

# JOURNAL

de Théorie des Nombres  
de BORDEAUX

*anciennement Séminaire de Théorie des Nombres de Bordeaux*

Eryk LIPKA

**Automaticity of the sequence of the last nonzero digits of  $n!$  in a fixed base**

Tome 31, n° 1 (2019), p. 283-291.

<[http://jtnb.cedram.org/item?id=JTNB\\_2019\\_\\_31\\_1\\_283\\_0](http://jtnb.cedram.org/item?id=JTNB_2019__31_1_283_0)>

© Société Arithmétique de Bordeaux, 2019, tous droits réservés.

L'accès aux articles de la revue « Journal de Théorie des Nombres de Bordeaux » (<http://jtnb.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://jtnb.cedram.org/legal/>). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>

## Automaticity of the sequence of the last nonzero digits of $n!$ in a fixed base

par ERYK LIPKA

RÉSUMÉ. En 2011, Deshouillers et Ruzsa [5] ont donné des arguments en faveur de la non-automaticité de la suite des derniers chiffres non nuls de  $n!$  en base 12. Cette assertion a été prouvée quelques années plus tard par Deshouillers [4]. Dans cet article, nous donnons une preuve alternative qui nous permet de généraliser le problème et donner une caractérisation complète des bases pour lesquelles la suite des derniers chiffres non nuls de  $n!$  est automatique.

ABSTRACT. In 2011 Deshouillers and Ruzsa [5] tried to argue that the sequence of the last nonzero digit of  $n!$  in base 12 is not automatic. This statement was proven a few years later by Deshouillers in [4]. In this paper we provide an alternate proof that lets us generalize the problem and give an exact characterization of the bases for which the sequence of the last nonzero digits of  $n!$  is automatic.

### 1. Introduction

Let  $(\ell_b(n!))_{n \in \mathbb{N}}$  be the sequence of last nonzero digits of  $n!$  in base  $b$ . In this paper we will answer the question for which values of  $b$  is this sequence automatic. It is known that  $(\ell_b(n!))_{n \in \mathbb{N}}$  is automatic in many cases including bases being primes or powers of primes. One can also prove that  $(\ell_b(n!))_{n \in \mathbb{N}}$  is automatic for some small bases that have more prime factors, like 6 or 10. In general, for a base of the form  $b = p_1^{a_1} p_2^{a_2}$  where  $p_1 \neq p_2$ ,  $p_1, p_2 \in \mathbb{P}$ ,  $a_1, a_2 \in \mathbb{N}_+$ , it can be shown that  $(\ell_b(n!))_{n \in \mathbb{N}}$  is automatic when  $a_1(p_1 - 1) \neq a_2(p_2 - 1)$ . The smallest base for which the answer is unclear is 12. This was the case analysed by Deshouillers and Ruzsa in [5]. They conjectured that  $(\ell_{12}(n!))_{n \in \mathbb{N}}$  cannot be automatic, despite the fact that it is equal to some automatic sequence nearly everywhere. An attempt to prove that conjecture was made by Deshouillers in his paper [3], and a few years later he answered the question by proving the following, stronger result

**Theorem 1.1** (Deshouillers [4]). *For  $a \in \{3, 6, 9\}$ , the characteristic sequence of  $\{n; \ell_{12}(n!) = a\}$  is not automatic.*

Another way of proving a similar fact (but for  $a \in \{4, 8\}$ ) was provided recently by Byszewski and Konieczny in [2]. It seems that both proofs can be generalized to all cases when  $a_1(p_1 - 1) = a_2(p_2 - 1)$ ; however it is not obvious if, or how, it can be extended to bases with more than two prime factors. This was our main motivation for writing this paper, and we provide a complete characterization of the bases for which this sequence is automatic, including those with many prime factors.

In this paper we will use the following notation: the string of digits of  $n$  in base  $k$  will be denoted  $[n]_k$ , or reversely we use  $\langle n_1 n_2 \dots n_l \rangle_b$  to describe the integer with digits  $n_1 \dots n_l$ ; by  $v_b(n)$  we mean the largest integer  $t$  such that  $b^t | n$ , and  $s_b(n)$  is the sum of digits of  $n$  in base  $b$ . This paper is composed of two main parts: first we recall some basic facts about automatic sequences for readers not familiar with the topic; in the latter part we present our results about the automaticity of  $(\ell_b(n!))_{n \in \mathbb{N}}$ .

