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## Greenberg algebras and ramified Witt vectors

par ALESSANDRA BERTAPELLE et MAURIZIO CANDILERA

RÉSUMÉ. Soit  $\mathcal{O}$  un anneau complet pour une valuation discrète, de caractéristique mixte et de corps résiduel fini  $\kappa$ . Dans cet article, on présente l'étude d'un morphisme naturel  $r: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}, \kappa}$  entre l'algèbre de Greenberg de  $\mathcal{O}$  et la fibre spéciale du schéma des vecteurs de Witt ramifiés sur  $\mathcal{O}$ . Ce morphisme est un homéomorphisme universel avec un noyau pro-infinitésimal qui, dans certains cas, peut être décrit explicitement.

ABSTRACT. Let  $\mathcal{O}$  be a complete discrete valuation ring of mixed characteristic and with finite residue field  $\kappa$ . We study a natural morphism  $r: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}, \kappa}$  between the Greenberg algebra of  $\mathcal{O}$  and the special fiber of the scheme of ramified Witt vectors over  $\mathcal{O}$ . It is a universal homeomorphism with pro-infinitesimal kernel that can be explicitly described in some cases.

### 1. Introduction

Let  $\mathcal{O}$  be a complete discrete valuation ring with field of fractions  $K$  of characteristic 0 and perfect residue field  $\kappa$  of positive characteristic  $p$ . We fix a uniformizing parameter  $\pi \in \mathcal{O}$ . It is known from [2, 8, 10] that for any  $n \in \mathbb{N}$  one can associate a *Greenberg algebra*  $\mathbb{R}_n$  to the artinian local ring  $\mathcal{O}/\pi^n\mathcal{O}$ , i.e., the algebraic  $\kappa$ -scheme that represents the fpqc sheaf associated to the presheaf

$$\begin{aligned} \{\text{affine } \kappa\text{-schemes}\} &\rightarrow \{\mathcal{O}/\pi^n\mathcal{O}\text{-algebras}\}, \\ \text{Spec}(A) &\mapsto W(A) \otimes_{W(\kappa)} \mathcal{O}/\pi^n\mathcal{O}, \end{aligned}$$

where  $W(A)$  is the ring of  $p$ -typical Witt vectors with coefficients in  $A$ . There are canonical morphisms  $\mathbb{R}_n \rightarrow \mathbb{R}_m$  for  $n \geq m$ , and passing to the limit one gets an affine ring  $\kappa$ -scheme  $\mathbb{R}_{\mathcal{O}}$  such that  $W(A) \otimes_{W(\kappa)} \mathcal{O} = \mathbb{R}_{\mathcal{O}}(A) := \text{Hom}_{\kappa}(\text{Spec}(A), \mathbb{R}_{\mathcal{O}})$  for any  $\kappa$ -algebra  $A$ ; see (2.1). The Greenberg algebra  $\mathbb{R}_n$  is the fundamental stone for the construction of the Greenberg realization  $\text{Gr}_n(X)$  of a  $\mathcal{O}/\pi^n\mathcal{O}$ -scheme  $X$ ; this is a  $\kappa$ -scheme whose set of  $\kappa$ -rational sections coincides with  $X(\mathcal{O}/\pi^n\mathcal{O})$  [8], [2, Lemma 7.1], and it plays a role in many results in Arithmetic Geometry.

Assume that  $\kappa$  is finite. One can define for any  $\mathcal{O}$ -algebra  $A$  the algebra  $W_{\mathcal{O}}(A)$  of *ramified Witt vectors with coefficients in  $A$*  [1, 6, 7, 9, 11]. These

algebras are important objects in  $p$ -adic Hodge theory. It is well-known that if  $A$  is a perfect  $\kappa$ -algebra, there is a natural isomorphism

$$W(A) \otimes_{W(\kappa)} \mathcal{O} \simeq W_{\mathcal{O}}(A)$$

(see [7, I.1.2], [1, §1.2], [11, Proposition 1.1.26]). Hence  $\mathbb{R}_{\mathcal{O}}(A) \simeq W_{\mathcal{O}}(A)$  if  $A$  is a perfect  $\kappa$ -algebra. For a general  $\kappa$ -algebra  $A$ , Drinfeld’s map  $u: W(A) \rightarrow W_{\mathcal{O}}(A)$  induces a unique homomorphism of  $\mathcal{O}$ -algebras  $\mathbb{R}_{\mathcal{O}}(A) \rightarrow W_{\mathcal{O}}(A)$ , functorial in  $A$ . Hence there is a morphism of ring schemes over  $\kappa$

$$\mathbf{r}: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec}(\kappa),$$

where  $\mathbb{W}_{\mathcal{O}}$  is the ring scheme of ramified Witt vectors over  $\text{Spec}(\mathcal{O})$ , that is,  $W_{\mathcal{O}}(A) = \text{Hom}_{\mathcal{O}}(\text{Spec}(A), \mathbb{W}_{\mathcal{O}})$  for any  $\mathcal{O}$ -algebra  $A$ . It is not difficult to check that the morphism  $\mathbf{r}$  is surjective with pro-infinitesimal kernel. Indeed Theorem 6.2 states that

*The morphism  $\mathbf{r}: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec}(\kappa)$  induces an isomorphism  $\mathbf{r}^{\text{pf}}: \mathbb{R}_{\mathcal{O}}^{\text{pf}} \rightarrow (\mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec}(\kappa))^{\text{pf}}$  on inverse perfection.*

Hence, up to taking inverse perfection, one can identify Greenberg algebra  $\mathbb{R}_{\mathcal{O}}$  with the special fiber of the scheme of ramified Witt vectors  $\mathbb{W}_{\mathcal{O}}$ .

The morphism  $\mathbf{r}$  is deeply related to the scheme-theoretic version  $\mathbf{u}$  of Drinfeld’s functor (Proposition 5.6) and a great part of the paper is devoted to the study of  $\mathbb{W}_{\mathcal{O}}$  and  $\mathbf{u}$ . In the unramified case the special fiber of  $\mathbf{u}$  is a universal homeomorphism and we can explicitly describe its kernel (Proposition 5.12). The ramified case requires ad hoc constructions (Proposition 5.14). All these results allow a better understanding of the kernel of  $\mathbf{r}$  (Lemma 6.3); in particular if  $\mathcal{O} = W(\kappa)$  the kernel of  $\mathbf{r}$  is isomorphic to

$$\text{Spec}\left(\kappa[X_0, X_1, \dots]/(X_0, X_1^{p^{h-1}}, \dots, X_i^{p^{i(h-1)}}, \dots)\right)$$

where  $p^h$  is the cardinality of  $\kappa$ .

**Notation.** For any morphism of  $\mathcal{O}$ -schemes  $f: X \rightarrow Y$  and any  $\mathcal{O}$ -algebra  $A$  (i.e., any homomorphism of commutative rings with unit  $\mathcal{O} \rightarrow A$ ) we write  $X(A)$  for  $\text{Hom}_{\mathcal{O}}(\text{Spec}(A), X)$  and  $f_A$  for the map  $X(A) \rightarrow Y(A)$  induced by  $f$ . For any  $\mathcal{O}$ -scheme  $X$ , we write  $X_{\kappa}$  for its special fiber. If  $f: \text{Spec}(B) \rightarrow \text{Spec}(A)$ ,  $f^*: A \rightarrow B$  denotes the corresponding morphism on global sections. We use bold math symbols for (ramified) Witt vectors and important morphisms.

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### 2. Greenberg algebras

Let  $\mathcal{O}$  be a complete discrete valuation ring with field of fractions  $K$  of characteristic 0 and perfect residue field  $\kappa$  of positive characteristic  $p$ . Let  $\pi \in \mathcal{O}$  denote a fixed uniformizing parameter and let  $e$  be the absolute ramification index so that  $\mathcal{O} \simeq \bigoplus_{i=0}^{e-1} W(\kappa)\pi^i$  as  $W(\kappa)$ -modules. Let  $\mathbb{W}$  (respectively,  $\mathbb{W}_m$ ) denote the ring scheme of  $p$ -typical Witt vectors of infinite length (respectively, length  $m$ ) over  $\text{Spec}(\mathbb{Z})$  and let  $\mathbb{W}_\kappa$  (respectively,  $\mathbb{W}_{m,\kappa}$ ) be its base change to  $\text{Spec}(\kappa)$ .

The *Greenberg algebra* attached to the artinian local ring  $\mathcal{O}/\pi^n\mathcal{O}$ ,  $n \geq 1$ , is the  $\kappa$ -ring scheme  $\mathbb{R}_n$  that represents the fpqc sheaf associated to the presheaf

$$\begin{aligned} \{\text{affine } \kappa\text{-schemes}\} &\rightarrow \{\mathcal{O}/\pi^n\mathcal{O}\text{-algebras}\}, \\ \text{Spec}(A) &\mapsto W(A) \otimes_{W(\kappa)} \mathcal{O}/\pi^n\mathcal{O}; \end{aligned}$$

it is unique up to unique isomorphism [10, Proposition A.1]. The explicit description of  $\mathbb{R}_n$  requires some work in general (we refer the interested reader to [2, 8, 10]) but is easy when considering indices that are multiple of  $e$ . Indeed  $\mathbb{R}_{me} \simeq \prod_{i=0}^{e-1} \mathbb{W}_{m,\kappa}$  as  $\kappa$ -group schemes and for any  $\kappa$ -algebra  $A$  it is

$$\mathbb{R}_{me}(A) \simeq \bigoplus W_m(A)\pi^i \simeq W_m(A)[T]/(f_\pi(T)) \simeq W_m(A) \otimes_{W_m(\kappa)} \mathcal{O}/\pi^{me}\mathcal{O},$$

where  $f_\pi(T) \in W(\kappa)[T]$  is the Eisenstein polynomial of  $\pi$ ; see [2, (3.6) and Remark 3.7(a)], where  $\mathcal{O}$  is denoted by  $R$ , and [2, Lemma 4.4] with  $R' = \mathcal{O}$ ,  $R = W(\kappa)$ ,  $m = n$  and  $\mathbb{R}_n$  denoted by  $\mathcal{R}_n$ . Hence the addition law on the  $\kappa$ -ring scheme  $\mathbb{R}_{me}$  is defined component-wise (via the group structure of  $\mathbb{W}_{m,\kappa}$ ) while the multiplication depends on  $f_\pi(T)$  and mixes indices.

