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$M(x) = o(x)$ Estimates for Beurling numbers

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Résumé. Dans la théorie des nombres premiers classique, certaines expressions sont considérées comme « équivalentes » au TNP (Théorème des nombres premiers). Parmi elles on trouve la borne $M(x) = o(x)$ pour la fonction sommatoire de Moebius. Dans le cas des nombres premiers, généralisés par Beurling, cette borne ne suit pas nécessairement du TNP sans exiger des hypothèses additionelles. Ici, deux conditions sont présentées, impliquant la version Beurling pour la borne sur $M(x)$ et quelques exemples sont construits, démontrant l'absence éventuelle de cette borne si ces conditions ne sont pas réalisées.

Abstract. In classical prime number theory there are several asymptotic formulas said to be "equivalent" to the PNT. One is the bound $M(x) = o(x)$ for the sum function of the Moebius function. For Beurling generalized numbers, this estimate is not an unconditional consequence of the PNT. Here we give two conditions that yield the Beurling version of the $M(x)$ bound, and examples illustrating failures when these conditions are not satisfied.

1. Introduction

Let $\mu(n)$ denote the Moebius arithmetic function and $M(x)$ its sum function. Von Mangoldt first established the estimate $M(x) = o(x)$, essentially going through the steps used in proving the Prime Number Theorem (PNT). A few years later, Landau showed by relatively simple real variable arguments that this and several other estimates followed from the PNT [13, §150]. Similarly, these relations imply each other and the PNT; thus they are said to be "equivalent" to the PNT.

In this article we consider an analog of the M bound for Beurling generalized (g-)numbers. A Beurling g-prime system is an unbounded sequence of real numbers $\mathcal{P} = \{p_i\}$ satisfying $1 < p_1 \leq p_2 \leq \dots$, and the multiplicative semigroup generated by $\mathcal{P}$ and 1 is called the associated collection $\mathcal{N}$ of g-integers (cf. [1], [9], [15]). The counting function $N(x)$ of $\mathcal{N}$ is the number

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of elements of $\mathcal{N}$ not exceeding x. The g-prime and g-prime-power counting functions of $\mathcal{N}$ and the g-Chebyshev function of $\mathcal{N}$ are (respectively)

$$\pi(x) = \sum_{p_i \le x} 1, \quad \Pi(x) = \sum_{p_i^{\alpha_i} \le x} 1/\alpha_i, \quad \psi(x) = \sum_{p_i^{\alpha_i} \le x} \log p_i.$$

(Here and below we give our Beurling functions the same names used in classical number theory.)

The PNT-related assertions are somewhat different for g-numbers: not all implications between the several corresponding assertions hold unconditionally, see e.g. [7], [9, Chap. 14], [16]. Here we study the Beurling version of the assertion $M(x) = o(x)$ and show this can be deduced

(a) from the PNT under an O-boundedness condition $N(x) = O(x)$, or
(b) from a Chebyshev-type upper bound assuming $N(x) \sim ax$ (for $a > 0$) and the integral condition (3.2) (below).

At the end, we give examples in which M estimates fail.

A word about the Beurling version of $M(x)$. The characteristic property of the Moebius function is its being the (multiplicative) convolution inverse of the 1 function. For g-numbers, we define the measure $\mathrm{d}M$ as the convolution inverse of $\mathrm{d}N$; by familiar Mellin transform properties (cf. [3], [9], [15]),

$$\int_{1-}^{\infty} u^{-s} \mathrm{d}M(u) = 1/\zeta(s) \quad \text{with} \quad \zeta(s) = \zeta_{\mathcal{N}}(s) = \int_{1-}^{\infty} u^{-s} \mathrm{d}N(u),$$

the Beurling number version of the Riemann zeta function.

We remark that these definitions make sense even when $\mathrm{d}N$ is not discrete or when factorization into primes is not unique; what is essential for us is just that $\Pi(x)$ (or equivalently $\psi(x)$) be a non-decreasing function on $(1, \infty)$.

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2. PNT hypothesis

In this section we show that the PNT together with the “O-density” condition $N(x) \ll x$ implies $M(x) = o(x)$.

Theorem 2.1. *Let $\mathcal{N}$ be a g-number system for which the PNT holds and $N(x) \ll x$. Then $M(x) = o(x)$.*

Other sufficient conditions are known to insure that a g-number system will satisfy an M estimate, e.g. [9, Prop. 14.10] if the PNT holds and the integer counting function satisfies the logarithmic density condition

$$\int_{1^-}^{x} \frac{\mathrm{d}N(t)}{t} \sim a \log x, \tag{2.1}$$

then $M(x) = o(x)$. The present result differs from the other in that log-density and O-density are conditions that do not imply one another; also, the proofs are very different.

The key to our argument is the following relation.

