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A transference principle for simultaneous rational approximation

par Ngoc Ai Van NGUYEN, Anthony POËLS et Damien ROY

Résumé. Nous établissons pour tout entier $n \geq 1$ un principe de transfert général concernant la mesure d'irrationalité des points de $\mathbb{R}^{n+1}$ dont les coordonnées sont linéairement indépendantes sur $\mathbb{Q}$. Partant de là nous retrouvons une inégalité importante de Marnat et Moshchevitin qui décrit le spectre conjoint des exposants ordinaire et uniforme d'approximation rationnelle pour ces points. Lorsque les exposants d'un point réalisent quasiment l'égalité, nous fournissons davantage d'informations sur la suite de ses meilleures approximations rationnelles. Nous concluons avec une application.

Abstract. We establish a general transference principle about the irrationality measure of points with $\mathbb{Q}$-linearly independent coordinates in $\mathbb{R}^{n+1}$, for any given integer $n \geq 1$. On this basis, we recover an important inequality of Marnat and Moshchevitin which describes the spectrum of the pairs of ordinary and uniform exponents of rational approximation to those points. For points whose pair of exponents are close to the boundary in the sense that they almost realize the equality, we provide additional information about the corresponding sequence of best rational approximations. We conclude with an application.

1. Introduction

Let n be a positive integer and let $\boldsymbol{\xi} = (\xi_0, \dots, \xi_n)$ be a point of $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$. For any integer point $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ we set

$$L_{\boldsymbol{\xi}}(\mathbf{x}) = \max_{1 \leq k \leq n} |\xi_0 x_k - \xi_k x_0|,$$

and for each $X \geq 1$ we define

$$\mathcal{L}_{\boldsymbol{\xi}}(X) = \min\{L_{\boldsymbol{\xi}}(\mathbf{x})\,;\, \mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\}, \|\mathbf{x}\| \leq X\}, \tag{1.1}$$

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where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^{n+1}$. The behavior of this irrationality measure $\mathcal{L}_{\boldsymbol{\xi}}$ is roughly captured by the quantities

$$
\begin{aligned}
\lambda(\boldsymbol{\xi}) &= \sup\Big\{\lambda\,;\ \liminf_{X\to\infty} X^{\lambda}\mathcal{L}_{\boldsymbol{\xi}}(X) < \infty\Big\},\\
\widehat{\lambda}(\boldsymbol{\xi}) &= \sup\Big\{\lambda\,;\ \limsup_{X\to\infty} X^{\lambda}\mathcal{L}_{\boldsymbol{\xi}}(X) < \infty\Big\}
\end{aligned}
\tag{1.2}
$$

which are called respectively the ordinary and the uniform exponents of rational approximation to $\boldsymbol{\xi}$. It is well known that they satisfy

$$
\frac{1}{n} \le \widehat{\lambda}(\boldsymbol{\xi}) \le 1 \quad\text{and}\quad \widehat{\lambda}(\boldsymbol{\xi}) \le \lambda(\boldsymbol{\xi}) \le \infty, \tag{1.3}
$$

the inequality $\widehat{\lambda}(\boldsymbol{\xi}) \ge 1/n$ coming from Dirichlet's box principle [13, Theorem 1A, Chapter II]. The study of such Diophantine exponents goes back to Jarník [1] and Khinchine [2, 3] and remains a topic of much research. Recently Marnat and Moshchevitin [6] proved the following inequality conjectured by Schmidt and Summerer [15, Section 3, p. 92].

Theorem 1.1 (Marnat–Moshchevitin)**.** *Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ be a point whose coordinates are linearly independent over $\mathbb{Q}$. We have*

$$
\widehat{\lambda}(\boldsymbol{\xi}) + \frac{\widehat{\lambda}(\boldsymbol{\xi})^2}{\lambda(\boldsymbol{\xi})} + \cdots + \frac{\widehat{\lambda}(\boldsymbol{\xi})^n}{\lambda(\boldsymbol{\xi})^{n-1}} \le 1, \tag{1.4}
$$

the ratio $\widehat{\lambda}(\boldsymbol{\xi})/\lambda(\boldsymbol{\xi})$ being interpreted as 0 *when $\lambda(\boldsymbol{\xi}) = \infty$.*

The formulation given by Marnat and Moshchevitin in [6] is slightly different and is complemented by a similar result for the dual pair of exponents which we omit here. These authors also show that (1.3) and (1.4) give a complete description of the set of values taken by $(\lambda, \widehat{\lambda})$ at points $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ with $\mathbb{Q}$-linearly independent coordinates. Previous to [6], the problem had been considered by several authors. The case $n = 1$ of Theorem 1.1 is classical, as it reduces to (1.3). The case $n = 2$ is a corollary of the work of Laurent [5]. The case $n = 3$ was established by Moshchevitin in [7], and revisited by Schmidt and Summerer using parametric geometry of numbers in [15]. For an alternative proof of the results of [6] based only on parametric geometry of numbers together with partial results towards a more general conjecture, see the PhD thesis of Rivard–Cooke [10, Chapter 2].

Given a subset S of $\mathbb{Z}^{n+1}$, we define for each $X \ge 1$

$$
\mathcal{L}_{\boldsymbol{\xi}}(X; S) = \min\{L_{\boldsymbol{\xi}}(\mathbf{x})\,;\ \mathbf{x} \in S \text{ and } 0 < \|\mathbf{x}\| \le X\},
$$

with the convention that $\min\emptyset = \infty$. When $S \not\subseteq \{0\}$, that function is eventually finite and monotonic decreasing. Then, upon replacing $\mathcal{L}_{\boldsymbol{\xi}}(X)$ by $\mathcal{L}_{\boldsymbol{\xi}}(X; S)$ in (1.2) we obtain two exponents $\lambda(\boldsymbol{\xi}; S)$, $\widehat{\lambda}(\boldsymbol{\xi}; S)$ which satisfy

$$
0 \le \widehat{\lambda}(\boldsymbol{\xi}; S) \le \widehat{\lambda}(\boldsymbol{\xi}) \le 1 \quad\text{and}\quad \lambda(\boldsymbol{\xi}, S) \le \lambda(\boldsymbol{\xi}). \tag{1.5}
$$

In particular, we have $\lambda(\boldsymbol{\xi};\mathbb{Z}^{n+1}) = \lambda(\boldsymbol{\xi})$ and $\widehat{\lambda}(\boldsymbol{\xi};\mathbb{Z}^{n+1}) = \widehat{\lambda}(\boldsymbol{\xi})$.

The next result gives further information about the behaviour of $\mathcal{L}_{\boldsymbol{\xi}}(X;S)$ as a function of X.

