JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 30, n° 3 (2018), p. 879-892.

<http://jtnb.cedram.org/item?id=JTNB_2018__30_3_879_0>

Journal de Théorie des Nombres de Bordeaux **30** (2018), 879–892

A note on the regularity of the Diophantine pair $\{k, 4k \pm 4\}$

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RÉSUMÉ. Soit $\varepsilon \in \{\pm 1\}$ et soit k un entier tel que $k \geq 2$ si $\varepsilon = -1$ et $k \geq 1$ si $\varepsilon = 1$. Pour tout eniter positif d, nous démontrons que si le produit de deux éléments distincts de l'ensemble

$$\{k, 4k + 4\varepsilon, 144k^3 + 240k^2\varepsilon + 124k + 20\varepsilon, d\}$$

augmenté de 1 est un carré parfait, alors $d = 9k + 6\varepsilon$ ou

$$d = 2304k^5 + 6144k^4\varepsilon + 6112k^3 + 2784k^2\varepsilon + 569k + 42\varepsilon.$$

Par conséquence, en combinant ce résultat avec un résultat récent de Filipin, Fujita et Togbé, nous provons que tous les quadruplets diophantiens de la forme $\{k, 4k + 4\varepsilon, c, d\}$ sont réguliers.

ABSTRACT. Let $\varepsilon \in \{\pm 1\}$ and let k be an integer such that $k \geq 2$ if $\varepsilon = -1$ and $k \geq 1$ if $\varepsilon = 1$. For positive integer d, we prove that if the product of any two distinct elements of the set

$$\{k, 4k + 4\varepsilon, 144k^3 + 240k^2\varepsilon + 124k + 20\varepsilon, d\}$$

increased by 1 is a perfect square, then $d = 9k + 6\varepsilon$ or

$$d = 2304k^5 + 6144k^4\varepsilon + 6112k^3 + 2784k^2\varepsilon + 569k + 42\varepsilon.$$

Consequently, combining this result with a recent result of Filipin, Fujita and Togbé, we show that all Diophantine quadruples of the form $\{k, 4k + 4\varepsilon, c, d\}$ are regular.

1. Introduction

A set $\{a_1, a_2, \ldots, a_m\}$ of m positive integers is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all i, j with $1 \leq i < j \leq m$. A folklore conjecture says that there does not exist a Diophantine quintuple. This conjecture was proved by the first, fourth authors and V. Ziegler [11].

Euler first proved that any Diophantine pair $\{a, b\}$ can be extended to a Diophantine triple $\{a, b, a + b + 2\sqrt{ab + 1}\}$. In 1979, Arkin, Hoggatt and

Manuscrit reçu le 9 juin 2017, révisé le 2 octobre 2017, accepté le 3 octobre 2017.
2010 *Mathematics Subject Classification.* 11D09, 11B37, 11J68, 11J86, 11Y65.
Mots-clefs. Diophantine m-tuples, Pell equations, Baker's method, Reduction method.

Strauss [1] showed that any Diophantine triple $\{a,b,c\}$ can be extended to a Diophantine quadruple

$$\left\{a,b,c,a+b+c+2abc+2\sqrt{(ab+1)(ac+1)(bc+1)}\right\}.$$

We call such a Diophantine quadruple *regular*. The following is a strong version of the folklore conjecture.

Conjecture 1.1. *Any Diophantine quadruple is regular.*

In 1969, by Baker and Davenport [2] who proved that the fourth element 120 in Fermat's quadruple uniquely extends the Diophantine triple $\{1,3,8\}$. In 2004, Dujella [6] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples. In 2014, Filipin, Fujita and Togbé [8], [9] studied the extendibility of some Diophantine pairs. They proved the following result.

Theorem 1.2 (cf. [9, Theorem 1.4])**.** *Let $\{a,b\}$ be a Diophantine pair with $a<b\le 8a$ and r the positive integer satisfying $ab+1=r^2$. Define an integer $c=c_\nu^\tau\,(\nu\in\{1,2,\ldots\},\tau\in\{\pm\})$ by*

$$c_\nu^\tau = \frac{1}{4ab}\Big\{(\sqrt{b}+\tau\sqrt{a})^2(r+\sqrt{ab})^{2\nu}+(\sqrt{b}-\tau\sqrt{a})^2(r-\sqrt{ab})^{2\nu}-2(a+b)\Big\}. \tag{1.1}$$

Suppose that $\{a,b,c,d\}$is a Diophantine quadruple with $d>c_\nu^{\tau+1}$ and that $\{a,b,c',c\}$is not a Diophantine quadruple for any c' with $0<c'<c_{\nu-1}^\tau$.