Lemma 2.2. *Under the hypotheses of the theorem,*

$$\frac{M(x)}{x} = \frac{-1}{\log x}\int_1^x \frac{M(t)}{t^2}\,\mathrm{d}t + o(1). \tag{2.2}$$

Proof. A variant of Chebyshev's identity for primes reads

$$L\mathrm{d}M = -\mathrm{d}M * \mathrm{d}\psi,$$

where L is the operator of multiplication by $\log t$ and $*$ is multiplicative convolution. This can be verified (in the classical or in the Beurling case) via a Mellin transform. Note that this transform carries convolutions into pointwise products and the L operator into $(-1)\times$ differentiation. The equivalent Mellin formula is the identity

$$\left\{\frac{1}{\zeta(s)}\right\}' = -\frac{1}{\zeta(s)}\cdot\frac{\zeta'(s)}{\zeta(s)}.$$

Now add and subtract a term $(\delta_1 + \mathrm{d}t) * \mathrm{d}M$ in the variant form of the Chebyshev relation, with δ_1 the point mass at 1 and $\mathrm{d}t$ the Lebesgue measure on $(1, \infty)$. Integrating, we find

$$\int_{1-}^x L\,\mathrm{d}M = \int_{1-}^x \mathrm{d}M * (\delta_1 + \mathrm{d}t - \mathrm{d}\psi) - \int_{1-}^x \mathrm{d}M * (\delta_1 + \mathrm{d}t). \tag{2.3}$$

Integrating by parts the left side of the last formula, we get

$$M(x)\log x - \int_1^x \frac{M(t)}{t}\,\mathrm{d}t = M(x)\log x + O(x),$$

since $|\mathrm{d}M| \le \mathrm{d}N$, and hence $|M(t)| \le N(t) \ll t$. If we evaluate the convolution integrals by the iterated integral formula

$$\int_{1-}^x \mathrm{d}A * \mathrm{d}B = \iint_{st\le x} \mathrm{d}A(s)\,\mathrm{d}B(t) = \int_{1-}^x A(x/t)\,\mathrm{d}B(t)$$

and use the PNT, we find the first term on the right side of (2.3) to be

$$\int_{1-}^x \{x/t - \psi(x/t)\}\,\mathrm{d}M(t) = \int_{1-}^x o(x/t)\,\mathrm{d}N(t) = o(x\log x).$$

Also, the last term of (2.3) is, upon integrating by parts,

$$\int_{1-}^x \frac{x}{t}\,\mathrm{d}M(t) = M(x) + x\int_1^x \frac{M(t)}{t^2}\,\mathrm{d}t.$$

Finally, we combine the bounds for the terms of (2.3), divide through by $x\log x$, and note that $M(x) \ll x$ to obtain (2.2). $\square$

Proof of Theorem 2.1. First, we can assume that $M(x)$ has an infinite number of sign changes. Otherwise, there is some number z such that $M(x)$ is of one sign for all $x \geq z$. By (2.2), as $x \to \infty$,

$$\frac{M(x)}{x} = \frac{-1}{\log x} \int_1^z \frac{O(t)}{t^2}\,\mathrm{d}t - \frac{1}{\log x} \int_z^x \frac{M(t)}{t^2}\,\mathrm{d}t + o(1).$$

Thus we have

$$\frac{M(x)}{x} + \frac{1}{\log x} \int_z^x \frac{M(t)}{t^2}\,\mathrm{d}t = \frac{O(\log z)}{\log x} + o(1) = o(1).$$

Since $M(x)$ and the integral are of the same sign, $M(x)/x \to 0$ as $x \to \infty$, and this case is done.

Now suppose that $M(x)$, which we regard as a right-continuous function, changes sign at x. We show that $M(x)/x = o(1)$. If $M(x)/x = 0$, there is nothing more to say here. If, on the other hand, $M(x) > 0$, then there is a number $y \in (x-1,\ x)$ with $M(y) \leq 0$. If we apply (2.2) again, we find

$$\begin{aligned}\frac{M(x)}{x} &= \frac{-\log y}{\log x}\,\frac{1}{\log y} \int_1^y \frac{M(t)}{t^2}\,\mathrm{d}t - \frac{1}{\log x} \int_y^x \frac{M(t)}{t^2}\,\mathrm{d}t + o(1)\\ &= \frac{\log y}{\log x}\,\frac{M(y)}{y} + \frac{1}{\log x} \int_y^x \frac{O(t)\,\mathrm{d}t}{t^2} + o(1).\end{aligned}$$

Thus $M(x)/x - M(y)/y = o(1)$, and since $M(x)/x > 0 \geq M(y)/y$, each is $o(1)$. A similar story holds if $M(x) < 0$.

