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The Fibonacci sequence and an elliptic curve

par Sungkon CHANG

Résumé. Les séries infinies impliquant les inverses des nombres de Fibonacci sont en général algébriquement indépendantes sur le corps des nombres rationnels. Dans la présente note, nous introduisons une identité qui révèle une relation de dépendance algébrique entre deux telles séries. L'identité a été découverte à partir d'une description spéciale d'une certaine fonction elliptique. Cette observation est généralisée pour produire des identités analogues pour une grande classe de suites définies par des récurrences linéaires portant sur trois termes consécutifs.

Abstract. Infinite series involving the reciprocal Fibonacci numbers may admit no algebraic dependence between each other over the rational numbers. In this note, we introduce an identity which reveals an algebraic dependence relation between two infinite series involving the reciprocal Fibonacci numbers. The identity was discovered from a peculiar description of an elliptic function, and this observation is generalized to produce similar identities on a large class of sequences defined by linear recurrences on three consecutive terms.

1. Introduction

In this note, we introduce an identity on infinite series that involves the reciprocal Fibonacci numbers. The identity was discovered from an elliptic curve, and the description of the elliptic curve is introduced in this paper as well. This observation is generalized to a large class of sequences defined by linear recurrences in three consecutive terms.

Let F_n be the Fibonacci sequence, which is defined by the linear recurrence $F_{n+2} = F_{n+1} + F_n$ for all indices $n \geq 0$, and $(F_0, F_1) = (0, 1)$. We may extend the definition for the negative indices as well, so that $F_{-n} = (-1)^{n+1} F_n$ for all integers n, and $F_{n+2} = F_{n+1} + F_n$ for all integers n. For example, $(F_{-4}, F_{-3}, F_{-2}, F_{-1}) = (-3, 2, -1, 1)$. By Binet's formula, $F_n = (\phi^n - \bar{\phi}^n)/\sqrt{5}$ for all integers n where $\phi := (1 + \sqrt{5})/2$ is the golden ratio and $\bar{\phi} := (1 - \sqrt{5})/2$. Let us use the notation $\sum_{n\neq 0} a_n$ for $\sum_{n=1}^{\infty} a_{-n} + \sum_{n=1}^{\infty} a_n$ when the infinite series are absolutely convergent.

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Stated below is the identity obtained from the description of a function on an elliptic curve.

Theorem 1.1.

$$\left(\sum_{n\neq 0}\frac{(-\phi)^{n/3}}{F_n}\right)^2=\sum_{n\neq 0}\frac{(-\phi)^{2n/3}}{F_n^2}. \tag{1.1}$$

We do not know if the value of (1.1) can be expressed in closed form, but the computer calculation suggests that it is approximately equal to 6.1469. If we use the identity $\phi^n = F_{n-1}+F_n\phi$, and break the series into two parts, we obtain the following equivalent version:

$$\left(\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt[3]{F_{n-1}+F_n\phi}-\sqrt[3]{F_{n-1}+F_n\bar{\phi}}}{F_n}\right)^2 = \sum_{n=1}^{\infty}\frac{\sqrt[3]{F_{2n-1}+F_{2n}\phi}+\sqrt[3]{F_{2n-1}+F_{2n}\bar{\phi}}}{F_n^2}$$

where $\sqrt[3]{x}$ for $x\in\mathbb{R}$ denotes the real root. The values of the reciprocal Fibonacci series such as $S_\ell := \sum_{n=1}^{\infty} 1/F_n^\ell$ have been studied; see [1, 2, 4, 6, 11, 14]. The irrationality of $S_1 \approx 3.3598$ is proved in [1], but in general, no closed formulas are known for S_ℓ. For certain series of the reciprocal Fibonacci numbers such as $S_2 := \sum_{n=1}^{\infty} 1/F_n^2$, the values can be formulated in terms of the values of theta functions. Studied in [11] are approximate values of S_1 and S_2 to arbitrary precision, and various similar results are found in the literature; see [14]. For example, if $n \geq 1$ is odd, proved in [11] is

$$\frac{1}{1+F_{n-1}F_n}<\sum_{k=n}^{\infty}\frac{1}{F_k^2}\leq\frac{1}{F_{n-1}F_n}.$$

The transcendence and algebraic independence of various reciprocal Fibonacci series have been studied in the literature; see [4, 6]. Using Nesterenko's theorem on certain Ramanujan functions [10], the authors of [4] prove that S_2 is transcendental over $\mathbb{Q}$, and the authors of [5] prove that $\{S_2, S_4, S_6\}$ are algebraically independent over $\mathbb{Q}$. In terms of algebraic independence, the two series in Theorem 1.1 can be viewed as an algebraic dependence relation, i.e., if $T_\ell := \sum_{n\neq 0}(-\phi)^{\ell n/3}/F_n^\ell$ for $\ell = 1, 2$, then $T_1^2 - T_2 = 0$.

