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Résumé. Soit p un nombre premier impair. Quels sont les polygones de Newton possibles pour les courbes en caractéristique p ? Autrement dit, quelles sont les strates de Newton qui s'intersectent avec le lieu de Torelli dans $\mathcal{A}_g$? Nous étudions les polygones de Newton de certaines courbes équipées d'une action du groupe fini $\mathbb{Z}/p\mathbb{Z}$. Plusieurs de ces courbes fournissent des exemples d'intersections improbables entre le lieu de Torelli et la stratification de Newton dans $\mathcal{A}_g$. Voici un exemple qui présente un intérêt particulier : en fixant un genre $g > 1$, nous montrons que pour tout k tel que $\frac{2g}{3} - \frac{2p(p-1)}{3} \geq 2k(p-1)$, il existe une courbe C de genre g telle que les pentes de Newton de C sont $\{0,1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$. Cela confirme une conjecture d'Oort selon laquelle l'amalgamation des polygones de Newton de deux courbes est aussi le polygone de Newton d'une courbe. Nous construisons aussi quelques familles de courbes $\{C_g\}_{g\geq 1}$ de genre g, dont les polygones asymptotiques de Newton sont intéressants. Par exemple, nous construisons une famille de courbes dont le polygone asymptotique de Newton est minoré par $y = \frac{x^2}{4g}$. Les outils principaux de l'article sont un résultat « polygone de Newton est situé au-dessus du polygone de Hodge » pour les courbes équipées d'une action de $\mathbb{Z}/p\mathbb{Z}$, dû à l'auteur, et un travail récent de Booher–Pries qui montre que cette borne de Hodge est atteinte.

Abstract. Let p be an odd prime. What are the possible Newton polygons for a curve in characteristic p? Equivalently, which Newton strata intersect the Torelli locus in $\mathcal{A}_g$? In this note, we study the Newton polygons of certain curves with $\mathbb{Z}/p\mathbb{Z}$-actions. Many of these curves exhibit unlikely intersections between the Torelli locus and the Newton stratification in $\mathcal{A}_g$. Here is one example of particular interest: fix a genus g. We show that for any k with $\frac{2g}{3} - \frac{2p(p-1)}{3} \geq 2k(p-1)$, there exists a curve of genus g whose Newton polygon has slopes $\{0,1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$. This provides evidence for Oort's conjecture that the amalgamation of the Newton polygons of two curves is again the Newton polygon of a curve. We also construct families of curves $\{C_g\}_{g\geq 1}$, where C_g is a curve of genus g, whose Newton polygons have interesting asymptotic properties. For example, we construct a family of curves whose Newton polygons are asymptotically bounded below by the graph $y = \frac{x^2}{4g}$.

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The proof uses a Newton-over-Hodge result for $\mathbb{Z}/p\mathbb{Z}$-covers of curves due to the author, in addition to recent work of Booher–Pries on the realization of this Hodge bound.

1. Introduction

Let p be an odd prime. By a curve, we will always mean a smooth proper irreducible curve. What are the possible Newton polygons for a curve of genus g in characteristic p? In general, it seems difficult to answer this question for all g. However, the following conjecture of Oort offers some guidance on what to expect.

Conjecture 1.1 (Oort, see [5, Conjecture 8.5.7])**.** *Let C (resp. C') be a curve of genus g (resp. g') with Newton polygon P (resp. P'). Then there exists a curve C'' of genus $g+g'$ whose Newton polygon P'' is the amalgamation of P and P' (i.e. the slopes of P'' are the disjoint union of the slopes of P and P').*

This conjecture implies, for example, that there exists irreducible supersingular curves of every genus. This is currently only know in characteristic 2, due to a theorem of van der Geer and van der Vlugt (see [8]). Another approach to studying Newton polygons of curves is to ask if there are curves C_g of every genus $g \in \mathbb{Z}_{\geq 0}$ whose Newton polygons approach some limit asymptotically. This motivates the following questions.

Question 1.2. Let $m_1, \dots, m_{2r} \in \mathbb{Q} \cap [0,1]$ such that $m_k = 1 - m_{2r-k}$. Does there exist a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where C_g has genus g, such that the Newton polygon of C_g consists only of the slopes $m_1, \dots, m_{2r}$ and each m_i occurs with multiplicity close to $\frac{2g}{2r}$?

Question 1.3. Let P be the graph of a continuous function $f : [0,2] \to [0,1]$. Does there exist a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where C_g has genus g, such that the Newton polygon of C_g, scaled by a factor of $\frac{1}{g}$, approaches or lies above P as $g \to \infty$?

In this article we study Conjecture 1.1, Question 1.2, and Question 1.3 by considering $\mathbb{Z}/p\mathbb{Z}$-covers of curves.

