JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Timo KELLER

On the p-torsion of the Tate–Shafarevich group of abelian varieties over higher dimensional bases over finite fields

Tome 34, nº 2 (2022), p. 497-513.
https://doi.org/10.5802/jtnb.1211

e-ISSN : 2118-8572

Journal de Théorie des Nombres
de Bordeaux **34** (2022), 497–513

On the p-torsion of the Tate–Shafarevich group of abelian varieties over higher dimensional bases over finite fields

par Timo KELLER

Résumé. Nous prouvons un théorème de finitude pour le premier groupe de cohomologie plate des schémas en groupes finis et plats sur les variétés intègres, normales et propres sur un corps fini. En conséquence, nous pouvons prouver l'invariance de la finitude de la partie p-primaire du groupe de Tate–Shafarevich des schémas abéliens sur des bases de dimension supérieure par isogénie et changement de base. Chemin faisant, nous généralisons certains des résultats précédents sur le groupe de Tate–Shafarevich dans ce contexte.

Abstract. We prove a finiteness theorem for the first flat cohomology group of finite flat group schemes over integral normal proper varieties over finite fields. As a consequence, we can prove the invariance of the finiteness of the Tate–Shafarevich group of Abelian schemes over higher dimensional bases under isogenies and alterations over/of such bases for the p-part. Along the way, we generalize previous results on the Tate–Shafarevich group in this situation.

1. Introduction

The Tate–Shafarevich group $Ш(\mathscr{A}/X)$ of an Abelian scheme $\mathscr{A}$ over a base scheme X is of great importance for the arithmetic of $\mathscr{A}$. It classifies everywhere locally trivial $\mathscr{A}$-torsors. Its finiteness is sufficient to establish our analogue of the conjecture of Birch and Swinnerton-Dyer [11] over higher dimensional bases over finite fields.

In [10, Section 4.3], we showed that finiteness of an ℓ-primary component of the Tate–Shafarevich group descends under generically étale alterations of generical degree prime to ℓ for ℓ invertible on the base scheme. This is used in [11, Corollary 5.11] to prove the finiteness of the Tate–Shafarevich group and an analogue of the Birch–Swinnerton-Dyer conjecture for certain

Manuscrit reçu le 24 novembre 2020, révisé le 12 avril 2021, accepté le 5 juin 2021.
2020 *Mathematics Subject Classification.* 11G10, 14F20, 14K15.
Mots-clefs. Tate–Shafarevich groups of abelian varieties over higher dimensional bases over finite fields, p-torsion in characteristic $p > 0$; Abelian varieties of dimension > 1; Étale and other Grothendieck topologies and cohomologies; Arithmetic ground fields for abelian varieties.
T. K. has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Projektnummer STO 299/18-1, AOBJ: 667349 while working on this article.

Abelian schemes over higher dimensional bases over finite fields under mild conditions. In [10, Section 4.4], we showed that finiteness of an ℓ-primary component of the Tate–Shafarevich group is invariant under étale isogenies. In this article, we prove these results also for the p^∞-torsion.

Recall that we defined the Tate–Shafarevich group of $\mathscr{A}/X$ for $\dim X > 1$ and $\mathscr{A}$ of good reduction as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$. In [10, Lemma 4.15] we proved as a hypothesis in [10, Theorem 4.5]:

Lemma 1.1. *Let X/k be a smooth variety and $\mathscr{C}/X$ a smooth proper relative curve. Assume $\dim X \leq 2$. Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension ≥ 2. Then*

$$\mathrm{H}^i_Z(X, \mathbf{Pic}^0_{\mathscr{C}/X}) = 0 \quad \text{for } i \leq 2.$$

If $\dim X > 2$, this holds at least up to p-torsion.

We weaken the hypothesis that $\mathscr{A}$ is a Jacobian in [10, Lemma 4.15] to arbitrary Abelian schemes:

Theorem (Corollary 2.5)**.** *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then the vanishing condition* [10, (4.4)] *holds for $\mathscr{A}/X$: $\mathrm{H}^i_Z(X, \mathscr{A})$ is torsion for all i. Furthermore, $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$, and for $i = 1, 2$, the only possible torsion is p-torsion for p not invertible on X.*

Our main results are now as follows:

Theorem (Theorem 3.14)**.** *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite.*

This theorem is proven by reduction to the finite flat simple group schemes $\mathbf{Z}/p$,μ_p and α_p over an algebraically closed field using de Jong's alteration theorem, