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On the number of prime factors of the composite numbers resulting after a change of digits of primes

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Résumé. Dans cette note, nous prouvons que pour tout entier fixé $K \geq 2$, pour tout $\epsilon > 0$ et pour tout x suffisamment grand, il existe au moins $x^{1-\epsilon}$ nombres premiers $x < p \leq (1+K^{-1})x$ tels que tous les nombres entiers de la forme $pj \pm a^h k$ avec $2 \leq a \leq K$, $0 < |k| \leq K$, $1 \leq j \leq K$, $0 \leq h \leq K \log x$ sont des nombres composés ayant au moins $(\log\log x)^{1-\epsilon}$ facteurs premiers distincts.

Abstract. In this note, we prove that for any fixed integer $K \geq 2$, for all $\epsilon > 0$ and for all sufficiently large x, there exist at least $x^{1-\epsilon}$ primes $x < p \leq (1+K^{-1})x$, such that all of the integers $pj \pm a^h k$, $2 \leq a \leq K$, $0 < |k| \leq K$, $1 \leq j \leq K$, $0 \leq h \leq K \log x$ are composite having at least $(\log\log x)^{1-\epsilon}$ distinct prime factors.

1. Introduction

In 1979, Erdős proved the following result, which appeared in the solution to a problem in Mathematics Magazine [3].

Theorem 1.1 (Erdős). *For all sufficiently large positive integers k, there exist primes p,*

$$p = \sum_{i=0}^{k} a_i 10^i,\ a_k > 0,\ 0 \leq a_i \leq 9,$$

such that all of the integers $p + t\,10^i$, $0 < |t| < 10$, $0 \leq i \leq k$ are composite.

In 2011, Tao [6] proved that for any integer $K \geq 2$, there exist at least $c_K \frac{x}{\log x}$ primes p in the interval $[x, (1+K^{-1})x]$ satisfying $|pj \pm a^h k|$ is composite for every $2 \leq a \leq K$, $1 \leq j, k \leq K$ and $1 \leq h \leq K \log x$, where $c_K > 0$ is a constant depending only on K. In a different direction, Hao Pan [5] proved the following theorem.

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Theorem 1.2. *Suppose that $K \geq 2$ is an integer and $\epsilon > 0$ is small number. Then for all sufficiently large (depending only on K and ϵ) x, there exist at least $x^{1-\epsilon}$ integers $n \in [x, (1+K^{-1})x]$ such that $\omega(nj \pm a^h k) \geq (\log\log\log x)^{\frac{1}{3}-\epsilon}$ for all $2 \leq a \leq K$, $1 \leq j, k \leq K$ and $0 \leq h \leq K \log x$. Here, as usual, $\omega(m)$ denotes the number of distinct prime factors of m.*

In [5], Pan also asked if one could improve the quoted lower bound by a log factor. (This is a natural question as the normal order of $\omega(n)$ is $\log\log n$.) In this note we give the affirmative answer. Indeed we prove the following result.

Theorem 1.3. *Let $K \geq 2$ be an integer, $\epsilon > 0$ be a small number. For all sufficiently large positive x, there exist at least $x^{1-\epsilon}$ primes $x < p \leq (1+K^{-1})x$, such that all of the integers $pj \pm a^h k$, $2 \leq a \leq K$, $0 < k \leq K$, $1 \leq j \leq K$, $0 \leq h \leq K \log x$ are composite having at least $(\log\log x)^{1-\epsilon}$ distinct prime factors.*

This result improves Theorem 1.2 in two ways; first the number of prime factors is improved by a log factor, secondly the numbers considered are prime numbers.

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2. Proof of Theorem 1.3

2.1. Lemmata. We first state the results which will be used in the proof of Theorem 1.3.

In [4], Linnik proved the following theorem.

Theorem 2.1 (Linnik's Theorem)**.** *Let a, q be two integers such that $q \geq 1$ and $(a, q) = 1$. There exists a prime p such that $p \equiv a \bmod q$, and $p \ll q^C$ for some positive absolute constant C.*

Following the proof of Linnik's Theorem in [2], one can obtain the following corollary. Due to lack of suitable reference we include the proof here.

