'\leqslant 2N}\Bigg|\sum_{\substack{N<n\leqslant N'\\ n\equiv m \mod q}} e(hn^{\delta})\Lambda(n)\Bigg|\\ &\ll N^{\delta}\exp(-c_1\sqrt{\log N})\end{aligned}$$

for an absolute constant $c_1 > 0$, and any fixed $1 < c < 12/11$.

Next, we deal with $T_{c,k}(\alpha, X)$. Writing $\beta = \alpha - a/q$

$$T_{c,k}(\alpha, X) = \sum_{\substack{1 \leqslant b \leqslant q \\ (b,q)=1}} e(ab^k/q) \sum_{\substack{p \leqslant X \\ p \equiv b \mod q}} \delta p^{\delta-1} e(\beta p^k) + O(\omega(q))$$

where $\omega(n)$ is the number of distinct prime divisors of n. It follows from Siegel–Walfisz theorem that

$$\sum_{\substack{p \leqslant x \\ p \equiv b \mod q}} 1 = \frac{1}{\varphi(q)} \int_2^x \frac{\mathrm{d}t}{\log t} + E(x),$$

uniformly for $q \leqslant \mathcal{L}^\kappa$, where $E(x) \ll x \exp(-c_2\sqrt{\log x})$ for an absolute constant $c_2 = c_2(\kappa) > 0$ and large x. By partial integration we derive that

$$\sum_{\substack{p \leqslant X \\ p \equiv b \mod q}} \delta p^{\delta-1} e(\beta p^k) = \int_{2^-}^X \frac{\delta x^{\delta-1} e(\beta x^k)}{\varphi(q) \log x} \, \mathrm{d}x + O\left(X^{\delta-1} E(X) + \mathcal{L}^\kappa \int_{2^-}^X |E(x)| x^{\delta-2} \, \mathrm{d}x \right).$$

Using $E(x) \ll x$ when x is small (say $x \leqslant \sqrt{X}$) and the above bound for large x in the last integral and inserting the result above we obtain

$$T_{c,k}(\alpha, X) = v(\alpha - a/q) + O_\kappa \left(X^\delta \exp(-c_3\sqrt{\log X}) \right),$$

for sufficiently large X and some positive absolute constant $c_3 < c_2$. Combining all the estimates above, the result follows. □

3. Proof of Theorem 1.1

Recall that, for a fixed $k \geqslant 3$ and $c > 1$, $R_{s,k}^{(c)}(\mathcal{N})$ is the number of representations of a positive integer $\mathcal{N}$ as in (1.1). It can be rewritten as

$$R_{s,k}^{(c)}(\mathcal{N}) = \int_{\mathcal{U}} S_{c,k}(\alpha, X)^s e(-\alpha \mathcal{N}) \, \mathrm{d}\alpha,$$

where $\mathcal{U}$ is any interval of unit length and $X = \left\lfloor \mathcal{N}^{1/k} \right\rfloor$.

Lemma 3.1 (Major Arcs)**.** *Assume that* $s \geqslant \max(5, k+1)$, *and that* $1 < c < \min\{\frac{12}{11}, \frac{s}{k}\}$. *Then, uniformly for integers* m *with* $1 \leqslant m \leqslant \mathcal{N}$, *and* $\kappa > \frac{4s}{2s-9}$,

$$\int_{\mathfrak{M}_\kappa} S_{c,k}(\alpha, X)^s e(-\alpha m) \, \mathrm{d}\alpha = \mathfrak{S}(m) m^{\delta s/k - 1} \frac{\Gamma(1+\delta/k)^s}{\Gamma(s\delta/k)\mathcal{L}^s} + o\left(\frac{\mathcal{N}^{\delta s/k-1}}{\mathcal{L}^s} \right).$$