Finally, suppose that $M(t)$ changes sign at $t = y$ (so that $M(y)/y = o(1)$) and $M(t)$ is of one sign for $y < t \leq z$. By yet another application of (2.2), we find for any $x \in (y,\ z]$

$$\frac{M(x)}{x} = \frac{-\log y}{\log x}\,\frac{1}{\log y} \int_1^y \frac{M(t)}{t^2}\,\mathrm{d}t - \frac{1}{\log x} \int_y^x \frac{M(t)}{t^2}\,\mathrm{d}t + o(1)$$

or

$$\frac{M(x)}{x} + \frac{1}{\log x} \int_y^x \frac{M(t)}{t^2}\,\mathrm{d}t = \frac{\log y}{\log x}\,\frac{M(y)}{y} + o(1) = o(1)$$

as $y \to \infty$. Since $M(x)/x$ and the integral are of the same sign, it follows that $M(x)/x = o(1)$. $\square$

3. Chebyshev hypothesis

What happens if the PNT hypothesis of the last theorem is weakened to just a Chebyshev upper bound? In the examples of Section 4 we show that even two-sided Chebyshev estimates by themselves are not strong enough to ensure that $M(x) = o(x)$ holds. Furthermore, this bound could fail even if, in addition to Chebyshev estimates, one also assumes that N satisfies both (2.1) and $N(x) \ll x$, see Example 4.2. We shall show, however, that if the regularity hypothesis on N is slightly augmented, then one can indeed deduce the desired M bound.

Theorem 3.1. *Suppose that a g-number system satisfies the following conditions:*

(a) *a Chebyshev upper bound, that is,*

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} < \infty, \tag{3.1}$$

(b) *for some positive constant a, $N(x) \sim ax$,*
(c) *for some $\beta \in (0, 1/2)$ and all $\sigma \in (1, 2)$*

$$\int_1^\infty \frac{|N(x) - ax|}{x^{\sigma+1}} \, \mathrm{d}x \ll (\sigma - 1)^{-\beta} . \tag{3.2}$$

Then, $M(x) = o(x)$ holds.

As a simple consequence we have an improvement of a result of W.-B. Zhang [16, Cor. 2.5]:

Corollary 3.2. *If a g-number system satisfies* (3.1) *and*

$$N(x) - ax \ll x \log^{-\gamma} x$$

for some $\gamma > 1/2$, then $M(x) = o(x)$ holds.

W.-B. Zhang had conjectured[1] that a Chebyshev bound along with the L^1 bound

$$\int_1^\infty \frac{|N(x) - ax|}{x^2} \, \mathrm{d}x < \infty \tag{3.3}$$

imply $M(x) = o(x)$. Naturally, (3.3) is included in (3.2) and $N(x) \sim ax$.

It is easy to show that $M(x) = o(x)$ is always implied by

$$m(x) = \int_{1^-}^x \frac{\mathrm{d}M(u)}{u} = o(1), \tag{3.4}$$

but the converse implication is not true in general [9]. On the other hand, it has recently been shown [7, Thm. 2] that the assertions become equivalent under the additional hypothesis (3.3). So, Theorem 3.1 and the quoted result yield at once:

Corollary 3.3. *Under* (3.1) *and* (3.3), *relation* (3.4) *holds as well.*

We can also strengthen another result of W.-B. Zhang [16, Thm. 2.3].

Corollary 3.4. *The condition* (3.3) *and*

$$\int_1^x \frac{(N(t) - at) \log t}{t} \, \mathrm{d}t \ll x \tag{3.5}$$

imply (3.4).

---

[1] Oral communication to the authors.

Proof. Chebyshev bounds are known to hold under the hypotheses (3.3) and (3.5) [9, Thm. 11.1]. The rest follows from Corollary 3.3. □

We shall prove Theorem 3.1 using several lemmas. Our method is inspired by W.-B. Zhang's proof of a Halász-type theorem for Beurling primes [16]. Our first step is to replace $M(x) = o(x)$ by an equivalent asymptotic relation.

Lemma 3.5. *Suppose that the integer counting function of a g-number system has a positive density, i.e., $N(x) \sim ax$ for some $a > 0$. Then, $M(x) = o(x)$ if and only if*

$$f(x) = \int_1^x \left(\int_1^u \log t \, \mathrm{d}M(t) \right) \frac{\mathrm{d}u}{u} = o(x \log x). \tag{3.6}$$

Proof. The direct implication is trivial. For the converse, we set

$$g(x) = \int_{1^-}^x \log t \, (\mathrm{d}N(t) + \mathrm{d}M(t)).$$

Notice that

$$\mathrm{d}N + \mathrm{d}M = 2 \sum_{n=0}^{\infty} \mathrm{d}\Pi^{*2n}/(2n)!$$

is a non-negative measure, so that g is non-decreasing. The hypotheses (3.6) and $N(x) \sim ax$ give

$$\int_1^x \frac{g(u)}{u} \, \mathrm{d}u \sim ax \log x.$$

Hence,

$$\int_1^x g(u) \, \mathrm{d}u = x \int_1^x \frac{g(u)}{u} \, \mathrm{d}u - \int_1^x \left(\int_1^t \frac{g(u)}{u} \, \mathrm{d}u \right) \mathrm{d}t \sim \frac{a}{2} x^2 \log x.$$

Since g is a non-decreasing function, we see by a simple differencing argument (see e.g. [12, p. 34]) that $g(x) \sim ax \log x$. But also we have $\int_{1^-}^x \log u \, \mathrm{d}N(u) \sim ax \log x$; consequently, $\int_{1^-}^x \log u \, \mathrm{d}M(u) = o(x \log x)$. Integration by parts now yields

$$\log x \, M(x) = \int_1^x \log u \, \mathrm{d}M(u) + \int_1^x \frac{M(u)}{u} \, \mathrm{d}u = o(x \log x) + O(x),$$

and the result then follows by dividing by $\log x$. □

The next lemma provides a crucial analytic estimate.