(1) *If $b<2a$, then $c\le c_3^+$.*
(2) *If $2a\le b\le 8a$, then $c\le c_2^+$.*

Let $\varepsilon\in\{\pm1\}$ and let k be an integer such that $k\ge 2$ if $\varepsilon=-1$ and $k\ge 1$ if $\varepsilon=1$. Define an integer $c=c_\nu^\tau\,(\nu\in\{1,2,\ldots\},\tau\in\{\pm\})$ by (1.2) with

$$a=k,\quad \text{and}\quad b=4k+4\epsilon.$$

In [9], Filipin, Fujita and Togbé proved that

Theorem 1.3 (cf. [9, Theorem 1.8])**.** *If $\{k,4k+4\varepsilon,c,d\}$ is a Diophantine quadruple with $c_2^+\ne c<d$, then $d=c_{\nu+1}^\tau$.*

However, it remains the case of the Diophantine triple

$$\{a,b,c_2^+\}=\{k,4k+4\varepsilon,144k^3+240k^2\varepsilon+124k+20\varepsilon\}.$$

Note that

$$c_1^+=9k+6\varepsilon,\quad c_3^+=2304k^5+6144k^4\varepsilon+6112k^3+2784k^2\varepsilon+569k+42\varepsilon,$$

such that $\{a,b,c_1^+,c_2^+\}$ and $\{a,b,c_2^+,c_3^+\}$ are both regular Diophantine quadruples. In this paper, we will show the following result.

Theorem 1.4. *If* $\{k, 4k + 4\varepsilon, c_2^+, d\}$ *is a Diophantine quadruple with* $c_2^+ < d$*, then* $d = c_3^+$.

Therefore, combining Theorem 1.3 and Theorem 1.4, we show that the Diophantine quadruples

$$\{k, 4k \pm 4, c, d\}$$

are regular. Moreover, with earlier works of Fujita [10], Bugeaud, Dujella and Mignotte [3] on Diophantine pairs $\{k-1, k+1\}$, we have

Corollary 1.5. *Any Diophantine quadruple which contains at least two elements in* $\{k-1, k+1, 4k\}$ *is regular.*

This also extends a result of Dujella [4] on the Diophantine triple $\{k-1, k+1, 4k\}$. It is interesting to mention that in this paper we study the extension of a Diophantine pair $\{a, b\}$ to a Diophantine triple $\{a, b, c\}$ with $c = c_2^+$. In general, it was very difficult to consider

$$c = c_2^{\pm} = 4r(r \pm a)(b \pm r).$$

This was done by Bugeaud, Dujella and Mignotte [3] when the pair is $\{k-1, k+1\}$. In [11], we have defined an operator on Diophantine triples by

$$\partial(\{a, b, c\}) = \{a, b, d_-(a, b, c)\}, \quad \text{for} \quad a < b < c,$$

where

$$d_- = d_-(a, b, c) = a + b + c + 2abc - 2\sqrt{(ab+1)(ac+1)(bc+1)}$$

and the degree of a given Diophantine triple is the number of iterations of ∂-operators to arrive at an Euler triple (a triple with $c = a + b + 2r$). For example, when $c = c_\nu^{\pm}$ as in (1.2), the triple $\{a, b, c\} = \{a, b, c_\nu^{\pm}\}$ has just degree $\nu - 1$. In particular, even though we remove the additional condition $b \leq 8a$, the form $\{a, b, c_2^{\pm}\}$ gives all Diophantine triples of degree 1.

The success here is due to the use of new congruences and a linear form in two logarithms. Moreover, the technique used for the proof of Theorem 1.4 can be used in the study of triples with $\deg(a, b, c) = 1$. Not only in some special case like $\{a, b\} = \{k-1, k+1\}, \{k, 4k \pm 4\}$, $\{A^2k + 2A, (A+1)^2k + 2(A+1)\}$, but also in general.

2. Preliminaries

Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d$. Then, there exist positive integers x, y, z such that $ad + 1 = x^2, bd + 1 = y^2, cd + 1 = z^2$. Eliminating d from these relations, we obtain

$$ay^2 - bx^2 = a - b, \tag{2.1}$$

$$az^2 - cx^2 = a - c, \tag{2.2}$$

$$bz^2 - cy^2 = b - c. \tag{2.3}$$

Assume that $a < b \le 8a$. If $\gcd(a,b)=1$, then [8, Lemma 4.1] implies that the positive solutions of the Diophantine equation (2.1) are given by

$$y\sqrt{a}+x\sqrt{b}=(\lambda\sqrt{a}+\sqrt{b})(r+\sqrt{ab})^{l},\ \ \lambda\in\{\pm 1\},\ \ l\ge 0,\ \ (l \text{ odd}). \tag{2.4}$$