The remainder of the paper is organized as follows. In Section 2, a complex analytic description of an elliptic curve is introduced, and in Section 3, peculiar descriptions of constant functions on the complex manifold are introduced along with our proof of Theorem 1.1. In Section 4, generalizations of this observation are introduced, and in Section 5, complex analytic isomorphisms to the classical description of elliptic curves are introduced.

2. The Fibonacci elliptic curve

An elliptic curve is often defined as a compact Riemann surface of genus 1; see [9]. For the complex analytic structure, we may use quotients of the multiplicative group $\mathbb{C}^* := \{z \in \mathbb{C} : z \neq 0\}$. Let $q \in \mathbb{C}^*$ be a complex number such that $|q| \neq 1$, and let $q^{\mathbb{Z}}$ denote the subgroup $\{q^n \in \mathbb{C}^* : n \in \mathbb{Z}\}$. Then, the quotient $\mathbb{C}^*/q^{\mathbb{Z}}$ is an elliptic curve; see [7, 13]. We find that the figures of fundamental domains of $\mathbb{C}^*/q^{\mathbb{Z}}$ are less popular, and sketch a sample in Figure A below.

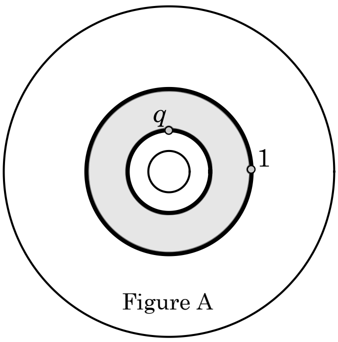

Figure A

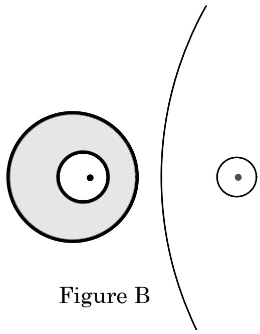

Figure B

Let C_n for $n \in \mathbb{Z}$ denote the circle in the complex plane passing through q^n and centered at 0. Then, the circle C_n is mapped to C_{n+1} when multiplied by q, preserving the counterclockwise orientation. For example, if $q = \frac{1}{2}i$, then the two circles C_0 and C_1 are sketched in thicker lines in Figure A. The region in between C_0 and C_1 forms the interior of a fundamental domain for $\mathbb{C}^*/q^{\mathbb{Z}}$, and the two circles are identified with each other in $\mathbb{C}^*/q^{\mathbb{Z}}$.

One of the key tools for studying Riemann surfaces has been meromorphic functions and their poles, and the Riemann–Roch Theorem is at the center of the subject; see [8, 9, 12]. For the remainder of this work, we hope that we will convince you that the values of meromorphic functions on the elliptic curves we consider in this work are inherently connected to identities that involve the infinite series similar to the one described in (1.1).

The subgroup $q^{\mathbb{Z}}$ may be considered as an orbit of 1 under the action of the multiplication-by-q on $\mathbb{C}^*$, i.e., $q^{\mathbb{Z}} = q^{\mathbb{Z}}1 := \{q^n \cdot 1 \in \mathbb{C} : n \in \mathbb{Z}\}$. This can be further generalized as the action of linear fractional transformations on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Given a matrix $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{C})$, the action of γ on a complex number z in the Riemann sphere is defined to be $\frac{az+b}{cz+d}$, which is denoted by γz, and the cyclic subgroup $\{\gamma^n \in \mathrm{GL}_2(\mathbb{C}) : n \in \mathbb{Z}\}$ shall be denoted by $\gamma^{\mathbb{Z}}$. For example, if $\gamma = \left[\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $\gamma z = qz$, and $\mathbb{C}^*/q^{\mathbb{Z}}$ is equal to the set of orbits $\gamma^{\mathbb{Z}}\backslash\mathbb{C}^* := \{\gamma^{\mathbb{Z}}z : z \in \mathbb{C}^*\}$. Also notice that $\gamma^n 1 \to 0$ as $n \to +\infty$ and $\gamma^n 1 \to \infty$ as $n \to -\infty$ if $|q| < 1$, and 0 and ∞ are the fixed points of the action γ on the Riemann sphere.

Let us use $\gamma = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ to demonstrate the construction of the Riemann surface. The action on the Riemann sphere is given by $\gamma z = 1/(z-1)$ where $z \in \widehat{\mathbb{C}}$. Its fixed points are determined by the equation $\gamma z = z$, i.e., $z^2 - z - 1 = 0$, and hence, they are the golden ratio ϕ and its Galois conjugate $\bar{\phi}$. As expected, circles converge toward a fixed point as $n \to \infty$. Illustrated in Figure B are circles $\gamma^n C$ for $n = 0, 1, 2, 3$ where C is a circle centered at ϕ, and they are converging toward the other fixed point $\bar{\phi}$ as $n \to \infty$. In Section 4, we shall prove that $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^\circ$ is an elliptic curve where $\widehat{\mathbb{C}}^\circ := \widehat{\mathbb{C}} - \{\phi, \bar{\phi}\}$. We call it *the Fibonacci elliptic curve*, and the shaded region in Figure B is a fundamental domain for the elliptic curve. All elliptic curves come with a group law that can be defined in terms of complex analytic maps, and let us introduce a group law on the Fibonacci elliptic curve. Let $\gamma^{\mathbb{Z}} z_1$ and $\gamma^{\mathbb{Z}} z_2$ be two orbits in $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^\circ$, and define