The canonical homomorphisms  $\mathcal{O}/\pi^{ne}\mathcal{O} \rightarrow \mathcal{O}/\pi^{me}\mathcal{O}$ ,  $n \geq m$ , induce morphisms of ring schemes  $\mathbb{R}_{ne} \rightarrow \mathbb{R}_{me}$  [10, Proposition A.1(iii)] and the Greenberg algebra associated with  $\mathcal{O}$  is then defined as the affine  $\kappa$ -ring scheme

$$\mathbb{R}_\mathcal{O} = \varprojlim \mathbb{R}_{me}$$

(see [2, §5] where  $\mathbb{R}_\mathcal{O}$  is denoted by  $\tilde{\mathcal{R}}$ ). By construction  $\mathbb{R}_\mathcal{O} \simeq \prod_{i=0}^{e-1} \mathbb{W}_\kappa$  as  $\kappa$ -group schemes and

$$(2.1) \quad \mathbb{R}_\mathcal{O}(A) = W(A)[T]/(f_\pi(T)) = W(A) \otimes_{W(\kappa)} \mathcal{O}$$

for any  $\kappa$ -algebra  $A$  [2, (5.4)]; note that by [2, Lemma 4.4] the hypothesis  $A = A^p$  in [2, (5.4)] is superfluous since  $\varprojlim_{m \in \mathbb{N}} \mathbb{R}_{me} = \varprojlim_{n \in \mathbb{N}} \mathbb{R}_n$ . We will say that  $\mathbb{R}_\mathcal{O}$  is an  $\mathcal{O}$ -algebra scheme over  $\text{Spec}(\kappa)$  since, as a functor on affine  $\kappa$ -schemes, it takes values on  $\mathcal{O}$ -algebras.

Note that if  $\mathcal{O} = W(\kappa)$ , then  $\mathbb{R}_\mathcal{O} \simeq \mathbb{W}_\kappa$ , the  $\kappa$ -scheme of  $p$ -typical Witt vectors.

### 3. Ramified Witt vectors

Let  $\mathcal{O}$  be a complete discrete valuation ring with field of fractions  $K$  and finite residue field  $\kappa$  of cardinality  $q = p^h$ .

For any  $\mathcal{O}$ -algebra  $B$  one defines the  $\mathcal{O}$ -algebra of *ramified Witt vectors*  $W_{\mathcal{O}}(B)$  as the set  $B^{\mathbb{N}_0}$  endowed with a structure of  $\mathcal{O}$ -algebra in such a way that the map

$$(3.1) \quad \begin{aligned} \Phi_B: W_{\mathcal{O}}(B) &\rightarrow B^{\mathbb{N}_0}, \\ \mathbf{b} = (b_n)_{n \in \mathbb{N}_0} &\mapsto (\Phi_0(\mathbf{b}), \Phi_1(\mathbf{b}), \Phi_2(\mathbf{b}), \dots), \end{aligned}$$

is a homomorphism of  $\mathcal{O}$ -algebras, where  $\Phi_n(\mathbf{b}) = b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^n b_n$  and the target  $\mathcal{O}$ -algebra  $B^{\mathbb{N}_0}$  is the product ring on which  $\mathcal{O}$  acts via multiplication in each component. Proving the existence of  $W_{\mathcal{O}}(B)$  with the indicated property requires some work; we refer to [11] for detailed proofs and to [4] for a generalized construction.

Note that if  $\pi$  is not a zero divisor in  $B$  then  $\Phi_B$  is injective and indeed bijective if  $\pi$  is invertible in  $B$ .

The above construction provides a ring scheme (and in fact an  $\mathcal{O}$ -algebra scheme)  $\mathbb{W}_{\mathcal{O}}$  such that  $W_{\mathcal{O}}(A) = \text{Hom}_{\mathcal{O}}(\text{Spec}(A), \mathbb{W}_{\mathcal{O}}) =: \mathbb{W}_{\mathcal{O}}(A)$  for any  $\mathcal{O}$ -algebra  $A$ , together with a morphism of  $\mathcal{O}$ -algebra schemes  $\Phi: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$  induced by the Witt polynomials

$$\Phi_n = \Phi_n(X_0, \dots, X_n) = X_0^{q^n} + \pi X_1^{q^{n-1}} + \dots + \pi^n X_n;$$

more precisely, if  $\mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}[Z_0, Z_1, \dots])$ , and  $\mathbb{W}_{\mathcal{O}} = \text{Spec}(\mathcal{O}[X_0, X_1, \dots])$  then  $\Phi^*(Z_n) = \Phi_n$ . Let  $\Phi_i: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{A}_{\mathcal{O}}^1$  denote the composition of  $\Phi$  with the projection onto the  $i$ th factor.

It is  $\mathbb{W}_{\mathbb{Z}_p} = \mathbb{W} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}_p)$ , the base change of the scheme of  $p$ -typical Witt vectors over  $\mathbb{Z}$ , but, despite the notation,  $\mathbb{W}_{\mathcal{O}}$  differs from  $\mathbb{W} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathcal{O})$  in general.

Let  $K'$  be a finite extension of  $K$  with residue field  $\kappa' = \mathbb{F}_{q^r}$  and ring of integers  $\mathcal{O}'$ , and let  $\varpi \in \mathcal{O}'$  be a fixed uniformizing parameter. We can repeat the above constructions with  $\varpi, q^r$  in place of  $\pi, q$  and then get a morphism of  $\mathcal{O}'$ -algebra schemes  $\Phi': \mathbb{W}_{\mathcal{O}'} \rightarrow \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$  defined by the Witt polynomials

$$(3.2) \quad \Phi'_n(X_0, \dots, X_n) = X_0^{q^{rn}} + \varpi X_1^{q^{r(n-1)}} + \dots + \varpi^n X_n.$$

By [6] there is a natural morphism of functors from the category of  $\mathcal{O}'$ -algebras to the category of  $\mathcal{O}$ -algebras

$$\mathbf{u} = \mathbf{u}_{(\mathcal{O}, \mathcal{O}')}: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}'} \quad (\text{Drinfeld's functor})$$

such that for any  $\mathcal{O}'$ -algebra  $B$  the following diagram

$$\begin{CD} W_{\mathcal{O}}(B) @>u>> W_{\mathcal{O}'}(B) \\ @V\Phi_BVV @VV\Phi'_BV \\ B^{\mathbb{N}_0} @>\Pi'>> B^{\mathbb{N}_0} \end{CD}$$

commutes, where the upper arrow is induced by  $u$  on  $B$ -sections and  $\Pi'$  maps  $(b_0, b_1, \dots)$  to  $(b_0, b_r, b_{2r}, \dots)$ . Further

$$u([b]) = [b], \quad u(F^r \mathbf{b}) = F(u(\mathbf{b})), \quad u(V\mathbf{b}) = \frac{\pi}{\varpi} V(u(F^{r-1} \mathbf{b})),$$

where  $[\cdot]$ ,  $F$ ,  $V$  denote, respectively, the Teichmüller map, the Frobenius and the Verschiebung both in  $W_{\mathcal{O}}(B)$  and in  $W_{\mathcal{O}'}(B)$ , and  $F^r$  is the  $r$ -fold composition of  $F$  with itself. By construction Drinfeld’s functor behaves well with respect to base change, i.e. if  $\mathcal{O}''/\mathcal{O}'$  is another extension, then

$$(3.3) \quad \mathbf{u}_{(\mathcal{O}, \mathcal{O}'')} = \mathbf{u}_{(\mathcal{O}', \mathcal{O}'')} \circ \mathbf{u}_{(\mathcal{O}, \mathcal{O}')}$$

as functors from the category of  $\mathcal{O}''$ -algebras to the one of  $\mathcal{O}$ -algebras. More details on  $\mathbf{u}$  and its scheme-theoretic interpretation will be given in Section 5.

### 4. Perfection

A  $\kappa$ -scheme  $X$  is called perfect if the absolute Frobenius endomorphism  $F_X$  is an isomorphism. For any  $\kappa$ -scheme  $X$  one constructs its (*inverse*) *perfection*  $X^{\text{pf}}$  as the inverse limit of copies of  $X$  with  $F_X$  as transition maps. It is known that the functor  $(\cdot)^{\text{pf}}$  is right adjoint of the forgetful functor from the category of perfect  $\kappa$ -schemes to the category of  $\kappa$ -schemes, i.e., if  $\rho: X^{\text{pf}} \rightarrow X$  denotes the canonical projection, there is a bijection

$$(4.1) \quad \text{Hom}_{\kappa}(Z, X^{\text{pf}}) \simeq \text{Hom}_{\kappa}(Z, X), \quad f \mapsto \rho \circ f$$

for any perfect  $\kappa$ -scheme  $Z$ ; see [3, Lemma 5.15 and (5.5)] for more details on this.

In the next sections we will need the following result.

**Lemma 4.1.** *Let  $\psi: X \rightarrow Y$  be a morphism of  $\kappa$ -schemes such that the map  $\psi_A: X(A) \rightarrow Y(A)$  is a bijection for any perfect  $\kappa$ -algebra  $A$ . Then  $\psi^{\text{pf}}: X^{\text{pf}} \rightarrow Y^{\text{pf}}$  is an isomorphism and  $\psi$  is a universal homeomorphism.*

*Proof.* By hypothesis

$$(4.2) \quad \text{Hom}_{\kappa}(Z, X) \simeq \text{Hom}_{\kappa}(Z, Y), \quad f \mapsto \psi \circ f,$$

for any perfect  $\kappa$ -scheme  $Z$ . In particular,

$$\text{Hom}_{\kappa}(Y^{\text{pf}}, X^{\text{pf}}) \simeq \text{Hom}_{\kappa}(Y^{\text{pf}}, X) \simeq \text{Hom}_{\kappa}(Y^{\text{pf}}, Y) \simeq \text{Hom}_{\kappa}(Y^{\text{pf}}, Y^{\text{pf}}),$$

where the first and third bijections follow from (4.1) and the second from (4.2). By standard arguments the inverse of  $\psi^{\text{pf}}$  is then the morphism associated with the identity on  $Y^{\text{pf}}$  via the above bijections. Consider further the following commutative square

$$\begin{CD} X^{\text{pf}} @>\sim>> Y^{\text{pf}} \\ @V{\rho}VV @VV{\rho}V \\ X @>{\psi}>> Y \end{CD}$$

Since the canonical morphisms  $\rho$  are universal homeomorphisms [3, Remark 5.4], the same is  $\psi$ . □

### 5. Results on ramified Witt vectors

In this section we study more closely ramified Witt vectors. Whenever possible, we take a scheme-theoretic approach which also makes the functorial properties more evident and shortens the proofs as well. Let notation be as in Section 3.

**5.1. Frobenius, Verschiebung, Teichmüller maps.** In this subsection we present classical constructions in the scheme-theoretic language. Their properties naturally descend from (5.1) and Remark 5.2.

Let  $B$  be an  $\mathcal{O}$ -algebra. If  $B$  admits an endomorphism of  $\mathcal{O}$ -algebras  $\sigma$  such that  $\sigma(b) \equiv b^q \pmod{\pi B}$  for any  $b \in B$ , then the image of the homomorphism  $\Phi_B$  in (3.1) can be characterized as follows

$$(5.1) \quad (a_n)_{n \in \mathbb{N}_0} \in \text{Im } \Phi_B \iff \sigma(a_n) \equiv a_{n+1} \pmod{\pi^{n+1} B} \quad \forall n \in \mathbb{N}_0;$$

see [11, Proposition 1.1.5]. We will apply this fact to polynomial rings  $\mathcal{O}[T_i, i \in I]$  with  $\sigma$  the endomorphism of  $\mathcal{O}$ -algebras mapping  $T_i$  to  $T_i^q$ .