1.1. Some previous results. There are two main approaches when trying to find curves with certain Newton polygons. The first is to consider $\mathbb{Z}/n\mathbb{Z}$-covers $f : X \to \mathbb{P}^1$. When n is a power of p and f is ramified at a single point, the Newton polygon of X has been studied extensively by Robba, Zhu, Blache–Ferard, and Liu–Wei, using p-adic methods pioneered by Dwork (see [2, 13, 16, 17]). Also, the work of van der Geer and van der Vlugt (see [8]) studies $\mathbb{Z}/p\mathbb{Z}$-covers with additional structure when $p = 2$ by analyzing their Jacobians. When n is coprime to p, the number of slope zero segments (i.e. the p-rank) was determined in many cases by Bouw (see [4]). Determining the larger slopes appears to be more difficult and is the subject of work by Li–Mantovan–Pries–Tang (see [11, 12]). The second approach is to use clutching morphisms between moduli spaces of curves. This technique was used by Achter–Pries in [1] to show that for $g \geq 4$, there exists a genus g curve whose Newton polygon has slopes $\{0,1\}^{g-4} \sqcup \{\frac{1}{4}, \frac{3}{4}\}^4$. More recently, these two techniques were combined in work of Li–Mantovan–Pries–Tang by studying clutching morphisms for tame covers of $\mathbb{P}^1$. Their work constructs many interesting families of curves whose Newton polygons are far from ordinary and follow certain patterns. However, most of these families do not include curves of every genus. Instead these families include curves whose genera satisfy certain congruence conditions. We describe some of these families in Remark 2.17 and Examples 4.5–4.6.

1.2. $\mathbb{Z}/p\mathbb{Z}$-covers with many branch points. In Section 4 we study $\mathbb{Z}/p\mathbb{Z}$-covers of curves with many branch points of fixed Swan conductor. Let us state a specific case of our result (see Theorem 4.1 for the general statement).

Theorem 1.4. *For any $g \geq 0$ and k with $\frac{2g}{3} - \frac{2p(p-1)}{3} \geq 2k(p-1)$, there exists a curve $C_{g,k}$ of genus g whose Newton polygon has slopes $\{0,1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$.*

From Theorem 1.4 we may deduce many interesting examples of Conjecture 1.1. Indeed, if $C = C_{g,k}$ and $C' = C_{g',k'}$, then the Newton polygon of $C'' = C_{g+g',k+k'}$ is the amalgamation of the Newton polygons of C and C'. When k is not small relative to g, we see that $C_{g,k}$ demonstrates an unlikely intersection between a Newton stratum and the Torelli locus in the Siegel modular variety (see Corollary 4.4). Furthermore, we may use Theorem 1.4 to study Question 1.2. By letting k be as large as possible, we see that there are curves of every genus consisting of slopes $0, \frac{1}{2}, 1$, where each slope occurs with multiplicity approximately $\frac{2g}{3}$. This was previously only known under the assumption of $p \equiv 2 \mod 3$ (see [11, Corollary 9.4]).

1.3. $\mathbb{Z}/p\mathbb{Z}$-covers of curves with large Swan conductor. In Section 5 we study curves with a small number of branch points and large Swan conductors. We construct a family of curves $\{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$, where C_g has genus g, such that their Newton polygons asymptotically lie on or above the graph $y = \frac{x^2}{4g}$ (see Definition 2.10 and Theorem 5.1 for a precise statement). To the best of our knowledge, it is unknown if there is a family of curves containing a curve of every genus whose Newton polygons asymptotically lie strictly above $y = \frac{x^2}{4g}$. This would certainly follow from Conjecture 1.1. However, all of the families constructed in Section 4 and the families constructed in [11] lie well below $y = \frac{x^2}{4g}$ when the genus is large.

Remark 1.5. It is natural to ask if $\frac{x^2}{4}$ has any particular significance. In this article, the lower bound in Theorem 3.1 comes from the irregular Hodge filtration defined by Deligne (see [7]). In the work of Li–Mantovan–Pries–Tang, they use the μ-ordinary lower bound, which is essentially a Hodge bound averaged over Frobenius orbits (see [11, Section 2.6.1]). In both cases, the graph $y = \frac{x^2}{4g}$ appears to be a natural limit for these Hodge bounds.

1.4. Method of proof. To prove Theorem 1.4 and Theorem 5.1, we use a Newton-over-Hodge result due to the author (see [10] or Theorem 3.1). This theorem gives a lower bound for the Newton polygon of a $\mathbb{Z}/p\mathbb{Z}$-cover $C \to X$ in terms of local Swan conductors. By considering covers of an arbitrary curve X instead of only $\mathbb{P}^1$, we are able to obtain curves of any genus (e.g. if X has genus i, the genus of C is of the form $i + k\frac{p-1}{2}$ by Riemann–Hurwitz). This is a clear advantage over earlier techniques. For the general statement of Theorem 4.1, we also need to use recent work of Booher–Pries (see [3]). This work shows the lower bound in [10] is realized when certain congruence conditions between p and the Swan conductors hold.

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2. Newton polygons and unlikely intersections in $\mathcal{A}_g$

2.1. Conventions on Newton polygons. Let $\alpha \in \mathbb{Z}_{\geq 0} \cup \infty$ and let $f : [0, \alpha] \to \mathbb{R}$ be a continuous convex function. We let $P = P(f, \alpha)$ refer to the graph of f in the xy-plane. We refer to the points $(0, f(0))$ and $(\alpha, f(\alpha))$ as the endpoints of P. We say that P is a *symmetric* if $f(\alpha - x) = f(\alpha) - f(x)$ for all $x \in [0, \alpha]$. The following two types of graphs are of particular interest.