$$\gamma^{\mathbb{Z}} z_1 \oplus \gamma^{\mathbb{Z}} z_2 := \gamma^{\mathbb{Z}} \, \frac{z_1 z_2 + 2(z_1 + z_2) - 1}{2 z_1 z_2 - z_1 - z_2 + 3}. \tag{2.1}$$

Since $\gamma^{\mathbb{Z}} \infty = \gamma^{\mathbb{Z}} 0$, we may use $\gamma^{\mathbb{Z}} 0$ to make sense out of the $\oplus$ operation that involves $\gamma^{\mathbb{Z}} \infty$, and the fact that the numerator and the denominator of (2.1) are not both zeros for any $z_1, z_2 \in \widehat{\mathbb{C}}^\circ$ makes the operation well-defined. Notice that the group law on $q^{\mathbb{Z}} \backslash \mathbb{C}^*$ is given by the multiplication of the complex numbers. In Section 4, we shall establish the isomorphism between the manifolds $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^\circ$ and $q^{\mathbb{Z}} \backslash \mathbb{C}^*$, and this isomorphism induces the operation (2.1). The identity element of the group law is $\gamma^{\mathbb{Z}}(1/2)$, so that $\gamma^{\mathbb{Z}} z \oplus \gamma^{\mathbb{Z}}(1/2) = \gamma^{\mathbb{Z}} z$, and the inverse of $\gamma^{\mathbb{Z}} z$ is given by $\gamma^{\mathbb{Z}}(1-z)$. The subgroup of the two-torsion points of an elliptic curve is isomorphic to the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the two elements $\gamma^{\mathbb{Z}}(i)$ and $\gamma^{\mathbb{Z}}(-i)$ generate the subgroup.

3. The Fibonacci elliptic functions

Let us introduce the standard elliptic functions on $\mathbb{C}^*$. Let q be a complex number such that $|q| \notin \{0, 1\}$. Let $f : \mathbb{C}^* \to \widehat{\mathbb{C}}$ be a function given by

$$f(z) = z \sum_{n \in \mathbb{Z}} \frac{q^n}{(1 - q^n z)^2} := z \left(\sum_{n=0}^{\infty} \frac{q^n}{(1 - q^n z)^2} + \sum_{n=1}^{\infty} \frac{q^{-n}}{(1 - q^{-n} z)^2} \right) \tag{3.1}$$

where the individual series indexed over $n \geq 1$ and $n \geq 0$ are absolutely convergent for each $z \in \mathbb{C}^* - q^{\mathbb{Z}}$. The function f is meromorphic, and it has poles at $\{q^n : n \in \mathbb{Z}\}$, each of which has order 2; they are often called *double poles*. Moreover, it satisfies the following invariant property. For all $z \in \mathbb{C}^*$,

$$f(qz) = qz \sum_{n \in \mathbb{Z}} \frac{q^n}{(1 - q^n (qz))^2} = z \sum_{n \in \mathbb{Z}} \frac{q^{n+1}}{(1 - q^{n+1} z)^2} = f(z). \tag{3.2}$$

This invariant property allows us to define a function on the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$. Let $\bar{f} : \mathbb{C}^*/q^{\mathbb{Z}} \to \widehat{\mathbb{C}}$ be a function defined by $\bar{f}(\gamma^n z) := f(z)$. Then, $\bar{f}$ becomes a *meromorphic function on the elliptic curve* $\mathbb{C}^*/q^{\mathbb{Z}}$; see [9] for the definition of a *meromorphic function on a Riemann surface.* For this reason, the meromorphic function f on $\mathbb{C}^*$ is called *an elliptic function.* We shall call it *an elliptic function under the action of* $q^{\mathbb{Z}}$.

Motivated from the classical elliptic function on $\mathbb{C}^*$, we define a similar version for the Fibonacci elliptic curve $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$. Let $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ be a function given by

$$f(z) := \sum_{n\in\mathbb{Z}} (\phi - \gamma^n z)(\bar{\phi} - \gamma^n z). \tag{3.3}$$

The RHS is an absolutely convergent series, and it defines a meromorphic function that has double poles only. This assertion follows immediately from Lemma 3.2, and we begin with the following to prove the lemma.