Corollary 2.2. *Let $K > 0$ be fixed. Let a, q be two integers such that $q \geq 1$ and $(a, q) = 1$, and let x be a real number so that $q^c \ll x$, for a sufficiently large constant $c > 0$. Then there are at least $\gg_K \frac{x}{q^2 \varphi(q) \log x}$ primes p such that $p \equiv a \bmod q$, and $x < p \leq (1+K^{-1})x$.*

Proof. The result in the case when $q \leq (\log x)^2$ follows by applying the Siegel–Walfisz Theorem. Suppose that $q > (\log x)^2$.We follow Bombieri's notation used in [2]. Here, $L(s, \chi)$ denotes a Dirichlet L-function for $s = \sigma + it$, where σ and t are real numbers, and χ is a Dirichlet character mod q. Let $c_1 > 0$ be the constant appearing in the Landau–Page Theorem (see [2, p. 39]), such that $L(s,\chi)\neq 0$ for $\sigma\geq 1-\frac{c_1}{\log T}$, $|t|\leq T$ for all primitive characters $\chi \bmod m$, $m\leq T$ except possibly for one exceptional real character. We let χ_1 denote a character modulo q, induced by an exceptional character, if it exists. In this case we let β_1 denote the exceptional zero of $L(s,\chi_1)$, and we also let $\delta_1:=1-\beta_1$.

We put $4A:=\frac{\log x}{\log q}$ so that $(1+K^{-1})x=q^{4c_0A}$, where $c_0=1+\frac{\log(1+1/K)}{4A\log q}$. Then $1<c_0<2$. Using the last equation in the proof of Linnik's Theorem in [2, p. 55], namely,

$$\sum_{\substack{p\leqslant x\\ p\equiv a \bmod q}}\log p=\frac{1}{\varphi(q)}\left(x-\chi_1(a)\frac{x^{\beta_1}}{\beta_1}\right)+O\left(\frac{x}{\varphi(q)}\delta_1(\log q)\exp\left(-c_1A\right)\right)+O\left(\frac{x\log x}{q^4}\right)+O\left(\frac{1}{\varphi(q)}x^{1/2}q^{20}\right),$$

we obtain

$$\sum_{\substack{x<p\leqslant(1+K^{-1})x\\ p\equiv a \bmod q}}\log p=\frac{1}{\varphi(q)}\left(K^{-1}x-\chi_1(a)\frac{((1+K^{-1})^{\beta_1}-1)x^{\beta_1}}{\beta_1}\right)+O\left(\frac{x}{\varphi(q)}\delta_1(\log q)\exp\left(-c'A\right)\right)+O\left(\frac{x\log x}{q^4}\right)+O\left(\frac{1}{\varphi(q)}x^{1/2}q^{20}\right). \tag{2.1}$$

Note that $\frac{1}{2}<\beta_1<1$, so $\frac{K^{-1}}{(2+K^{-1})}<(1+K^{-1})^{\beta_1}-1<K^{-1}$. So we have

$$\frac{K^{-1}}{(2+K^{-1})}\frac{x^{\beta_1}}{\beta_1}<\left|\chi_1(a)\frac{((1+K^{-1})^{\beta_1}-1)x^{\beta_1}}{\beta_1}\right|<K^{-1}\frac{x^{\beta_1}}{\beta_1}.$$

If $\chi_1(a)>0$, then

$$K^{-1}x-\chi_1(a)\frac{((1+K^{-1})^{\beta_1}-1)x^{\beta_1}}{\beta_1}>K^{-1}x-\chi_1(a)K^{-1}\frac{x^{\beta_1}}{\beta_1}.$$

If $\chi_1(a)<0$, then

$$K^{-1}x-\chi_1(a)\frac{((1+K^{-1})^{\beta_1}-1)x^{\beta_1}}{\beta_1}>K^{-1}x-\frac{K^{-1}}{(2+K^{-1})}\chi_1(a)\frac{x^{\beta_1}}{\beta_1}>\frac{K^{-1}}{(2+K^{-1})}x-\frac{K^{-1}}{(2+K^{-1})}\chi_1(a)\frac{x^{\beta_1}}{\beta_1}.$$