Proof. By Lemma 2.12 below,

$$S_{c,k}(\alpha,X)-v(\alpha-a/q)\ll X^{\delta}E(X),$$

where $E(X)=\exp(-c_2\sqrt{\log X})$, uniformly for $\alpha\in\mathfrak{M}_\kappa(a,q)$ with $(a,q)=1$ and $1\leqslant a\leqslant q\leqslant\mathcal{L}^\kappa$. Put $\beta=\alpha-a/q$. Then,

$$S_{c,k}(\alpha,X)^s-v(\beta)^s\ll(X^{\delta}E(X))^s+X^{\delta}E(X)|v(\beta)|^{s-1}.$$

Therefore,

$$\sum_{q\leqslant\mathcal{L}^\kappa}\sum_{\substack{a\leqslant q\\(a,q)=1}}\int_{\mathfrak{M}_\kappa(a,q)}(S_{c,k}(\alpha,X)^s-v(\alpha-a/q)^s)e(-\alpha m)\,\mathrm{d}\alpha=o\left(X^{\delta s-k}\mathcal{L}^{-s}\right).$$

Furthermore,

$$\sum_{q\leqslant\mathcal{L}^\kappa}\sum_{\substack{a\leqslant q\\(a,q)=1}}\int_{\mathfrak{M}_\kappa(a,q)}v(\beta)^se(-\alpha m)\,\mathrm{d}\alpha$$
$$=\sum_{q\leqslant\mathcal{L}^\kappa}S_m(q)\int_{|q\beta|\leqslant\frac{\mathcal{L}^\kappa}{X^k}}\mathcal{J}(\beta)^se(-\beta m)\,\mathrm{d}\beta.$$

Using Lemma 2.8 together with (2.5) and the bound $S_m(q)\ll q^{1-s/2+\varepsilon}$ (which follows from (2.4)), we see that replacing the integral $\mathcal{J}(\beta)$ above by $\mathcal{L}^{-1}\mathcal{I}(\beta)$ introduces an error of size $o(X^{\delta s-k}/\mathcal{L}^s)$. We can then extend the integral over β to $\mathbb{R}$ with another permissible error. By [2, Lemma 8],

$$\int_{\mathbb{R}}\mathcal{I}(\beta)^se(-\beta m)\,\mathrm{d}\beta=m^{\delta s/k-1}\frac{\Gamma(1+\delta/k)^s}{\Gamma(s\delta/k)}.$$

Finally, using

$$\sum_{q\leqslant\mathcal{L}^\kappa}S_m(q)=\mathfrak{S}(m)+O\left(\mathcal{L}^{\kappa(2-s/2+\varepsilon)}\right)$$

completes the proof. □

Lemma 3.2 (Minor Arcs)**.** *Assume that $\lambda>0$, and t is an integer for which (1.2) holds. Then,*

$$\int_{\mathfrak{m}_\kappa}|S_{c,k}(\alpha,X)|^s\,\mathrm{d}\alpha\ll X^{\delta s-k}\mathcal{L}^{2t-1-\lambda\delta(s-2t)+\eta}+X^{(s-2t)\theta+2t-k+\varepsilon},$$

provided that $\kappa\geqslant 2^{6k}(2+\lambda)$, where θ is the exponent of X in the error term in Lemma 2.11.

Proof. Using equation (2.8) we obtain

$$\int_{\mathfrak{m}_\kappa}|S_{c,k}(\alpha,X)|^s\,\mathrm{d}\alpha\ll I_1+I_2+O\left((\log\log X)^s\right)$$

where

$$I_1 = \int_{\mathfrak{m}_\kappa} |T_{c,k}(\alpha, X)|^s \, \mathrm{d}\alpha, \qquad I_2 = \int_{\mathfrak{m}_\kappa} \Big| \sum_{p \leqslant X} e(\alpha p^k) \Delta\psi(p) \Big|^s.$$

We first bound I_1. Let $2 < J \leqslant X$ be a constant to be determined. By partial integration

$$T_{c,k}(\alpha, X) \ll J^\delta + J^{\delta-1} \sup_{J < t \leqslant X} \Big| \sum_{J < p \leqslant t} e(\alpha p^k) \Big|.$$