Lemma 3.1. *For* $n \in \mathbb{Z}$,

$$\gamma^n = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}.$$

Proof. The characteristic polynomial of γ is $t^2 + t - 1$, and by the Cayley–Hamilton theorem, $\gamma^2 = -\gamma + I$ where I is the 2×2 identity matrix, i.e., $\gamma^{n+2} = -\gamma^{n+1} + \gamma^n$ for all $n \in \mathbb{Z}$. Thus, the two matrices γ^0 and γ^1 completely determines γ^n for all $n \in \mathbb{Z}$, and we leave to the reader the task of checking if the formula satisfies the recursion on γ^n and matches the two matrices. □

Then, the meromorphic function $f(z)$ satisfies the invariant property $f(\gamma z) = f(z)$ for all $z \in \widehat{\mathbb{C}}^\circ$, and it shall be rightly called *a Fibonacci elliptic function.* The expression of $f(z)$ given in the following lemma is rather prefered for that name.

Lemma 3.2. *Let* $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ *be the meromorphic function defined in* (3.3). *Then,*

$$f(z) = (z^2 - z - 1) \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(F_{n+1} - F_n z)^2}. \tag{3.4}$$

Proof. Recall the formula of γ^n from Lemma 3.1, and let $D_n(z) := F_{n+1} - F_n z$. Notice that $(\phi - \gamma^n z)(\bar{\phi} - \gamma^n z)$ vanishes at $z = \phi$ and $z = \bar{\phi}$, and that

$$\begin{aligned} (\phi - \gamma^n z)&(\bar{\phi} - \gamma^n z) \\ &= \frac{(\phi D_n(z) - (F_{n-1} z - F_n))(\bar{\phi} D_n(z) - (F_{n-1} z - F_n))}{D_n(z)^2}. \end{aligned} \tag{3.5}$$

Since $D_n(z)$ does not vanish at ϕ or $\bar{\phi}$, the numerator of the summand must vanish at the two values. Since it is a quadratic polynomial in z, it must be equal to $a(z-\phi)(z-\bar{\phi}) = a(z^2-z-1)$ for some constant a. The leading coefficient of the numerator is $(-\phi F_n - F_{n-1})(-\bar{\phi}F_n - F_{n-1}) = -F_n^2 + F_nF_{n-1} + F_{n-1}^2$. This is equal to $\det\gamma^n = F_{n-1}F_{n+1} - F_n^2 = F_{n-1}(F_n + F_{n-1}) - F_n^2$. Since $\det(\gamma^n) = (\det\gamma)^n = (-1)^n$, it follows that $(\phi - \gamma^n z)(\bar{\phi} - \gamma^n z) = (-1)^n(z^2-z-1)/D_n(z)^2$, which proves the assertion. □

We interpret the meaning of the convergence of $f(z)$ at $z = \infty \in \widehat{\mathbb{C}}$ as the convergence of the analytic continuation of $f(1/z)$ at $z = 0$. Let $h(z) = z^2 - z - 1$. Then, $h(\gamma z) = -h(z)D_1(z)^{-2}$ for all $z \in \widehat{\mathbb{C}}^\circ$, and we leave to the readers the fun task of checking if $f(\gamma z) = f(z)$ holds for all $z \in \widehat{\mathbb{C}}^\circ$ according to the description given in (3.4). It became unexpectedly interesting when we looked for an elliptic function that is given in a form similar to (3.1) or (3.4), but has poles of order < 2. Experts would not be interested in such a quest because the genus of $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$ is 1, and the Riemann–Roch Theorem [8, 9, 12] implies that there will not be a meromorphic function $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ such that $f(\gamma z) = f(z)$ and the poles of f are *simple*, i.e., of order 1. This can also be proven more directly using the First and Second Liouville's Theorem; see [7]. Let us state the result below.

Theorem 3.3. *Let $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ be an elliptic function under the cyclic group $\gamma^{\mathbb{Z}}$, i.e., f is meromorphic on $\widehat{\mathbb{C}}^\circ$ and satisfies $f(\gamma z) = f(z)$ for all $z \in \widehat{\mathbb{C}}^\circ$. Then, either f is a constant function, or has poles of order ≥ 2.*

The interesting turn of the event was that we found the description of a meromorphic function that is invariant under $\gamma^{\mathbb{Z}}$, but has no poles of order ≥ 2. Let us use the notation $\sum_{P(\mathbf{v})}$ for the summation over tuples of integers $\mathbf{v} = (n_1, n_2, \ldots, n_\ell)$ that satisfy conditions $P(\mathbf{v})$. For example $\sum_{n\neq m}$ means the summation over all integer pairs (n, m) such that $n \neq m$. Let $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ be a meromorphic function given by

$$f(z) = (z^2 - z - 1)\sum_{n\neq m} \frac{i^{n+m}}{(F_{n+1} - F_n z)(F_{m+1} - F_m z)}. \tag{3.6}$$

Then, it is an absolutely convergent series for all $z \in \widehat{\mathbb{C}}^\circ$ not equal to any F_{n+1}/F_n, and satisfies the invariant property $f(\gamma z) = f(z)$. Since the denominator of the summand of (3.6) does not have a repeated linear factor, this meromorphic function does not have poles of order ≥ 2, and hence, by Theorem 3.3, it analytically continues to a constant elliptic function! Notice that if we consider the summation in (3.6) over the indices (n, m) such that $n = m$, it will be equal to the elliptic function (3.4). In fact, the function (3.6) was first found after we considered the meromorphic function

$$(z^2 - z - 1)\left(\sum_{n\in\mathbb{Z}} \frac{i^n}{F_{n+1} - F_n z}\right)^2.$$

It is a Fibonacci elliptic function with poles of order 2, and if we expand the squared part of the function, it is decomposed into the terms described in (3.4) and the terms described in (3.6).