**Lemma 5.1.** *Let  $\sigma : B \rightarrow B$  be an endomorphism of  $\mathcal{O}$ -algebras,  $\varpi \in B$  an element such that  $\pi \in \varpi B$  and  $f \in \mathbb{N}$ . If  $\sigma(b) \equiv b^{q^f} \pmod{\varpi B}$  for any  $b \in B$ , then*

$$\sigma(\Phi_{f_n}(\mathbf{b})) \equiv \Phi_{f_{(n+1)}}(\mathbf{b}) \pmod{\varpi^{f_{n+1}} B}$$

for all  $\mathbf{b} = (b_0, b_1, \dots) \in W_{\mathcal{O}}(B)$  and  $n \geq 0$ .

*Proof.* Let  $\mathbf{b} = (b_0, b_1, \dots)$ . Since  $\Phi_{f_{(n+1)}}(\mathbf{b}) \equiv \Phi_{f_n}(b_0^{q^f}, b_1^{q^f}, \dots) \pmod{\pi^{n f_{n+1}} B}$ , we are left to prove that

$$\sigma(\Phi_{f_n}(\mathbf{b})) \equiv \Phi_{f_n}(b_0^{q^f}, b_1^{q^f}, \dots) \pmod{\varpi^{f_{n+1}} B}.$$

We first note that  $\sigma(b) \equiv b^{q^f} \pmod{\varpi B}$  implies that

$$(5.2) \quad \sigma(b^{q^s}) \equiv b^{q^{f+s}} \pmod{\varpi^{s+1}}, \quad \forall s \geq 0,$$

(cf. [11, Lemma 1.1.1]). Hence by (5.2)

$$\begin{aligned} \sigma(\Phi_{f_n}(\mathbf{b})) &= \sigma\left(b_0^{q^{f_n}} + \pi b_1^{q^{f_n-1}} + \dots + \pi^{f_n} b_{f_n}\right) \\ &= \sigma(b_0)^{q^{f_n}} + \pi \sigma(b_1)^{q^{f_n-1}} + \dots + \pi^{f_n} \sigma(b_{f_n}) \\ &\equiv b_0^{q^{(n+1)f}} + \pi b_1^{q^{f(n+1)-1}} + \dots + \pi^{f_n} b_{f_n}^{q^f} = \Phi_{f_n}(\mathbf{b}^{q^f}), \end{aligned}$$

where the equivalence holds modulo  $\varpi^{f_{n+1}}B$ . □

Let  $B$  be an  $\mathcal{O}$ -algebra,  $\sigma: B \rightarrow B$  an endomorphism of  $\mathcal{O}$ -algebras such that  $\sigma(b) \equiv b^q \pmod{\pi B}$ , and let  $h: \text{Spec}(B) \rightarrow \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}[Z_0, Z_1, \dots])$  be a morphism of  $\mathcal{O}$ -schemes; the latter is uniquely determined by  $(h_0, \dots) \in B^{\mathbb{N}_0}$  with  $h_i = h^*(Z_i)$ . The morphism  $h$  factors through  $\Phi: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$  if and only if  $(h_0, h_1, \dots) \in \text{Im } \Phi_B$ . Hence we can rephrase (5.1) as follows:

(5.3)  $h$  factors through  $\Phi$

$$\iff \sigma(h^*(Z_n)) \equiv h^*(Z_{n+1}) \pmod{\pi^{n+1}B} \quad \forall n \in \mathbb{N}_0.$$

**Remarks 5.2.**

- (a) Note that if  $\pi$  is not a zero divisor in  $B$  and  $h$  factors through  $\Phi$ , then it factors uniquely. Indeed, let  $g, g': \text{Spec}(B) \rightarrow \mathbb{W}_{\mathcal{O}}$  be such that  $\Phi \circ g = h = \Phi \circ g'$  and let  $\mathbf{b}, \mathbf{b}' \in W_{\mathcal{O}}(B) = \mathbb{W}_{\mathcal{O}}(B)$  be the corresponding sections. Then  $\Phi_B(\mathbf{b}) = \Phi_B(\mathbf{b}')$  and one concludes that  $g = g'$  by the injectivity of  $\Phi_B$  [11, Lemma 1.1.3].
- (b) Since the above constructions depend on  $\pi$ , it seems that one should write  $\Phi_{\pi}$  and  $\mathbb{W}_{\mathcal{O},\pi}$  above. However, if  $\varpi$  is another uniformizing parameter of  $\mathcal{O}$ , let  $\sigma$  be the  $\mathcal{O}$ -algebra endomorphism on  $B = \mathcal{O}[X_0, X_1, \dots]$  mapping  $X_i$  to  $X_i^q$ . Then  $\sigma(\Phi_{\varpi,n}(X_i)) \equiv \Phi_{\varpi,n+1}(X_i) \pmod{\varpi^{n+1}B} = \pi^{n+1}B$ ; hence by (5.3) and (a) one deduces the existence of a unique morphism  $h_{\pi,\varpi}: \mathbb{W}_{\mathcal{O},\pi} \rightarrow \mathbb{W}_{\mathcal{O},\varpi}$  such that  $\Phi_{\pi} = \Phi_{\varpi} \circ h_{\pi,\varpi}$ . Similarly one constructs  $h_{\varpi,\pi}: \mathbb{W}_{\mathcal{O},\varpi} \rightarrow \mathbb{W}_{\mathcal{O},\pi}$  and (a) implies that  $h_{\varpi,\pi} \circ h_{\pi,\varpi}$  and  $h_{\pi,\varpi} \circ h_{\varpi,\pi}$  are the identity morphisms.
- (c) Note that if  $h: \mathbb{G}_{\mathcal{O}} \rightarrow \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$  is a morphism of group (respectively, ring) schemes with  $\mathbb{G}_{\mathcal{O}} \simeq \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$  or  $\mathbb{G}_{\mathcal{O}} \simeq \mathbb{A}_{\mathcal{O}}^m$  as schemes, and there exists a morphism  $g: \mathbb{G}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$ , unique by point (a), such that  $h = \Phi \circ g$ , then  $g$  is a morphism of group (respectively, ring) schemes. Indeed let  $\mu_{\mathbb{G}}, \mu_{\mathbb{W}}, \mu_{\mathbb{A}}$  be the group law on  $\mathbb{G}_{\mathcal{O}}, \mathbb{W}_{\mathcal{O}}$  and  $\mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$  respectively. Since  $\mathbb{G}_{\mathcal{O}} \times_{\mathcal{O}} \mathbb{G}_{\mathcal{O}} = \text{Spec}(C)$  with  $C$  reduced, in order to prove that  $g \circ \mu_{\mathbb{G}} = \mu_{\mathbb{W}} \circ (g \times g): \mathbb{G}_{\mathcal{O}} \times_{\mathcal{O}} \mathbb{G}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$ , it suffices to prove that  $\Phi \circ g \circ \mu_{\mathbb{G}} = \Phi \circ \mu_{\mathbb{W}} \circ (g \times g)$ . Now  $\Phi \circ g \circ \mu_{\mathbb{G}} = h \circ \mu_{\mathbb{G}} =$

$\mu_{\mathbb{A}} \circ (h \times h) = \mu_{\mathbb{A}} \circ (\Phi \times \Phi) \circ (g \times g) = \Phi \circ \mu_{\mathbb{W}} \circ (g \times g)$ . Similar arguments work for the multiplication law when considering morphisms of ring schemes.

As applications of (5.1) and (5.3) one proves the existence of the Frobenius, Verschiebung and Teichmüller morphisms as well as of endomorphisms  $\lambda: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$  for any  $\lambda \in \mathcal{O}$ .

We now see how to deduce the existence of classical group/ring endomorphisms of  $\mathbb{W}_{\mathcal{O}}$  from endomorphisms of  $\mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$ .

**Proposition 5.3.**

- (i) Let  $\mathbb{A}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}[Z_0, Z_1, \dots])$  and let  $f$  be the endomorphism of  $\mathbb{A}^{\mathbb{N}_0}$  such that  $f^*(Z_n) = Z_{n+1}$ . There exists a unique morphism of ring schemes  $\mathbf{F}: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$  such that  $\Phi \circ \mathbf{F} = f \circ \Phi$ .
- (ii) Let  $v$  be the endomorphism of  $\mathbb{A}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}[Z_0, Z_1, \dots])$  such that  $v^*(Z_0) = 0$  and  $v^*(Z_{n+1}) = \pi Z_n$  for  $n \geq 0$ . Then there exists a unique morphism of  $\mathcal{O}$ -group schemes  $\mathbf{V}: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$  such that  $\Phi \circ \mathbf{V} = v \circ \Phi$ .
- (iii) For  $\lambda \in \mathcal{O}$  let  $f_{\lambda}$  be the group endomorphism of  $\mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}[Z_0, \dots])$  such that  $f_{\lambda}^*(Z_n) = \lambda Z_n$ . Then there exists a unique morphism of  $\mathcal{O}$ -group schemes  $\lambda: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}}$  such that  $\Phi \circ \lambda = f_{\lambda} \circ \Phi$ .
- (iv) Let  $\sigma: \mathbb{A}_{\mathcal{O}}^1 \rightarrow \mathbb{A}_{\mathcal{O}}^1 = \text{Spec}(\mathcal{O}[T])$  be the morphism of  $\mathcal{O}$ -schemes such that  $\sigma^*(T) = T^q$  and let  $\sigma = (\text{id}, \sigma, \sigma^2, \dots): \mathbb{A}_{\mathcal{O}}^1 \rightarrow \mathbb{A}_{\mathcal{O}}^{\mathbb{N}_0}$ . Then there exists a unique morphism of  $\mathcal{O}$ -schemes  $\tau: \mathbb{A}_{\mathcal{O}}^1 \rightarrow \mathbb{W}_{\mathcal{O}}$  such that  $\Phi \circ \tau = \sigma$ . It is a multiplicative section of the projection onto the first component  $\Phi_0: \mathbb{W}_{\mathcal{O}} \rightarrow \mathbb{A}_{\mathcal{O}}^1$ .

*Proof.* For proving (i)–(iii) we use (5.3) with  $B = \mathcal{O}[X_0, X_1, \dots]$ , the ring of global sections of  $\mathbb{W}_{\mathcal{O}}$ , endowed with its unique lifting of Frobenius, more precisely with the morphism of  $\mathcal{O}$ -algebras  $\sigma$  mapping  $X_i$  to  $X_i^q$ . Let  $\mathbf{X}$  denote the vector  $(X_0, X_1, \dots) \in W_{\mathcal{O}}(\mathcal{O}[X_0, X_1, \dots])$  and set  $\mathbf{X}^{\sigma} = (X_0^q, X_1^q, \dots)$ .

The morphism  $\mathbf{F}$  exists as soon as the condition in (5.3) is satisfied for  $h = f \circ \Phi$ , i.e., if  $\Phi_{n+1}(\mathbf{X}^{\sigma}) \equiv \Phi_{n+2}(\mathbf{X}) \pmod{\pi^{n+1}}$  for any  $n$ . This is evident since  $\Phi_{n+2}(\mathbf{X}) = \Phi_{n+1}(\mathbf{X}^{\sigma}) + \pi^{n+2} X_{n+2}$ .