**Acknowledgements.** We would like to thank Piotr Miska and Maciej Ulas for proofreading and helpful suggestions while preparing this paper

## 2. Basics of automatic sequences

In this section we will give a short summary of topics from automatic sequence theory that we will be using later. If the reader is interested in getting more insight into this topic, we strongly recommend the book of Allouche and Shallit [1] that covers all important topics in this area.

**Definition 2.1.** *Deterministic finite automaton with output* is a 6-tuple  $(Q, \Sigma, \rho, q_0, \Delta, \tau)$  such that

- $Q$  is a *finite* set of states;
- $\Sigma$  is an input alphabet;
- $\rho : Q \times \Sigma \rightarrow Q$  is a transition function;
- $q_0 \in Q$  is an initial state;
- $\Delta$  is an output alphabet (finite set);
- $\tau : Q \rightarrow \Delta$  is an output function.

The transition function can be generalized to take strings of characters instead of single ones. For a string  $s_1 s_2 s_3 \dots$  we define  $\rho(q, s_1 s_2 s_3 \dots) = \rho(\dots \rho(\rho(q, s_1), s_2) \dots)$ .

**Definition 2.2.** For any finite alphabet  $\Sigma$ , a function  $f : \Sigma^* \rightarrow \Delta$  is called a *finite-state function* if there exists a deterministic finite automaton with output  $(Q, \Sigma, \rho, q_0, \Delta, \tau)$  such that  $f(\omega) = \tau(\rho(q_0, \omega))$ .

**Lemma 2.3.** *If  $f : \Sigma^* \rightarrow \Delta$  is a finite-state function then the function  $g : \Sigma^* \rightarrow \Delta$  defined by  $g(\omega) = f(\omega^R)$  is also finite-state. ( $R$  denotes taking the everse of a word).*

*Proof (Sketch).* Let  $(Q, \Sigma, \rho, q_0, \Delta, \tau)$  be the automaton that is related to  $f$ , we will define another automaton  $(Q', \Sigma, \rho', q'_0, \Delta, \tau')$ . Let  $Q' = \Delta^Q$  be all functions from  $Q$  to  $\Delta$  and  $q'_0 \equiv \tau$ . For all  $g \in Q'$  we define  $\tau'(g) = g(q_0)$ , and for all  $\sigma \in \Sigma, q \in Q$  we put  $\rho'(g, \sigma)(q) = g(\rho(q, \sigma))$ . By induction on length of the word  $\omega \in \Sigma^*$  one can prove that the equation

$$\rho'(g, \omega)(q) = g(\rho(q, \omega^R))$$

holds for any  $g \in Q', q \in Q$ . And finally

$$g(\omega) = \tau'(\rho'(q'_0, \omega)) = \rho'(q'_0, \omega)(q_0) = g(\rho(q_0, \omega^R)) = f(\omega^R). \quad \square$$

**Definition 2.4.**  $(a(n))_{n \in \mathbb{N}}$  is a  $k$ -automatic sequence if the function  $[n]_k \rightarrow a_n$  is finite-state. By Lemma 2.3 it is not important whether we read the representation of  $n$  from the right or from the left side.

Now we present some simple examples of sequences that are automatic.

**Example 2.5.** The sequence  $a_n = n \pmod m$  is  $k$ -automatic for any  $k \geq 2, m \in \mathbb{Z}_+$ . In order to see this, it is enough to take  $Q = \{0, 1, \dots, m - 1\}$ ,  $\rho(q, \sigma) = kq + \sigma \pmod m$  and read input “from left to right”.

The sequence  $a_n = s_k(n) \pmod m$  is  $k$ -automatic for any  $k \geq 2, m \in \mathbb{Z}_+$ . Take  $Q = \{0, 1, \dots, m - 1\}$  and  $\rho(q, \sigma) = q + \sigma \pmod m$ .

For any  $k \geq 2$  and  $x \in \mathbb{N}$ , the characteristic sequence  $a_n = \delta_x(n)$  is  $k$ -automatic. The automaton that computes it can be constructed by taking  $\lceil \log_k(x) \rceil$  states that counts how many digits were correct plus one “sinkhole” state that accepts all numbers other than  $x$ .

We can also obtain automatic sequences by modifying the existing ones.