Applying this idea to the elliptic function defined in (3.1), the author of this note had obtained a unique set of results [3] on divisor functions, and for the work presented in this note, he pursued descriptions of constant Fibonacci elliptic functions, hoping to find an interesting identity on the Fibonacci sequence. In [3], the real surprise came when the triple product summand was considered instead of the double product summand in (3.6). Let $f : \widehat{\mathbb{C}}^\circ \to \widehat{\mathbb{C}}$ be a meromorphic function given by

$$f(z) := (z^2 - z - 1)(z - \phi) \sum_{\substack{n,m,\ell \\ \text{dist.}}} \frac{(-\phi)^{(n+m+\ell)/3}}{D_n(z)D_m(z)D_\ell(z)} \tag{3.7}$$

where the summation runs over all triples of distinct integers. The factor $(z - \phi)$ has the following property under the action of γ:

$$(\gamma z - \phi) = \frac{1}{z-1}(1 - \phi(z-1)) = \frac{-\phi}{z-1}(z - \phi).$$

Recalling the property of $h(z) = z^2 - z - 1$ under the action of γ, we see that $f(z)$ becomes an elliptic function under the cyclic subgroup $\gamma^{\mathbb{Z}}$. As in the double product case, by Theorem 3.3, it analytically continues to a constant elliptic function. The surprise was what happens to the expression (3.7) near $z = 0$. For (3.6), there was not a surprise, i.e., the Laurent expansion was $\frac{1}{z}(a_0 + a_1 z + a_2 z^2 + \cdots)$ and it was straightforward to show that $a_0 = 0$ as the infinite sum description of a_0 cancels itself out. However, for (3.7), there is no apparent symmetry that forces the constant a_0 to be zero, but rather produced an identity, which was presented in (1.1). Let us use the elliptic function (3.7) to prove Theorem 1.1. The triples (n, m, ℓ) considered in (3.7) for which $n = -1$ generate an infinite partial sum of (3.7), and due to its symmetric construction, the infinite partial sums for $m = -1$ and $\ell = -1$ generate the same value. Thus, $D_{-1}(z) = -z$ implies

$$\begin{aligned} f(z) &= (z^2 - z - 1)(z - \phi) \\ &\quad \times \left(\frac{3(-\phi)^{-1/3}}{-z} \sum_{\substack{n,m\neq -1 \\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{D_n(z)D_m(z)} + \sum_{\substack{n,m,\ell\neq -1 \\ \text{dist.}}} \frac{(-\phi)^{(n+m+\ell)/3}}{D_n(z)D_m(z)D_\ell(z)} \right). \end{aligned}$$

Notice that $D_n(0) \neq 0$ for all integers $n \neq -1$, and hence, both infinite sums in the parentheses are analytic at $z = 0$. Let $a_0 + a_1 z + a_2 z^2 + \cdots$ be the MaClaurin series of the first sum. Then,

$$\frac{3(-\phi)^{-1/3}}{-z} \sum_{\substack{n,m\neq -1\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{D_n(z)D_m(z)} = 3(+\phi)^{-1/3}\left(\frac{a_0}{z} + a_1 + a_2 z + \cdots\right). \tag{3.8}$$

Since $f(z)$ is analytically continued to $z = 0$, and $(z^2 - z - 1)(z - \phi)$ does not vanish at $z = 0$, it follows that $a_0 = 0$, i.e.,

$$\begin{aligned} 0 = a_0 = \sum_{\substack{n,m\neq -1\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{D_n(0)D_m(0)} &= \sum_{\substack{n,m\neq -1\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{F_{n+1}F_{m+1}} \\ &= \sqrt[3]{-\phi}^{\,-2} \sum_{\substack{n,m\neq 0\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{F_n F_m} \end{aligned} \tag{3.9}$$

$$\text{which implies} \quad \sum_{\substack{n,m\neq 0\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{F_n F_m} = 0; \tag{3.10}$$

$$\begin{aligned} \Rightarrow \quad \left(\sum_{n\neq 0} \frac{(-\phi)^{n/3}}{F_n}\right)^2 &= \sum_{n\neq 0} \frac{(-\phi)^{n/3}}{F_n} \sum_{m\neq 0} \frac{(-\phi)^{m/3}}{F_m} \\ &= \sum_{n=m\neq 0} \frac{(-\phi)^{n/3}}{F_n}\frac{(-\phi)^{m/3}}{F_m} + \sum_{\substack{n,m\neq 0\\ \text{dist.}}} \frac{(-\phi)^{(n+m)/3}}{F_n F_m} \\ &= \sum_{n\neq 0} \frac{(-\phi)^{2n/3}}{F_n^2} + 0. \end{aligned}$$

4. Linear recurrences and elliptic curves

Theorem 1.1 holds for a larger class of linear recurrences.