Definition 2.1. A graph $P = P(f, 2)$ is called *basic* if P is symmetric and the endpoints of P are $(0, 0)$ and $(2, 1)$.

Definition 2.2. We say $P(f, 2g)$ is a *Newton polygon of height* $2g$ if f satisfies the following:

(1) f is symmetric and the endpoints are $(0, 0)$ and $(2g, g)$.
(2) For any integer $i \in (0, 2g]$, the function $f(x)$ is linear on the domain $x \in [i - 1, i]$ with slope $m_i \geq 0$.

We will refer to the multiset $\{m_i\}_{1 \leq i \leq 2g}$ as the *slope-set* of P and its elements as the slopes of P. Note that the slope-set completely determines the Newton polygon.

Remark 2.3. It is often required that the vertices of a Newton polygon have integer coordinates. We drop this requirement, since some of our lower bounds (see e.g. Theorem 3.1) will have non-integer vertices.

For $i = 1, 2$, consider graphs $P_i = P(f_i, \alpha)$. We write $P_1 \succeq P_2$ if $f_1(x) \geq f_2(x)$ for all $0 \leq x \leq \alpha$. When $P_1 \succeq P_2$, we say that P_1 lies above P_2. If P_2 is a Newton polygon with slope-set N, we will occasionally write $P_1 \succeq N$ instead of $P_1 \succeq P_2$. Finally, for any $c \geq 0$, we define the scaled graph $cP_i = \{(cx, cy) | (x, y) \in P_i\}$.

2.2. The Newton stratification of $\mathcal{A}_g$. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g and let $\mathcal{X} \to \mathcal{A}_g$ be the universal abelian scheme. For each closed $x \in \mathcal{A}_g$, we obtain an Abelian variety $\mathcal{X}_x$. Let NP_x denote the Newton polygon of height $2g$ associated to $\mathcal{X}_x$ (see [14, (1.2)]). We remark that the vertices of NP_x have integer coordinates. This greatly restricts the possibilities for NP_x.

Definition 2.4. Let P be a Newton polygon of height $2g$. Let $W_P \subset \mathcal{A}_g$ (resp. W_P^0) denote the locus of principally polarized abelian varieties $\mathcal{X}_x$ with $NP_x \succeq P$ (resp. $NP_x = P$). Note that W_P is closed in $\mathcal{A}_g$ and W_P^0 is open in W_P (see [14, Section 4]).

Theorem 2.5. *The codimension of W_P in $\mathcal{A}_g$ is at least $\#\Omega(P)$, where*

$$\Omega(P) = \{(x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid 0 \leq x \leq g \text{ and } (x, y) \text{ lies strictly below } P\}.$$

Furthermore, if P has integer vertices, the codimension of W_P in $\mathcal{A}_g$ is exactly $\#\Omega(P)$.

Proof. This follows from a theorem of Oort (see [14, Theorem 4.1]), by noting that $W_P = \cup_{P'} W_{P'}$, where the union runs over all Newton polygons of height $2g$ with integer vertices lying above P. We warn the reader that the ordering we put on Newton polygons differs from the ordering found in Oort's article. □

2.3. The Torelli locus. The Torelli map $\iota : \mathcal{M}_g \to \mathcal{A}_g$ sends a curve C of genus g to its Jacobian. Let $\mathcal{T}_g$ denote the image of the Torelli map. It is a closed subscheme of dimension $3g - 3$. The Newton polygon $NP(C)$ of C is defined to be the Newton polygon of the corresponding point in $\mathcal{A}_g$. If C is defined over a finite field $\mathbb{F}_q$, then $NP(C)$ is equal to the q-adic Newton polygon of the numerator of its zeta function. We define the *scaled Newton polygon* to be

$$sNP(C) = \frac{1}{g} NP(C).$$

Note that $sNP(C)$ is a basic graph (see Definition 2.1). We are interested in the following questions.

Question 2.6. How does the Torelli locus interact with the Newton stratification? More specifically, let P be a Newton polygon of height $2g$. Does there exist a curve C of genus g with $NP(C) = P$ (resp. $NP(C) \succeq P$). Equivalently, is $W_P^0 \cap \mathcal{T}_g$ (resp. $W_P \cap \mathcal{T}_g$) nonempty?

This question appears to be very difficult in general. Instead, one may ask for asymptotic behavior as g gets large. One way to do this is to study the behavior of $sNP(C)$ as C varies over a collection of curves. This prompts the following definitions.

Definition 2.7. A *family of curves* is a collection $\mathcal{C} = \{C_g\}_{g \in S}$ where $S \subset \mathbb{Z}_{\geq 0}$ and C_g is a curve of genus g. We say that $\mathcal{C}$ is *full* if $S = \mathbb{Z}_{\geq 0}$ and we say that $\mathcal{C}$ is *arithmetic* if S is the union of finitely many arithmetic progressions.

Definition 2.8. A *family of Newton polygons (resp. basic graphs)* is a collection $\mathbf{P} = \{P_g\}_{g \in S}$ (resp. $\mathbf{sP} = \{sP_g\}_{g \in S}$) where $S \subset \mathbb{Z}_{\geq 0}$ and P_g (resp. sP_g) is a Newton polygon of height $2g$ (resp. a basic graph). If $\mathcal{C} = \{C_g\}_{g \in S}$ is a family of curves, we may associate a family of Newton polygons (resp. basic graphs) by $\mathbf{NP}(\mathcal{C}) = \{NP(C_g)\}_{g \in S}$ (resp. $\mathbf{sNP}(\mathcal{C}) = \{sNP(C_g)\}_{g \in S}$).