Theorem 1.2. *Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ with $\mathbb{Q}$-linearly independent coordinates and let $S \subseteq \mathbb{Z}^{n+1}$. Suppose that there exist positive real numbers a, b, α, β such that*

$$bX^{-\beta} \le \mathcal{L}_{\boldsymbol{\xi}}(X;S) \le aX^{-\alpha} \tag{1.6}$$

for each large enough real number X. Then α and β satisfy

$$\alpha + \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}} \le 1. \tag{1.7}$$

In case of equality in (1.7)*, we have*

$$\limsup_{X\to\infty} X^{\alpha}\mathcal{L}_{\boldsymbol{\xi}}(X;S) > 0 \quad \text{and} \quad \liminf_{X\to\infty} X^{\beta}\mathcal{L}_{\boldsymbol{\xi}}(X;S) < \infty, \tag{1.8}$$

thus $\alpha = \widehat{\lambda}(\boldsymbol{\xi};S)$ and $\beta = \lambda(\boldsymbol{\xi};S)$.

Assuming that $\widehat{\lambda}(\boldsymbol{\xi};S) > 0$, the first part of Theorem 1.2 implies that

$$\widehat{\lambda}(\boldsymbol{\xi};S) + \frac{\widehat{\lambda}(\boldsymbol{\xi};S)^2}{\lambda(\boldsymbol{\xi};S)} + \cdots + \frac{\widehat{\lambda}(\boldsymbol{\xi};S)^n}{\lambda(\boldsymbol{\xi};S)^{n-1}} \le 1, \tag{1.9}$$

which gives Theorem 1.1 by choosing $S = \mathbb{Z}^{n+1}$. Indeed, if $\lambda(\boldsymbol{\xi};S) < \infty$, then (1.6) holds for X large enough with $a = b = 1$ and any choice of α, β with $0 < \alpha < \widehat{\lambda}(\boldsymbol{\xi};S)$ and $\beta > \lambda(\boldsymbol{\xi};S)$. Inequality (1.7) then gives (1.9) by letting α tend to $\widehat{\lambda}(\boldsymbol{\xi};S)$ and β to $\lambda(\boldsymbol{\xi};S)$. Otherwise, we have $\lambda(\boldsymbol{\xi};S) = \infty$ and (1.9) holds trivially since $\widehat{\lambda}(\boldsymbol{\xi};S) \le 1$. Another application of Theorem 1.2 is given in Section 6.

Rather than taking monomials to control the function $\mathcal{L}_{\boldsymbol{\xi}}$, we now turn to a more general setting in the spirit of [1]. The following transference principle is our main result. As we will see, it implies Theorem 1.2.

Theorem 1.3. *Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ with $\mathbb{Q}$-linearly independent coordinates and let $S \subseteq \mathbb{Z}^{n+1}$. Suppose that there exist an unbounded subinterval I of $(0,\infty)$, a point $A \in I$ and continuous functions $\varphi, \psi, \vartheta : I \to (0,\infty)$ with the following properties.*

(i) *We have $\psi(X) \le \mathcal{L}_{\boldsymbol{\xi}}(X;S) \le \varphi(X)$ for each $X \ge A$.*

(ii) *The functions φ and ψ are strictly decreasing, whereas ϑ is increasing with*

$$\lim_{X\to\infty} \varphi(X) = \lim_{X\to\infty} \psi(X) = 0 \quad \text{and} \quad \lim_{X\to\infty} \vartheta(X) = \infty.$$

(iii) *For each $k = 1, \dots, n-1$, the k-th iterate ϑ^k of ϑ maps $[A,\infty)$ to I.*

(iv) *We have $\varphi(X) = \psi(\vartheta(X))$ for each $X \ge A$.*

(v) *The functions* $\varphi_0, \dots, \varphi_{n-1}$, $\Phi_0, \dots, \Phi_{n-1}$ *defined on* $[A, \infty)$ *by*

$$\varphi_0(X) = \varphi(X) \tag{1.10}$$

$$\varphi_k(X) = \varphi(\vartheta^k(X)) \cdots \varphi(\vartheta(X))\varphi(X) \quad (1 \le k < n), \tag{1.11}$$

$$\Phi_k(X) = X\varphi_k(X) \quad (0 \le k \le n-1) \tag{1.12}$$

have the property that Φ_0 *is monotonically increasing and that the remaining* $\Phi_1, \dots, \Phi_{n-1}$ *are monotonic (either decreasing or increasing).*

Then $\Phi_0, \dots, \Phi_{n-2}$ *are monotonically increasing and we have*

$$\Phi_{n-1} \ge c, \tag{1.13}$$

for some constant $c > 0$ *depending only on* $\boldsymbol{\xi}$.

Note that since φ is decreasing and ϑ is increasing, each function φ_k is decreasing and tends to 0. The most natural choice for the functions $\varphi, \varphi, \vartheta$ is to take monomials in X as below. In doing so, we now prove that Theorem 1.3 implies Theorem 1.2. With the notation of Theorem 1.2, the functions ψ, φ, ϑ defined for each $X > 0$ by

$$(\psi, \varphi, \vartheta)(X) = \Big(bX^{-\beta}, aX^{-\alpha}, \Big(\frac{a}{b}\Big)^{-1/\beta} X^{\alpha/\beta}\Big) \tag{1.14}$$

satisfy $\varphi = \psi \circ \vartheta$. Moreover since $\alpha \le \widehat{\lambda}(\boldsymbol{\xi}, S) \le 1$ by (1.5), the product $\Phi_0(X) = X\varphi(X) = aX^{1-\alpha}$ is monotonically increasing for $X > 0$. For each k with $0 \le k \le n-1$ and $X > 0$ we have

$$\varphi(\vartheta^k(X)) = a\Big(\frac{a}{b}\Big)^{(\alpha/\beta)+\cdots+(\alpha/\beta)^k} X^{-\alpha^{k+1}/\beta^k}, \tag{1.15}$$

and so the functions $\Phi_1, \dots, \Phi_{n-1}$ defined by (1.12) are monotonic. Thus φ, ψ, ϑ satisfy Conditions (ii) to (v) of Theorem 1.3, and Condition (i) amounts to Condition (1.6) of Theorem 1.2. Furthermore note that there is a positive number $\delta > 0$ (which is a polynomial in α/β) such that for each $X > 0$ we have

$$\Phi_{n-1}(X) = a^n\Big(\frac{a}{b}\Big)^{\delta} X^{\varepsilon}, \quad \text{where } \varepsilon = 1 - \Big(\alpha + \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}}\Big). \tag{1.16}$$

By (1.13) we then get (1.7), namely $\varepsilon \ge 0$. This in turn implies (1.9) as explained after Theorem 1.2. Suppose now that $\varepsilon = 0$. Since $\alpha \le \widehat{\lambda}(\boldsymbol{\xi}; S)$ and $\beta \ge \lambda(\boldsymbol{\xi}; S)$, we thus have

$$1 = \alpha + \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}} \le \widehat{\lambda}(\boldsymbol{\xi}; S) + \frac{\widehat{\lambda}(\boldsymbol{\xi}; S)^2}{\lambda(\boldsymbol{\xi}; S)} + \cdots + \frac{\widehat{\lambda}(\boldsymbol{\xi}; S)^n}{\lambda(\boldsymbol{\xi}; S)^{n-1}} \le 1,$$

and we conclude that $\alpha = \widehat{\lambda}(\boldsymbol{\xi}; S)$ and $\beta = \lambda(\boldsymbol{\xi}; S)$. Moreover, by using once again (1.13), (1.16) and the hypothesis that $\varepsilon = 0$, we obtain

$$a^n\Big(\frac{a}{b}\Big)^\delta \ge c,$$

where c is given by (1.13). It means that in (1.6), we cannot replace a by a constant strictly smaller that $a' = (cb^\delta)^{1/(n+\delta)}$ and b by a constant strictly larger than $b' = (a^{n+\delta}/c)^{1/\delta}$. This proves (1.8) with the superior limit $\ge a'$ and the inferior limit $\le b'$.