Theorem 4.1. *Let* $\gamma = \left[\begin{smallmatrix} a & b \\ 1 & d \end{smallmatrix}\right]$ *be a matrix in* $\mathrm{GL}_2(\mathbb{C})$ *with eigenvalues* ψ_1 *and* ψ_2 *such that* $|\psi_1/\psi_2| \neq 1$. *Then,* $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$ *is an elliptic curve.*

Let ρ *be a fixed point of* γ. *If* $\gamma^n = \left[\begin{smallmatrix} P_n & Q_n \\ R_n & S_n \end{smallmatrix}\right]$, *then for all* $n \in \mathbb{Z}$,

$$\begin{aligned} R_{n+2} &= (a+d)R_{n+1} + (b - ad)R_n, \ (R_0, R_1) = (0,1), \\ S_{n+2} &= (a+d)S_{n+1} + (b - ad)S_n, \ (S_0, S_1) = (1,d), \end{aligned}$$

$$\textit{and} \quad \left(\sum_{n\neq 1} \frac{\big((a-\rho)\det\gamma\big)^{n/3}}{S_n - dR_n}\right)^2 = \sum_{n\neq 1} \frac{\big((a-\rho)\det\gamma\big)^{2n/3}}{(S_n - dR_n)^2}. \tag{4.1}$$

For example, let $\gamma = \left[\begin{smallmatrix} -2 & -5 \\ 1 & 3 \end{smallmatrix}\right]$. Then, it has eigenvalues ϕ and $\bar{\phi}$, and $\phi - 3$ is a fixed point of γ. Then, Theorem 4.1 implies

$$\left(\sum_{n \neq 1} \frac{(-\bar{\phi})^{n/3}}{L_{n+1} - 3F_n} \right)^2 = \sum_{n \neq 1} \frac{(-\bar{\phi})^{2n/3}}{(L_{n+1} - 3F_n)^2}$$

where $L_n = \phi^n + \bar{\phi}^n$ for $n \in \mathbb{Z}$ are Lucas numbers. The proof of (4.1) is obtained by repeating calculations similar to the Fibonacci elliptic curve. Below we only summarize the key results, and leave the proof to the reader. However, we shall prove that $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^{\circ}$ is an elliptic curve in this section.

Let γ be the matrix described in Theorem 4.1, and let $\{\rho, \rho'\}$ be the two (distinct) fixed points of γ. Then, the following is a meromorphic function on the elliptic curve $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^{\circ}$ with double poles at $\{\gamma^n(-d) : n \in \mathbb{Z}\}$:

$$f(z) := \sum_{n \in \mathbb{Z}} (\rho - \gamma^n z)(\rho' - \gamma^n z) = (z^2 + (d-a)z - b) \sum_{n \in \mathbb{Z}} \frac{\det \gamma^n}{(S_n + R_n z)^2},$$

and the following analytically continues to a constant function:

$$(4.2) \quad (z^2 + (d-a)z - b)(z - \rho) \sum_{\substack{n,m,\ell \\ \text{dist.}}} \frac{\big((a-\rho)\det\gamma\big)^{(n+m+\ell)/3}}{(S_n + R_n z)(S_m + R_m z)(S_\ell + R_\ell z)}.$$

The identity (4.1) is obtained by setting aside the factor $S_1 + R_1 z = d + z$ of the denominator as done in (3.8). Let us prove that $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^{\circ}$ is an elliptic curve. Let $\alpha = \left[\begin{smallmatrix} \rho_1 & \rho_2 \\ 1 & 1 \end{smallmatrix}\right]$ be a matrix consisting of eigenvectors of eigenvalues ψ_1 and ψ_2, respectively, as column vectors. Since $\psi_1 \neq \psi_2$ and $\psi_1\psi_2 \neq 0$, the matrix is invertible, and defines a linear fractional transformation α from $\widehat{\mathbb{C}}^*$ to $\widehat{\mathbb{C}}^{\circ}$ where $\widehat{\mathbb{C}}^* := \widehat{\mathbb{C}} - \{0, \infty\}$ and $\widehat{\mathbb{C}}^{\circ} := \widehat{\mathbb{C}} - \{\rho_1, \rho_2\}$. Since $\alpha(0) = \rho_2$ and $\alpha(\infty) = \rho_1$, the linear fractional transformation α defines a complex analytic isomorphism from $\widehat{\mathbb{C}}^* \to \widehat{\mathbb{C}}^{\circ}$. Notice that $M := \left[\begin{smallmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{smallmatrix}\right] = \alpha^{-1}\gamma\alpha$, and the linear fractional transformation defined by M is equal to the multiplication-by-q where $q = \psi_1/\psi_2$; we denote the transformation simply by q, so $q = \alpha^{-1}\gamma\alpha$, i.e., $\alpha \circ q = \gamma\alpha$.