Definition 2.9. Let $\mathbf{sP} = \{sP_g\}_{g \in S}$ be a family of basic graphs with $sP_g = (f_g, 2)$ and let $P = (f, 2)$ be a basic graph. We say that $\mathbf{sP}$ converges uniformly to P if the f_g converge uniformly to f.

Definition 2.10. Let $\mathcal{C} = \{C_g\}_{g \in S}$ be a family of curves and let $P = P(f, 2)$ be a basic graph. For each $g \in S$, let f_g be the piecewise linear function such that $sNP(C_g) = P(f_g, 2)$. We write $\mathbf{sNP}(\mathcal{C}) \succeq P$ if

$$\liminf_{g \in S} \min_{x \in [0,2]} \{f_g(x) - f(x)\} \geq 0.$$

Question 2.11. Let P be a basic graph. Does there exist a family of curves $\mathcal{C}$ such that $\mathbf{sNP}(\mathcal{C}) \succeq P$? Can we take this family to be arithmetic or full?

The following lemma will be helpful when studying Question 2.11.

Lemma 2.12. *Let $\mathcal{C} = \{C_g\}_{g \in S}$ be a family of curves and let $\mathbf{sP} = \{sP_g\}_{g \in S}$ be a family of basic graphs $sP_g = P(h_g, 2)$ such that $sNP(C_g) \succeq sP_g$. If $\mathbf{sP}$ converges to $P = P(f, 2)$, then $\mathbf{sNP}(\mathcal{C}) \succeq P$.*

Proof. Let f_g be the piecewise linear function such that $sNP(C_g) = P(f_g, 2)$, so that $f_g(x) \geq h_g(x)$ for all $x \in [0, 2]$. Then we have

$$\liminf_{g \in S} \min_{x \in [0,2]} \{f_g(x) - f(x)\} \geq \liminf_{g \in S} \min_{x \in [0,2]} \{h_g(x) - f(x)\} = 0. \qquad \square$$

Another natural question is to ask for families of curves where certain slopes occur with some specified frequency. For example, we may ask for a family of curves $\{C_g\}_{g \in S}$ where $NP(C_g)$ only has slopes $0, \frac{1}{2}, 1$, and each slope occurs with approximately equal frequency. This prompts the following definition.

Definition 2.13. Let $\mathbf{P} = \{P_g\}_{g \in S}$ be a family of Newton polygons. Let $\mathcal{N} = \{m_i\}_{i=1}^{2r}$ be the slope-set of a Newton polygon with $m_i \in [0,1] \cap \mathbb{Q}$. We write $\mathbf{P} \sim \mathcal{N}$ if there exists $\epsilon > 0$ such that

$$P_g = \bigsqcup_{i=1}^{2r} \{m_i\}^{\frac{g}{r}+e_i(g)},$$

with $|e_i(g)| < \epsilon$ for all $g \in S$. Informally, this means that P_g has slopes $m_1, \ldots, m_{2r}$, each occurring with approximately the same frequency.

Question 2.14. Let $\mathcal{N} = \{m_i\}_{i=1}^{2r}$ be as in Definition 2.13. Is there a full or arithmetic family $\mathcal{C}$ with $\mathbf{NP}(\mathcal{C}) \sim \mathcal{N}$?

By the following lemma, Question 2.14 and Question 2.11 are closely related.

Lemma 2.15. *Let $\mathbf{P} = \{P_g\}_{g \in S}$ be a family of Newton polygons and define the family of basic graphs by $\mathbf{sP} = \{\frac{1}{g}P_g\}_{g \in S}$. If $\mathbf{P} \sim \mathcal{N}$, where $\mathcal{N} = \{m_i\}_{i=1}^{2r}$, then $\mathbf{sP}$ converges uniformly to $\frac{1}{r}\mathcal{N}$.*

Proof. Without loss of generality assume that $m_{i+1} \geq m_i$. Let ϵ be as in Definition 2.13. Take $\mathcal{R}_g$ (resp. $\mathcal{S}_g$) to be the graph $\frac{g}{r}\mathcal{N}$ shifted down (resp. up) by $2r\epsilon$. That is, each $\mathcal{R}_g$ (resp. $\mathcal{S}_g$) is the lower convex hull of the points $(n\frac{g}{r}, \sum_{i=1}^{n} \frac{g}{r}m_i - 2r\epsilon)$ (resp. $(n\frac{g}{r}, \sum_{i=1}^{n} \frac{g}{r}m_i + 2r\epsilon)$). By definition, we know $\mathcal{S}_g \succeq NP(C_g) \succeq \mathcal{R}_g$, and thus $\frac{1}{g}\mathcal{S}_g \succeq \frac{1}{g}NP(C_g) \succeq \frac{1}{g}\mathcal{R}_g$. The lemma follows by observing that both $\frac{1}{g}\mathcal{R}_g$ and $\frac{1}{g}\mathcal{S}_g$ converge uniformly on $[0,2]$ to $\frac{1}{r}\mathcal{N}$. $\square$

Remark 2.16. If one does not require the family of curves to be arithmetic or full, it is much easier to find families with interesting asymptotic properties. For example, in [15, Corollary 2.6] the authors construct an infinite family of supersingular curves. However, this family is much too sparse to be arithmetic.