Thus, we may write $x=p_l$, $y=V_l$, where

$$p_0=1,\quad p_1=r+\lambda a,\quad p_{l+2}=2rp_{l+1}-p_l, \tag{2.5}$$

$$V_0=\lambda,\quad V_1=b+\lambda r,\quad V_{l+2}=2rV_{l+1}-V_l. \tag{2.6}$$

Moreover, by Lemma 1 in [6] the positive solutions of Diophantine equations (2.2) and (2.3) are respectively given by

$$z\sqrt{a}+x\sqrt{c}=(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^{m},\quad m\ge 0, \tag{2.7}$$

$$z\sqrt{b}+y\sqrt{c}=(z_1\sqrt{b}+y_1\sqrt{c})(t+\sqrt{bc})^{n},\quad n\ge 0, \tag{2.8}$$

where m,n are non-negative integers, and (z_0,x_0), (z_1,y_1) are fundamental solutions of (2.2), (2.3), respectively. We have $z=v_m=w_n$, where

$$v_0=z_0,\quad v_1=sz_0+cx_0,\quad v_{m+2}=2sv_{m+1}-v_m, \tag{2.9}$$

$$w_0=z_1,\quad w_1=tz_1+cy_1,\quad w_{n+2}=2tw_{n+1}-w_n. \tag{2.10}$$

We may also write $x=q_m$, $y=W_n$, where

$$q_0=x_0,\quad q_1=sx_0+az_0,\quad q_{m+2}=2sq_{m+1}-q_m, \tag{2.11}$$

$$W_0=y_1,\quad W_1=ty_1+bz_1,\quad W_{n+2}=2tW_{n+1}-W_n. \tag{2.12}$$

In our case,

$$a=k,\quad b=4k+4\varepsilon,\quad c=c_2^{+}=144k^3+240\varepsilon k^2+124k+20\varepsilon,$$

$$r=2k+\varepsilon,\quad s=12k^2+10\varepsilon k+1,\quad t=24k^2+32\varepsilon k+9.$$

We have some special relations in our case.

Lemma 2.1. *If* $(a,b,c)=(k,4k+4\varepsilon,c_2^{+})$, *then* $s\equiv t\equiv -1 \pmod{2r}$ *and* $c\equiv 0 \pmod{4r}$.

Proof. The results directly come from

$$s+1=2(2k+\varepsilon)(3k+\varepsilon)=2r(3k+\varepsilon),\ \ t+1=2(2k+\varepsilon)(6k+5\varepsilon)=2r(6k+5\varepsilon),$$

and

$$c=4(2k+\varepsilon)(3k+\varepsilon)(6k+5\varepsilon)=4r(3k+\varepsilon)(6k+5\varepsilon).$$ □

The following result is just Lemma 3.1 of [9].

Lemma 2.2 ([9, Lemma 3.1(4)])**.** *If* $(a,b,c)=(k,4k+4\varepsilon,c_2^{+})$, *then* $v_{2m+1}\neq w_{2n}$ *and* $v_{2m}\neq w_{2n+1}$. *Moreover, there are two types of fundamental solution to equation* (2.2) *and* (2.3)*:*

1. *If* $v_{2m}=w_{2n}$, *then* $z_0=z_1=\lambda_1\in\{\pm 1\}$.
2. *If* $v_{2m+1}=w_{2n+1}$, *then* $z_0=\lambda_2 t$ *and* $z_1=\lambda_2 s$ *with* $\lambda_2\in\{\pm 1\}$.

We prove the following results.

Lemma 2.3. *We have* $\lambda = 1$*. Moreover,*

(1) *If* $v_{2m} = w_{2n}$*, then* l *is even.*
(2) *If* $v_{2m+1} = w_{2n+1}$*, then* l *is odd.*

Proof. By Lemma 2.2, when $v_{2m} = w_{2n}$, then $|z_1| = 1$ implies $y_1 = 1$. When $v_{2m+1} = w_{2n+1}$, the fact $|z_1| = s$ provides $y_1 = r$. From (2.12) and $t \equiv 1 \pmod{b}$, we have

$$(W_n \mod b)_{n\geq 0} = \begin{cases} (1,1,1,1,\dots), & \text{if } v_{2m} = w_{2n}, \\ (r,r,r,r,\dots), & \text{if } v_{2m+1} = w_{2n+1}. \end{cases}$$

On the other hand, from (2.6), we have

$$(V_l \mod b)_{l\geq 0} = (\lambda, \lambda r, \lambda, \lambda r, \dots).$$

Since $y = V_l = W_n$, consider the two cases. Therefore, the lemma is proved. □

Lemma 2.4. *We have*

(1) *If* $v_{2m} = w_{2n}$*, then* $2m \equiv 2n \equiv 0 \pmod{r}$ *or* $m \equiv -4n \equiv -2\varepsilon\lambda_1 \pmod{r}$*.*
(2) *If* $v_{2m+1} = w_{2n+1}$*, then* $2m+1 \equiv 2n+1 \equiv \pm 1 \pmod{r}$*.*

Proof. In our proof, we will use the congruences $s \equiv t \equiv -1 \pmod{r}$ and $c \equiv 0 \pmod{4r}$ (cf. Lemma 2.1).