Remark 1.4. Clearly, Conditions (ii) to (v) apply to many more general classes of functions φ and ψ. For example, we can take

$$\varphi(X) = aX^{-\alpha}\log^\sigma(X) \quad \text{and} \quad \psi(X) = bX^{-\beta}\log^\rho(X)$$

for suitable positive numbers a, b, α, β and real numbers σ, ρ.

The next result complements Theorem 1.2.

Theorem 1.5. *Let $n > 1$, let $\boldsymbol{\xi}$ be a point of $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ and let $S \subseteq \mathbb{Z}^{n+1}$. Suppose that there are positive real numbers a, b, α, β such that*

$$bX^{-\beta} \le \mathcal{L}_{\boldsymbol{\xi}}(X; S) \le aX^{-\alpha} \tag{1.17}$$

for each sufficiently large real number X. Then we have $\alpha \le \beta$ and

$$\varepsilon := 1 - \Big(\alpha + \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}}\Big) \ge 0. \tag{1.18}$$

Moreover, there exists a constant $C > 0$ which depends only on $\boldsymbol{\xi}, a, b, \alpha, \beta$ with the following property. If

$$\varepsilon \le \frac{1}{4n}\Big(\frac{\alpha}{\beta}\Big)^n \min\{\alpha, \beta - \alpha\}, \tag{1.19}$$

then there is an unbounded sequence $(\mathbf{y}_i)_{i\ge 0}$ of non-zero integer points in S which for each $i \ge 0$ satisfies the following conditions:

- (i) $\big|\alpha \log\|\mathbf{y}_{i+1}\| - \beta \log\|\mathbf{y}_i\|\big| \le C + 4\varepsilon(\beta/\alpha)^n \log\|\mathbf{y}_{i+1}\|$;
- (ii) $\big|\log L_{\boldsymbol{\xi}}(\mathbf{y}_i) + \beta \log\|\mathbf{y}_i\|\big| \le C + 4\varepsilon(\beta/\alpha)^2 \log\|\mathbf{y}_i\|$;
- (iii) $\det(\mathbf{y}_i, \dots, \mathbf{y}_{i+n}) \ne 0$;
- (iv) *there exists no* $\mathbf{x} \in S \setminus \{0\}$ *with* $\|\mathbf{x}\| < \|\mathbf{y}_i\|$ *and* $L_{\boldsymbol{\xi}}(\mathbf{x}) \le L_{\boldsymbol{\xi}}(\mathbf{y}_i)$.

For a point $\boldsymbol{\xi}$ of the form $\boldsymbol{\xi} = (1, \xi, \xi^2)$ with $\xi \in \mathbb{R}$ not algebraic of degree at most 2 over $\mathbb{Q}$, satisfying (1.17) with $S = \mathbb{Z}^3$, $\beta = 1$ and $\varepsilon = 0$, we recover a construction of the third author [11, Theorem 5.1] dealing with extremal numbers. For a point $\boldsymbol{\xi} = (1, \vartheta, \dots, \vartheta^n, \xi)$ with $\vartheta \in \mathbb{R}$ algebraic of degree n over $\mathbb{Q}$ and $\xi \in \mathbb{R} \setminus \mathbb{Q}(\vartheta)$, satisfying (1.17) with $S = \mathbb{Z}^{n+1}$, $\beta = 1/(n-1)$ and $\varepsilon = 0$, the result is due to the first author [8, Theorem 2.4.3].

Remark 1.6. As the proof will show, the upper bound for ε in (1.19) and the coefficients of ε in (i) and (ii) can easily be improved.

This paper is organized as follows. In Section 2 we set the notation and we recall the definition of minimal points. Section 3 is devoted to our main tool which is a construction of subspaces of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$, together with inequalities relating their heights. The proofs of Theorems 1.3 and 1.5 follow in Sections 4 and 5 respectively. Finally, some applications of our results are presented in the last section.

2. Notation, heights and minimal points

Given points $\mathbf{y}_1, \mathbf{y}_2, \dots$ of $\mathbb{R}^{n+1}$, we denote by $\langle \mathbf{y}_1, \mathbf{y}_2, \dots \rangle_{\mathbb{R}}$ the vector subspace of $\mathbb{R}^{n+1}$ that they span. Recall that we endow $\mathbb{R}^{n+1}$ with its usual structure of inner product space and that we denote by $\|\cdot\|$ the corresponding Euclidean norm. In general, for any integer $k = 1, \dots, n+1$, we endow the vector space $\bigwedge^k(\mathbb{R}^{n+1})$ with the unique structure of inner product space such that, for any orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^{n+1}$, the products $e_{i_1} \wedge \cdots \wedge e_{i_k}$ $(i_1 < \cdots < i_k)$ form an orthonormal basis of $\bigwedge^k(\mathbb{R}^{n+1})$. We still denote by $\|\cdot\|$ the associated norm.

If W is a subspace of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$, we define its *height* $H(W)$ as the co-volume in W of the lattice of integer points $W \cap \mathbb{Z}^{n+1}$. If $\dim W = k$, this is given by

$$H(W) = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\|$$

for any $\mathbb{Z}$-basis $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of $W \cap \mathbb{Z}^{n+1}$. Schmidt proved the following result [14, Chapter 1, Lemma 8A].

Theorem 2.1 (Schmidt)**.** *There exists a positive constant c which depends only on n such that for any subspaces A, B of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$, we have*

$$H(A+B)\,H(A\cap B) \le cH(A)\,H(B).$$

If $f, g : I \to [0, +\infty)$ are two functions on a set I, we write $f = \mathcal{O}(g)$ or $f \ll g$ or $g \gg f$ to mean that there is a positive constant c such that $f(x) \le cg(x)$ for each $x \in I$. We write $f \asymp g$ when both $f \ll g$ and $g \ll f$.