Thus we have

$$K^{-1}x-\chi_1(a)\frac{((1+K^{-1})^{\beta_1}-1)x^{\beta_1}}{\beta_1}\gg_K x-\chi_1(a)\frac{x^{\beta_1}}{\beta_1}.$$

For A large enough, the term $x - \chi_1(a)\frac{x^{\beta_1}}{\beta_1}$ is $\gg (\delta_1 \log q)x$. So the main term is $\gg_K \frac{x}{\varphi(q)} q^{-2}$ and the first error term on the right hand side of (2.1) is negligible compared to the main term for large A. Moreover, it follows from the argument given in [2] that $x - \chi_1(a)\frac{x^{\beta_1}}{\beta_1} \gg \frac{x}{q^2}$. Now we note that for A large enough, the sum of the last two error terms on the right of (2.1) is also negligible. Thus we obtain

$$\sum_{\substack{x < p \leqslant (1+K^{-1})x \\ p \equiv a \bmod q}} \log p \gg_K \frac{x}{\varphi(q)} q^{-2}. \tag{2.2}$$

Since each term of the sum in (2.2) is $\leq \log x$, the result follows. □

The following is a well known result special cases of which have been discovered by many mathematicians independently. See [1], for example.

Theorem 2.3 (Zsigmondy's Theorem, 1892)**.** *Let a and n be two integers greater than 1. Then there exists a prime q such that a has order n* mod q*, except exactly in the following cases:*

(1) *$n = 2$ and $a = 2^k - 1$, where $k \geq 2$.*
(2) *$n = 6$ and $a = 2$.*

The idea Erdős used for the proof of Theorem 1.1 was to find small prime numbers q and a prime p so that each $p + t10^i$ is divisible by one of the primes q. In order to do that effectively (using as few small primes as possible), he used Zsigmondy's Theorem to choose primes $\{q\}$ so that most of the powers 10^i of 10 fall into the same congruence class for some prime q. This made the argument effective enough to obtain several congruence conditions (whose simultaneous solution exists by Chinese remainder theorem) with a common solution to a small enough modulus so that Linnik's Theorem provides a solution which is a prime number. As in the case of the proof of Theorem 1.1, Zsigmondy's Theorem is going to be the key in our argument to prove Theorem 1.3. Before we start giving the proof of the theorem, we state the following technical lemma.

Lemma 2.4. *Let A be a finite set of consecutive positive integers. For each $a \in A$, and each integer $i \geq 2$ for which the pair (a, i) is not an exception to Zsigmondy's theorem, let $q_{a,i}$ be a prime for which the order of a* mod $q_{a,i}$ *is i. We can choose a family of disjoint sets $\{Q_a\}_{a \in A}$ such that if we write $Q_a = \{q_{a,i_1} < q_{a,i_2} < q_{a,i_3} < \ldots\}$, then each difference $i_{j+1} - i_j \leq 1 + \#A$.*

Proof. We construct the sets Q_a greedily. Proceed through the elements $a \in A$ in order. For each a, add to Q_a the prime $q_{a,i}$, where i is chosen as small as possible subject to the conditions that

(1) $q_{a,i}$ is defined, and
(2) $q_{a,i}$ has not already been included in any of set $Q_{a'}$ $(a' \in A)$.

After having gone through the entire list of a's, we start over and repeat the process. We continue this indefinitely to construct the sets Q_a.