Case (1). We have $v_{2m} = w_{2n}$. From (2.4), we have

$$\begin{aligned} y\sqrt{a} + x\sqrt{b} &= (\sqrt{a}+\sqrt{b})(r+\sqrt{ab})^{2l} \\ &\equiv (\sqrt{a}+\sqrt{b})(2r^2-1+2r\sqrt{ab})^{l} \\ &\equiv \pm(\sqrt{a}+\sqrt{b}) \pmod{2r}. \end{aligned} \tag{2.13}$$

Thus, by (2.5) we deduce

$$x = p_{2l} \equiv \pm 1 \pmod{2r}. \tag{2.14}$$

From (2.7) and Lemma 2.1, we obtain

$$\begin{aligned} z\sqrt{a} + x\sqrt{c} &= (\lambda_1\sqrt{a}+\sqrt{c})(s+\sqrt{ac})^{2m} \\ &\equiv (\lambda_1\sqrt{a}+\sqrt{c})(2ac+1+2s\sqrt{ac})^{m} \\ &\equiv (\lambda_1\sqrt{a}+\sqrt{c})(1-2\sqrt{ac})^{m} \\ &\equiv (\lambda_1\sqrt{a}+\sqrt{c})(1-2m\sqrt{ac}) \\ &\equiv \lambda_1\sqrt{a} + (1-2\lambda_1 am)\sqrt{c} \pmod{2r}. \end{aligned}$$

Thus, from (2.11) we get

$$x = q_{2m} \equiv 1 - 2\lambda_1 am \pmod{2r}. \tag{2.15}$$

Using (2.14) and (2.15), we have $\pm 1 \equiv 1 - 2\lambda_1 am \pmod{2r}$. This implies $2\lambda_1 am \equiv 0, 2 \pmod{r}$. Since $a = k$, $r = 2k + \varepsilon$, then $2a \equiv -\varepsilon \pmod{r}$. This implies $-\varepsilon\lambda_1 m \equiv 0, 2 \pmod{r}$. Thus, we have

$$m \equiv 0, -2\varepsilon\lambda_1 \pmod{r}. \tag{2.16}$$

Similarly, from (2.13) we have

$$y = V_{2l} \equiv \pm 1 \pmod{2r}. \tag{2.17}$$

Equation (2.8) and Lemma 2.1 imply

$$\begin{aligned} z\sqrt{b} + y\sqrt{c} &= (\lambda_1\sqrt{b} + \sqrt{c})(t + \sqrt{bc})^{2n} \\ &\equiv (\lambda_1\sqrt{b} + \sqrt{c})(2bc + 1 + 2t\sqrt{bc})^n \\ &\equiv (\lambda_1\sqrt{b} + \sqrt{c})(1 - 2\sqrt{bc})^n \\ &\equiv (\lambda_1\sqrt{b} + \sqrt{c})(1 - 2n\sqrt{bc}) \\ &\equiv \lambda_1\sqrt{b} + (1 - 2\lambda_1 bn)\sqrt{c} \pmod{2r}. \end{aligned}$$

Thus, we get

$$y = W_{2n} \equiv 1 - 2\lambda_1 bn \pmod{2r}. \tag{2.18}$$

From (2.17) and (2.18), we have $\pm 1 \equiv 1 - 2\lambda_1 bn \pmod{2r}$. It follows that $\lambda_1 bn \equiv 0, 1 \pmod{r}$. By $b = 4k + 4\varepsilon$, $r = 2k + \varepsilon$, we have $b \equiv 2\varepsilon \pmod{2r}$.

$$2n \equiv 0, \varepsilon\lambda_1 \pmod{r}. \tag{2.19}$$

Combining (2.16) and (2.19), the first part of the lemma is proved.

Case (2). Now, we consider $v_{2m+1} = w_{2n+1}$. It has been shown by Lemma 2.3 that l is odd. From (2.4), we have

$$\begin{aligned} y\sqrt{a} + x\sqrt{b} &= (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2l+1} \\ &\equiv (\sqrt{a} + \sqrt{b})(\sqrt{ab})^{2l+1} \\ &\equiv (-1)^l(\sqrt{a} + \sqrt{b})\sqrt{ab} \\ &\equiv (-1)^l b\sqrt{a} + (-1)^l a\sqrt{b} \pmod{r}. \end{aligned} \tag{2.20}$$

Thus, we see that

$$x = p_{2l+1} \equiv (-1)^l a \pmod{r}. \tag{2.21}$$

From (2.7) and Lemma 2.1, we have

$$\begin{aligned} z\sqrt{a} + x\sqrt{c} &= (\lambda_2 t\sqrt{a} + r\sqrt{c})(s + \sqrt{ac})^{2m+1} \\ &\equiv -\lambda_2\sqrt{a}(-1 + \sqrt{ac})^{2m+1} \\ &\equiv -\lambda_2\sqrt{a}(-1 + (2m+1)\sqrt{ac}) \\ &\equiv \lambda_2\sqrt{a} - \lambda_2(2m+1)a\sqrt{c} \pmod{r}. \end{aligned}$$