Let us prove that the linear fractional transformation α reduces to a bijective function $\bar{\alpha}$ from $q^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^*$ to $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^{\circ}$ given by $\bar{\alpha}(q^{\mathbb{Z}} z) = \gamma^{\mathbb{Z}}(\alpha z)$. First, let us prove that $\bar{\alpha}$ is well-defined. Notice that $\gamma^{\mathbb{Z}}(\alpha \circ q)z = \gamma^{\mathbb{Z}}((\gamma\alpha)z) = \gamma^{\mathbb{Z}}\alpha z$. If $qz_1 = z_2$, then

$$\bar{\alpha}(q^{\mathbb{Z}} z_2) = \gamma^{\mathbb{Z}}(\alpha z_2) = \gamma^{\mathbb{Z}}(\alpha \circ q)\, z_1 = \gamma^{\mathbb{Z}}(\gamma\alpha z_1) = \gamma^{\mathbb{Z}}(\alpha z_1) = \bar{\alpha}(q^{\mathbb{Z}} z_1).$$

The function from $\gamma^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^{\circ}$ to $q^{\mathbb{Z}} \backslash \widehat{\mathbb{C}}^*$ given by $\gamma^{\mathbb{Z}} z \mapsto q^{\mathbb{Z}}(\alpha^{-1} z)$ defines an inverse function of $\bar{\alpha}$, and we leave the proof to the reader. This concludes the proof of the bijectivity of $\bar{\alpha}$. Moreover, the following is a commutative diagram of functions between sets, and the projections π^* and π° make it a commutative diagrams of continuous maps between topological spaces, two of which are equipped with the quotient topology:

$$\begin{array}{ccc} \widehat{\mathbb{C}}^* & \xrightarrow{\alpha} & \widehat{\mathbb{C}}^\circ \\ \pi^*\big\downarrow & & \big\downarrow\pi^\circ \\ q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^* & \xrightarrow{\bar{\alpha}} & \gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ \end{array} . \tag{4.3}$$

It is well-known that $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$ is an elliptic curve [7, 13]. Let us prove below that the projection π^* serves as charts of the Riemann surface $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$.

Lemma 4.2. *The projection π^* is a unramified continuous map, i.e., for each point $z_0 \in \widehat{\mathbb{C}}^*$, there are open neighborhoods V of z_0 in $\widehat{\mathbb{C}}^*$ and U of $w_0 := \pi^*(z_0) = q^{\mathbb{Z}} z_0$ in $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$ such that π^* restricts to a homeomorphism from V to U.*

Proof. Without loss of generality, let us assume that $|q| < 1$. Let z_0 be a complex number in $\widehat{\mathbb{C}}^*$, and suppose that there is an integer n such that $|q|^{n+1} < |z_0| < |q|^n$. Let V be an open annulus $\{z \in \widehat{\mathbb{C}}^* : |q|^{n+1} < |z| < |q|^n\}$. Then, V is an open neighborhood of z_0, and for any $m \in \mathbb{Z}$, the subset $q^m V = \{z \in \widehat{\mathbb{C}}^* : |q|^{m+n+1} < |z| < |q|^{m+n}\}$ is open. Let us show that $q^m V$ for $m \in \mathbb{Z}$ form a disjoint union of open subsets. If $m_1 < m_2$ are two integers, then $|q|^{m_2 - m_1} \le |q| = \frac{|q|^{n+1}}{|q|^n} \Rightarrow |q|^{m_2+n} \le |q|^{m_1+n+1}$. If $z_1 \in q^{m_1} V$ and $z_2 \in q^{m_2} V$, then $|z_2| < |q|^{m_2+n} \le |q|^{m_1+n+1} < |z_1|$, and hence, $z_2 \ne z_1$. This proves that the two open subsets are disjoint. If z_0 is a complex number such that $|z_0| = |q|^n$ for some integer n, then the open subset $V = \{z \in \widehat{\mathbb{C}}^* : |q|^{n-\frac{1}{2}} < |z| < |q|^{n+\frac{1}{2}}\}$ will have the property that $q^m V$ for $m \in \mathbb{Z}$ form a disjoint union of open subsets, and we leave the proof to the reader.