Remark 2.17. The work of [11] proves the existence of many families $\mathcal{C}$ and many interesting slope-sets $\mathcal{N}$ satisfying $\mathbf{NP}(\mathcal{C}) \sim \mathcal{N}$. Most of these families are arithmetic, although a few special cases are full. Here is one particularly interesting example: assume that $p \equiv 4 \mod 5$. They prove the existence of a family $\mathcal{C} = \{C_g\}_{g \in S}$ where $S = \{10n - 4\}_{n \geq 1}$ and

$$\mathbf{NP}(\mathcal{C}) \sim \{0\}^3 \sqcup \{\frac{1}{2}\}^4 \sqcup \{1\}^3.$$

The key technical aspect of their work is an analysis of clutching morphisms for moduli of tame cyclic covers of $\mathbb{P}^1$. Using this analysis, they give an inductive process to construct arithmetic families of curves with prescribed Newton polygons. Combining this with their previous results on special subvarieties of Shimura varieties (see [12]) gives many interesting examples.

2.4. Unlikely intersections on $\mathcal{A}_g$. Let X be a variety of dimension d. Let V_1 and V_2 be subvarieties of X with codimensions c_1 and c_2. If $c_1 + c_2 \leq d$, then we expect the intersection $V_1 \cap V_2$ to be a k-cycle, where $k = d - c_1 - c_2$. However, if $c_1 + c_2 > d$, then $V_1 \cap V_2$ will typically be empty. We say that V_1 and V_2 have an *unlikely intersection* if $V_1 \cap V_2$ is nonempty and $c_1 + c_2 > d$. For example, let $\mathcal{A}_g^{s.s.} \subset \mathcal{A}_g$ denote the supersingular locus. We know that $\dim(\mathcal{A}_g^{s.s.}) = \lfloor \frac{g^2}{4} \rfloor$ and $\dim(\mathcal{T}_g) = 3g - 3$. Thus, the existence of a supersingular curve of genus $g > 3$ implies $\mathcal{A}_g^{s.s.}$ and $\mathcal{T}_g$ have an unlikely intersection. More generally, a high genus curve that is sufficiently far from being ordinary implies an unlikely intersection between the Torelli locus and a Newton stratum. We point the reader to Oort's article in [5] for more background.

Definition 2.18. Let C be a curve of genus g and let $P = NP(C)$. We say that C has an *unlikely Newton polygon* if W_P^0 and $\mathcal{T}_g$ have an unlikely intersection. We say a family of curves $\mathcal{C} = \{C_g\}_{g \in S}$ is *unlikely* if for $g \gg 0$ the curve C_g has an unlikely Newton polygon.

Lemma 2.19. *Let $\mathcal{C} = \{C_g\}_{g \in S}$ be a family of curves and let $P = P(f, 2)$ be a basic graph. If $f(1) > 0$ and $\mathbf{sNP}(\mathcal{C}) \succeq P$, then $\mathcal{C}$ is an unlikely family.*

Proof. We may replace P with a slightly lower basic graph so that $sNP(C_g) \succeq P$ for large g. By lowering P more we may assume that P consists of three line segments with slopes 0, $\frac{1}{2}$, and 1. The codimension of $\mathcal{T}_g$ in $\mathcal{A}_g$ is $\frac{g(g+1)}{2} - 3g + 3$, so it suffices to show that the codimension of W_{gP} in $\mathcal{A}_g$ grows quadratically in g. Define $R_g = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 0 \leq x \leq g \text{ and } 0 \leq y < gf(\frac{x}{g})\}$. By Theorem 2.5, it suffices to show that $\#((\mathbb{Z} \times \mathbb{Z}) \cap R_g)$ grows quadratically in g. This follows by observing that the R_g are similar triangles whose side lengths grow linearly in g. □

Corollary 2.20. *Let $\mathcal{C}$ be a family of curves and let $\mathcal{N}$ be a Newton polygon of height $2g$ that is non-ordinary. If $\mathbf{NP}(\mathcal{C}) \sim \mathcal{N}$, then $\mathcal{C}$ is an unlikely family.*

Proof. Let $P(f, 2)$ be the basic graph $\frac{1}{g}\mathcal{N}$. By Lemma 2.12 and Lemma 2.15 we know that $\mathbf{sNP}(\mathcal{C}) \succeq P(f, 2)$. The non-ordinary condition implies $f(1) > 0$, so the result follows from Lemma 2.19. □

3. $\mathbb{Z}/p\mathbb{Z}$-covers of curves

Let X be a curve of genus g over a finite field $\mathbb{F}_q$ and let $r : C \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-cover. Let $\tau_1, \dots, \tau_m \in X$ be the points where r is ramified. Let d_i be the Swan conductor of r at τ_i. We may describe d_i as follows. Let t_i be a local parameter at τ_i. Locally the cover r is given by an Artin–Schreier equation $Y^p - Y = g_i$, where $g_i \in \mathbb{F}_q((t_i))$. We may assume that g_i has a pole whose order is coprime to p. The order of this pole is equal to the Swan conductor. That is, $g_i = \sum_{n \geq -d_i} a_n t_i^n$ and $a_{-d_i} \neq 0$. Now, let f be a rational function on X and assume that C is given by the equation $Y^p - Y = f$. Then $-\mathrm{ord}_{\tau_i}(f) \geq d_i$ with equality if and only if $\gcd(\mathrm{ord}_{\tau_i}(f), p) = 1$.