Thus, we have

$$x = q_{2m+1} \equiv -\lambda_2(2m+1)a \pmod{r}. \tag{2.22}$$

Using (2.21) and (2.22), we deduce that $(2m+1)a \equiv (-1)^{l+1}\lambda_2 a \pmod{r}$. Since $\gcd(a, r) = 1$, thus we get

$$2m + 1 \equiv (-1)^{l+1}\lambda_2 \pmod{r}. \tag{2.23}$$

Similarly, from (2.13) we have

$$y = V_{2l+1} \equiv (-1)^l b \pmod{r}. \tag{2.24}$$

We see that equation (2.8) and Lemma 2.1 imply

$$\begin{aligned} z\sqrt{b} + y\sqrt{c} &= (\lambda_2 s\sqrt{b} + r\sqrt{c})(t + \sqrt{bc})^{2n+1} \\ &\equiv -\lambda_2\sqrt{b}(-1 + \sqrt{bc})^{2n+1} \\ &\equiv -\lambda_2\sqrt{b}(-1 + (2n+1)\sqrt{bc}) \\ &\equiv \lambda_2\sqrt{b} - \lambda_2(2n+1)b\sqrt{c} \pmod{r}. \end{aligned}$$

Thus, we have

$$y = W_{2n+1} \equiv -\lambda_2(2n+1)b \pmod{r}. \tag{2.25}$$

From (2.24) and (2.25), we have $(-1)^l b \equiv -\lambda_2(2n+1)b \pmod{r}$. Since $\gcd(b, r) = 1$, then

$$2n + 1 \equiv (-1)^{l+1}\lambda_2 \pmod{r}. \tag{2.26}$$

Therefore, from (2.23) and (2.26) we have

$$2m + 1 \equiv 2n + 1 \equiv (-1)^{l+1}\lambda_2 \equiv \pm 1 \pmod{r}. \tag{2.27}$$

This completes the proof of Lemma 2.4. □

The following computational result can help us to have information about "very small" cases.

Lemma 2.5 (cf. [9, Lemma 1.3 (2)]). *Suppose that* $\{a, b, c, d\}$ *is a Diophantine quadruple with* $a < b < c < d_+ < d$. *If* $2a \le b \le 8a$, *then* $b > 1.3 \cdot 10^5$.

Therefore, in order to proof our main theorem, we assume that $k \ge 32499$.

3. Proof of Theorem 1.4 for large k

In this section, our goal is proof Theorem 1.4 for $k \ge 7.84 \cdot 10^6$. Let us denote

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_3 = \frac{\sqrt{b}(\sqrt{c} + \lambda_1\sqrt{a})}{\sqrt{a}(\sqrt{c} + \lambda_1\sqrt{b})},$$

$$\alpha_2 = t + \sqrt{bc}, \quad \alpha_4 = \frac{\sqrt{b}(r\sqrt{c} + \lambda_2 t\sqrt{a})}{\sqrt{a}(r\sqrt{c} + \lambda_2 s\sqrt{b})}.$$

By formula (60) of [6], if $v_{m'} = w_{n'}$ has a solution with $m', n' > 0$, then we have

$$(3.1) \qquad 0 < m' \log \alpha_1 - n' \log \alpha_2 + \log \alpha_{3,4} < \frac{8}{3} ac \alpha_1^{-2m'}.$$

Define

$$\begin{aligned} \Lambda_1 &= 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3, && \text{for } v_{2m} = w_{2n}, \\ \Lambda_2 &= (2m+1) \log \alpha_1 - (2n+1) \log \alpha_2 + \log \alpha_4, && \text{for } v_{2m+1} = w_{2n+1}. \end{aligned}$$

Then, we have

$$0 < \Lambda_1 < \frac{8}{3} ac \alpha_1^{-4m} \quad \text{and} \quad 0 < \Lambda_2 < \frac{8}{3} ac \alpha_1^{-4m-2}.$$

We will transform the forms $\Lambda_{1,2}$ into linear forms in two logarithms in order to apply the following result due to Laurent that we recall. See Corollary 1 in [12]. For any non-zero algebraic number γ of degree D over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $A \prod_{j=1}^{D} (X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{D} \left(\log A + \sum_{j=1}^{D} \log \max \left(1, \left| \gamma^{(j)} \right| \right) \right)$$

its absolute logarithmic height.

Lemma 3.1. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $\gamma_1 > 1$, $\gamma_2 > 1$, $\log \gamma_1$, $\log \gamma_2$ are real and positive, b_1 and b_2 are positive integers and*

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \geq \max \left\{ h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\} \quad \textit{for} \quad i = 1, 2$$

and

$$b' \geq \frac{|b_1|}{D\, h_2} + \frac{|b_2|}{D\, h_1}.$$

Then

$$\log |\Lambda| \geq -17.9 \cdot D^4 \left(\max \left\{ \log b' + 0.38, \frac{30}{D}, \frac{1}{2} \right\} \right)^2 h_1 h_2.$$

Remark 3.2. One can also use Theorem 2 of [12] to get a better result than the use of the above lemma. However, we still need to run a program of the Baker–Davenport reduction method. So we just choose this lemma.