Lemma 3.6. *Suppose that $N(x) \sim ax$, with $a > 0$. Then,*

$$\frac{1}{\zeta(\sigma + it)} = o\left(\frac{1}{\sigma - 1} \right), \quad \sigma \to 1^+, \tag{3.7}$$

uniformly for t on compact intervals.

Proof. We first show that (3.7) holds pointwise, i.e., for each fixed $t \in \mathbb{R}$, without the uniformity requirement. If $t = 0$ this is clear because $\zeta(\sigma) \sim a/(\sigma - 1)$ and thus $1/\zeta(\sigma) = o(1)$. Note that $N(x) \sim ax$ implies

$$\zeta(s) = \frac{a}{s-1} + o\left(\frac{|s|}{\sigma - 1}\right)$$

uniformly. If $t \neq 0$, we obtain, $\zeta(\sigma + 2it) = o_t(1/(\sigma - 1))$. Applying the 3-4-1 inequality, we conclude that

$$1 \leq |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| = |\zeta(\sigma + it)|^4 \, o_t\left(\frac{1}{(\sigma - 1)^4}\right),$$

which shows our claim. Equivalently, we have

$$\exp\left(-\int_1^\infty x^{-\sigma}(1 + \cos(t \log x)) \,\mathrm{d}\Pi(x)\right) = \frac{1}{|\zeta(\sigma)\zeta(\sigma + it)|} = o_t(1).$$

The left-hand side of the last formula is a net of continuous monotone functions in the variable σ that tend pointwise to 0 as $\sigma \to 1^+$; Dini's theorem then asserts that it must also converge to 0 uniformly for t on compact sets, as required. □

The next lemma is very simple but useful.

Lemma 3.7. *Let F be a right continuous function of local bounded variation with support in $[1, \infty)$ satisfying the bound $F(x) = O(x)$. Set $\widehat{F}(s) = \int_{1^-}^\infty x^{-s} \,\mathrm{d}F(x)$, $\Re s > 1$. Then,*

$$\int_{\Re s = \sigma} \left|\frac{\widehat{F}(s)}{s}\right|^2 |\mathrm{d}s| \ll \frac{1}{\sigma - 1}.$$

Proof. Indeed, by the Plancherel theorem,

$$\begin{aligned} \int_{\Re s = \sigma} \left|\frac{\widehat{F}(s)}{s}\right|^2 |\mathrm{d}s| &= 2\pi \int_0^\infty e^{-2\sigma x} |F(e^x)|^2 \,\mathrm{d}x \\ &\ll \int_0^\infty e^{-2(\sigma - 1)x} \,\mathrm{d}x = \frac{1}{2(\sigma - 1)}. \end{aligned}$$ □

If a g-number system satisfies a Chebyshev upper bound, then Lemma 3.7 implies

$$\int_{\Re s = \sigma} \left|\frac{\zeta'(s)}{s\zeta(s)}\right|^2 |\mathrm{d}s| \ll \frac{1}{\sigma - 1}. \tag{3.8}$$

Similarly, $N(x) = O(x)$ yields

$$\int_{\Re s = \sigma} \left|\frac{\zeta(s)}{s}\right|^2 |\mathrm{d}s| \ll \frac{1}{\sigma - 1}. \tag{3.9}$$

Also, we shall need the following version of the Wiener–Wintner theorem [2, 14].

Lemma 3.8. *Let F_1 and F_2 be right continuous functions of local bounded variation with support in $[1,\infty)$. Suppose that their Mellin–Stieltjes transforms $\widehat{F}_j(s) = \int_{1^-}^{\infty} x^{-s}\mathrm{d}F_j(x)$ are convergent on $\Re s > \alpha$, that F_1 is non-decreasing, and $|\mathrm{d}F_2| \le \mathrm{d}F_1$. Then, for all $b \in \mathbb{R}$, $c > 0$, and $\sigma > \alpha$,*

$$\int_b^{b+c} |\widehat{F}_2(\sigma+it)|^2 \,\mathrm{d}t \le 2\int_{-c}^{c} |\widehat{F}_1(\sigma+it)|^2 \,\mathrm{d}t.$$

We are ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. In view of Lemma 3.5, it suffices to show (3.6). The Mellin–Stieltjes transform of the function f is

$$-\frac{1}{s}\left(\frac{1}{\zeta(s)}\right)'.$$

Given $x > 1$, it is convenient to set $\sigma_o = 1+1/\log x$. By the Perron inversion formula, the Cauchy–Schwarz inequality, and (3.8), we have

$$\begin{aligned}\frac{|f(x)|}{x} &= \frac{1}{2\pi}\left|\int_{\Re s=\sigma_o} \frac{x^{s-1}\zeta'(s)}{s^2\zeta^2(s)}\,\mathrm{d}s\right| \le \frac{e}{2\pi}\int_{\Re s=\sigma_o}\left|\frac{\zeta'(s)}{s^2\zeta^2(s)}\right| |\mathrm{d}s| \\ &\ll \log^{1/2} x \left(\int_{\Re s=\sigma_o}\left|\frac{1}{s\zeta(s)}\right|^2 |\mathrm{d}s|\right)^{1/2}.\end{aligned}$$