Let U be the subset $\pi^*(V)$ of $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$. Then, $(\pi^*)^{-1}(U)$ is the disjoint union of $\bigcup_{m\in\mathbb{Z}} q^m V$. Since $q^m V$ for $m \in \mathbb{Z}$ are open, the subset U is open in $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$ by the definition of the quotient topology. Since $q^m V$ for $m \in \mathbb{Z}$ form mutually disjoint open subsets, the projection π^* restricts to a homeomorphism from V to U. □

Since the domain of π^* is an open subset of $\widehat{\mathbb{C}}$, the unramified projection serves as complex charts of the Riemann surface $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$. Moreover, (4.3) is a commutative diagram of continuous maps where α and $\bar{\alpha}$ are homeomorphisms, and hence, the projection $\pi^\circ = \bar{\alpha}\circ\pi^*\circ\alpha^{-1}$ serves as a complex chart for $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$, making the topological space a compact Riemann surface. Also, the homeomorphism α in (4.3) serves as complex analytic isomorphisms between charts of the two Riemann surfaces, and hence, it reduces to a complex analytic isomorphism between $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^*$ and $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$, making $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^\circ$

an elliptic curve. We may use the isomorphism $\bar{\alpha}$ to define a group law on the elliptic curve $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^{\circ}$ as follows:

$$\gamma^{\mathbb{Z}} z_1 \oplus \gamma^{\mathbb{Z}} z_2 := \gamma^{\mathbb{Z}} \alpha((\alpha^{-1} z_1) \cdot (\alpha^{-1} z_2)). \tag{4.4}$$

Since the group law on $q^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^{*}$ is given by the usual multiplication of complex numbers, the above shall define a group law on $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^{\circ}$. For the case of the Fibonacci elliptic curve, the formula simplifies to the one given in (2.1).

5. Elliptic functions

Let γ and q be a matrix and a complex number, respectively, described in Theorem 4.1. The $\mathbb{C}$-vector space of meromorphic functions on the elliptic curve $\gamma^{\mathbb{Z}}\backslash\widehat{\mathbb{C}}^{\circ}$ arise from meromorphic functions on $\widehat{\mathbb{C}}^{\circ}$ invariant under the action of γ, which we called elliptic functions under the action of γ. Let $\mathcal{M}^*$ be the $\mathbb{C}$-vector space of elliptic functions under q, and $\mathcal{M}^{\circ}$, the $\mathbb{C}$-vector space of elliptic functions under γ. Since (4.3) is a commutative diagram of complex analytic maps, we have the pull-back on the elliptic functions under q to the elliptic functions under γ, i.e., $(\alpha^{-1})^* : \mathcal{M}^* \to \mathcal{M}^{\circ}$ given by

$$(\alpha^{-1})^*(f) := f \circ \alpha^{-1}. \tag{5.1}$$

It is an isomorphism between the vector spaces, and the description of the constant elliptic function described in (4.2) can be obtained as a pull-back of the description of a constant elliptic function $f \in \mathcal{M}^*$. Given below is the description of a constant elliptic function under q, which is similar to (3.7):

$$f(z) = z \sum_{\substack{n,m,\ell \\ \text{dist.}}} \frac{q^{(n+m+\ell)/3}}{(1-q^n z)(1-q^m z)(1-q^\ell z)}.$$

If we set aside the factor of $(1-z)$ from the denominator as done in (3.8), we obtain the following result.

Theorem 5.1. *Let q be a complex number such that $|q| \notin \{0,1\}$. Then,*

$$\left(\sum_{n\neq 0} \frac{q^{n/3}}{1-q^n} \right)^2 = \sum_{n \neq 0} \frac{q^{2n/3}}{(1-q^n)^2}. \tag{5.2}$$

By specializing the identity (5.2) at $q = \psi_1/\psi_2$, we also obtain the identity (4.1) as well.

6. Concluding remarks

The identity (5.2) is simpler than (4.1), and we wonder whether this identity can be proved rather directly. We would like to note, though, that such an attempt naturally leads us to an identity on the Fourier expansion of the Eisenstein series of weight 2, i.e., $E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n$ where

$\sigma(n)$ is the sum of divisors of n, and the identity seems to be new in the literature of the Eisenstein series as well. First, notice that the identity (5.2) reduces to the one given below in (6.1), which is a general version of (3.10). If we rearrange the summation over positive indices, we obtain the identity (6.2); see [3] for technical details. Using the derivative of the geometric series formula, one can easily show that the left-hand side of (6.2) is equal to $(1 - E_2(q))/12$, and the identity seems to be new in the literature of Eisenstein series:

$$\sum_{\substack{n,m\neq 0\\ \text{dist.}}} \frac{q^{(n+m)/3}}{(1-q^n)(1-q^m)} = 0 \tag{6.1}$$

$$\Rightarrow \quad 2\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{\substack{n,m\geq 1\\ \text{dist.}}} \frac{q^{n/3}-q^{2n/3}}{1-q^n}\,\frac{q^{m/3}-q^{2m/3}}{1-q^m}. \tag{6.2}$$

Notice also that comparing the q-coefficients of the series in (6.2) yields implications on the divisor functions, and they are discussed in [3].

Recall that the identity (1.1) can be interpreted as an algebraic dependence on $T_\ell := \sum_{n\neq 0}(-\phi)^{n/3}/F_n^\ell$, i.e., $T_1^2 - T_2 = 0$. It is reasonable to ask whether there are algebraic numbers ψ_ℓ for positive even integers ℓ such that the series $\sum_{n\neq 0} {\psi_\ell}^n/F_n^\ell$ for even positive integers ℓ admit more algebraic dependence.

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