We will consider two cases: $v_{2m} = w_{2n}$ and $v_{2m+1} = w_{2n+1}$.

Even case, i.e. $v_{2m} = w_{2n}$. By Lemma 2.4(1), if $v_{2m} = w_{2n}$ has a solution, then $2m \equiv 2n \equiv 0 \pmod{r}$ or $m \equiv -4n \equiv -2\varepsilon\lambda_1 \pmod{r}$. So we set

$$2m = m_1 r - 4\mu_1 \quad \text{and} \quad 2n = n_1 r + \mu_1,$$

with some positive integers m_1, n_1 and $\mu_1 \in \{0, \pm 1\}$. Then, we rewrite Λ_1 into the form

$$\begin{aligned}\Lambda_1 &= (m_1 r - 4\mu_1)\log\alpha_1 - (n_1 r + \mu_1)\log\alpha_2 + \log\alpha_3 \\ &= r\log\left(\frac{\alpha_1^{m_1}}{\alpha_2^{n_1}}\right) - \log\left(\frac{(\alpha_1^4\alpha_2)^{\mu_1}}{\alpha_3}\right). \end{aligned} \tag{3.2}$$

In order to apply Lemma 3.1, we set

$$D = 4, \quad b_1 = 1, \quad b_2 = r, \quad \gamma_1 = \frac{(\alpha_1^4\alpha_2)^{\mu_1}}{\alpha_3}, \quad \gamma_2 = \frac{\alpha_1^{m_1}}{\alpha_2^{m_2}}.$$

The multiplicative independence of γ_1 and γ_2 is easy to check, so we omit it. To ensure that $\log\gamma_1$ and $\log\gamma_2$ are positive, if $\log\gamma_1 < 0$ and $\log\gamma_2 < 0$, we use $1/\gamma_1$, $1/\gamma_2$ instead of γ_1, γ_2, respectively. Then, we work on $-\Lambda_1$ and exchange the indexes. Or, if one of $\log\gamma_i$ $(i = 1, 2)$ is negative and the other is positive, then we have a contradiction to

$$\begin{aligned}4 < 5\log\alpha_1 - 1 &< \Big|\log(\alpha_1^4\alpha_2) - |\log\alpha_3|\Big| \\ &\le |\log\gamma_1| < |\Lambda_1| < \frac{8}{3}ac\alpha_1^{-4m} \le \frac{1}{6ac},\end{aligned}$$

for $\mu_1 = \pm 1$ or

$$\begin{aligned}\frac{1}{4} &< \left(1 - \sqrt{\frac{a}{b}}\right)\cdot\frac{\sqrt{c}}{\sqrt{c}+\sqrt{a}} = \frac{\sqrt{bc}-\sqrt{ac}}{\sqrt{bc}+\sqrt{ab}} < \log\left(1 + \frac{\sqrt{bc}-\sqrt{ac}}{\sqrt{ac}+\sqrt{ab}}\right) \\ &= \log\frac{\sqrt{b}(\sqrt{c}+\sqrt{a})}{\sqrt{a}(\sqrt{c}+\sqrt{b})} \le |\log\alpha_3| = |\log\gamma_1| < |\Lambda_1| < \frac{1}{6ac},\end{aligned}$$

for $\mu_1 = 0$, where we used $|\log\alpha_3| < 1$ and $\log(1+x) > \frac{x}{1+x}$ for $x > -1$.

We have $h(\alpha_1) = \frac{1}{2}\log\alpha_1$, $h(\alpha_2) = \frac{1}{2}\log\alpha_2$. Since the absolute values of the conjugates of α_3 greater than one are

$$\frac{\sqrt{b}(\sqrt{c}+\sqrt{a})}{\sqrt{a}(\sqrt{c}+\sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c}+\sqrt{a})}{\sqrt{a}(\sqrt{c}-\sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c}-\sqrt{a})}{\sqrt{a}(\sqrt{c}+\sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c}-\sqrt{a})}{\sqrt{a}(\sqrt{c}-\sqrt{b})},$$

then

$$h(\alpha_3) \le \frac{1}{4}\log\left((ac-ab)^2\cdot\frac{b^2}{a^2}\cdot\frac{(c-a)^2}{(c-b)^2}\right) < \frac{1}{2}\log(bc) < \log\alpha_2.$$

It follows that

$$h(\gamma_1) \leq 4h(\alpha_1) + h(\alpha_2) + h(\alpha_3) < 2\log\alpha_1 + \frac{1}{2}\log\alpha_2 + \log\alpha_2 < 3.5\log\alpha_2. \tag{3.3}$$