**Example 2.6.** If  $(a(n))_{n \in \mathbb{N}}$  is a  $k$ -automatic sequence then so is  $b_n = f(a_n)$  for any function  $f$  taking values from the image of  $a_n$ . The difference will be only in the output function of the related automaton.

If  $(a(n))_{n \in \mathbb{N}}, (b(n))_{n \in \mathbb{N}}$  are  $k$ -automatic sequences, then so is  $c_n = f(a_n, b_n)$  for any function  $f$  as long as it is well defined on all possible pairs  $(a_n, b_n)$ . To obtain such an automaton  $(Q_c, \Sigma, \rho_c, q_c, \Delta_c, \tau_c)$  we can take the “product” of automata  $(Q_a, \Sigma, \rho_a, q_a, \Delta_a, \tau_a), (Q_b, \Sigma, \rho_b, q_b, \Delta_b, \tau_b)$  defined by

- $Q_c = Q_a \times Q_b$ ;
- $\rho_c(a, b) = (\rho_a(a), \rho_b(b))$ ;
- $q_c = (q_a, q_b)$ ;
- $\Delta_c = f(\Delta_a \times \Delta_b)$ ;
- $\tau_c(a, b) = f(\tau_a(a), \tau_b(b))$ .

This can be easily generalized to the case with  $f$  taking any finite number of sequences as an input.

By combining the above examples together we can also observe the following.

**Lemma 2.7.** *Let  $k \in \mathbb{N}_{\geq 2}$  be fixed, then:*

- *the characteristic sequence of a finite set is  $k$ -automatic;*
- *if sequence  $(a(n))_{n \in \mathbb{N}}$  differs from  $(b(n))_{n \in \mathbb{N}}$  only on finitely many terms and one of them is  $k$ -automatic then so is the other one;*
- *a periodic sequence is  $k$ -automatic;*
- *an ultimately periodic sequence is  $k$ -automatic;*

Of course this does not exhaust all possible automatic sequences, but is enough to give some insight and be useful in our work. We should also notice what is the relation between automaticity in different bases.

**Lemma 2.8.** *Sequence  $(a(n))_{n \in \mathbb{N}}$  is  $k$ -automatic if and only if it is  $k^m$ -automatic for all  $m \in \mathbb{N}_{\geq 2}$ .*

*Proof (Sketch).* If we have a  $k$ -automaton generating a sequence, then we can easily manipulate it to create a  $k^m$ -automaton generating the same sequence. The main idea is to take the transition function to be the  $m$ -th composition of the original transition function with itself (a digit in base  $k^m$  can be seen as  $m$  digits in base  $k$ ).

On the other hand, let  $Q$  be the set of states of the  $k^m$ -automaton generating a sequence, and  $\rho$  be its transition function. We take  $Q' = Q \times \{0, 1, \dots, k^{m-1} - 1\} \times \{0, 1, \dots, m - 1\}$  and

$$\rho'((q, r, s), \sigma) = \begin{cases} (q, kr + \sigma, s + 1) & \text{if } s < m - 1 \\ (\rho(q, kr + \sigma), 0, 0) & \text{if } s = m - 1; \end{cases}$$

this way we accumulate base  $k$  digits until we collect  $m$  of them and then use the original transition function. □

### 3. New results

Let's start with some facts that we will be using in our proof:

**Proposition 3.1** (Legendre's formula [6]). *For any prime  $p$  and positive integers  $a, n$ , we have*

$$v_{p^a}(n!) = \left\lfloor \frac{n - s_p(n)}{a(p - 1)} \right\rfloor.$$

**Proposition 3.2** (Result from [7]). *For any positive integers  $b, c$  such that  $\frac{\ln(b)}{\ln(c)} \notin \mathbb{Q}$  there exists a constant  $d$  such that for each integer  $n > 25$  we have*

$$s_b(n) + s_c(n) > \frac{\log \log n}{\log \log \log n + d} - 1.$$

The next proposition is a known fact, but we haven't found it clearly stated anywhere; it can be easily proven using Dirichlet's approximation theorem or Equidistribution theorem.