The morphism  $\mathbf{V}$  exists as soon as the condition in (5.3) is satisfied for  $h = v \circ \Phi$ , i.e., if  $0 \equiv \pi X_0 \pmod{\pi B}$  and  $\pi \Phi_{n-1}(\mathbf{X}^{\sigma}) \equiv \pi \Phi_n(\mathbf{X}) \pmod{\pi^{n+1} B}$  for any  $n \geq 1$ . The first fact is trivial while the second is evident since  $\Phi_n(\mathbf{X}^{\sigma}) = \Phi_{n-1}(\mathbf{X}^q) + \pi^n X_n$ .

The morphism  $\lambda$  exists as soon as the condition in (5.3) is satisfied for  $h = f_{\lambda} \circ \Phi$ , i.e., if  $\lambda \Phi_n(\mathbf{X}^{\sigma}) \equiv \lambda \Phi_{n+1}(\mathbf{X}) \pmod{\pi^{n+1}}$  for any  $n$ . This is evident since  $\Phi_n(\mathbf{X}^{\sigma}) \equiv \Phi_{n+1}(\mathbf{X}) \pmod{\pi^{n+1}}$  by Lemma 5.1 with  $f = 1$ ,  $\varpi = \pi$ .

Uniqueness of  $\mathbf{F}, \mathbf{V}, \boldsymbol{\lambda}$  follows by Remark 5.2(a). The fact that they are group/ring scheme morphisms follows by Remark 5.2(c).

For (iv), we consider condition (5.3) for  $B = \mathcal{O}[T]$  and  $h = \boldsymbol{\sigma}$ . It is satisfied since  $h^*(Z_n) = T^{q^n}$ ; whence  $\boldsymbol{\tau}$  exists. Uniqueness follows again by Remark 5.2(a) and multiplicativity of  $\boldsymbol{\tau}$  follows from multiplicativity of  $\boldsymbol{\sigma}$  as in Remark 5.2(c). Finally, by construction,  $\boldsymbol{\tau}$  is a section of  $\Phi_0$ .  $\square$

The ring scheme endomorphism  $\mathbf{F}$  is called *Frobenius* and the  $\mathcal{O}$ -scheme endomorphism  $\mathbf{V}$  is called *Verschiebung*. By a direct computation one checks that for any  $\mathcal{O}$ -algebra  $A$ , the induced homomorphism  $F_A: W_{\mathcal{O}}(A) \rightarrow W_{\mathcal{O}}(A)$  satisfies

$$(5.4) \quad F_A(a_0, a_1, \dots) \equiv (a_0, a_1, \dots)^q \pmod{\pi W_{\mathcal{O}}(A)},$$

and, if  $A$  is a  $\kappa$ -algebra,

$$(5.5) \quad F_A(a_0, a_1, \dots) = (a_0^q, a_1^q, \dots)$$

holds. Further, both  $F_A$  and  $V_A$  are  $\mathcal{O}$ -linear [11, §1] and

$$(5.6) \quad F_A V_A = \pi \cdot \text{id}_{W_{\mathcal{O}}(A)},$$

$$(5.7) \quad \begin{aligned} V_A F_A &= \pi \cdot \text{id}_{W_{\mathcal{O}}(A)}, & \text{if } \pi A = 0, \\ \mathbf{a} \cdot V_A(\mathbf{c}) &= V_A(F_A(\mathbf{a}) \cdot \mathbf{c}), & \text{for all } \mathbf{a}, \mathbf{c} \in W_{\mathcal{O}}(A). \end{aligned}$$

Finally  $V_A^n W_{\mathcal{O}}(A)$  is an ideal of  $W_{\mathcal{O}}(A)$  for any  $n > 0$  where  $V_A^n$  denotes the  $n$ -fold composition of  $V_A$ . Note that  $W_{\mathcal{O}}(A) = \varprojlim_{n \in \mathbb{N}} W_{\mathcal{O}}(A)/V_A^n W_{\mathcal{O}}(A)$  and if  $A$  is a semiperfect  $\kappa$ -algebra, i.e., the Frobenius is surjective on  $A$ , then  $V_A^n W_{\mathcal{O}}(A) = \pi^n W_{\mathcal{O}}(A)$ .

The morphism  $\boldsymbol{\tau}$  is called *Teichmüller map*. For any  $\mathcal{O}$ -algebra  $B$ , we have  $\tau_B: B \rightarrow W_{\mathcal{O}}(B), b \mapsto [b] := (b, 0, 0, \dots)$ , since  $\Phi_B([b]) = (b, b^q, b^{q^2}, \dots)$ . Note that  $\boldsymbol{\sigma}$  is not a morphism of  $\mathcal{O}$ -group schemes and hence we can not expect that  $\boldsymbol{\tau}$  is a morphism of group schemes.

**Remark 5.4.** For any subset  $I \subset \mathbb{N}_0$  and any  $\mathcal{O}$ -algebra  $A$ , let  $W_{\mathcal{O},I}(A)$  denote the subset of  $W_{\mathcal{O}}(A)$  consisting of vectors  $\mathbf{b} = (b_0, \dots)$  such that  $b_i = 0$  if  $i \notin I$ . If  $J \subset \mathbb{N}_0$  satisfies  $I \cap J = \emptyset$ , then the sum in  $W_{\mathcal{O}}(A)$  of a vector  $\mathbf{b} = (b_0, \dots) \in W_{\mathcal{O},I}(A)$  and a vector  $\mathbf{c} = (c_0, \dots) \in W_{\mathcal{O},J}(A)$  is simply obtained by “gluing” the two vectors, i.e.,  $\mathbf{b} + \mathbf{c} = \mathbf{d} = (d_0, \dots) \in W_{\mathcal{O},I \cup J}(A)$  with  $d_i = b_i$  if  $i \in I$  and  $d_i = c_i$  if  $i \in J$ . For proving this fact, since  $A$  can be written as quotient of a polynomial algebra over  $\mathcal{O}$  with possibly infinitely many indeterminates, we may assume that  $A$  is  $\pi$ -torsion free. In this case  $\mathbf{d}$  is uniquely determined by the condition  $\sum_{i=0}^n \pi^i d_i^{q^{n-i}} = \Phi_n(d_0, \dots) = \Phi_n(b_0, \dots) + \Phi_n(c_0, \dots) = \sum_{i=0}^n \pi^i b_i^{q^{n-i}} + \sum_{i=0}^n \pi^i c_i^{q^{n-i}}$ ; since for any index  $i$  either  $b_i$  or  $c_i$  (or both) is zero, the above choice of  $d_i$  works. More generally, if  $I_0, \dots, I_r$ , are disjoint subsets of  $\mathbb{N}_0$ , and  $\mathbf{b}_j$  are vectors

in  $W_{\mathcal{O}, I_j}(A)$ , then the sum  $\mathbf{b}_0 + \dots + \mathbf{b}_r$  is obtained by “gluing” the vectors  $\mathbf{b}_j$ . As immediate consequence, any element in  $W_{\mathcal{O}}(A)$  can be written as

$$(5.8) \quad (a_0, a_1, \dots) = \sum_{i=0}^{\infty} V_A^i [a_i],$$

since  $V^i[b] = (0, \dots, 0, b, 0, \dots) \in W_{\mathcal{O}, \{i\}}$ .

**Lemma 5.5.** *Let  $B$  be a  $\kappa$ -algebra and consider the map*

$$B^n \rightarrow W_{\mathcal{O}, n}(B) := W_{\mathcal{O}}(B)/V_B^n W_{\mathcal{O}}(B),$$

$$(b_0, \dots, b_{n-1}) \mapsto \sum_{j=0}^{n-1} [b_j] \pi^j.$$

*If  $B$  is reduced (respectively, semiperfect, perfect) the above map is injective (respectively, surjective, bijective). Hence if  $B$  is semiperfect (respectively, perfect), any element of  $W_{\mathcal{O}}(B) = \varprojlim W_{\mathcal{O}, n}(B)$  can be written (respectively, uniquely written) in the form  $\sum_{j=0}^{\infty} [b_j] \pi^j$ .*

*Proof.* By (5.6), (5.7), (5.4) and Remark 5.4 it is

$$\sum_{j=0}^{n-1} [b_j] \pi^j = \sum_{j=0}^{n-1} \pi^j [b_j] = \sum_{j=0}^{n-1} V^j F^j [b_j] = \sum_{j=0}^{n-1} V^j [b_j^{q^j}] = (b_0, \dots, b_{n-1}^{q^{n-1}}, 0, \dots),$$

where we have omitted the subscript  $B$  on  $F$  and  $V$ . Injectivity is clear when  $B$  is reduced. Assume now  $B$  semiperfect and let  $\mathbf{b} = (b_0, b_1, \dots) \in W_{\mathcal{O}}(B)$ . Then by Remark 5.4

$$\mathbf{b} = (b_0, \dots, b_{n-1}, 0, \dots) + (0, \dots, 0, b_n, \dots) \in (b_0, \dots, b_{n-1}, 0, \dots) + V^n W_{\mathcal{O}}(B),$$

and by (5.8) and (5.5)

$$(b_0, \dots, b_{n-1}, 0, \dots) = \sum_{j=0}^{n-1} V^j [b_j] = \sum_{j=0}^{n-1} V^j F^j [b_j^{1/q^j}] = \sum_{j=0}^{n-1} \pi^j [b_j^{1/q^j}]$$

where  $b_i^{1/q^i}$  denotes any  $q^i$ th root of  $b_i$ , which exists since  $B$  is semiperfect. Hence surjectivity is clear too. □

**5.2. The Drinfeld morphism.** Let  $K'$  denote a finite extension of  $K$  with residue field  $\kappa' = \mathbb{F}_{q^r}$ , ring of integers  $\mathcal{O}'$  and ramification degree  $e$ ; since we don't work with absolute ramification indices in this section, there is no risk of confusion with notation of Section 2. Let  $\varpi \in \mathcal{O}'$  be a uniformizing parameter and write  $\pi = \alpha \varpi^e$  with  $\alpha$  a unit in  $\mathcal{O}'$ . Let  $\Phi'_n(X_0, \dots, X_n) = X_0^{q^{rn}} + \varpi X_1^{q^{r(n-1)}} + \dots + \varpi^n X_n$  be the polynomials as in (3.2) that define the morphism  $\Phi': \mathbb{W}_{\mathcal{O}'} \rightarrow \mathbb{A}_{\mathcal{O}'}$ .