Suppose that the prime added to Q_a at a certain stage is $q_{a,i}$. By the next time we are to add a prime to Q_a, we have used (in the worst case) $\#A$ possible candidates. Since there is at most one index $j \geq 2$ for which $q_{a,j}$ is undefined, the prime we add at this next stage, say $q_{a,i'}$, necessarily satisfies $i' - i \leq \#A + 1$, as desired. □

2.2. Proof of Theorem 1.3. Our proof strategy is as follows: First, note that for an integer m coprime to j, if $p \equiv \frac{-a^h k}{j} \bmod m$ then $pj + a^h k \equiv 0 \bmod m$. In order to find primes p with the desired property, we attempt to find residue classes to many (at least $(\log\log x)^{1-\epsilon}$) different prime moduli in order for the numbers $\frac{a^h k}{j}$ to be "covered". One way to do this could be assigning congruence conditions to each one of those numbers using different moduli at each step. However this naive choice is not efficient enough for our purpose: If we apply Chinese Remainder Theorem after writing down lots of congruence conditions, the modulus to which we can ensure a simultaneous solution would end up being too large to be able to find small enough primes p in our range. Thus we would like to use the same congruence classes for different $\frac{a^h k}{j}$, whenever there is no obstruction to do so. Here, knowing that we can always find moduli for which a is far from being a primitive root (by Zsigmondy's Theorem) allows us to have an efficient way to decrease the number of moduli we use at the end, and the modulus we find the simultaneous solution for becomes much smaller, allowing us to ensure that we can find prime solutions as small as we need for our purpose.

We let $K \geq 2$ be a given integer, and let $\epsilon > 0$ be a fixed small real number. For a given large x, we put $t = \lfloor K \log x \rfloor$. Define the set

$$\mathcal{D}_{K,t} := \{-K, -K+1, \ldots, -1, 1, \ldots, K\} \times \{0, 1, \ldots, t\} \times \{1, 2, \ldots, K\},$$

so that $\#\mathcal{D}_{K,t} = 2K^2(t+1)$.

First, put $r := K\lfloor (\log x)^{\frac{1}{3}} \rfloor$. By Lemma 2.4, we can construct pairwise disjoint sets $\{\mathcal{Q}_a\}_{a \in \{2,3,\ldots K\}}$ as follows: each $\mathcal{Q}_a = \{q_{a,i_1} < q_{a,i_2} < \ldots\}$ and the indices i_l satisfy $K \leq i_1 < i_2 < \cdots \leq r$, and $i_l - i_{l-1} \leq K$, for all $l > 1$. We enforce $i_1 \geq K$, so that $\mathcal{Q}_a$ has no element $\leq K$, while including only elements indexed by $i_l \leq r$ ensures that the number of elements in $\mathcal{Q}_a$ is at most $r - K + 1$. We put $\mathcal{I}_a := \{i_1, i_2, \ldots : q_{a,i_l} \in \mathcal{Q}_a\}$.

Now, let $n = \lceil (\log\log x)^{1-\epsilon} \rceil$ and let $1 \leq d \leq n$ be an integer. Here, n is the number of times we will repeat our argument, and we will divide the process into n pieces associated to the congruence classes modulo n (we use congruence classes as a bookkeeping measure, this division may have been done in a different way without changing the result). We determine several congruence classes $\pmod{q_{a,i_l}}$, for $i_l \in \mathcal{I}_a$ such that $l = d + en$, inductively on e. Fix any congruence class $-u_{a,i_d} \pmod{q_{a,i_d}}$, then suppose that we have determined the congruences $-u_{a,i_{d+en}} \pmod{q_{a,i_{d+en}}}$, for each $0 \le e \le c-1$. Let $\mathscr{C}^a_{i_{d+cn}}$ be the set of numbers of the form $\frac{k\cdot a^h}{j}$, $(k,h,j) \in \mathcal{D}_{K,t}$ which are not congruent to any of $-u_{a,i_{d+en}} \bmod q_{a,i_{d+en}}$, for any $0 \le e \le c-1$. (Here $\frac{k\cdot a^h}{j} \equiv -u$ is equivalent to saying that $k \cdot a^h \equiv -ju$.) Now, since the powers of a take exactly i_{d+cn} distinct values $\pmod{q_{a,i_{d+cn}}}$, numbers of the form $\frac{k\cdot a^h}{j}$ can occupy at most $2K^2\, i_{d+cn}$ residue classes $\pmod{q_{a,i_{d+cn}}}$. So by the Pigeonhole Principle, there exists $-u_{i_{d+cn}}$ for which the congruence class $-u_{a,i_{d+cn}} \pmod{q_{a,i_{d+cn}}}$ is occupied by at least $\left\lceil \frac{\#\mathscr{C}^a_{i_{d+cn}}}{2K^2 i_{d+cn}} \right\rceil$ elements of $\mathscr{C}^a_{i_{d+cn}}$.