When $S \subseteq \mathbb{Z}^{n+1}$ is such that $\lim_{X\to\infty} \mathcal{L}_{\boldsymbol{\xi}}(X; S) = 0$, there exists a sequence $(\mathbf{x}_i)_{i\ge 0}$ of non-zero points in S satisfying

(a) $\|\mathbf{x}_0\| < \|\mathbf{x}_1\| < \|\mathbf{x}_2\| < \dots$
(b) $L_{\boldsymbol{\xi}}(\mathbf{x}_0) > L_{\boldsymbol{\xi}}(\mathbf{x}_1) > L_{\boldsymbol{\xi}}(\mathbf{x}_2) > \dots$
(c) For any $i \ge 0$ and any non-zero point $\mathbf{z} \in S$ with $\|\mathbf{z}\| < \|\mathbf{x}_{i+1}\|$, we have $L_{\boldsymbol{\xi}}(\mathbf{z}) \ge L_{\boldsymbol{\xi}}(\mathbf{x}_i)$.

We say that such a sequence is a sequence of *minimal points* for $\boldsymbol{\xi}$ with respect to S. Minimal points are a standard tool for studying rational approximation. The usual choice is to take $S = \mathbb{Z}^{n+1}$.

3. Families of vector subspaces

The goal of this section is to prove the following key-theorem established by the first author in her thesis [8, Section 2.3] in the case where $S = \mathbb{Z}^{n+1}$. The proof in the general case is the same. In this section n is an integer > 1.

Theorem 3.1. *Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ with $\mathbb{Q}$-linearly independent coordinates. Suppose that for some $S \subseteq \mathbb{Z}^{n+1}$ we have $\lim_{X\to\infty} \mathcal{L}_{\boldsymbol{\xi}}(X;S) = 0$. Let $(\mathbf{x}_i)_{i\geq 0}$ be a sequence of minimal points for $\boldsymbol{\xi}$ with respect to S. For each $i \geq 0$, set*

$$X_i = \|\mathbf{x}_i\| \quad \textit{and} \quad L_i = \mathcal{L}_{\boldsymbol{\xi}}(X_i;S) = L_{\boldsymbol{\xi}}(\mathbf{x}_i).$$

Fix also an index $i_0 \geq 0$. Then for each $t = 1,\dots,n-1$ there exists a largest integer i_t with $i_t \geq i_0$ such that

$$\dim\langle \mathbf{x}_{i_0}, \mathbf{x}_{i_0+1}, \dots, \mathbf{x}_{i_t}\rangle_{\mathbb{R}} = t+1. \tag{3.1}$$

For these indices $i_0 < i_1 < \cdots < i_{n-1}$, we have

$$X_{i_1}\cdots X_{i_{n-1}} \leq c L_{i_0} X_{i_0+1} \cdots L_{i_{n-1}} X_{i_{n-1}+1}$$

with a constant $c > 0$ depending only on $\boldsymbol{\xi}$ and not on i_0.

We first note that under the conditions of Theorem 3.1, each subsequence $(\mathbf{y}_i)_{i\in\mathbb{N}}$ of $(\mathbf{x}_i)_{i\in\mathbb{N}}$ spans $\mathbb{R}^{n+1}$. Indeed, suppose by contradiction that a subsequence $(\mathbf{y}_i)_{i\in\mathbb{N}}$ spans a proper subspace W of $\mathbb{R}^{n+1}$. Since $(\mathbf{y}_i)_{i\in\mathbb{N}}$ converges to $\boldsymbol{\xi}$ projectively, we deduce that $\boldsymbol{\xi} \in W$, which is impossible since W is defined by linear equations with coefficients in $\mathbb{Q}$ while the coordinates of $\boldsymbol{\xi}$ are linearly independent over $\mathbb{Q}$. In particular, $(\mathbf{x}_i)_{i\geq i_0}$ spans $\mathbb{R}^{n+1}$ for the given index i_0, and the existence of $i_1,\dots,i_{n-1}$ follows.

Clearly we have $i_0 < i_1 < \cdots < i_{n-1}$. For simplicity, we set

$$\mathcal{V}[i,j] := \langle \mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_j\rangle_{\mathbb{R}}$$

for each pair of integers i, j with $0 \leq i \leq j$. Then, for each $t = 0,1,\dots,n-1$, we have

$$\dim \mathcal{V}[i_0,i_t] = t+1 \quad \text{and} \quad \dim \mathcal{V}[i_0, i_t+1] = t+2,$$

thus $\mathbf{x}_{i_t+1} \notin \mathcal{V}[i_0,i_t]$. By comparing dimensions, we deduce that

$$\mathbb{R}^{n+1} = \mathcal{V}[i_0, i_{n-1}+1] \quad \text{and} \quad \mathcal{V}[i_0, i_{t-1}+1] = \mathcal{V}[i_0,i_t] \quad (1 \leq t \leq n-1).$$

For each $(t,k) \in \mathbb{N}^2$ with $1 \leq k \leq t+1 \leq n$, we define

$$V_t^{k+1} = \mathcal{V}[s(t,k), i_t+1] \quad \text{and} \quad U_t^k = \mathcal{V}[s(t,k), i_t],$$

where $s(t,k)$ is the largest integer with $s(t,k) \leq i_t$ for which we have $\dim V_t^{k+1} = k+1$. By varying k for fixed t, we obtain a decreasing sequence

$$s(t,1) = i_t > s(t,2) > \cdots > s(t,t+1) \geq i_0.$$

Thus U_t^k is contained in $\mathcal{V}[i_0, i_t]$, so $\mathbf{x}_{i_t+1} \notin U_t^k$ and therefore $\dim U_t^k = k$. Moreover, when $2 \le k \le t+1$, we have $s(t,k) < s(t,k-1) \le i_t$, thus

$$V_t^{k+1} = U_t^k + V_t^k \tag{3.2}$$

is the sum of two distinct k-dimensional subspaces. Since U_t^k and V_t^k both contain U_t^{k-1}, we deduce that

$$U_t^{k-1} = U_t^k \cap V_t^k, \tag{3.3}$$

as both sides have dimension $k-1$. Finally, we note that, for $t = 1, \dots, n-1$, the subspaces U_t^{t+1} and V_{t-1}^{t+1} are both contained in $\mathcal{V}[i_0, i_t] = \mathcal{V}[i_0, i_{t-1}+1]$. Since all of these have dimension $t+1$, we conclude that

$$U_t^{t+1} = \mathcal{V}[i_0, i_{t-1}+1] = V_{t-1}^{t+1}. \tag{3.4}$$

The proof of Theorem 3.1 relies on the following lemma relating the heights of the above families of subspaces.

Lemma 3.2. *For each $k = 1, \dots, n-1$, we have*

$$H(U_k^k)H(U_{k+1}^k)\cdots H(U_{n-1}^k) \ll H(V_{k-1}^{k+1})H(V_k^{k+1})\cdots H(V_{n-1}^{k+1}) \tag{3.5}$$

with an implicit constant depending only on n.