Next, we take a large number λ, fixed for the while. We split the integration line $\{\Re s = \sigma_o\}$ of the last integral into two parts, $\{\sigma_o + it : |t| \ge \lambda\}$ and $\{\sigma_o + it : |t| \le \lambda\}$, and we denote the corresponding integrals over these sets as $I_1(x)$ and $I_2(x)$ respectively, so that

$$\frac{|f(x)|}{x} \ll ((I_1(x))^{1/2} + (I_2(x))^{1/2}) \log^{1/2} x. \tag{3.10}$$

To estimate $I_1(x)$, we apply Lemma 3.8 to $|\mathrm{d}M| \le \mathrm{d}N$ and employ (3.9),

$$\left(\int_{-\lambda-m-1}^{-\lambda-m} + \int_{\lambda+m}^{\lambda+m+1}\right)\left|\frac{1}{\zeta(\sigma_o+it)}\right|^2 \mathrm{d}t \le 4\int_{-1}^{1} |\zeta(\sigma_o+it)|^2\,\mathrm{d}t \ll \log x.$$

Therefore, we have the bound

$$I_1(x) \ll \log x \sum_{m=0}^{\infty} \frac{1}{1+m^2+\lambda^2} \ll \frac{\log x}{\lambda}. \tag{3.11}$$

To deal with $I_2(x)$, we need to derive further properties of the zeta function. Using the hypothesis (3.2), we find that

$$\begin{aligned}\zeta(s) - \frac{a}{s-1} &= s\int_1^{\infty} x^{-s}\frac{N(x)-ax}{x}\,\mathrm{d}x + a \\ &\ll |s|\int_1^{\infty} x^{-\sigma}\frac{|N(x)-ax|}{x}\,\mathrm{d}x = O\left(\frac{1+|t|}{(\sigma-1)^{\beta}}\right).\end{aligned}$$

Hence, we obtain

$$\zeta(\sigma_o + it) = \frac{a}{\sigma_o - 1 + it} + O((1+|t|)\log^\beta x) \tag{3.12}$$

for some number $\beta \in (0,\, 1/2)$, uniformly in t. We are ready to estimate $I_2(x)$. Set $\eta = (1-2\beta)/(1-\beta)$ and note that $\eta \in (0,\, 1)$. Then, using Lemma 3.6,

$$I_2(x) \le \int_{-\lambda}^{\lambda} \left|\frac{1}{\zeta(\sigma_o+it)}\right|^2 \mathrm{d}t \le \left(\int_{-\lambda}^{\lambda} \left|\frac{1}{\zeta(\sigma_o+it)}\right|^{2-\eta} \mathrm{d}t\right) o_\lambda(\log^\eta x).$$

On the other hand, applying Lemma 3.8 to $\mathrm{d}F_1 = \exp^*(-(1-\eta/2)\,\mathrm{d}\Pi)$ and $\mathrm{d}F_2 = \exp^*((1-\eta/2)\,\mathrm{d}\Pi)$ and using (3.12), we find

$$\begin{aligned}\int_{-\lambda}^{\lambda} \left|\frac{1}{\zeta(\sigma_o+it)}\right|^{2-\eta} \mathrm{d}t &\le 4\int_{-\lambda}^{\lambda} |\zeta(\sigma_o+it)|^{2-\eta}\,\mathrm{d}t \\ &\ll \int_{-\lambda}^{\lambda} \frac{\mathrm{d}t}{((\sigma_o-1)^2+t^2)^{1-\eta/2}} + \lambda\,\lambda^{2-\eta}\log^{(2-\eta)\beta} x \\ &\ll \lambda^{3-\eta}\log^{1-\eta} x,\end{aligned}$$

which implies $I_2(x) = o_\lambda(\log x)$. Inserting this and the bound (3.11) into (3.10), we arrive at $|f(x)|/(x\log x) \ll \lambda^{-1/2} + o_\lambda(1)$. Taking first the limit superior as $x \to \infty$ and then $\lambda \to \infty$, we have shown that

$$\lim_{x\to\infty} \frac{f(x)}{x\log x} = 0.$$

By Lemma 3.5, $M(x) = o(x)$. This completes the proof of Theorem 3.1. $\square$

4. Three examples

The examples of this section center on the importance of $N(x)$ being close to ax in Theorem 3.1. In the first example, $N(x)/x$ has excessive wobble and in the second one, excessive growth; in both cases $M(x) = o(x)$ fails. The third example shows that condition (3.2) is not sufficient to insure the convergence of $N(x)/x$, whence the introduction of this hypothesis.