**Proposition 3.3.** *For any positive integers  $a, b, c$  such that  $\frac{\ln(b)}{\ln(c)} \notin \mathbb{Q}$  there exist infinitely many triples of non-negative integers  $d, e, f$  with  $1 \leq f < b^e$  such that*

$$c^d = a \cdot b^e + f.$$

*In other words, there are infinitely many powers of  $c$  with base  $b$  notation starting with given string of digits.*

After such an introduction we can finally state our results. The following lemma and theorem are the main steps in proving when  $(\ell_b(n!))_{n \in \mathbb{N}}$  is not automatic.

**Lemma 3.4.** *Let  $P$  be a non-empty finite set of prime numbers and  $p$  be its greatest element. Let  $a > 0, k > 1$  be integers. Then there exists an integer  $a'$  such that  $\max_{i \in P} \{s_i(a')\} = s_p(a')$  and  $[a']_k$  is prefix of  $[a']_k$ .*

*Proof.* If  $k$  is not a power of  $p$ , then by Proposition 3.3 there exist infinitely many triples  $(d, e, f)$  of non-negative integers with  $1 \leq f + 1 < k^e$  such that

$$p^d = a \cdot k^e + (f + 1).$$

Furthermore we have

$$s_p(p^d - 1) = d(p - 1) > (p - 1) \frac{\ln(p^d - 1)}{\ln(p)},$$

and from the definition of  $s_q$ , for any prime  $q$  the following holds

$$s_q(p^d - 1) < (q - 1) \left( \frac{\ln(p^d - 1)}{\ln(q)} + 1 \right).$$

Because  $p$  is the greatest number in  $P$ , then for any  $q \in P, q \neq p$ , we have

$$s_q(p^d - 1) - s_p(p^d - 1) < \ln(p^d - 1) \left( \frac{q - 1}{\ln(q)} - \frac{p - 1}{\ln(p)} \right) + q - 1.$$

The right side of this inequality is negative for  $d$  big enough, so because  $0 \leq f < k^e$  we can take  $a' = p^d - 1$ . When  $k = p^t$  we can notice that for any integer  $d$

$$\begin{aligned} s_p(a \cdot p^{td} + p^{td} - 1) &= s_p(a) + td(p - 1) \\ &> (p - 1) \left( \frac{\ln(a \cdot p^{td} + p^{td} - 1)}{\ln(p)} - \frac{\ln(a)}{\ln(p)} - 1 \right), \end{aligned}$$

and hence it is enough to take  $a' = a \cdot p^{td} + p^{td} - 1$  for  $d$  sufficiently large.  $\square$

**Theorem 3.5.** *Let  $P$  be a finite set of prime numbers with at least two elements and  $p$  be its greatest element, also let  $c > 0$  be a real number. Let us define sets*

$$A_- = \left\{ n \in \mathbb{Z}_+ : \max_{i \in P} \{s_i(n)\} = s_p(n) \right\},$$

$$A_+ = \left\{ n \in \mathbb{Z}_+ : \max_{i \in P} \{s_i(n)\} - s_p(n) \geq c \right\}.$$

*Then there does not exist a deterministic finite automaton with output that assigns one value to integers in  $A_-$  and another value to those in  $A_+$ .*

*Proof.* Let's suppose that we have such an automaton  $(Q, \Sigma_k, \rho, q_0, \Delta, \tau)$  for some  $k$ . Because  $Q$  is finite, there exists an internal state  $\mathcal{S} \in Q$  such that for infinitely many positive integers  $c_1 < c_2 < \dots$  we have  $\rho(q_0, [p^{c_i}]_k) = \mathcal{S}$ . Now, by Lemma 3.4, there exists an integer  $a' \in A_-$  which can be obtained from  $p^{c_1}$  by appending some suffix. Hence we can fix positive integers  $e, f < k^e$  such that  $a' = p^{c_1} \cdot k^e + f$ . Let the sequence of digits  $(f_1, f_2, \dots, f_e)$  be a representation of  $f$  in base  $k$ , possibly with added leading zeros. By  $\mathcal{T} \in Q$  we denote an internal state such that

$$\mathcal{T} = \rho(q_0, [a']_k) = \rho(\mathcal{S}, f_1 f_2 \dots f_e).$$

This means that for every  $i \in \mathbb{N}_+$  we have

$$\rho(q_0, [p^{c_i} \cdot k^e + f]_k) = \rho(q_0, [p^{c_i}]_k f_1 f_2 \dots f_e) = \rho(\mathcal{S}, f_1 f_2 \dots f_e) = \mathcal{T},$$

and this implies that  $\tau(\rho(q_0, [p^{c_i} \cdot k^e + f]_k)) = \tau(\mathcal{T})$  does not depend on the value of  $i$ .