Theorem 3.1. *We have*

$$NP(C) \succeq \{0,1\}^{pg+m(p-1)} \bigsqcup \left\{ \frac{1}{d_1}, \ldots, \frac{d_1 - 1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m} \right\}^{p-1}.$$

Proof. This is an earlier theorem of the author. See [10, Corollary 1.3]. The proof uses the Monsky trace formula and some delicate p-adic analysis. □

Corollary 3.2. *Assume X is ordinary and each Swan conductor is equal to 2. Then*

$$NP(C) = \{0,1\}^{pg+m(p-1)} \bigsqcup \left\{ \frac{1}{2} \right\}^{m(p-1)}. \tag{3.1}$$

Proof. From Theorem 3.1 we know $NP(C) \succeq \{0,1\}^{pg+m(p-1)} \sqcup \{\frac{1}{2}\}^{m(p-1)}$. By the Deuring–Shafarevich formula (see e.g. [6]), we see that $\{0,1\}^{pg+m(p-1)} \sqcup \{\frac{1}{2}\}^{m(p-1)}$ has the correct number of slope zero segments and slope one segments. As the remaining slopes are $1/2$, the two Newton polygons must be equal. □

In general, the bound in Theorem 3.1 will not be attained. However, if $p \equiv 1 \mod d_i$ for each i, recent work of Booher–Pries shows this bound is optimal.

Theorem 3.3 (Booher–Pries)**.** *Assume X is ordinary and let $d_1, \ldots, d_m \in \mathbb{Z}_{\geq 1}$ such that $p \equiv 1 \mod d_i$ for each i. There exists a $\mathbb{Z}/p\mathbb{Z}$-cover of X, which is ramified at the points $\tau_1, \ldots, \tau_m$ with Swan conductor d_i, such that*

$$NP(C) = \{0,1\}^{pg+m(p-1)} \bigsqcup \left\{ \frac{1}{d_1}, \ldots, \frac{d_1 - 1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m} \right\}^{p-1}. \tag{3.2}$$

Proof. See [3, Corollary 4.3]. The main idea is as follows: work of Blache–Ferard computes the Newton polygon of a generic $\mathbb{Z}/p\mathbb{Z}$-cover of $\mathbb{P}^1$ ramified only at ∞ when the Swan conductor is less than $3p$. Booher and Pries use this to construct a $\mathbb{Z}/p\mathbb{Z}$-cover of singular curves $C_0 \to X_0$ and calculate the Newton polygon of C_0. Using a formal patching argument, they show that $C_0 \to X_0$ deforms to a family of $\mathbb{Z}/p\mathbb{Z}$-covers $\mathcal{C}_0 \to \mathcal{X}_0$ that generically gives a cover of X. By Grothendieck's specialization theorem (see [9]), this gives an upper bound for the Newton polygon of a generic cover in this family. This upper bound is precisely the lower bound of Theorem 3.1 when $p \equiv 1 \mod d_i$. □

Remark 3.4. In forthcoming work of James Upton and the author, we prove that all (3.2) holds for all such $\mathbb{Z}/p\mathbb{Z}$-covers of X.

4. Letting the number of branch points tend to infinity

Theorem 4.1. *Let $d \geq 2$ with $p \nmid d$. Set δ to be 1 if d is odd and 2 if d is even. For $g \geq 1$ and k satisfying*

$$\frac{2g}{d+1} - \frac{2p(p-1)}{d+1} \geq k\delta(p-1), \tag{4.1}$$

there exists a curve $C_{g,k}$ of genus g such that

$$NP(C_{g,k}) \succeq \{0,1\}^{g-\frac{k\delta(p-1)(d-1)}{2}} \bigsqcup \left\{\frac{1}{d}, \dots, \frac{d-1}{d}\right\}^{k\delta(p-1)}. \tag{4.2}$$

If $p \equiv 1 \mod d$, then we may choose $C_{g,k}$ so that (4.2) is an equality of Newton polygons.