In preparation for treating the first two examples, we give a necessary condition for $M(x) = o(x)$. An analytic function $G(s)$ on the half-plane $\{s : \Re s > \alpha\}$ is said to have a right-hand zero of order $\beta > 0$ at $s = it_0 + \alpha$ if $\lim_{\sigma\to\alpha^+}(\sigma-\alpha)^{-\beta}G(it_0+\sigma)$ exists and is non-zero. Our examples violate the following necessary condition:

Lemma 4.1. *If $M(x) = o(x)$, then $\zeta(s)$ does not have any right-hand zero of order ≥ 1 on $\{\Re s = 1\}$.*

Proof. We must have

$$\frac{1}{\zeta(\sigma+it)} = s\int_1^\infty x^{-it-\sigma}\frac{M(x)}{x}\,\mathrm{d}x = o\Big(\frac{1}{\sigma-1}\Big), \quad \sigma\to 1^+,$$

uniformly for t on compact intervals. □

Example 4.2. We consider

$$\Pi(x) = \sum_{2^{k+1/2}\le x}\frac{2^{k+1/2}}{k}\,.$$

This satisfies the Chebyshev bounds: we have

$$\psi(x) = 2^{\lfloor \log x/\log 2 + 1/2\rfloor + 1/2}\log 2 + O(x/\log x).$$

Thus

$$\liminf_{x\to\infty}\frac{\psi(x)}{x} = \log 2 \quad\text{and}\quad \limsup_{x\to\infty}\frac{\psi(x)}{x} = 2\log 2.$$

Further, the zeta function of this g-number system can be explicitly computed:

$$\log\zeta(s) = 2^{-(s-1)/2}\sum_{k=1}^\infty\frac{2^{-k(s-1)}}{k} = -2^{-(s-1)/2}\log(1-2^{-(s-1)}).$$

We conclude that

$$\zeta(\sigma)\sim\frac{1}{(\sigma-1)\log 2}, \quad \sigma\to 1^+;$$

therefore, by the Hardy–Littlewood–Karamata Tauberian Theorem [9], [12], N has logarithmic density

$$\int_{1^-}^x\frac{\mathrm{d}N(u)}{u}\sim\frac{\log x}{\log 2}. \tag{4.1}$$

Furthermore, $\zeta(s)$ has infinitely many right-hand zeros of order 1 at the points $s = 1\pm i2\pi(2n+1)/\log 2$, $n\in\mathbb{N}$, because

$$\zeta\left(\sigma\pm i\frac{2\pi(2n+1)}{\log 2}\right) = \frac{1}{\zeta(\sigma)}\sim(\sigma-1)\log 2.$$

It follows, by Lemma 4.1, that

$$M(x) = \Omega(x).$$

To show the wobble of $F(x) = N(x)/x$, we apply an idea of Ingham [11] that is based on a finite form of the Wiener–Ikehara method. We use (essentially) the result given in [3, Thm. 11.12]. For $\Re s > 0$,

$$\int_1^\infty x^{-s-1}F(x)\,\mathrm{d}x = \frac{\zeta(s+1)}{s+1} = G(s).$$

The discontinuities of $G(s)$ on the line segment $(-8\pi i/\log 2, 8\pi i/\log 2)$, which provide a measure of the wobble, occur at $s=0$ and $s=\pm 4\pi i/\log 2$.

We analyze the behavior of G near these points. Let s be a complex number with $\Re s \geq 0$ (to avoid any logarithmic fuss) and $0 < |s| \leq 1/2$ (so $\log s \ll \log|1/s|$ is valid). For $n = 0, \pm 1$, a small calculation shows that

$$\zeta(1 + s + 4\pi ni/\log 2) = \frac{1}{s\log 2} + O(\log|1/s|).$$

For $n = -1, 0, 1$ set

$$\gamma_n = 4\pi n/\log 2, \qquad \alpha_n = 1/(\log 2 + 4\pi in).$$

Take T a number between $4\pi/\log 2$ and $8\pi/\log 2$, e.g. $T = 36$, and set

$$G^*(s) = \sum_{-1\leq n\leq 1} \frac{\alpha_n}{s - i\gamma_n}, \quad F_T^*(u) = \sum_{-1\leq n\leq 1} \alpha_n \left(1 - \frac{|\gamma_n|}{T}\right) e^{i\gamma_n u}.$$

Now $G - G^*$ has a continuation to the closed strip $\{s\colon \sigma \geq 0,\ |t| \leq T\}$ as a function that is continuous save logarithmic singularities at γ_{-1}, γ_0, γ_1. In particular, $G - G^*$ is integrable on the imaginary segment $(-iT, iT)$. The result in [3] is stated for an extension that is continuous at all points of such an interval, but integrability is a sufficient condition for the result to hold.