On the other hand, when  $c_i > \lceil \log_p(f) \rceil$  we have  $s_p(p^{c_i} \cdot k^e + f) = s_p(k^e) + s_p(f)$  which is a constant. However, due to Proposition 3.2 we know that for any  $q \in P, q \neq p$ , the value of  $s_q(p^{c_i} \cdot k^e + f)$  is increasing with  $c_i$ . Hence for  $c_i$  big enough we have  $p^{c_i} \cdot k^e + f \in A_+$ .

All but finitely many integers of the form  $p^{c_i} \cdot k^e + f$  are elements of  $A_+$  but at least one (namely  $p^{c_1} \cdot k^e + f$ ) is an element of  $A_-$ . This proves that such an automaton cannot assign different values to members of those two sets. □

Now we will show that  $l_b(n!)$  can be automatic for some  $b$ .

**Lemma 3.6.** *If  $b = p^a, p \in \mathbb{P}, a \in \mathbb{N}$  then the sequence  $(l_b(n!))_{n \in \mathbb{N}}$  is  $b$ -automatic.*

*Proof.* First, we notice that  $l_b(xy) = l_b(l_b(x) l_b(y))$ , so

$$l_b((bn)!) = l_b \left( l_b(n!) \prod_{i=n+1}^{bn} l_b(i) \right).$$

Because  $\ell_b(bx) = \ell_b(x)$  we can rewrite the product in the following way:

$$\ell_b((bn)!) = \ell_b \left( \ell_b(n!) \prod_{\substack{i=1 \\ b \nmid i}}^{bn} \ell_b(i) \right) = \ell_b \left( \ell_b(n!) \prod_{i=1}^n \ell_b \left( \prod_{j=1}^{j=b-1} j \right) \right).$$

We denote  $m_i = \ell_b(i!)$  and obtain  $\ell_b((bn)!) = \ell_b(\ell_b(n!)m_{b-1}^n)$ . Now we take the string of digits  $n_1n_2 \dots n_l = [n]_b$  and obtain the following formula

$$\begin{aligned} \ell_b(n!) &= \ell_b(\langle n_1n_2 \dots n_l \rangle_b!) \\ &= \ell_b((b \cdot \langle n_1n_2 \dots n_{l-1} \rangle_b + n_l)!) = \\ &= \ell_b \left( (b \cdot \langle n_1n_2 \dots n_{l-1} \rangle_b)! \prod_{i=1}^{n_l} (b \cdot \langle n_1n_2 \dots n_{l-1} \rangle_b + i) \right) \\ &= \ell_b(\ell_b((b \cdot \langle n_1n_2 \dots n_{l-1} \rangle_b)!) \ell_b(n_l!)) \\ &= \ell_b(m_{n_l} \ell_b(\langle n_1n_2 \dots n_{l-1} \rangle_b!) m_{b-1}^{\langle n_1n_2 \dots n_{l-1} \rangle_b}), \end{aligned}$$

which by iteration leads to

$$(3.1) \quad \ell_b(\langle n_1n_2 \dots n_l \rangle_b!) = \ell_b(m_{n_1}m_{n_2} \dots m_{n_l} \ell_b(m_{b-1}^r)),$$

where  $r = \langle n_1n_2 \dots n_{l-1} \rangle_b + \dots + \langle n_1n_2 \rangle_b + \langle n_1 \rangle_b$ . Now, by Euler's Theorem  $m_{b-1}^{\varphi(b)} \equiv m_{b-1}^{p^a - p^{a-1}} \equiv 1 \pmod{b}$  so we only need to know the value of  $r \pmod{p^a - p^{a-1}}$ .

$$\begin{aligned} r &= \sum_{i=1}^{l-1} \left( b^{i-1} \sum_{j=1}^{l-i} n_j \right) \\ &= \sum_{j=1}^{l-1} n_j + \sum_{i=2}^{l-1} \left( p^{a(i-1)} \sum_{j=1}^{l-i} n_j \right) \\ &\equiv \sum_{i=1}^{l-1} n_i + p^{a-1} \sum_{i=2}^{l-1} \left( \sum_{j=1}^{l-i} n_j \right) \pmod{p^a - p^{a-1}}. \end{aligned}$$