We use the bounds for $\mathscr{C}^a_{i_l}$ for various $l \equiv d \mod n$ iteratively to obtain that for given integers a and d, $1 \le d \le n$, $2 \le a \le K$, the number $R_{d,a}$ of triples $(k,h,j) \in \mathcal{D}_{K,t}$ for which $\frac{ka^h}{j}$ is not $\equiv -u_{a,i_l} \bmod q_{a,i_l}$ for any $K \le i_l \le r$, $l \equiv d \bmod n$ is

$$\le 2K^2(t+1) \prod_{\substack{i_l \in \mathcal{I}_a \\ l \equiv d \bmod n}} \left(1 - \frac{1}{2K^2\, i_l}\right). \tag{2.3}$$

In order to assign congruence classes for the remainders from each step of this process, we now list the numbers labeled by the triples counted by $\sum_{d=1}^n R_{d,a}$, meaning that the remaining elements of the form $\frac{ka^h}{j}$ not covered by the chosen residues classes for each $1 \le d \le n$. We introduce the notation given by the list: $\{v_{a,d,f} : 1 \le f \le R_{d,a},\, 2 \le a \le K,\, 1 \le d \le n\}$. For each element in this list we assign a prime number among the first $\sum_{a=2}^K \sum_{d=1}^n R_{d,a} + rK$ primes which are not in $\cup_{a=2}^K \mathcal{Q}_a$, denoted by the elements of the following list: $\{Q_{a,d,f} : 1 \le f \le R_{d,a},\, 2 \le a \le K,\, 1 \le d \le n\}$. Note that the number of primes in $\cup_{a=2}^K \mathcal{Q}_a$ is at most $(K-1)(r-K+1) \le rK$.

Using the construction above, we consider the following system of congruences:

$$\begin{aligned} &p \equiv u_{a,i_l} \bmod q_{a,i_l},\ i_l \in \mathcal{I}_a,\ 2 \le a \le K, \\ &p \equiv -v_{a,d,f} \bmod Q_{a,d,f},\ 1 \le f \le R_{d,a},\ 2 \le a \le K,\ 1 \le d \le n. \end{aligned} \tag{2.4}$$

By the Chinese Remainder Theorem, the solution to the system of congruences (2.4) is unique modulo $(\prod q_{a,i_l} \prod Q_{a,d,f})$. If a prime p is a solution to (2.4), then for all triples $(k,h,j) \in \mathcal{D}_{K,t}$, each $pj + a^h k$ is $\equiv 0$ modulo at least n distinct primes. Indeed, let $(k,h,j) \in \mathcal{D}_{K,t}$. For every $1 \le d \le n$, in the above construction we determine a congruence class modulo a prime among either $\{q_{a,i_l}\}$ or $\{Q_{a,d,f}\}$ occupied by $\frac{ka^h}{j}$, call this class $-u \bmod q$. Then $p \equiv u \bmod q \equiv -\frac{ka^h}{j} \bmod q$ which is equivalent to the congruence $pj + a^h k \equiv 0 \mod q$. So for each $pj + a^h k$, we have at least n distinct primes q (one for each choice of d) dividing $pj + a^h k$.

Next, we show that the modulus $(\prod q_{a,i_l} \prod Q_{a,d,f})$ is not too large. First note that, by the construction of $\mathcal{Q}_a$, for any $a \in \{2, 3, \ldots K\}$, and $l \geq 1$ such that $q_{a,i_l} \in \mathcal{Q}_a$, we have $K \leq i_l$ and $0 < i_{l+1} - i_l \leq K$. Moreover, using the construction given in Lemma 2.2 for the sets $\mathcal{Q}_a$, we have $i_1 \leq 2K + 2$. So for any $2 \geq d \geq n$, we have $i_d \leq i_1 + dK \leq (d+2)K + 2$, and similarly we have $i_{d+en} \leq (d + en + 2)K + 2$. Note that there will be at least $\left\lfloor \frac{r-(d+4)K}{nK} \right\rfloor$ elements of the form i_{d+en}, since the i_l only go up to r.