Take $\alpha \in \mathfrak{m}_\kappa$. By Dirichlet's approximation theorem, one can find integers a, q with $1 \leqslant a \leqslant q \leqslant \mathcal{L}^{-\kappa} X^k$ such that $|\alpha - a/q| \leqslant q^{-1} \mathcal{L}^\kappa X^{-k}$. Since $\alpha \in \mathfrak{m}_\kappa$, we have $q > \mathcal{L}^\kappa$. Writing $\alpha = a/q + \beta$, and using partial integration we obtain

$$\sum_{J < p \leqslant t} e(\alpha p^k) \ll \sup_{J < y \leqslant t} \Big| \sum_{J < p \leqslant y} e(a p^k / q) \Big| \left(1 + |\beta| t^k\right).$$

Following the proof of Lemma 2.5 and recalling that $y \leqslant t \leqslant X$,

$$\sum_{J < p \leqslant y} e(a p^k / q) \ll \log X \sup_{\gamma \in [0,1)} \Big| \sum_{p \leqslant X} e(a p^k / q + \gamma p) \Big|,$$

so that

$$T_{c,k}(\alpha, X) \ll J^\delta + J^{\delta-1} \sup_{\gamma \in [0,1)} \Big| \sum_{p \leqslant X} e(a p^k / q + \gamma p) \Big| \log X.$$

By [7, Theorem 10] it follows for arbitrary $\lambda > 0$ and any $\gamma \in \mathbb{R}$ that whenever $\kappa \geqslant 2^{6k}(2 + \lambda)$,

$$\sum_{p \leqslant X} e(a p^k / q + \gamma p) \ll X \mathcal{L}^{-\lambda - 1}.$$

Choosing $J = X \mathcal{L}^{-\lambda}$ we conclude that

$$T_{c,k}(\alpha, X) \ll X^\delta \mathcal{L}^{-\lambda\delta}.$$

Using this bound together with Hölder's inequality yields

$$\begin{aligned} I_1 &\leqslant \sup_{\alpha \in \mathfrak{m}_\kappa} |T_{c,k}(\alpha, X)|^{s-2t} \int_{\mathfrak{m}_\kappa} \Big| \sum_{2^l = N \leqslant X} \sum_{p \sim N} \delta p^{\delta - 1} e(\alpha p^k) \Big|^{2t} \mathrm{d}\alpha \\ &\ll (X^\delta \mathcal{L}^{-\lambda\delta})^{(s-2t)} \mathcal{L}^{2t-1} \sum_{N \leqslant X} \int_0^1 \Big| \sum_{p \sim N} \delta p^{\delta-1} e(\alpha p^k) \Big|^{2t} \mathrm{d}\alpha. \end{aligned}$$

By considering the underlying Diophantine equations we see that the last integral is

$$\ll N^{2t(\delta-1)} \int_0^1 \Big| \sum_{n \leqslant N} e(\alpha n^k) \Big|^{2t} \mathrm{d}\alpha.$$

Using (1.2) we conclude that for some $\eta \geqslant 0$,

$$I_1 \ll X^{\delta s-k}\mathcal{L}^{2t-1-\lambda\delta(s-2t)+\eta}. \tag{3.1}$$

Next, we deal with I_2. Note that

$$I_2 \ll \sup_{\alpha\in\mathfrak{m}_\kappa}\left|\sum_{p\leqslant X} e(\alpha p^k)\Delta\psi(p)\right|^{s-2t}\int_0^1\left|\sum_{n\leqslant X} e(\alpha n^k)\right|^{2t}\mathrm{d}\alpha.$$

Using (2.8) and then applying Lemma 2.11 together with (1.2) we obtain

$$I_2 \ll X^{(s-2t)\theta+2t-k+\varepsilon}. \tag{3.2}$$

Combining (3.1) and (3.2), the proof is completed. □

The proof of Theorem 1.1 can now be completed by taking $m=\mathcal{N}$ in Lemma 3.1 and observing that taking λ (and thus κ) sufficiently large in Lemma 3.2 ensures that the contribution from minor arcs is $o(X^{\delta s-k}\mathcal{L}^{-s})$ under the additional assumption in (1.3).

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