**Proposition 5.6.** *There exists a unique morphism of  $\mathcal{O}'$ -ring schemes  $\mathbf{u} = \mathbf{u}_{(\mathcal{O}, \mathcal{O}')}$  such that the following diagram*

$$(5.9) \quad \begin{array}{ccc} \mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec } \mathcal{O}' & \xrightarrow{\mathbf{u}} & \mathbb{W}_{\mathcal{O}'} \\ \Phi \times \text{id}_{\mathcal{O}'} \downarrow & & \downarrow \Phi' \\ \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0} & \xrightarrow{\Pi'} & \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0} \end{array}$$

commutes, where  $\Phi \times \text{id}_{\mathcal{O}'}$  is the base change of  $\Phi$  to  $\text{Spec}(\mathcal{O}')$  and  $\Pi'$  is the morphism mapping  $(x_0, x_1, \dots)$  to  $(x_0, x_r, x_{2r}, \dots)$ . For any  $\lambda \in \mathcal{O}$  it is  $\lambda \circ \mathbf{u} = \mathbf{u} \circ (\lambda \times \text{id}_{\mathcal{O}'})$ , i.e.,  $\mathbf{u}$  induces homomorphisms of  $\mathcal{O}$ -algebras  $u_B: W_{\mathcal{O}}(B) \rightarrow W_{\mathcal{O}'}(B)$  for any  $\mathcal{O}'$ -algebra  $B$ .

*Proof.* (cf. [6, Proposition 1.2]) Let  $B = \mathcal{O}'[X_0, \dots]$  be the ring of global sections of  $\mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec } \mathcal{O}'$  and let  $\sigma$  be the endomorphism the  $\mathcal{O}'$ -algebra  $B$  mapping  $X_i$  to  $X_i^{q^r}$ . Let  $h = \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'}) : \text{Spec}(B) \rightarrow \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$ . Then by (5.3) the morphism of  $\mathcal{O}'$ -schemes  $\mathbf{u}$  exists as soon as  $\sigma(h^*(Z_n)) \equiv h^*(Z_{n+1}) \pmod{\varpi^{n+1}B}$ . By definition of  $h$ , this condition is equivalent to  $\sigma(\Phi_{nr}) \equiv \Phi_{(n+1)r} \pmod{\varpi^{n+1}B}$ , and the latter holds by Lemma 5.1 with  $f = r$  and  $\mathbf{b} = (X_0, X_1, \dots) \in W_{\mathcal{O}'}(B)$ . Hence  $\mathbf{u}$  exists as morphism of schemes. Uniqueness follows by Remark 5.2(a). Since  $\Phi \times \text{id}_{\mathcal{O}'}$  and  $\Pi'$  are morphism of ring schemes, the same is  $\mathbf{u}$  by the commutativity of (5.9) and Remark 5.2(c).

Finally, since both  $\lambda \circ \mathbf{u}$  and  $\mathbf{u} \circ (\lambda \times \text{id}_{\mathcal{O}'})$  correspond to the endomorphism of  $\mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$  mapping  $Z_n$  to  $\lambda Z_{rn}$  on algebras, the result is clear.  $\square$

The morphism  $\mathbf{u}$  is called the *Drinfeld morphism*. Note that the commutativity of (5.9) says that for any  $\mathcal{O}'$ -algebra  $B$  and any  $\mathbf{b} \in W_{\mathcal{O}}(B)$  it is

$$\Phi'_n(u_B(\mathbf{b})) = \Phi_{nr}(\mathbf{b}).$$

**Lemma 5.7.** *Let  $\tau, \tau'$  be the Teichmüller maps of  $\mathbb{W}_{\mathcal{O}}, \mathbb{W}_{\mathcal{O}'}$  respectively. Then  $\tau' = \mathbf{u} \circ (\tau \times \text{id}_{\mathcal{O}'})$ .*

*Proof.* Let  $\mathbb{A}_{\mathcal{O}}^1 = \text{Spec } \mathcal{O}[T]$  and  $\mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0} = \text{Spec } \mathcal{O}'[Z_0, Z_1, \dots]$ . Let  $\sigma: \mathbb{A}_{\mathcal{O}}^1 \rightarrow \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$  be the morphism in Proposition 5.3 mapping  $Z_n$  to  $T^{q^n}$  on algebras, and let  $\sigma': \mathbb{A}_{\mathcal{O}'}^1 \rightarrow \mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$  be the analogous morphism for  $\mathcal{O}'$  mapping  $Z_n$  to  $T^{q^{rn}}$  on algebras. Then  $\tau'$  is uniquely determined by the property  $\Phi' \circ \tau' = \sigma'$ . Since  $\Phi' \circ \mathbf{u} \circ (\tau \times \text{id}_{\mathcal{O}'}) = \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'}) \circ (\tau \times \text{id}_{\mathcal{O}'}) = \Pi' \circ (\sigma \times \text{id}_{\mathcal{O}'}) = \sigma'$ , the conclusion follows.  $\square$

Let  $B$  be an  $\mathcal{O}'$ -algebra  $B$ . As a consequence of the above lemma and  $\mathcal{O}$ -linearity of the Drinfeld map  $u_B$ , it is  $u_B(\sum_{i=0}^n [b_i] \pi^i) = \sum_{i=0}^n [b_i] \pi^i$  and

hence

$$(5.10) \quad u_B \left( \sum_{i=0}^{\infty} [b_i] \pi^i \right) = \sum_{i=0}^{\infty} [b_i] \pi^i,$$

where  $[b_i]$  in the left-hand side (respectively, in the right-hand side) is the Teichmüller representative of  $b_i$  in  $W_{\mathcal{O}}(B)$  (respectively, in  $W_{\mathcal{O}'}(B)$ ) and  $\pi$  in the right-hand side is viewed as element of  $\mathcal{O}'$ .

**Lemma 5.8.** *Let  $\mathbf{F}, \mathbf{F}'$  be the Frobenius maps on  $\mathbb{W}_{\mathcal{O}}$  and  $\mathbb{W}_{\mathcal{O}'}$  respectively. Then  $\mathbf{u} \circ (\mathbf{F}^r \times \text{id}_{\mathcal{O}'}) = \mathbf{F}' \circ \mathbf{u}$ , where  $\mathbf{F}^r$  is the  $r$ -fold composition of  $\mathbf{F}$ .*

*Proof.* Let  $f$  also denote the endomorphism of  $\mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0} = \text{Spec}(\mathcal{O}'[Z_0, Z_1, \dots])$  associated with  $\mathbf{F}'$  as in Proposition 5.3(i), which maps  $Z_n$  to  $Z_{n+1}$  on algebras, and let  $f^r$  denote the  $r$ -fold composition of  $f$ . Since  $\Phi' \circ \mathbf{F}' \circ \mathbf{u} = f \circ \Phi' \circ \mathbf{u} = f \circ \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'}) = \Pi' \circ f^r \circ (\Phi \times \text{id}_{\mathcal{O}'}) = \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'}) \circ (\mathbf{F}^r \times \text{id}_{\mathcal{O}'}) = \Phi' \circ \mathbf{u} \circ (\mathbf{F}^r \times \text{id}_{\mathcal{O}'})$ , the conclusion follows by Remark 5.2(a).  $\square$

**Lemma 5.9.** *Let  $\frac{\pi}{\varpi}$  denote the group homomorphism of  $\mathbb{W}_{\mathcal{O}'}$  associated with  $\frac{\pi}{\varpi} \in \mathcal{O}'$  as in Proposition 5.3(iv). Then  $\mathbf{u} \circ (\mathbf{V} \times \text{id}_{\mathcal{O}'}) = \frac{\pi}{\varpi} \circ \mathbf{V}' \circ \mathbf{u} \circ (\mathbf{F}^{r-1} \times \text{id}_{\mathcal{O}'})$ .*

*Proof.* We keep notation as in Proposition 5.3:  $v$  is the endomorphism of the affine space  $\mathbb{A}_{\mathcal{O}'}^{\mathbb{N}_0}$  associated with  $\mathbf{V}$ , similarly for  $v', \mathbf{V}'$  over  $\mathcal{O}'$ ;  $f$  is the endomorphism associated with  $\mathbf{F}$  and  $f_{\frac{\pi}{\varpi}}$  the one associated with  $\frac{\pi}{\varpi}$ .

Note that  $\Pi' \circ (v \times \text{id}_{\mathcal{O}'})$  maps  $Z_0$  to  $0$  and  $Z_n$  to  $\pi Z_{rn-1}$  if  $n > 0$ . Now  $\Phi' \circ \mathbf{u} \circ (\mathbf{V} \times \text{id}_{\mathcal{O}'}) = \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'}) \circ (\mathbf{V} \times \text{id}_{\mathcal{O}'}) = \Pi' \circ (v \times \text{id}_{\mathcal{O}'}) \circ (\Phi \times \text{id}_{\mathcal{O}'})$ .

On the other hand,

$$\Phi' \circ \frac{\pi}{\varpi} \circ \mathbf{V}' \circ \mathbf{u} = f_{\frac{\pi}{\varpi}} \circ \Phi' \circ \mathbf{V}' \circ \mathbf{u} = f_{\frac{\pi}{\varpi}} \circ v' \circ \Phi' \circ \mathbf{u} = f_{\frac{\pi}{\varpi}} \circ v' \circ \Pi' \circ (\Phi \times \text{id}_{\mathcal{O}'})$$

Hence

$$\Phi' \circ \frac{\pi}{\varpi} \circ \mathbf{V}' \circ \mathbf{u} \circ (\mathbf{F}^{r-1} \times \text{id}_{\mathcal{O}'}) = f_{\frac{\pi}{\varpi}} \circ v' \circ \Pi' \circ (f^{r-1} \times \text{id}_{\mathcal{O}'}) \circ (\Phi \times \text{id}_{\mathcal{O}'})$$

Since both  $\Pi' \circ (v \times \text{id}_{\mathcal{O}'})$  and  $f_{\frac{\pi}{\varpi}} \circ v' \circ \Pi' \circ (f^{r-1} \times \text{id}_{\mathcal{O}'})$  induce the endomorphism of  $\mathcal{O}'[Z_0, Z_1, \dots]$  mapping  $Z_0$  to  $0$  and  $Z_n$  to  $\pi Z_{rn-1}$  for  $n > 0$ , they coincide and the conclusion follows by Remark 5.2(a).  $\square$

We now discuss properties of the Drinfeld morphism.

**Lemma 5.10.** *Let  $B$  be a reduced  $\kappa'$ -algebra. Then Drinfeld morphism induces an injective map  $u_B: W_{\mathcal{O}}(B) \rightarrow W_{\mathcal{O}'}(B)$  on  $B$ -sections.*

*Proof.* Let  $B^{\text{pf}}$  denote the perfect closure of  $B$ , i.e.,  $B^{\text{pf}} = \varinjlim_{i \in \mathbb{N}_0} B_i$  with  $B_i = B$  and Frobenius  $b \mapsto b^p$  as transition maps. Since  $B$  is reduced, the canonical map  $\phi: B = B_0 \rightarrow B^{\text{pf}}$  is injective and thus the same is  $\mathbb{W}_{\mathcal{O}}(\phi)$ .