Proof. Write $g = i + m(p-1)$, where $0 \leq i < p-1$, and define

$$A = g - ip + (p-1) - \frac{k\delta(p-1)(d+1)}{2}.$$

By (4.1) we know $A \geq 0$. Also, we see that $(p-1)|A$. Let $j \geq 0$ with $A = j(p-1)$. Choose an ordinary curve X_i with genus i. Then choose an Artin–Schreier cover $C_{g,k} \to X_i$ ramified at $j + \delta k$ points, such that j points have Swan conductor 1 and δk points have Swan conductor d. By Riemann–Hurwitz we know $C_{g,k}$ has genus g. We then apply Theorem 3.1 to obtain the bound (4.2). If $p \equiv 1 \mod d$ we can use Theorem 3.3 (or Corollary 3.2 when $d = 2$) to make sure the Newton polygons in (4.2) are equal. ☐

Corollary 4.2. *Let $d \geq 2$ with $p \nmid d$. Let $\mathcal{N}$ be the Newton polygon with slopes $\{0,1\}^u \sqcup \{\frac{1}{d}, \dots, \frac{d-1}{d}\}^v$ where $u \geq v$ and let $\mathcal{P} = \frac{1}{2u+(d-1)v}\mathcal{N}$ be the scaled Newton polygon. There exists a full family of curves $\mathcal{C}$ such that $\mathbf{sNP}(\mathcal{C}) \succeq \mathcal{P}$. In particular, $\mathcal{C}$ is an unlikely family. If $p \equiv 1 \mod d$, we may choose $\mathcal{C}$ so that $\mathbf{NP}(\mathcal{C}) \sim \mathcal{N}$.*

Proof. By replacing u and v with $2u$ and $2v$, we may assume they are even. In particular, $\mathcal{N}$ has $2r = 2u + (d-1)v$ slopes for a whole number r. Use the Euclidean algorithm to write

$$2g = 2r\delta(p-1)l + \alpha_g, \text{ where} \tag{4.3}$$

$$2p(p-1) \leq \alpha_g \leq 2p(p-1) + 2r\delta(p-1). \tag{4.4}$$

Dividing (4.3) by $d+1$ and rearranging gives

$$lv\delta(p-1) = \frac{2g}{d+1} - \frac{\alpha_g + 2(u-v)\delta(p-1)l}{d+1}.$$

By (4.4) and our assumption that $u \geq v$, we see that $k = lv$ satisfies (4.1). We then take C_g to be the curve $C_{g,lv}$ from Theorem 4.1 and let $\mathcal{C} = \{C_g\}_{g\in\mathbb{Z}_{\geq 0}}$. We claim that $\mathcal{C}$ is the desired family of curves. By rearranging (4.3) we obtain two identities

$$\begin{aligned} \delta(p-1)l &= \frac{g}{r} - \frac{\alpha_g}{2r}, \\ g - \frac{v\delta(p-1)l(d-1)}{2} &= u\delta(p-1)l + \frac{\alpha_g}{2}. \end{aligned} \tag{4.5}$$

Then (4.2) and (4.5) give

$$NP(C_g) \succeq \Big\{0,1\Big\}^{\frac{g}{r}u - \frac{\alpha_g}{2r}u + \frac{\alpha_g}{2}} \bigsqcup \Big\{\frac{1}{d}, \dots, \frac{d-1}{d}\Big\}^{\frac{g}{r}v - \frac{\alpha_g}{2r}v}, \tag{4.6}$$

with equality when $p \equiv 1 \mod d$. Let P_g be the Newton polygon on the right side of (4.6) and set $\mathbf{P} = \{P_g\}_{g\in\mathbb{Z}_{\geq 0}}$. Since $\frac{\alpha_g}{2r}v$ and $\frac{\alpha_g}{2r}u - \frac{\alpha_g}{2}$ are bounded independently from g, we see that $\mathbf{P} \sim \mathcal{N}$. Then by Lemma 2.12 and Lemma 2.15 we have $\mathbf{sNP}(\mathcal{C}) \succeq \mathcal{P}$. Furthermore, by Lemma 2.19 and Corollary 2.20 we know $\mathcal{C}$ is an unlikely family. □

Corollary 4.3. *Let $d \geq 2$ and assume $p \equiv 1 \mod d$. Let $g, g' \geq 1$. Let $C_{g,k}$ and $C_{g',k'}$ be curves as in Theorem 4.1. Then Oort's conjecture (see Conjecture 1.1) holds for $C_{g,k}$ and $C_{g'k'}$. That is, there exists a curve C of genus $g + g'$ whose Newton polygon is $NP(C_{g,k}) \sqcup NP(C_{g',k'})$.*

Proof. Take C to be $C_{g+g',k+k'}$. □

Corollary 4.4. *Set $d = 2$. Let g and k be as in Theorem 4.1. Define $m = \frac{p-1}{2}$. Let $\mathcal{N}_{g,k}$ denote the Newton polygon with slopes $\{0,1\}^{g-k(p-1)} \sqcup \{\frac{1}{2}\}^{2k(p-1)}$. If $km(km-1) > 3g - g$, then $W_{\mathcal{N}_{g,k}}$ and $\mathcal{T}_g$ have an unlikely intersection.*

Proof. By Theorem 4.1 we know $W_{\mathcal{N}_{g,k}}$ and $\mathcal{T}_g$ have a nonempty intersection. We will compute the codimension of $W_{\mathcal{N}_{g,k}}$ in $\mathcal{A}_g$ using Theorem 2.5. Let $T_{g,k}$ be the triangle whose vertices are $(g - k(p-1), 0)$, $(g, 0)$ and $(g, \frac{k(p-1)}{2})$. We compute $\#(T_{g,k} \cap \mathbb{Z} \times \mathbb{Z}) = (km)^2$. Also, there are km lattice points on the hypotenuse of $T_{g,k}$. The codimension of $W_{\mathcal{N}_{g,k}}$ in $\mathcal{A}_g$ is thus $km(km-1)$. The codimension of $\mathcal{T}_g$ in $\mathcal{A}_g$ is $\frac{g(g-1)}{2} - 3g - 3$, which proves the corollary. □