Hence

$$(3.2) \quad r \equiv \sum_{i=1}^{l-1} n_i + p^{a-1} \sum_{i=1}^{l-2} (l-1-i) n_i \pmod{p^a - p^{a-1}}.$$

Finally, we can define an automaton  $(Q, \Sigma_b, \rho, q_0, \Delta, \tau)$  generating the sequence  $(\ell_b(n!))_{n \in \mathbb{N}}$  in the following way:

- the input alphabet  $\Sigma_b = \{0, 1, 2, \dots, b-1\}$ ;
- the output alphabet  $\Delta = \{1, 2, \dots, b-1\}$ ;
- the set of states  $Q = \Delta \times \Sigma_{p^a - p^{a-1}} \times \Sigma_{p-1}$ ;
- the initial state  $q_0 = (1, 0, 0)$ ;



- the output function  $\tau(u, v, w) = \ell_b(u \cdot m_{b-1}^{v+p^{a-1}w})$ ;
  - the transition function  $\rho((u, v, w), s)$
- $$= \left( \ell_b(u \cdot m_s), (v + s) \pmod{(p^a - p^{a-1})}, (w + v) \pmod{(p - 1)} \right).$$

With this definition we have  $\rho(q_0, [n]_b) = (u, v, w)$  where

- $u = \ell_b(m_{n_1} m_{n_2} \dots m_{n_l})$ ;
- $v = \sum_{i=1}^{l-1} n_i \pmod{(p^a - p^{a-1})}$ ;
- $w = \sum_{i=1}^{l-2} (l - 1 - i) n_i \pmod{(p - 1)}$ .

Hence using equations (3.1) and (3.2) we see, that  $\ell_b(n!) = \tau(u, v, w)$ .  $\square$

Now we are ready to prove the main result.

**Theorem 3.7.** *Let  $b = p_1^{a_1} p_2^{a_2} \dots$  with  $a_1(p_1 - 1) \geq a_2(p_2 - 1) \geq \dots$ . The sequence  $(\ell_b(n!))_{n \in \mathbb{N}}$  is  $p_1$ -automatic if  $a_1(p_1 - 1) > a_2(p_2 - 1)$  or  $b = p_1^{a_1}$  and not automatic otherwise.*

*Proof.* Let  $n \gg 0$ . For  $b = p_1^{a_1}$  the sequence is  $b$ -automatic from Lemma 3.6; by Lemma 2.8 it is also  $p_1$ -automatic. If  $b$  has more than one prime factor and  $a_1(p_1 - 1) > a_2(p_2 - 1)$  we take  $b' = \frac{b}{p_1^{a_1}}$  so  $p_1 \nmid b'$ . From Proposition 3.1 and the definition of  $v_{b'}$  we have

$$\begin{aligned} v_{b'}(n!) &= \min_{i>1} v_{p_i^{a_i}}(n!) \\ &= \min_{i>1} \left\lfloor \frac{n - s_{p_i}(n)}{a_i(p_i - 1)} \right\rfloor \\ &> \min_{i>1} \left\lfloor \frac{n - (\log_{p_i}(n) + 1)(p_i - 1)}{a_i(p_i - 1)} \right\rfloor \\ &\geq \min_{i>1} \left\lfloor \frac{n}{a_i(p_i - 1)} - \log_{p_i}(n) - 1 \right\rfloor \\ &> \left\lfloor \frac{n}{a_1(p_1 - 1)} \right\rfloor > \left\lfloor \frac{n - s_{p_1}(n)}{a_1(p_1 - 1)} \right\rfloor = v_{p_1^{a_1}}(n!), \end{aligned}$$

which leads to  $b' | \ell_b(n!)$ . Thus  $\ell_b(n!) \in \{b', 2b', 3b', \dots, (p_1^{a_1} - 1)b'\}$ , so the value of  $\ell_b(n!)$  can be computed from the value of  $\ell_b(n!) \pmod{p_1^{a_1}}$ . Now, we can be sure that there exist integers  $c, d$  satisfying the equations

$$\begin{cases} n! = b^{(v_b(n!))} \ell_b(n!) + b^{(v_b(n!)+1)} c \\ n! = p_1^{a_1(v_{p_1^{a_1}}(n!))} \ell_{p_1^{a_1}}(n!) + p_1^{a_1(v_{p_1^{a_1}}(n!)+1)} d. \end{cases}$$