Hence, it suffices to consider the case where  $B$  is perfect. By Lemma 5.5, any element  $\mathbf{b}$  of  $W_{\mathcal{O}}(B)$  is of the form  $\sum_{i=0}^{\infty} [b_i] \pi^i$ ,  $b_i \in B$ , and hence  $u_B(\mathbf{b}) = \sum_{i=0}^{\infty} [b_i] \pi^i$  by (5.10). Injectivity of  $u_B$  is then clear since  $\pi^i \in (V'_B)^{ei} W_{\mathcal{O}'}(B)$  and  $W_{\mathcal{O}'}(B)$  has no  $\pi$ -torsion.  $\square$

Note that if  $\mathcal{O}' \neq \mathcal{O}$  and  $B$  is a non-reduced  $\kappa'$ -algebra, then  $u_B$  is not injective. Indeed let  $0 \neq b \in B$  such that  $b^p = 0$ . Then by (5.6) and (5.7) with  $\mathcal{O}'$  in place of  $\mathcal{O}$  and Lemmas 5.7 and 5.9 we have

$$u_B(V_B[b]) = \alpha \varpi^{e-1} V'_B([b]) = \alpha (V'_B)^e (F'_B)^{e-1}([b]) = \alpha (V'_B)^e(0) = 0$$

if  $r = 1$  and  $e > 1$ , and  $u_B(V_B[b]) = \alpha \varpi^{e-1} V'_B(u_B(0)) = 0$  if  $r > 1$ .

More precise statements can be given in the unramified or totally ramified cases.

**5.2.1. The unramified case.**

**Lemma 5.11.** *Let  $\mathcal{O}'/\mathcal{O}$  be an unramified extension and let  $B$  be a  $\kappa'$ -algebra. Then  $u_B: W_{\mathcal{O}}(B) \rightarrow W_{\mathcal{O}'}(B)$  is injective (respectively, surjective, bijective) if  $B$  is reduced (respectively, semiperfect, perfect).*

*Proof.* Let  $\mathbb{W}_{\mathcal{O}} = \text{Spec } \mathcal{O}[X_0, X_1, \dots]$ ,  $\mathbb{W}_{\mathcal{O}'} = \text{Spec } \mathcal{O}'[Y_0, Y_1, \dots]$  and set  $u_i = u^*(Y_i) \in \mathcal{O}'[X_0, \dots]$ , so that  $\Phi'_m(u_0, u_1, \dots) = \Phi_{mr}(X_0, X_1, \dots)$  by commutativity of (5.9). We claim that

$$u_0 = X_0, \quad u_m \equiv X_m^{q^{m(r-1)}} \pmod{(\pi)} \quad \text{for } m > 0.$$

Since  $u_0 = X_0$  is clear by construction, only the second equivalence has to be proved. We proceed by induction on  $m$ . First note that for any  $m \geq 0$

$$\begin{aligned} & \Phi_{(m+1)r}(X_0, \dots) \\ & \equiv X_0^{q^{(m+1)r}} + \dots + \pi^m X_m^{q^{(m+1)r-m}} + \pi^{m+1} X_{m+1}^{q^{(m+1)r-m-1}} \pmod{(\pi^{m+2})} \end{aligned}$$

and

$$\Phi'_{m+1}(Y_0, \dots) = Y_0^{q^{(m+1)r}} + \dots + \pi^m Y_m^{q^r} + \pi^{m+1} Y_{m+1}.$$

Assume that  $u_i \equiv X_i^{q^{i(r-1)}} \pmod{(\pi)}$  for  $0 \leq i \leq m$ , then

$$\pi^i u_i^{q^{(m+1-i)r}} \equiv \pi^i X_i^{q^{(m+1)r-i}} \pmod{(\pi^{i+1+(m+1-i)r})},$$

where  $i + 1 + (m + 1 - i)r = m + r + 1 + (r - 1)(m - i) \geq m + 2$ . Hence

$$\begin{aligned} 0 &= \Phi'_{m+1}(u_0, \dots) - \Phi_{(m+1)r}(X_0, \dots) \\ & \equiv \pi^{m+1} u_{m+1} - \pi^{m+1} X_{m+1}^{q^{(m+1)r-m-1}} \pmod{(\pi^{m+2})}, \end{aligned}$$

thus the claim.

Now, if  $B$  is any  $\mathcal{O}$ -algebra,  $\mathbf{b} = (b_0, \dots) \in W_{\mathcal{O}}(B)$  and  $u_B(\mathbf{b}) = \mathbf{c} = (c_0, c_1, \dots)$  it is  $c_0 = b_0$  and  $c_m \equiv b_m^{q_m^{m(r-1)}} \pmod{\pi B}$ . In particular, if  $B$  is a  $\kappa'$ -algebra, it is

$$(5.11) \quad c_m = b_m^{q_m^{m(r-1)}}, \quad \forall m \geq 0.$$

This implies that  $u_B$  is injective if  $B$  is reduced (as already seen in Lemma 5.10), surjective if  $B$  is semiperfect and bijective if  $B$  is perfect.  $\square$

The above lemma has the following geometric interpretation.

**Proposition 5.12.** *Assume that the extension  $\mathcal{O}'/\mathcal{O}$  is unramified. Then Drinfeld’s morphism  $\mathbf{u}$  restricted to special fibers is a universal homeomorphism with pro-infinitesimal kernel isomorphic to*

$$\text{Spec}(\kappa'[X_0, X_1, \dots]/(X_0, \dots, X_i^{q_i^{i(r-1)}}, \dots))$$

where  $q^r$  is the cardinality of  $\kappa'$ .

*Proof.* The first assertion follows from Lemmas 4.1 and 5.11. By the very explicit description of  $\mathbf{u}_\kappa$  in (5.11) one gets the assertion on the kernel.  $\square$

**5.2.2. The totally ramified case.** Let  $\mathcal{O}'/\mathcal{O}$  be a totally ramified extension of degree  $e > 1$ . Then  $\kappa' = \kappa$ ,  $\mathcal{O}' = \bigoplus_{i=0}^{e-1} \mathcal{O}\varpi^i$  as  $\mathcal{O}$ -module, and  $\pi = \alpha\varpi^e$  with  $\alpha$  a unit in  $\mathcal{O}'$ . Let  $B$  be a  $\mathcal{O}'$ -algebra. We can not expect  $u_B: W_{\mathcal{O}}(B) \rightarrow W_{\mathcal{O}'}(B)$  to be surjective, even if  $B$  is a perfect  $\kappa$ -algebra; indeed (5.10) shows that  $\varpi$  is not in the image of  $u_B$ . Note that  $u_B$  is a morphism of  $\mathcal{O}$ -algebras and hence we can extend it to a morphism of  $\mathcal{O}'$ -algebras

$$(5.12) \quad \begin{aligned} u_B^{\text{ra}} = u_B \otimes \text{id}: W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}' &\rightarrow W_{\mathcal{O}'}(B), \\ \sum_{i=0}^{e-1} \mathbf{b}_i \otimes \varpi^i &\mapsto \sum_{i=0}^{e-1} u_B(\mathbf{b}_i)\varpi^i, \end{aligned}$$

with  $\mathbf{b}_i \in W_{\mathcal{O}}(B)$ . Since for any  $\mathcal{O}$ -algebra  $A$  it is

$$(5.13) \quad W_{\mathcal{O}}(A) \otimes_{\mathcal{O}} \mathcal{O}' = W_{\mathcal{O}}(A) \otimes_{\mathcal{O}} \bigoplus_{i=0}^{e-1} \mathcal{O}\varpi^i = \bigoplus_{i=0}^{e-1} W_{\mathcal{O}}(A)\varpi^i,$$

forgetting about the multiplication on  $W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}'$ ,  $u_B^{\text{ra}}$  is the group homomorphism making the following diagram commute

$$(5.14) \quad \begin{array}{ccc} \prod_{i=0}^{e-1} \mathbb{W}_{\mathcal{O}}(B) & \xrightarrow{u_B^{\text{ra}}} & \mathbb{W}_{\mathcal{O}'}(B) \\ \downarrow \prod \Phi_B & & \downarrow \Phi'_B \\ \prod_{i=0}^{e-1} B^{\mathbb{N}_0} & \xrightarrow{\quad} & B^{\mathbb{N}_0} \end{array}$$

$$\begin{array}{ccc} (\mathbf{b}_i)_i & \mapsto & \sum_i u_B(\mathbf{b}_i) \varpi^i \\ \downarrow & & \downarrow \\ (\Phi_B(\mathbf{b}_i)) & \mapsto & \sum_i \Phi_B(\mathbf{b}_i) \varpi^i = \sum_i \Phi'_B(u_B(\mathbf{b}_i)) \varpi^i \end{array}$$

We deduce from (5.13) that the product group scheme  $\prod_{i=0}^{e-1} \mathbb{W}_{\mathcal{O}}$ , whose group of  $A$ -sections is  $\bigoplus_{i=0}^{e-1} W_{\mathcal{O}}(A)$ , for any  $\mathcal{O}$ -algebra  $A$ , can be endowed with a ring scheme structure that depends on the Eisenstein polynomial of  $\varpi$  and mixes components. We denote by  $\prod^{\varpi} \mathbb{W}_{\mathcal{O}}$  the resulting ring scheme over  $\mathcal{O}$ . In particular the functoriality of the maps  $u_B^{\text{ra}}$  says the existence of a morphism of ring schemes over  $\mathcal{O}'$

$$\mathbf{u}^{\text{ra}}: \prod^{\varpi} \mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec}(\mathcal{O}') \rightarrow \mathbb{W}_{\mathcal{O}'}$$

which induces  $u_B^{\text{ra}}$  on  $B$ -sections. More precisely,  $\mathbf{u}^{\text{ra}}$  is a morphism of schemes of  $\mathcal{O}'$ -algebras. Let

$$(5.15) \quad \mathbf{u}_{\kappa}^{\text{ra}}: \prod^{\varpi} \mathbb{W}_{\mathcal{O}, \kappa} \rightarrow \mathbb{W}_{\mathcal{O}', \kappa}$$

be the restriction of  $\mathbf{u}^{\text{ra}}$  to special fibers.

We can not expect that results in Lemma 5.11 and Proposition 5.12 hold in the totally ramified case, but they hold for  $\mathbf{u}^{\text{ra}}$  in place of  $\mathbf{u}$ .

**Lemma 5.13.** *Let  $\mathcal{O}'/\mathcal{O}$  be a totally ramified extension of degree  $e$  and let  $B$  be a  $\kappa$ -algebra. If  $B$  is reduced (respectively, semiperfect, perfect) then the homomorphism  $u_B^{\text{ra}} = u_B \otimes \text{id}$  in (5.12) is injective (respectively, surjective, bijective).*

*Proof.* For the injectivity, as in the proof of Lemma 5.10, we may assume that  $B$  is perfect. Let  $\mathbf{x} = \sum_{i=0}^{e-1} \mathbf{b}_i \otimes \omega^i$  with  $\mathbf{b}_i = \sum_{j=0}^{\infty} [b_{i,j}] \pi^j \in W_{\mathcal{O}}(B)$  by Lemma 5.5. Then by (5.10) it is  $u_B^{\text{ra}}(\mathbf{x}) = \sum_{i=0}^{e-1} \sum_{j=0}^{\infty} [b_{i,j}] \pi^j \varpi^i = \sum_{i=0}^{e-1} \sum_{j=0}^{\infty} \alpha_j (V'_B)^{ej+i} [\tilde{b}_{i,j}]$  with  $\alpha = \pi/\varpi^e$  a unit in  $\mathcal{O}'$ ,  $\tilde{b}_{i,j}$  the  $q^{ej+i}$ th power of  $b_{i,j}$  and  $V'_B$  the Verschiebung on  $W_{\mathcal{O}'}(B)$ . Hence injectivity follows.