Proof. We proceed by descending induction on k. By (3.2), we have

$$\mathbb{R}^{n+1} = V_{n-1}^{n+1} = U_{n-1}^n + V_{n-1}^n.$$

Since (3.3) gives $U_{n-1}^{n-1} = U_{n-1}^n \cap V_{n-1}^n$, it follows from Schmidt's Theorem 2.1 that

$$H(U_{n-1}^{n-1}) \ll H(U_{n-1}^n)H(V_{n-1}^n)$$

because $H(\mathbb{R}^{n+1}) = 1$.As (3.4) gives $H(U_{n-1}^n) = H(V_{n-2}^n)$, this proves (3.5) for $k = n-1$.

Assume that (3.5) holds for some k with $1 < k \le n-1$. By Theorem 2.1, the relations (3.2) and (3.3) imply that

$$H(V_t^{k+1}) \ll \frac{H(U_t^k)H(V_t^k)}{H(U_t^{k-1})}$$

for each $t = k-1, \dots, n-1$. Combining this with the induction hypothesis, we obtain

$$H(U_k^k)\cdots H(U_{n-1}^k) \ll \frac{H(U_{k-1}^k)H(V_{k-1}^k)}{H(U_{k-1}^{k-1})}\cdots\frac{H(U_{n-1}^k)H(V_{n-1}^k)}{H(U_{n-1}^{k-1})}.$$

After simplification, this leads to

$$H(U_{k-1}^{k-1})\cdots H(U_{n-1}^{k-1}) \ll H(U_{k-1}^k)H(V_{k-1}^k)\cdots H(V_{n-1}^k).$$

Since $U_{k-1}^k = V_{k-2}^k$ by (3.4), this yields (3.5) with k replaced by $k-1$. Thus, by induction, (3.5) holds for all $k = 1, \dots, n-1$. □

Proof of Theorem 3.1. By Lemma 3.2 applied with $k = 1$, we have

$$H(U_1^1)H(U_2^1)\cdots H(U_{n-1}^1) \ll H(V_0^2)H(V_1^2)\cdots H(V_{n-1}^2)$$

where $U_t^1 = \langle \mathbf{x}_{i_t}\rangle_{\mathbb{R}}$ and $V_t^2 = \langle \mathbf{x}_{i_t}, \mathbf{x}_{i_t+1}\rangle_{\mathbb{R}}$ for $t = 0, \ldots, n-1$. The conclusion follows since

$$H(U_t^1) = \|\mathbf{x}_{i_t}\| = X_{i_t} \quad \text{and} \quad H(V_t^2) \le \|\mathbf{x}_{i_t} \wedge \mathbf{x}_{i_t+1}\| \ll X_{i_t+1}L_{i_t}$$

for $t = 0, \ldots, n-1$, with implicit constants depending only on $\boldsymbol{\xi}$. □

4. Proof of Theorem 1.3

Suppose that $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$, $S \subseteq \mathbb{R}^{n+1}$, $A \in I$ and $\varphi, \psi, \vartheta : I \to (0, \infty)$ satisfy the hypotheses of Theorem 1.3, and let $(\mathbf{x}_i)_{i\ge 0}$ be a sequence of minimal points for $\boldsymbol{\xi}$ with respect to S. Since Φ_0 is monotonically increasing, the case $n = 1$ of Theorem 1.3 is trivial. Thus we may suppose that $n > 1$. As in Section 3, we write $X_i = \|\mathbf{x}_i\|$ and $L_i = L_{\boldsymbol{\xi}}(\mathbf{x}_i) = \mathcal{L}_{\boldsymbol{\xi}}(X_i; S)$ for each $i \ge 0$. Choose $k_0 \ge 0$ such that $X_{k_0} \ge A$. Then, for each $i \ge k_0$ and $\varepsilon \in (0, 1]$ we have

$$\psi(X_i) \le L_i = \mathcal{L}_{\boldsymbol{\xi}}(X_i; S) = \mathcal{L}_{\boldsymbol{\xi}}(X_{i+1} - \varepsilon; S) \le \varphi(X_{i+1} - \varepsilon)$$

by definition of minimal points. Letting ε tend to 0, we deduce that

$$L_i \le \varphi(X_{i+1}) \quad \text{and} \quad X_i \ge \vartheta(X_{i+1}) \quad (i \ge k_0), \tag{4.1}$$

because $\varphi = \psi \circ \vartheta$ is continuous and ψ is strictly decreasing. Then, for each $i_0 \ge k_0$, the sequence of integers $i_0 < \cdots < i_{n-1}$ given by Theorem 3.1 satisfies

$$X_{i_1}\cdots X_{i_{n-1}} \le c\Phi_0(X_{i_0+1})\cdots\Phi_0(X_{i_{n-1}+1}), \tag{4.2}$$

where $c = c(\boldsymbol{\xi}) > 0$ and $\Phi_0(X) = X\varphi(X)$ as in (1.12).

Lemma 4.1. *Suppose that the functions $\Phi_0, \ldots, \Phi_{m-2}$ are monotonically increasing for some integer m with $2 \le m \le n$, and let $j_0, \ldots, j_{m-1}$ be integers with $k_0 \le j_0 < \cdots < j_{m-1}$. Then we have*

$$\Phi_0(X_{j_0+1})\cdots\Phi_0(X_{j_{m-1}+1}) \le X_{j_1}\cdots X_{j_{m-1}}\Phi_{m-1}(X_{j_{m-1}+1}). \tag{4.3}$$

Proof. For simplicity set $Y_k = X_{j_k}$ and $Z_k = X_{j_k+1}$ for $k = 0, \ldots, m-1$. By induction on k, we show that

$$\prod_{\ell=0}^{m-1}\Phi_0(Z_\ell) \le \left(\prod_{\ell=1}^{k-1}Y_\ell\right)\Phi_{k-1}(Z_{k-1})\left(\prod_{\ell=k}^{m-1}\Phi_0(Z_\ell)\right) \quad (k = 1, \ldots, m). \tag{4.4}$$

The case $k = 1$ is an equality; there is nothing to prove. Suppose that (4.4) holds for some k with $1 \le k < m$. We have $Z_{k-1} \le Y_k$ since $j_{k-1} < j_k$. We also have $\vartheta(Z_k) \le Y_k$ by (4.1). Since Φ_{k-1} is monotonically increasing and φ_{k-1} is monotonically decreasing, we deduce that

$$\Phi_{k-1}(Z_{k-1}) \le \Phi_{k-1}(Y_k) = Y_k\varphi_{k-1}(Y_k) \le Y_k\varphi_{k-1}(\vartheta(Z_k)). \tag{4.5}$$

Since $\varphi_{k-1}(\vartheta(Z_k))\Phi_0(Z_k) = \Phi_k(Z_k)$, we conclude that (4.4) holds as well with k replaced by $k+1$. The inequality (4.3) corresponds to the case $k = m$. □

Lemma 4.2. *The functions $\Phi_0, \dots, \Phi_{n-2}$ are monotonically increasing.*