Example 4.5. Consider the case where $d = 2$ and let $\mathcal{N} = \{0, \frac{1}{2}, 1\}$. Corollary 4.2 tells us there exists a full family of curves $\mathcal{C} = \{C_g\}$, where $NP(C_g)$ only has slopes 0,1, and $\frac{1}{2}$, and each occurs about a third of the time (with a constant error term). This was previously only known under the assumption $p \equiv 2 \mod 3$ (see [11, Corollary 9.4]). More generally, let $\epsilon \leq \frac{1}{3}$ be a rational number. There exists a full family $\mathcal{C} = \{C_g\}$, such that the Newton polygon of C_g consists only of slopes $0, 1$, and $\frac{1}{2}$ and the multiplicity of $\frac{1}{2}$ is approximately $2g\epsilon$ (with a constant error term).

Example 4.6. Consider the case where $d = 3$ and $p \equiv 1 \mod 3$. Let $\mathcal{N} = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Corollary 4.2 gives a full family of curves $\mathcal{C} = \{C_g\}$, where $NP(C_g)$ only has slopes $0, \frac{1}{3}, \frac{2}{3}$, and 1. Furthermore, each slope occurs with approximately equal frequency. To the best of our knowledge, there were no previous examples of full or arithmetic families of curves with this property. There were, however, examples of arithmetic families of curves where the slopes $\frac{1}{3}, \frac{2}{3}$ occurred with smaller frequencies (see [11, Section 9.2]).

5. Letting the ramification break tend to infinity

Theorem 5.1. *There exists a full family $\mathcal{C} = \{C_g\}$ such that*

$$\mathbf{sNP}(\mathcal{C}) \succeq P\Big(\frac{x^2}{4}, 2\Big).$$

In particular $\mathcal{C}$ is an unlikely family.

Proof. Let $m = \frac{p-1}{2}$ and for each $i = 0, \ldots, m-1$, choose u_i, v_i such that

$$pu_i - (p-1) = i + mv_i.$$

Let X_i be a smooth ordinary curve of genus u_i. Let $g = i + km$. For $g \gg 0$ we define a genus g curve C_g as follows:

(a) If $d_g = k+1-v_i$ is relatively prime to p, we choose a rational function f_g on X_i that has exactly one pole of order d (since g and hence d is sufficiently large, this is possible by Riemann–Roch). We let C_g be the curve defined by the Artin–Schreier equation $y^p - y = f_g$. By the Riemann–Hurwitz formula we know C_g has genus g.

(b) If $d_g = k + 1 - v_i$ is divisible by p, we choose a rational function f_g that has exactly two poles: one pole of of order $d - 2$ and one pole of order 2 (again, this is possible by Riemann–Roch). We let C_g be the curve defined by the Artin–Schreier equation $y^p - y = f_g$. By the Riemann–Hurwitz formula we know C_g has genus g.

Let $\mathcal{C}'_i$ (resp. $\mathcal{C}''_i$) be the subfamily of $\mathcal{C}$ consisting of curves C_g with $g \equiv i \mod m$ defined in case (a) (resp. (b)). It suffices to show $\mathbf{sNP}(\mathcal{C}'_i) \succeq P(\frac{x^2}{4}, 2)$ and $\mathbf{sNP}(\mathcal{C}''_i) \succeq P(\frac{x^2}{4}, 2)$ for each i. We will prove the result for $\mathcal{C}'_i$, as the proof for $\mathcal{C}''_i$ is almost identical.

Let $C_g \in \mathcal{C}'_i$. Let Q_g be the Newton polygon with slope-set $\{0,1\}^{pu_i+p-1} \sqcup \{\frac{1}{d_g}, \ldots, \frac{d_g-1}{d_g}\}^{p-1}$. By Theorem 3.1 we know $NP(C_g) \succeq Q_g$. Since the number of slope zero and one segments in Q_g is independent of g, we see that $\frac{1}{g}Q_g$ converges uniformly to $\frac{x^2}{4}$ on the interval $[0, 2]$. The theorem follows from Lemma 2.12. ☐

Question 5.2. Theorem 5.1 allows for the possibility that $sNP(C_g)$ lies well above $P(\frac{x^2}{4}, 2)$. Does there exist a full family of curves $\mathcal{C} = \{C_g\}_{g \in \mathbb{Z}_{\geq 0}}$ such that $sNP(C_g)$ converges uniformly to $P(\frac{x^2}{4}, 2)$?

Question 5.3. Does there exist a full or arithmetic family of curves $\mathcal{C}$ and a basic graph P_0 that lies strictly above $P(\frac{x^2}{4}, 2)$ such that $\mathbf{sNP}(\mathcal{C}) \succeq P_0$?

To the best of our knowledge, both questions are unknown even for arithmetic families. The arithmetic families constructed in [11] have scaled Newton polygons whose limits are well below $P(\frac{x^2}{4}, 2)$. Similarly, the bounds in Theorem 4.2 are well below $P(\frac{x^2}{4}, 2)$.

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