Now we prove surjectivity in the case where  $B$  is semiperfect. By Lemma 5.5 any element of  $W_{\mathcal{O}'}(B)$  can be written in the form

$$\sum_{j=0}^{\infty} [a_j] \varpi^j = \sum_{i=0}^{e-1} \sum_{h=0}^{\infty} [a_{he+i}] \pi^h \varpi^i / \alpha^h.$$

It suffices to check that  $\sum_{h=0}^{\infty} [a_{he+i}] \pi^h \alpha^{-h}$  is in the image of  $u_B^{\text{ra}}$  for all  $i$ . Note that the series  $\sum_{h=0}^{\infty} [a_{he+i}] \otimes \pi^h \alpha^{-h}$  is in  $W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}'$  since

$$\begin{aligned} W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}' &\simeq \left( \varprojlim_m W_{\mathcal{O}}(B) / \pi^m W_{\mathcal{O}}(B) \right) \otimes_{\mathcal{O}} \mathcal{O}' \\ &\simeq \varprojlim_m \left( (W_{\mathcal{O}}(B) / \pi^m W_{\mathcal{O}}(B)) \otimes_{\mathcal{O}} \mathcal{O}' \right) \\ &= \varprojlim_m W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}' / \pi^m (W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}'), \end{aligned}$$

where the first isomorphism follows by Lemma 5.5 and the second by the fact that  $\mathcal{O}'$  is a finite free  $\mathcal{O}$ -module. Now by  $\mathcal{O}'$ -linearity of  $u_B^{\text{ra}}$  and Lemma 5.7

$$u_B^{\text{ra}} \left( \sum_{h=0}^{\infty} [a_{he+i}] \otimes \pi^h \alpha^{-h} \right) = \sum_{h=0}^{\infty} [a_{he+i}] \pi^h \alpha^{-h},$$

and we are done. □

We now study morphisms  $\mathbf{u}$  and  $\mathbf{u}^{\text{ra}}$ .

**Proposition 5.14.** *Let  $\mathcal{O}'/\mathcal{O}$  be a totally ramified extension of degree  $e$ . Then the morphism  $\mathbf{u}_{\kappa}^{\text{ra}}$  in (5.15) is a universal homeomorphism with pro-infinitesimal kernel isomorphic to*

$$\text{Spec} \left( \kappa[X_{n,i}; n \in \mathbb{N}_0, 0 \leq i < e] / (X_{n,i}^{q^{n(e-1)+i}}) \right).$$

*Proof.* The first assertion follows from Lemmas 4.1 and 5.13.

We now describe the kernel of the morphism of  $\mathcal{O}'$ -group schemes

$$\prod^{\varpi} \mathbb{W}_{\mathcal{O}'} = \text{Spec } \mathcal{O}'[X_{n,i}, n \in \mathbb{N}_0, 0 \leq i < e] \xrightarrow{\mathbf{u}^{\text{ra}}} \mathbb{W}_{\mathcal{O}'} = \text{Spec } \mathcal{O}'[Y_0, \dots].$$

Set  $u_m^{\text{ra}} = u^{\text{ra}*}(Y_m) \in \mathcal{O}'[X_{n,i}, n \in \mathbb{N}_0, 0 \leq i < e]$  where  $u^{\text{ra}*}$  is the homomorphism induced by  $\mathbf{u}^{\text{ra}}$  on global sections. The kernel of  $\mathbf{u}^{\text{ra}}$  is the closed subscheme of  $\prod^{\varpi} \mathbb{W}_{\mathcal{O}}$  whose ideal  $I$  is generated by the polynomials  $u_m^{\text{ra}}, m \geq 0$ . Let  $J$  be the ideal generated by the monomials  $X_{n,i}^{q^{n(e-1)+i}}$ . We want to prove that  $I$  coincides with  $J \bmod \varpi$ . Both ideals admit a filtration by subideals  $I_s \subset I, J_s \subset J$  where  $I_s$  is generated by those  $u_m^{\text{ra}}$  with  $m \leq s$  and  $J_s$  is generated by monomials  $X_{m,j}^{q^{m(e-1)+j}}$  such that  $me + j \leq s$ . It is sufficient to check that  $I_s$  coincides with  $J_s \bmod \varpi$  for any  $s$ . We prove it by induction on  $s$ .

Clearly  $I_0 = (u_0^{\text{ra}}) = (X_{0,0}) = J_0$ . Assume  $s > 0$ , write it as  $s = ne + i$  with  $0 \leq i < e$  and assume  $(\varpi, I_m) = (\varpi, J_m)$  for all  $m < s$ . Note that  $I_s = (I_{s-1}, u_s^{\text{ra}})$  and  $J_s = (J_{s-1}, X_{n,i}^{q^{n(e-1)+i}})$ . Hence it is sufficient to prove that

$$(5.16) \quad u_{ne+i}^{\text{ra}} = u_s^{\text{ra}} \equiv \alpha^n X_{n,i}^{q^{n(e-1)+i}} \pmod{(\varpi^s, J_{s-1})}$$

since  $\alpha := \pi/\varpi^e$  is a unit in  $\mathcal{O}'$ .

Note that

$$(5.17) \quad \sum_{j=0}^{e-1} \varpi^j \Phi_s(X_{\cdot,j}) = \Phi'_s(u_0^{\text{ra}}, \dots) = \Phi'_{s-1}((u_0^{\text{ra}})^q, \dots) + \varpi^s u_s^{\text{ra}},$$

where the first equality follows by the commutativity of diagram (5.14) and the second one by definition of the polynomials  $\Phi'_m$ . The left hand side of (5.17) is sum of monomials of the form

$$\varpi^j \pi^m X_{m,j}^{q^{s-m}} = \varpi^{me+j} \alpha^m X_{m,j}^{q^{s-m}}$$

with  $m \leq s = ne + i$  and  $0 \leq j < e$ .

If  $m > n$ , the  $\varpi$ -order of the coefficient is bigger than  $s$ ; similarly if  $m = n$  and  $j > i$ . Hence

$$\sum_{j=0}^{e-1} \varpi^j \Phi_{s,j} \equiv \varpi^s \alpha^n X_{n,i}^{q^{s-n}} + \sum_{me+j < s} \varpi^{me+j} \alpha^m X_{m,j}^{q^{s-m}} \pmod{(\varpi^{s+1})},$$

and one concludes that

$$(5.18) \quad \sum_{j=0}^{e-1} \varpi^j \Phi_{s,j} \equiv \varpi^s \alpha^n X_{n,i}^{q^{s-n}} \pmod{(\varpi^{s+1}, J_{s-1})}.$$

since  $X_{m,j}^{q^{s-m}} = X_{m,j}^{q^{ne+i-m}}$  is a power of  $X_{m,j}^{m(e-1)+j} \in J_{s-1}$  when  $me + j < ne + i = s$ .

We now discuss the right hand side in (5.17).

$$(5.19) \quad \begin{aligned} & \Phi'_{s-1}((u_0^{\text{ra}})^q, \dots) + \varpi^s u_s^{\text{ra}} \\ &= \sum_{l=0}^{s-1} \varpi^l (u_l^{\text{ra}})^{q^{s-l}} + \varpi^s u_s^{\text{ra}} \equiv \varpi^s u_s^{\text{ra}} \pmod{(\varpi^{s+1}, J_{s-1})}, \end{aligned}$$

where the last equivalence follows from the fact that  $u_l^{\text{ra}} \equiv 0 \pmod{(\varpi, J_{s-1})}$  by inductive hypothesis. We conclude then by (5.17), (5.18) and (5.19) that

$$\varpi^s u_s^{\text{ra}} \equiv \varpi^s \alpha^n X_{n,i}^{q^{s-n}} \pmod{(\varpi^{s+1}, J_{s-1})},$$

whence claim (5.16) is true and the proof is finished.  $\square$

**5.2.3. The general case.** Let  $\mathcal{O}^{\text{un}}$  be the maximal unramified extension of  $\mathcal{O}$  in  $\mathcal{O}'$ . Then by (3.3)  $u_B = u_{(\mathcal{O}, \mathcal{O}'), B}$  is the composition

$$(5.20) \quad W_{\mathcal{O}}(B) \xrightarrow{u_{(\mathcal{O}, \mathcal{O}^{\text{un}}), B}} W_{\mathcal{O}^{\text{un}}}(B) \xrightarrow{u_{(\mathcal{O}^{\text{un}}, \mathcal{O}'), B}} W_{\mathcal{O}'}(B)$$

and results on  $u_B$  are usually deduced by a dévissage argument. We see here below an example.

**Lemma 5.15.** *Set  $\mathcal{O}_0 = W(\kappa)$  and let  $B$  be a reduced (respectively semiperfect, perfect)  $\kappa$ -algebra. Then the homomorphism*

$$r_B: W(B) \otimes_{\mathcal{O}_0} \mathcal{O} \rightarrow W_{\mathcal{O}}(B)$$

*induced by the Drinfeld functor is an injective (respectively, surjective, bijective). In particular the natural map  $\mathcal{O} \rightarrow W_{\mathcal{O}}(\kappa)$  is an isomorphism.*

*Proof.* Recall that  $\mathcal{O}_0/\mathbb{Z}_p$  is unramified,  $W(B) = W_{\mathbb{Z}_p}(B)$  and the extension  $\mathcal{O}/\mathcal{O}_0$  is totally ramified. The homomorphism in the lemma is then the composition

$$(5.21) \quad W(B) \otimes_{\mathcal{O}_0} \mathcal{O} \xrightarrow{u^{\text{un}} \otimes \text{id}_{\mathcal{O}}} W_{\mathcal{O}_0}(B) \otimes_{\mathcal{O}_0} \mathcal{O} \xrightarrow{u^{\text{ra}}} W_{\mathcal{O}}(B),$$

where  $u^{\text{un}}: u_{(\mathbb{Z}_p, \mathcal{O}_0), B}: W(B) \rightarrow W_{\mathcal{O}_0}(B)$  and  $u^{\text{ra}} = u_{(\mathcal{O}_0, \mathcal{O}), B} \otimes \text{id}_{\mathcal{O}}$ . Since  $\mathcal{O}$  is a free  $\mathcal{O}_0$ -module, it suffices to check the indicated properties for  $u^{\text{un}}$  and  $u^{\text{ra}}$ . These follow by Lemmas 5.11 and 5.13.  $\square$

**5.2.4. The case  $\pi = \varpi^e$ .** The description of  $u_B$  is particularly nice under the assumption that  $\pi = \varpi^e$ . Note that if  $\mathcal{O}'/\mathcal{O}$  is tamely ramified the hypothesis is satisfied up to enlarging  $\mathcal{O}'$ .