Moreover, we have

$$|\log\gamma_1| \leq 4\log\alpha_1 + \log\alpha_2 + |\log\alpha_3| < 5\log\alpha_2 + 1.$$

Put $T_{m_1} + K_{m_1}\sqrt{ac} := \alpha_1^{m_1}$, $P_{n_1} + Q_{n_1}\sqrt{bc} := \alpha_2^{n_1}$. One can check that the leading coefficient of the irreducible polynomial of $\alpha_1^{m_1}/\alpha_2^{n_1}$ is 1. If $\alpha_1^{m_1} > \alpha_2^{n_1}$, then the absolute values of conjugates of $\alpha_1^{m_1}/\alpha_2^{n_1}$ greater than one are

$$\frac{T_{m_1} + K_{m_1}\sqrt{ac}}{P_{n_1} + Q_{n_1}\sqrt{bc}}, \quad \frac{T_{m_1} + K_{m_1}\sqrt{ac}}{P_{n_1} - Q_{n_1}\sqrt{bc}}.$$

We deduce that $h(\gamma_2) = \frac{m_1}{2}\log\alpha_1$. Similarly, if $\alpha_1^{m_1} < \alpha_2^{n_1}$, then $h(\gamma_2) = \frac{n_1}{2}\log\alpha_2$. By Lemma 2.5, we have $r > 6.49 \cdot 10^4$. We use (3.1) and (3.2) to get

$$|\log\gamma_2| = \left|\frac{m_1}{2}\log\alpha_1 - \frac{n_1}{2}\log\alpha_2\right| < \frac{1}{2r}\left(|\log\gamma_1| + \frac{8}{3}ac\alpha_1^{-4}\right) < \frac{1}{2r}\left(5\log\alpha_2 + 1 + 0.001\right) < 0.001.$$

So we have

$$h(\gamma_2) < \frac{m_1}{2}\log\alpha_1 + 0.001. \tag{3.4}$$

We set

$$h_1 = 3.5\log\alpha_2, \quad h_2 = \frac{m_1}{2}\log\alpha_1 + 0.001$$

and

$$\frac{b_1}{4h_2} + \frac{b_2}{4h_1} = \frac{r}{14\log\alpha_2} + \frac{1}{2m_1\log\alpha_1 + 0.004} < \frac{r}{14\log\alpha_2} + 0.03 =: b'.$$

We have

$$b' > \frac{r}{14\log\alpha_2} > \frac{2k-1}{14\log(48k^2 + 64k + 18)} > 188.$$

Applying Lemma 3.1, it results

$$\log|\Lambda_1| \geq -17.9 \cdot 4^4\left(\log b' + 0.38\right)^2 h_1 h_2.$$

This and $|\Lambda_1| < \frac{8}{3}ac\alpha_1^{-4m}$ give

$$4m\log\alpha_1 < 17.9 \cdot 4^4\left(\log b' + 0.38\right)^2 h_1 h_2 + \log\left(\frac{8}{3}ac\right).$$

Then, we get

$$m < 17.9 \cdot 4^3\left(\log b' + 0.38\right)^2 \left(3.5\log\alpha_2\right)\left(\frac{m_1}{2} + 0.001\right) + 0.5.$$

As $2m = m_1 r - 4\mu_1 \geq m_1 r - 4$, we have

$$0.998r < 17.9 \cdot 4^3 \left(\log b' + 0.38\right)^2 (3.5 \log \alpha_2) + 5$$

and so

$$b' - 0.03 = \frac{r}{14 \log \alpha_2} < 286.974 \left(\log b' + 0.38\right)^2 + \frac{5.011}{14 \log \alpha_2}.$$

We simplify it to have

$$b' < 286.974 \left(\log b' + 0.38\right)^2 + 0.05. \tag{3.5}$$

By a straightforward computation, we get $b' < 33461.2$. Therefore, we get the inequality

$$r < 468456.4 \log \alpha_2.$$

Recall that $r = 2k + \varepsilon$ and $\alpha_2 = t + \sqrt{bc} < 2t = 2(24k^2 + 32\varepsilon k + 9)$, we have

$$2k - 1 < 468456.4 \log(48k^2 + 64k + 18).$$

This gives $k < 8.38 \cdot 10^6$.

Odd case, i.e. $v_{2m+1} = w_{2n+1}$. Also, from Lemma 2.4(2), if $v_{2m+1} = w_{2n+1}$, then $2m + 1 \equiv 2n + 1 \equiv \pm 1 \pmod r$. Let $2m + 1 = m_2 r + \mu_2$, $2n + 1 = n_2 r + \mu_2$, for some nonnegative integers m_2, n_2 and $\mu_2 \in \{\pm 1\}$. We have

$$\begin{aligned} \Lambda_2 &= (m_2 r + \mu_2) \log \alpha_1 - (n_2 r + \mu_2) \log \alpha_2 + \log \alpha_4 \\ &= \log \left(\alpha_4 \left(\frac{\alpha_1}{\alpha_2} \right)^{\mu_2} \right) - r \log \left(\frac{\alpha_2^{n_2}}{\alpha_1^{m_2}} \right). \end{aligned} \tag{3.6}$$