Proof. Otherwise there is a largest integer m with $2 \le m < n$ such that $\Phi_0, \dots, \Phi_{m-2}$ are monotonically increasing. By our choice of m, the function Φ_{m-1} is monotonically decreasing. It is thus bounded from above. Let $i_0 < i_1 < \cdots < i_{n-1}$ be integers satisfying (4.2) for a choice of $i_0 \ge k_0$. For simplicity we write $Y_k = X_{i_k}$ and $Z_k = X_{i_k+1}$ $(k = 0, \dots, n-1)$. Then, Lemma 4.1 applied to $j_0 = i_{n-m}, \dots, j_{m-1} = i_{n-1}$ implies that

$$\begin{aligned}\Phi_0(Z_{n-m}) \cdots \Phi_0(Z_{n-1}) &\le Y_{n-m+1} \cdots Y_{n-1}\Phi_{m-1}(Z_{n-1})\\ &= \mathcal{O}(Y_{n-m+1} \cdots Y_{n-1}),\end{aligned}$$

with an implicit constant depending only on Φ_{m-1}, not on i_0. Furthermore for $k = 0, \dots, n-m-1$ we have $\Phi_0(Z_k) = \varphi(Z_k)Z_k \le \varphi(Z_k)Y_{k+1} = o(Y_{k+1})$ as i_0 tends to infinity. Putting these inequalities together yields

$$\Phi_0(Z_0) \cdots \Phi_0(Z_{n-1}) = o(Y_1 \cdots Y_{n-1})$$

as i_0 tends to infinity. This contradicts (4.2). □

Proof of Theorem 1.3. Fix $i_0 < \cdots < i_{n-1}$ satisfying (4.2) for some $i_0 \ge k_0$. According to Lemma 4.2, we may apply Lemma 4.1 with $m = n$ and $j_0 = i_0, \dots, j_{m-1} = i_{n-1}$. This gives

$$\Phi_0(X_{i_0+1}) \cdots \Phi_0(X_{i_{n-1}+1}) \le X_{i_1} \cdots X_{i_{n-1}}\Phi_{n-1}(X_{i_{n-1}+1}),$$

which together with (4.2) yields $\Phi_{n-1}(X_{i_{n-1}+1}) \ge c^{-1}$. Since the function Φ_{n-1} is monotonic, we deduce that $\Phi_{n-1}(X) \ge c^{-1}$ for each X large enough, by letting i_0 go to infinity. □

5. Proof of Theorem 1.5

First, note that (1.18) follows from Theorem 1.2. So it only remains to prove the second part of Theorem 1.5. Let $(\mathbf{x}_i)_{i\ge 0}$ be a sequence of minimal points for $\boldsymbol{\xi}$ with respect to S. For each $i \ge 0$, we write

$$X_i = \|\mathbf{x}_i\| \quad \text{and} \quad L_i = \mathcal{L}_{\boldsymbol{\xi}}(X_i; S) = L_{\boldsymbol{\xi}}(\mathbf{x}_i).$$

The sequence $(\mathbf{y}_i)_{i\ge 0}$ will be constructed as a subsequence of $(\mathbf{x}_i)_{i\ge 0}$ so that Condition (iv) of Theorem 1.5 will be automatically satisfied. In this section, all implicit constants depend only on $\boldsymbol{\xi}, a, b, \alpha, \beta$. For each $X > 0$, we set

$$(\psi, \varphi, \vartheta)(X) = \Big(bX^{-\beta}, aX^{-\alpha}, \Big(\frac{a}{b}\Big)^{-1/\beta} X^{\alpha/\beta}\Big),$$

as in (1.14). Then, for $k = 0, \dots, n-1$, we denote by φ_k and Φ_k the functions defined on $(0, \infty)$ by the formulas (1.10)–(1.12) from Theorem 1.3. We also fix an index ℓ_0 such that the main hypothesis (1.17) is satisfied for each $X \geq X_{\ell_0}$.

Consider the sequence $i_0 < i_1 < \cdots < i_{n-1}$ given by Theorem 3.1 for a choice of $i_0 \geq \ell_0$. For each $k = 0, \dots, n-1$, we set

$$(\mathbf{y}_k, Y_k) = (\mathbf{x}_{i_k}, X_{i_k}) \quad \text{and} \quad (\mathbf{z}_k, Z_k) = (\mathbf{x}_{i_k+1}, X_{i_k+1}). \tag{5.1}$$

By construction, we have

$$\langle \mathbf{y}_0, \mathbf{z}_0 \rangle_{\mathbb{R}} = \langle \mathbf{y}_0, \mathbf{y}_1 \rangle_{\mathbb{R}} \quad \text{and} \quad \langle \mathbf{y}_0, \dots, \mathbf{y}_{n-1}, \mathbf{z}_{n-1} \rangle_{\mathbb{R}} = \mathbb{R}^{n+1}. \tag{5.2}$$

Using (1.15), we also find that

$$\Phi_k(X) = X\varphi_k(X) = c_k X^{\varepsilon_k} \quad \text{with } \varepsilon_k = 1 - \alpha - \cdots - \frac{\alpha^{k+1}}{\beta^k}, \tag{5.3}$$

for each $k = 0, \dots, n-1$ and each $X > 0$, where $c_k > 0$ depends only on a, b, α, β. Note that

$$\varepsilon_0 > \cdots > \varepsilon_{n-1} = \varepsilon \geq 0$$

where ε is given by (1.18). We find

$$\begin{aligned}
c^{-1} \prod_{k=1}^{n-1} Y_k &\leq \prod_{k=0}^{n-1} Z_k L_{\boldsymbol{\xi}}(\mathbf{y}_k) && \text{by Theorem 3.1,}\\
&\leq \prod_{k=0}^{n-1} \Phi_0(Z_k) && \text{by (4.1),}\\
&\leq \left(\prod_{k=1}^{n-1} Y_k \right) \Phi_{n-1}(Z_{n-1}) && \text{by Lemma 4.1 with } m = n,\\
&= \left(\prod_{k=1}^{n-1} Y_k \right) c_{n-1} Z_{n-1}^{\varepsilon} && \text{by (5.3).}
\end{aligned}$$

This uses sequentially the inequalities

$$L_{\boldsymbol{\xi}}(\mathbf{y}_k) \leq \varphi(Z_k) \quad (0 \leq k < n),$$

coming from (4.1) as well as the inequalities

$$\Phi_{k-1}(Z_{k-1}) \leq \Phi_{k-1}(Y_k) \quad \text{and} \quad \varphi_{k-1}(Y_k) \leq \varphi_{k-1}(\vartheta(Z_k)) \quad (1 \leq k < n)$$

coming from (4.5) in the proof of Lemma 4.1 with $m = n$. In each of these inequalities the ratio of the right-hand side divided by the left-hand side is therefore at most $cc_{n-1}Z_{n-1}^{\varepsilon}$. Using (5.3) and the fact that for $k = 1, \dots, n-1$ we have

$$\varepsilon_{k-1} = \varepsilon + \frac{\alpha^{k+1}}{\beta^k} + \cdots + \frac{\alpha^n}{\beta^{n-1}} \geq \alpha \left(\frac{\alpha}{\beta} \right)^k \quad \text{and} \quad 1 - \varepsilon_{k-1} \geq \alpha,$$

we thus get the following estimates

$$|\log L_{\boldsymbol{\xi}}(\mathbf{y}_k) + \alpha \log Z_k| \le \mathcal{O}(1) + \varepsilon \log Z_{n-1} \qquad (0 \le k < n), \tag{5.4}$$

$$|\log Y_k - \log Z_{k-1}| \le \mathcal{O}(1) + \frac{\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^k \log Z_{n-1} \quad (1 \le k < n), \tag{5.5}$$

$$\left|\log Y_k - \frac{\alpha}{\beta}\log Z_k\right| \le \mathcal{O}(1) + \frac{\varepsilon}{\alpha}\log Z_{n-1} \qquad (1 \le k < n). \tag{5.6}$$

Suppose from now on that ϵ satisfies the inequality (1.19) of Theorem 1.5. We distinguish two cases.