**Lemma 5.16.** *Let  $B$  be a  $\kappa'$ -algebra and assume  $\pi = \varpi^e$ . Then the map  $u_B: W_{\mathcal{O}}(B) \rightarrow W_{\mathcal{O}'}(B)$  factors through the subset  $W_{\mathcal{O}', e\mathbb{N}_0}(B)$  consisting of vectors  $\mathbf{b} = (b_0, \dots)$  such that  $b_j = 0$  if  $e \nmid j$ . If  $B$  is semiperfect its image is  $W_{\mathcal{O}', e\mathbb{N}_0}(B)$ , thus in this hypothesis,  $W_{\mathcal{O}', e\mathbb{N}_0}(B)$  is a subring of  $W_{\mathcal{O}'}(B)$ . If  $B$  is perfect then  $W_{\mathcal{O}}(B)$  is isomorphic to  $W_{\mathcal{O}', e\mathbb{N}_0}(B)$ .*

*Proof.* By Lemma 5.11 the case  $e = 1$  is clear. Since  $u_B = u_{(\mathcal{O}, \mathcal{O}'), B}$  is the composition of the maps in (5.20) we may assume that  $\mathcal{O}'/\mathcal{O}$  is totally ramified. Let  $\mathbb{W}_{\mathcal{O}} = \text{Spec } \mathcal{O}[X_0, \dots]$ ,  $\mathbb{W}_{\mathcal{O}'} = \text{Spec } \mathcal{O}'[Y_0, \dots]$  and set  $u_i = u^*(Y_i)$ . It is

$$\begin{aligned} \Phi_n(X_0, X_1, \dots) &= \Phi_{n-1}(X_0^q, \dots) + \pi^n X_n && \text{for } n \geq 1, \\ \Phi'_n(Y_0, Y_1, \dots) &= \Phi'_{n-1}(Y_0^q, \dots) + \varpi^n Y_n && \text{for } n \geq 1, \\ \Phi'_m(u_0, u_1, \dots) &= \Phi_m(X_0, X_1, \dots) && \text{for } m \geq 0. \end{aligned}$$

One checks recursively that  $u_0 = X_0$ ,  $u_i \equiv 0 \pmod{\varpi}$  if  $e \nmid i$  and  $u_{ne} \equiv X_n^{q^{n(e-1)}} \pmod{\varpi}$ . Hence, if  $B$  is any  $\mathcal{O}'$ -algebra and  $\mathbf{b} = (b_0, \dots) \in W_{\mathcal{O}}(B)$ , then  $u_B(\mathbf{b}) = \mathbf{c} = (c_0, c_1, \dots)$  with  $c_0 = b_0$ ,  $c_{ne} \equiv b_n^{q^{e(n-1)}} \pmod{\varpi B}$  and

$c_j \in \varpi B$  otherwise. In particular, if  $B$  is a  $\kappa'$ -algebra, it is  $c_{ne} = b_n^{q^{e(n-1)}}$  for any  $n \geq 0$  and zero otherwise. This implies that  $u_B(W_{\mathcal{O}}(B)) \subseteq W_{\mathcal{O}', e\mathbb{N}_0}(B)$  with equality if  $B$  is semiperfect.  $\square$

If  $\kappa = \kappa'$  we have a better understanding of  $\mathbf{u}^{\text{ra}}$ .

**Lemma 5.17.** *Let  $\mathcal{O}'/\mathcal{O}$  be a totally ramified extension of degree  $e$ . Assume  $\pi = \varpi^e$  and let  $B$  be a  $\kappa'$ -algebra. Then the homomorphism*

$$u_B^{\text{ra}}: W_{\mathcal{O}}(B) \otimes_{\mathcal{O}} \mathcal{O}' = \bigoplus_{i=0}^{e-1} W_{\mathcal{O}}(B)\varpi^i \rightarrow W_{\mathcal{O}'}(B),$$

$$\sum_{i=0}^{e-1} \mathbf{b}_i \varpi^i \mapsto \sum_{i=0}^{e-1} u_B(\mathbf{b}_i) \varpi^i,$$

$\mathbf{b}_i \in W_{\mathcal{O}}(B)$ , maps the module  $W_{\mathcal{O}}(B)\varpi^i$  into  $W_{\mathcal{O}', i+e\mathbb{N}_0}(B)$  and it is injective (respectively, surjective, bijective) if  $B$  is reduced (respectively, semiperfect, perfect).

*Proof.* We have seen in Lemma 5.16 that for any  $\mathbf{b} \in W_{\mathcal{O}}(B)$  it is  $u_B(\mathbf{b}) \in W_{\mathcal{O}', e\mathbb{N}_0}(B)$ ; hence by (5.6) and (5.5)  $u_B(\mathbf{b})\varpi = VFu_B(\mathbf{b}) \in VW_{\mathcal{O}', e\mathbb{N}_0}(B) = W_{\mathcal{O}', 1+e\mathbb{N}_0}(B)$  and recursively  $u_B(\mathbf{b})\varpi^i \in W_{\mathcal{O}', i+e\mathbb{N}_0}(B)$ . Note further that the subsets  $e\mathbb{N}_0, 1+e\mathbb{N}_0, \dots, e-1+e\mathbb{N}_0$  form a partition of  $\mathbb{N}_0$  so that the sum  $\sum_{i=0}^{e-1} u_B(\mathbf{b}_i)\varpi^i$  is simply obtained by “glueing” the components of each vector  $u_B(\mathbf{b}_i)\varpi^i = V^i F^i u_B(\mathbf{b}_i)$  (see Remark 5.4). As a consequence the injectivity (respectively, surjectivity) statement follows from Lemma 5.10.  $\square$

### 6. The comparison result

Let  $\mathcal{O}$  be a complete discrete valuation ring with residue field  $\kappa$  of cardinality  $q = p^h$ , and absolute ramification  $e$ . Set  $\mathcal{O}_0 = W(\kappa)$ . As seen in Lemma 5.15 we may consider the Drinfeld map  $u: W(A) \rightarrow W_{\mathcal{O}}(A)$  for any  $\kappa$ -algebra  $A$  and hence we extend it to a natural homomorphism of  $\mathcal{O}$ -algebras

$$r_A := u \otimes \text{id}_{\mathcal{O}}: W(A) \otimes_{W(\kappa)} \mathcal{O} \rightarrow W_{\mathcal{O}}(A).$$

In other words, due to the description of  $A$ -sections of  $\mathbb{R}_{\mathcal{O}}$  in (2.1), there exists a morphism of  $\kappa$ -ring schemes

$$(6.1) \quad \mathbf{r}: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O}, \kappa} := \mathbb{W}_{\mathcal{O}} \times_{\mathcal{O}} \text{Spec}(\kappa)$$

that coincides with  $r_A$  on  $A$ -sections. Then Lemma 5.15 can be rewritten as follows.

**Lemma 6.1.** *If  $A$  is a reduced (respectively, semiperfect, perfect)  $\kappa$ -algebra then  $r_A: \mathbb{R}_{\mathcal{O}}(A) \rightarrow \mathbb{W}_{\mathcal{O}}(A)$  is injective (respectively, surjective, bijective).*

We can now prove the comparison result announced in the introduction.

**Theorem 6.2.** *The morphism  $\mathbf{r}: \mathbb{R}_{\mathcal{O}} \rightarrow \mathbb{W}_{\mathcal{O},\kappa}$  defined in (6.1) induces an isomorphism  $\mathbf{r}^{\text{Pf}}: \mathbb{R}_{\mathcal{O}}^{\text{Pf}} \rightarrow \mathbb{W}_{\mathcal{O},\kappa}^{\text{Pf}}$  on perfection. Hence  $\mathbf{r}$  is a universal homeomorphism, thus surjective, and it has pro-infinitesimal kernel.*

*Proof.* By Lemma 6.1 and 4.1 the morphism  $\mathbf{r}^{\text{Pf}}$  is invertible and  $\mathbf{r}$  is a universal homeomorphism. Further,  $\mathbf{r}$  is a morphism of affine  $\kappa$ -group schemes and

$$\begin{aligned} \ker(\mathbf{r})(\bar{\kappa}) &= \ker(\mathbb{R}_{\mathcal{O}}(\bar{\kappa}) \rightarrow \mathbb{W}_{\mathcal{O}}(\bar{\kappa})) \simeq \ker(\mathbb{R}_{\mathcal{O}}^{\text{Pf}}(\bar{\kappa}) \rightarrow \mathbb{W}_{\mathcal{O}}^{\text{Pf}}(\bar{\kappa})) \\ &= \ker(\mathbf{r}^{\text{Pf}})(\bar{\kappa}) = \{0\}, \end{aligned}$$

where  $\bar{\kappa}$  denotes an algebraic closure of  $\kappa$  and the bijection in the middle follows by (4.1). Hence  $\ker(\mathbf{r})$  is pro-infinitesimal by [5, V §3 Lemme 1.4]. □

We can say something more on the kernel of  $\mathbf{r}$ .

**Lemma 6.3.**

(a) *If  $\mathcal{O} = W(\kappa)$  then  $\mathbf{r} = \mathbf{u}_{(\mathbb{Z}_p, W(\kappa)), \kappa}$  and*

$$\ker \mathbf{r} \simeq \text{Spec}\left(\kappa[X_0, X_1, \dots] / (X_0, X_1^{p^{h-1}}, \dots, X_i^{p^{i(h-1)}}, \dots)\right).$$

(b) *If  $\kappa = \mathbb{F}_p$ , then  $\mathbf{r} = \mathbf{u}_{\kappa}^{\text{ra}}$  and*

$$\ker \mathbf{r} \simeq \text{Spec}\left(\mathbb{F}_p[X_{n,i}; n \in \mathbb{N}_0, 0 \leq i < e] / (X_{n,i}^{p^{n(e-1)+i}}; n \in \mathbb{N}_0, 0 \leq i < e)\right).$$

(c) *In general,  $\ker(\mathbf{r})$  is extension of a pro-infinitesimal group scheme as in Proposition 5.14 by the product of  $e$  pro-infinitesimal group schemes as in (a).*

*Proof.* Consider the extension  $\mathcal{O}/\mathbb{Z}_p$ . Statements (a) and (b) follow from Propositions 5.12 and 5.14. For the general case, note that (5.21) implies that  $\mathbf{r}$ , as morphism of  $\kappa$ -group schemes, is the composition

$$\prod_{i=0}^{e-1} \mathbb{W}_{\kappa} \xrightarrow{\prod_i \mathbf{u}_{\kappa}} \prod_{i=0}^{e-1} \mathbb{W}_{\mathcal{O}_0, \kappa} \xrightarrow{\mathbf{u}_{\kappa}^{\text{ra}}} \mathbb{W}_{\mathcal{O}, \kappa}$$

where  $\mathbf{u}_k$  on the first arrow stays for  $\mathbf{u}_{(\mathbb{Z}_p, \mathcal{O}_0), \kappa}$ , whose kernel was described in a), and  $\mathbf{u}_{\kappa}^{\text{ra}}$  is the morphism in Proposition 5.14 for the ramified extension  $\mathcal{O}/\mathcal{O}_0$ . Hence the conclusion follows. □

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