We set (by replacing γ_1 and γ_2 by their reciprocals, if necessary)

$$D = 4, \quad b_1 = r, \quad b_2 = 1, \quad \gamma_1 = \frac{\alpha_2^{n_2}}{\alpha_1^{m_2}}, \quad \gamma_2 = \alpha_4 \left(\frac{\alpha_1}{\alpha_2} \right)^{\mu_2}.$$

Similarly to the proof in the even case,

$$h(\gamma_1) < \frac{m_2}{2} \log \alpha_1 + 0.001. \tag{3.7}$$

Since the absolute values of conjugates of α_4 greater than one are

$$\frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} + s\sqrt{b})}, \quad \frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})}, \quad \frac{\sqrt{b}(r\sqrt{c} - t\sqrt{a})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})},$$

then

$$\begin{aligned} h(\alpha_4) &\leq \frac{1}{4} \log \left(a^2 (c-b)^2 \cdot \frac{b^{3/2}}{a^{3/2}} \cdot \frac{c-a}{c-b} \cdot \frac{r\sqrt{c} + t\sqrt{a}}{r\sqrt{c} - s\sqrt{b}} \right) \\ &< \frac{1}{4} \log \left(4a^{1/2} b^{3/2} c^2 r^2 \right) < \frac{3}{2} \log \alpha_2. \end{aligned}$$

So we get

$$h(\gamma_2) \le h(\alpha_1) + h(\alpha_2) + h(\alpha_4) \le 2.5 \log \alpha_2. \tag{3.8}$$

One can see that the values of $h(\gamma_i)$ are not exceeding those in the even case. Hence, after applying Lemma 3.1, we get that the upper bound of k is not exceeding $8.38 \cdot 10^6$. We summarize it here.

Proposition 3.3. *If $\{k, 4k + 4\varepsilon, c_2^+, d\}$ is a Diophantine quadruple with $c_2^+ < d$, then $d = c_3^+$ for $k \ge 8.38 \cdot 10^6$.*

4. Final Computation

In order to deal with the remaining cases $32499 \le k < 8.38 \cdot 10^6$, we will use a Diophantine approximation algorithm called the Baker–Davenport reduction method. The following lemma is a slight modification of the original version of the Baker–Davenport reduction method (see [7, Lemma 5a]).

Lemma 4.1. *Assume that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of a real number κ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then the inequality

$$0 < J\kappa - K + \mu < AB^{-J}$$

has no solutions in integers J and K with

$$\frac{\log(Aq/\eta)}{\log B} \le J \le M.$$

To apply the above lemma, we use

$$\Lambda = m' \log \alpha_1 - n' \log \alpha_2 + \log \alpha_{3,4}$$

with

$$\Lambda = \Lambda_1 = 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3, \quad \text{for } v_{2m} = w_{2n},$$
$$\Lambda = \Lambda_2 = (2m+1) \log \alpha_1 - (2n+1) \log \alpha_2 + \log \alpha_4, \quad \text{for } v_{2m+1} = w_{2n+1}.$$

We set

$$J = m', \quad K = n', \quad \kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_{3,4}}{\log \alpha_2}.$$

Since $0 < \Lambda < \frac{8}{3} ac\alpha_1^{-2m'}$, then we take

$$A = \frac{8ac/3}{\log \alpha_2}, \quad B = \alpha_1^2.$$

Before running the program, we need to determine the value of M. This is an absolute upper bound of m'. From formula (40) of [5], we have

$$\frac{m'}{\log m'} < 2.867 \cdot 10^{15} \log^2 c.$$

As $c \leq 144k^3 + 240k^2 + 124k + 20$ and $k < 8.38 \cdot 10^6$, we have $m' < 4 \cdot 10^{20} =: M$. We ran a GP program in 8 hours to check no more than $8 \cdot 8.38 \cdot 10^6$ cases. We obtained $m' \leq 2$. Thus we have

Proposition 4.2. *If* $\{k, 4k + 4\varepsilon, c_2^+, d\}$ *is a Diophantine quadruple with* $c_2^+ < d$*, then* $d = c_3^+$ *for* $k \leq 8.38 \cdot 10^6$.

Combining Proposition 3.3 and Proposition 4.2, we complete the proof of Theorem 1.4.

Acknowledgements

The authors are grateful to the referee whose constructive comments help us to seriously improve the earlier manuscript. The first and the second authors were supported by Natural Science Foundation of China (Grant No. 11301363), and Sichuan provincial scientific research and innovation team in Universities (Grant No. 14TD0040), and the Natural Science Foundation of Education Department of Sichuan Province (Grant No. 16ZA0371). The first and the third authors were supported by Natural Science Foundation of China (Grant No. 11561021). The fourth author thanks Purdue University Northwest for the partial support.

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