We notice that  $v_b(n!) = \min_{i \geq 1} v_{p_i^{a_i}}(n!) = v_{p_1^{a_1}}(n!)$ , which leads to

$$b^{(v_{p_1^{a_1}}(n!))} \ell_b(n!) + b^{(v_{p_1^{a_1}}(n!)+1)} c = p_1^{a_1(v_{p_1^{a_1}}(n!))} \ell_{p_1^{a_1}}(n!) + p_1^{a_1(v_{p_1^{a_1}}(n!)+1)} d.$$

After division of the above equality by  $p_1^{a_1(v_{p_1^{a_1}}(n!))}$  we obtain the following

$$(b')^{(v_{p_1^{a_1}}(n!))} \ell_b(n!) + p_1^{a_1} (b')^{(v_{p_1^{a_1}}(n!)+1)} c = \ell_{p_1^{a_1}}(n!) + p_1^{a_1} d.$$

From the above equation, we can notice that  $\ell_b(n!)(b')^{v_{p_1^{a_1}}(n!)} \equiv \ell_{p_1^{a_1}}(n!) \pmod{p_1^{a_1}}$ , hence to finish this part of the proof we just need to construct a  $p_1$ -automaton that returns the value of  $v_{p_1^{a_1}}(n!) \pmod{\varphi(p_1^{a_1})}$ . By Proposition 3.1 this value can be computed from  $(n - s_{p_1}(n)) \pmod{\varphi(p_1^{a_1}) \cdot a_1(p_1 - 1)}$ , and this expression is  $p_1$ -automatic as we already mentioned in Example 2.5.

Now, in the last case, when  $a_1(p_1 - 1) = a_2(p_2 - 1)$ , let  $I = \{i : a_i(p_i - 1) = a_1(p_1 - 1)\}$ . Without loss of generality we can assume  $p_1 = \max_{i \in I} p_i$ . By Legendre's formula (Proposition 3.1) we have

$$\begin{aligned} \max_{i \in I} s_{p_i}(n) = s_{p_1}(n) &\Rightarrow v_{p_1^{a_1}}(n!) = \min_{i \in I} v_{p_i^{a_i}}(n!) = v_b(n!) \\ &\Rightarrow p_1^{a_1} \nmid \ell_b(n!), \\ \max_{i \in I} s_{p_i}(n) > a_1(p_1 - 1) + s_{p_1}(n) &\Rightarrow v_{p_1^{a_1}}(n!) > \min_{i \in I} v_{p_i^{a_i}}(n!) = v_b(n!) \\ &\Rightarrow p_1^{a_1} \mid \ell_b(n!). \end{aligned}$$

Hence, by Theorem 3.5, there is no finite automaton that can, for given  $n$ , tell whether  $p_1^{a_1}$  divides  $\ell_b(n!)$  or not. This completes the proof, as any finite automaton generating the sequence  $(\ell_b(n!))_{n \in \mathbb{N}}$  should distinguish those two sets. □

### References

- [1] J.-P. ALLOUCHE & J. SHALLIT, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, 2003.
- [2] J. BYSZEWSKI & J. KONIECZNY, "A density version of Cobham's theorem", <https://arxiv.org/abs/1710.07261>, 2017.
- [3] J.-M. DESHOULLERS, "A footnote to The least non zero digit of  $n!$  in base 12", *Unif. Distrib. Theory* **7** (2012), no. 1, p. 71-73.
- [4] ———, "Yet another footnote to The least non zero digit of  $n!$  in base 12", *Unif. Distrib. Theory* **11** (2016), no. 2, p. 163-167.
- [5] J.-M. DESHOULLERS & I. RUZSA, "The least non zero digit of  $n!$  in base 12", *Publ. Math.* **79** (2011), no. 3-4, p. 395-400.
- [6] A.-M. LEGENDRE, *Théorie des nombres*, Firmin Didot frères, 1830.
- [7] C. L. STEWART, "On the representation of an integer in two different bases", *J. Reine Angew. Math.* **319** (1980), p. 63-72.

Eryk LIPKA  
 Potok 448  
 38-404 Krosno, Poland  
*Current address* Institute of Mathematics  
 Pedagogical University of Cracow; Poland  
*E-mail:* [erykclipka@gmail.com](mailto:erykclipka@gmail.com)