We find that

$$\liminf_{u\to\infty} F(u) \leq \inf_u F_T^*(u) < \sup_u F_T^*(u) \leq \limsup_{u\to\infty} F(u).$$

By a little algebra,

$$\begin{aligned}
\sup_u F_T^*(u) &= \frac{1}{\log 2} + \frac{2}{|\log 2 - 4\pi i|}\left(1 - \frac{4\pi}{36\log 2}\right) > 1.52,\\
\inf_u F_T^*(u) &= \frac{1}{\log 2} - \frac{2}{|\log 2 - 4\pi i|}\left(1 - \frac{4\pi}{36\log 2}\right) < 1.37.
\end{aligned}$$

Thus $N(x)/x$ has no asymptote as $x \to \infty$.

It is interesting to note that ζ is $4\pi i/\log 2$ periodic. So $\zeta(s)$ has an analytic continuation to $\{s\colon \Re s = 1,\ s \neq 1 + 4n\pi i/\log 2,\ n \in \mathbb{N}\}$. One could show a larger oscillation by a more elaborate analysis exploiting additional singularities.

Finally, we discuss $m(x) = \int_{1-}^{x} u^{-1}\mathrm{d}M(u)$ for this example. Since $M(x) = \Omega(x)$, we necessarily have $m(x) = \Omega(1)$. We will prove that

$$m(x) = O(1). \tag{4.2}$$

This shows that, in general, having Chebyshev bounds, log-density (2.1), $N(x) \ll x$, and (4.2) together are not enough to establish $M(x) = o(x)$.

To prove (4.2), we first need to improve (4.1) to

$$\int_{1-}^{x} \frac{\mathrm{d}N(u)}{u} = \frac{\log x}{\log 2} - \frac{\log\log x}{2} + O(1). \tag{4.3}$$

This estimate can be shown by applying a Tauberian theorem of Ingham–Fatou–Riesz type [6] (cf. [10, 12]). In fact, the Laplace transform of the non-decreasing function $\tau_1(x) = \int_{1^-}^{e^x} u^{-1}\mathrm{d}N(u)$ is analytic on $\Re s > 0$ and

$$\begin{aligned} G(s) &= \mathcal{L}\{\tau_1; s\} - \frac{1}{s^2\log 2} + \frac{\log(1/s)}{2s} = \frac{\zeta(s+1)}{s} - \frac{1}{s^2\log 2} + \frac{\log(1/s)}{2s} \\ &= \frac{1+\log\log 2}{2s} + O(\log^2|1/s|), \quad |s| < 1/2, \end{aligned}$$

as a small computation shows. In the terminology of [6], $G(s)$ has local pseudomeasure boundary behavior on the imaginary segment $(-i/2, i/2)$; in fact, its boundary value on that segment is the sum of a pseudomeasure and a locally integrable function. One then deduces (4.3) directly from [6, Thm. 3.7]. Similarly, we use the fact that $(s\,\zeta(s+1))^{-1}$ has a continuous extension to the same imaginary segment and we apply the same Tauberian result to the non-decreasing function $\tau_2(x) = \int_{1^-}^{e^x} u^{-1}(\mathrm{d}M(u)+\mathrm{d}N(u))$, whose Laplace transform is $\mathcal{L}\{\tau_2; s\} = \mathcal{L}\{\tau_1; s\} + (s\zeta(s+1))^{-1}$. The conclusion is again the asymptotic formula $\tau_2(\log x) = \log x/\log 2 - (\log\log x)/2 + O(1)$. One then obtains (4.2) upon subtracting (4.3) from this formula.

Example 4.3. As a second example, we consider a modification of the Beurling–Diamond examples from [4, 8] (see also [5]), namely, the continuous prime measure $\mathrm{d}\Pi_B$ given by

$$\Pi_B(x) = \int_1^x \frac{1-\cos(\log u)}{\log u}\,\mathrm{d}u$$

and the discrete g-prime system

$$q_k = \Pi_B^{-1}(k), \quad k = 1, 2, \ldots$$

with g-prime and g-integer counting functions $\pi_D(x)$ and $N_D(x)$.

We study here the continuous prime measure $\mathrm{d}\Pi_C = 2\mathrm{d}\Pi_B$ and the discrete g-primes formed by taking each q_k twice, that is, the g-prime system

$$\mathcal{P} = \{q_1,\, q_1,\, q_2,\, q_2,\, q_3,\, q_3, \ldots\}.$$

The associated number-theoretic functions will be denoted as $\pi_{\mathcal{P}}(x)$, $\Pi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$, $M_{\mathcal{P}}(x)$, and $\zeta_{\mathcal{P}}(s)$, and those corresponding to $\mathrm{d}\Pi_C$ we denote by $N_C(x)$, $M_C(x)$, and $\zeta_C(s)$.