First case: $\alpha < \beta$. We start by noting that

$$\log Z_{n-1} \le \mathcal{O}(1) + 2\left(\frac{\beta}{\alpha}\right)^{n-k} \log Z_{k-1} \quad (1 \le k < n). \tag{5.7}$$

Indeed, (5.5) and (5.6) imply that

$$\log Z_k \le \mathcal{O}(1) + \frac{\beta}{\alpha}\log Z_{k-1} + \frac{2\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^{k+1} \log Z_{n-1} \quad (1 \le k < n),$$

and by descending induction starting with $k = n-1$, we obtain

$$\log Z_{n-1} \le \mathcal{O}(1) + \left(\frac{\beta}{\alpha}\right)^{n-k} \log Z_{k-1} + \frac{2(n-k)\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n} \log Z_{n-1}$$

for $k = 1, \dots, n-1$. This yields (5.7) since by (1.19) the coefficient of $\log Z_{n-1}$ in the right-hand side is less than $1/2$.

Combining (5.6) and (5.7) together with $Z_{k-1} \le Y_k$, we obtain

$$|\alpha \log Z_k - \beta \log Y_k| \le \mathcal{O}(1) + 2\varepsilon\left(\frac{\beta}{\alpha}\right)^{n-k+1} \log Y_k \quad (1 \le k < n). \tag{5.8}$$

Thus there exists a constant $C > 0$ (depending only on ξ, a, b, α, β) such that

$$\left|\log X_{i+1} - \frac{\beta}{\alpha}\log X_i\right| \le C + 2\frac{\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n} \log X_i \tag{5.9}$$

for each i among $\{i_1, i_2, \dots, i_{n-1}\}$. By (5.5) and (5.7), we also have

$$|\log Y_k - \log Z_{k-1}| \le \mathcal{O}(1) + 2\frac{\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n} \log Z_{k-1} \quad (1 \le k < n). \tag{5.10}$$

For the intermediate indices i with $i_{k-1} < i < i_k$ for some $k \in \{1, \dots, n-1\}$, we have $Z_{k-1} \le X_i < X_{i+1} \le Y_k$, and the above estimate yields

$$|\log X_{i+1} - \log X_i| \le C + 2\frac{\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n} \log X_i, \tag{5.11}$$

at the expense of replacing C by a larger constant if necessary.

By the hypothesis (1.19) on ϵ and the fact that $\beta/\alpha > 1$, the inequalities (5.9) and (5.11) cannot hold simultaneously for any sufficiently large integer i, say for any $i \geq \ell_1$ where $\ell_1 \geq \ell_0$. Define I to be the set of all integers $i \geq \ell_1$ for which (5.9) holds. Then, for a sequence $i_0 < i_1 < \cdots < i_{n-1}$ as above, with $i_0 \geq \ell_1$, we have $I \cap (i_0, i_{n-1}] = \{i_1, i_2, \dots, i_{n-1}\}$. In particular, the set I is infinite and, if we choose $i_0 \in I$, then $i_0, i_1, \dots, i_{n-1}$ are n consecutive elements of I.

Denote by $i_0 < i_1 < \cdots$ the elements of I and define $\mathbf{y}_k$, Y_k, $\mathbf{z}_k$ and Z_k by (5.1) for each $k \geq 0$. By the above, the relations (5.2) extend to

$$\langle \mathbf{y}_k, \mathbf{z}_k \rangle_{\mathbb{R}} = \langle \mathbf{y}_k, \mathbf{y}_{k+1} \rangle_{\mathbb{R}} \quad \text{and} \quad \langle \mathbf{y}_k, \dots, \mathbf{y}_{k+n-1}, \mathbf{z}_{k+n-1} \rangle_{\mathbb{R}} = \mathbb{R}^{n+1}$$

for each $k \geq 0$. Thus $\{\mathbf{y}_k, \dots, \mathbf{y}_{k+n-1}, \mathbf{y}_{k+n}\}$ spans $\mathbb{R}^{n+1}$ for each $k \geq 0$ and so $(\mathbf{y}_k)_{k \geq 0}$ satisfies Condition (iii) of the theorem. Applying (5.4), (5.7), (5.8) and (5.10) with $k = n-1$ (which is possible since $n \geq 2$), we also obtain that

$$\left|\log L_{\boldsymbol{\xi}}(\mathbf{y}_k) + \alpha \log Z_k\right| \leq \mathcal{O}(1) + \varepsilon \log Z_k, \tag{5.12}$$

$$\log Z_k \leq \mathcal{O}(1) + 2\left(\frac{\beta}{\alpha}\right) \log Z_{k-1}, \tag{5.13}$$

$$\left|\alpha \log Z_k - \beta \log Y_k\right| \leq \mathcal{O}(1) + 2\varepsilon \left(\frac{\beta}{\alpha}\right)^2 \log Y_k, \tag{5.14}$$

$$\left|\log Y_k - \log Z_{k-1}\right| \leq \mathcal{O}(1) + 2\frac{\varepsilon}{\alpha}\left(\frac{\beta}{\alpha}\right)^n \log Z_{k-1}, \tag{5.15}$$

for each $k \geq n-1$. Combining the first three inequalities (5.12)–(5.14), we find

$$\begin{aligned}
\left|\log L_{\boldsymbol{\xi}}(\mathbf{y}_k) + \beta \log Y_k\right| &\leq \mathcal{O}(1) + 2\varepsilon\left(\frac{\beta}{\alpha}\right) \log Z_{k-1} + 2\varepsilon\left(\frac{\beta}{\alpha}\right)^2 \log Y_k \\
&\leq \mathcal{O}(1) + 4\varepsilon\left(\frac{\beta}{\alpha}\right)^2 \log Y_k
\end{aligned}$$

since $Z_{k-1} \leq Y_k$. Thus Condition (ii) is fulfilled. Finally, replacing k by $k+1$ in (5.15) and using (5.14), we find

$$\begin{aligned}
\left|\alpha \log Y_{k+1} - \beta \log Y_k\right| &\leq \mathcal{O}(1) + 2\varepsilon\left(\frac{\beta}{\alpha}\right)^n \log Z_k + 2\varepsilon\left(\frac{\beta}{\alpha}\right)^2 \log Y_k \\
&\leq \mathcal{O}(1) + 4\varepsilon\left(\frac{\beta}{\alpha}\right)^n \log Y_{k+1}
\end{aligned}$$

since $Y_k \leq Z_k \leq Y_{k+1}$. Thus Condition (i) is satisfied as well.