Therefore,

$$\sum_{\substack{i_l \in \mathcal{I}_a \\ l \equiv d \bmod n}} \frac{1}{i_l} \geq \sum_{e=0}^{\left\lfloor \frac{r-(d+4)K}{nK} \right\rfloor} \frac{1}{(d + en + 2)K + 2}$$
$$\geq \frac{1}{K} \sum_{e=0}^{\left\lfloor \frac{r-(d+4)K}{nK} \right\rfloor} \frac{1}{d + en + 3} \geq \frac{1}{nK} \sum_{e=0}^{\left\lfloor \frac{r-(d+4)K}{nK} \right\rfloor} \frac{1}{e + 5}.$$

Since

$$\frac{r - (d + 4)K}{(n + 1)K} \geq (\log x)^{\frac{1}{4}},$$

for large x, we have that

$$\sum_{\substack{i_l \in \mathcal{I}_a \\ l \equiv d \bmod n}} \frac{1}{i_l} \geq \frac{1}{nK} \left(\frac{1}{5} \log \log x \right) \gg (\log \log x)^{\frac{\epsilon}{2}}.$$

Thus,

$$\prod_{\substack{i_l \in \mathcal{I}_a \\ l \equiv d \bmod n}} \left(1 - \frac{1}{2K^2 i_l}\right) \leq \exp\left\{ \frac{-1}{2K^2} (\log \log x)^{\frac{\epsilon}{2}} \right\} \ll \exp\left\{ -(\log \log x)^{\frac{\epsilon}{3}} \right\}.$$

Hence, recalling the upper bound in (2.3), we obtain

$$\sum_{a=2}^{K} \sum_{d=1}^{n} R_{d,a} \ll_K K^2 t \sum_{a=2}^{K} \sum_{d=1}^{n} \exp\left\{ -(\log \log x)^{\frac{\epsilon}{3}} \right\}$$
$$\ll_K n \log x \exp\left\{ -(\log \log x)^{\frac{\epsilon}{3}} \right\}$$
$$\ll_K \frac{\log x \log \log x}{(\log \log x)^{\epsilon} \exp\{(\log \log x)^{\frac{\epsilon}{3}}\}} \ll_K \frac{\log x}{\exp\{(\log \log x)^{\frac{\epsilon}{4}}\}}.$$

Since the product of the first ℓ primes is $\exp\{(1+o(1))\ell\log\ell\}$, and since the primes labeled by $Q_{a,d,f}$ lie in the first $\sum_{a=2}^{K}\sum_{d=1}^{n} R_{d,a}+rK$ primes, we have the following upper bound for the product $\prod Q_{a,d,f}$:

$$\prod Q_{a,d,f} \ll_K \exp\left\{C'\frac{\log x\log\log x}{\exp\left\{(\log\log x)^{\frac{\epsilon}{4}}\right\}}\right\} \ll_K x^{\frac{\epsilon}{2}}. \tag{2.5}$$

On the other hand, as for each $q_{a,i_l}\in\mathcal{Q}_a,\quad q_{a,i_l}\le a^{i_l}$,

$$\prod_{a=2}^{K}\prod_{i_l\in\mathcal{I}_a} q_{a,i_l} \le \prod_{a=2}^{K}\prod_{h=1}^{r} a^h \le (K!)^{\frac{r(r+1)}{2}} \ll_K \exp\{C(\log x)^{\frac{2}{3}}\} \ll_K x^{\frac{\epsilon}{2}}. \tag{2.6}$$

Thus, combining (2.5) and (2.6), we seek for primes in a certain arithmetic progression where the modulus is $\ll x^{\epsilon}$. Hence we apply Corollary 2.2 to finish the proof of Theorem 1.3. □

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