It is easy to verify that

$$\Pi_C(x) = \frac{x}{\log x}\left(2 - \sqrt{2}\cos\left(\log x - \frac{\pi}{4}\right)\right) + O\left(\frac{x}{\log^2 x}\right)$$

and, since $\pi_{\mathcal{P}}(x) = 2\pi_D(x) = 2\lfloor \Pi_B(x) \rfloor = \Pi_C(x) + O(1)$,

$$\pi_P(x) = \frac{x}{\log x}\left(2 - \sqrt{2}\cos\left(\log x - \frac{\pi}{4}\right)\right) + O\left(\frac{x}{\log^2 x}\right), \tag{4.4}$$

whence both Π_C and π_P satisfy lower and upper Chebyshev bounds. The zeta function $\zeta_C(s)$ can be explicitly computed:

$$\log \zeta_C(s) = -2\log(s-1) + \log(s-1-i) + \log(s-1+i),$$

and so

$$\zeta_C(s) = \frac{(s-1)^2+1}{(s-1)^2} = 1 + \frac{1}{(s-1)^2}\,.$$

Now $\zeta_C(s)$ has right-hand zeros of order 1 located at $1 \pm i$, and so Lemma 4.1 implies that $M_C(x) = \Omega(x)$. For the discrete g-number system, we have

$$\zeta_{\mathcal{P}}(s) = \zeta_C(s)\exp(G(s)),$$

where $G(s) = \int_1^\infty x^{-s}\mathrm{d}(\Pi_{\mathcal{P}} - \Pi_C)(x)$ is analytic on $\Re s > 1/2$ because

$$\Pi_{\mathcal{P}}(x) = \pi_{\mathcal{P}}(x) + O(\sqrt{x}) = \Pi_C(x) + O(\sqrt{x}).$$

Thus, the same argument yields $M_{\mathcal{P}}(x) = \Omega(x)$.

Note that ζ_C is the Mellin transform of the measure $\mathrm{d}N_C = \delta_1 + \log u\,\mathrm{d}u$, and therefore we have the exact formula

$$N_C(x) = x\log x - x + 2, \quad x \geq 1.$$

We now show that $N_{\mathcal{P}}$ satisfies a lower bound of a similar type. It is well-known [5, 8] that $N_D(x) = cx + O(x\log^{-3/2}x)$ with $c > 0$. Thus $N_D(x) \geq c'x$ for some $c' > 0$ and all $x \geq 1$ and so

$$\begin{aligned} N_{\mathcal{P}}(x) &= \int_1^x \mathrm{d}N_D * \mathrm{d}N_D = \int_1^x N_D(x/t)\,\mathrm{d}N_D(t) \geq \int_1^x c'x/t\,\mathrm{d}N_D(t) \\ &\geq c'\int_1^x N_D(x/t)\,\mathrm{d}t \geq c'^2\int_1^x x/t\,\mathrm{d}t = c'^2 x\log x \neq O(x). \end{aligned}$$

Applying the Dirichlet hyperbola method, one can actually obtain the sharper asymptotic estimate

$$N_{\mathcal{P}}(x) = c^2 x\log x + bx + O\left(\frac{x}{\log^{1/2}x}\right) \tag{4.5}$$

with certain constants b, $c \in \mathbb{R}$. We leave the verification of (4.5) to the reader.

Example 4.4. There exists a non-decreasing function N on $[1, \infty)$ for which

$$\int_1^\infty \frac{|N(x) - x|}{x^{\sigma+1}}\,\mathrm{d}x \ll (\sigma - 1)^{-1/3}\,. \tag{3.2 bis}$$

holds for $1 < \sigma < 2$, but $\limsup_{x\to\infty} N(x)/x = \infty$. (The exponent 1/3 could be replaced by any positive number less than 1/2.)

To see this, set $f(n)=e^{e^n}$ and take

$$N(x)=\begin{cases} x,\ x\geq 1, & x\notin\bigcup_{m=1}^{\infty}\left[f(m),\,e^{m/3}f(m)\right] \\ e^{n/3}f(n), & f(n)\leq x\leq e^{n/3}f(n). \end{cases}$$

Clearly, $N(e^{e^n})/e^{e^n}=e^{n/3}\to\infty$.

On the other hand, for each n we have

$$\begin{aligned}\int_{f(n)}^{e^{n/3}f(n)}\frac{|N(x)-x|}{x^{\sigma+1}}\,\mathrm{d}x &< e^{n/3}f(n)\int_{f(n)}^{e^{n/3}f(n)}\frac{\mathrm{d}x}{x^{\sigma+1}} \\ &< e^{n/3}f(n)\int_{f(n)}^{\infty}\frac{\mathrm{d}x}{x^{\sigma+1}}<e^{n/3}f(n)^{-(\sigma-1)}.\end{aligned}$$

Instead of summing $e^{n/3}f(n)^{-(\sigma-1)}$, we calculate the corresponding integral:

$$\int_0^\infty e^{u/3}e^{-(\sigma-1)e^u}\mathrm{d}u=\int_0^\infty v^{1/3}e^{-(\sigma-1)v}\frac{\mathrm{d}v}{v}=\Gamma(1/3)(\sigma-1)^{-1/3}.$$

Also, $\{e^{n/3}f(n)^{-(\sigma-1)}\}$ is a unimodal sequence whose maximal term is of size at most $(3e(\sigma-1))^{-1/3}\ll(\sigma-1)^{-1/3}$. Thus (3.2 bis) holds.

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