Second case: $\alpha = \beta$. Then we have $\varepsilon = 0$ and $\alpha = \beta = 1/n$. Moreover, the hypothesis (1.17) implies that

$$L_{\boldsymbol{\xi}}(\mathbf{x}_i) \asymp X_i^{-1/n} \quad (i \geq 0). \tag{5.16}$$

Thus the estimate (5.4) with $k = 0$ yields $Y_0 \asymp Z_0$, while (5.5) and (5.6) simplify to

$$Z_0 \asymp Y_1 \asymp Z_1 \asymp \cdots \asymp Y_{n-1} \asymp Z_{n-1}.$$

Thus $\{\mathbf{x}_{i_0}, \mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_{n-1}}, \mathbf{x}_{i_{n-1}+1}\}$ is a basis of $\mathbb{R}^{n+1}$ with

$$\|\mathbf{x}_{i_0}\| \asymp \|\mathbf{x}_{i_1}\| \asymp \cdots \asymp \|\mathbf{x}_{i_{n-1}}\| \asymp \|\mathbf{x}_{i_{n-1}+1}\|. \tag{5.17}$$

We now construct recursively a subsequence $(\mathbf{y}_k)_{k\geq 0}$ of $(\mathbf{x}_i)_{i\geq 0}$ such that

$$\|\mathbf{y}_k\| \asymp \|\mathbf{y}_{k+1}\| \quad \text{and} \quad \langle \mathbf{y}_k, \dots, \mathbf{y}_{k+n}\rangle_{\mathbb{R}} = \mathbb{R}^{n+1}$$

for each $k \geq 0$. To start, we simply choose $i_0 = \ell_0$ and set

$$(\mathbf{y}_0, \dots, \mathbf{y}_n) = (\mathbf{x}_{i_0}, \dots, \mathbf{x}_{i_{n-1}}, \mathbf{x}_{i_{n-1}+1}).$$

Now suppose that $\mathbf{y}_0, \dots, \mathbf{y}_k$ have been constructed for an index $k \geq n$. Then $W = \langle \mathbf{y}_{k-n+1}, \dots, \mathbf{y}_k\rangle_{\mathbb{R}}$ is a subspace of $\mathbb{R}^{n+1}$ of dimension n. We take i_0 to be the index for which $\mathbf{y}_k = \mathbf{x}_{i_0}$. By the above there exists a point $\mathbf{y}_{k+1}$ among $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_{n-1}}, \mathbf{x}_{i_{n-1}+1}$ which lies outside of W. Then $\{\mathbf{y}_{k-n+1}, \dots, \mathbf{y}_{k+1}\}$ spans $\mathbb{R}^{n+1}$, and by (5.17) we have $\|\mathbf{y}_{k+1}\| \asymp \|\mathbf{y}_k\|$.

This sequence $(\mathbf{y}_k)_{k\geq 0}$ has all the requested properties since it also satisfies $L_{\boldsymbol{\xi}}(\mathbf{y}_k) \asymp \|\mathbf{y}_k\|^{-1/n}$ for each $k \geq 0$ by (5.16).

6. Applications

The following result is implicit in the thesis of the first author. It follows from the proof of Theorem 2.1.3 of [8] although the theorem by itself is a weaker assertion. We give a short proof based on Theorem 1.2.

Theorem 6.1. *Let ϑ be a real algebraic number of degree $n \geq 2$ and let $\xi \in \mathbb{R} \setminus \mathbb{Q}(\vartheta)$. Then the point $\boldsymbol{\xi} = (1, \vartheta, \dots, \vartheta^{n-1}, \xi) \in \mathbb{R}^{n+1}$ satisfies*

$$\widehat{\lambda}(\boldsymbol{\xi}) \leq \lambda_n \tag{6.1}$$

where λ_n is the unique positive solution of

$$x + (n-1)x^2 + \cdots + (n-1)^{n-1}x^n = 1.$$

Moreover precisely, we have

$$\limsup_{X\to\infty} X^{\lambda_n} \mathcal{L}_{\boldsymbol{\xi}}(X) > 0. \tag{6.2}$$

Proof. By Liouville's inequality, there exists a constant $c_1 = c_1(\vartheta) > 0$ such that the system

$$\max_{1\leq k\leq n-1} |y_k| \leq X^{1/(n-1)} \quad \text{and} \quad |y_0 + \vartheta y_1 + \cdots + \vartheta^{n-1} y_{n-1}| \leq c_1 X^{-1}$$

admits no non-zero integer solution $(y_0, \dots, y_{n-1})$ for any $X \geq 1$. By Khintchine's transference principle [13, Theorem 5A], there is therefore a constant $c_2 = c_2(\vartheta) > 0$ such that the dual system

$$|x_0| \leq X \quad \text{and} \quad \max_{1 \leq k \leq n-1} |x_k - \vartheta^k x_0| \leq c_2 X^{-1/(n-1)} \tag{6.3}$$

admits no non-zero integer solution $(x_0, \dots, x_{n-1})$ for each $X \geq 1$. Thus, we have

$$c_2 X^{-1/(n-1)} \leq \mathcal{L}_{\boldsymbol{\xi}}(X) \tag{6.4}$$

for each $X \geq 1$. If $\mathcal{L}_{\boldsymbol{\xi}}(X) \geq X^{-\lambda_n}$ for arbitrarily large values of X, then (6.2) is immediate. Otherwise, Condition (1.6) of Theorem 1.2 is fulfilled with $\alpha = \lambda_n$ and $\beta = 1/(n-1)$. As this yields an equality in (1.7), we again get (6.2) as a consequence of (1.8). ☐

In the case $n = 2$, the number $\lambda_2 \cong 0.618$ is the inverse of the golden ratio and it follows from [12] – which more generally deals with approximation to real points on conics in $\mathbb{P}^2(\mathbb{R})$– that the upper bound (6.1) is best possible: for any quadratic number $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$, there exists $\xi \in \mathbb{R} \setminus \mathbb{Q}(\vartheta)$ such that $\boldsymbol{\xi} = (1, \vartheta, \xi)$ satisfies $\limsup X^{\lambda_2} \mathcal{L}_{\boldsymbol{\xi}}(X) < \infty$ and $\widehat{\lambda}(\boldsymbol{\xi}) = \lambda_2$. For $n \geq 3$ the optimal upper bound is not known.

In [9], the second and the third authors apply Theorems 1.2 and 1.5 to extend the results of [4] and [12] to points on general quadratic hypersurfaces of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$.

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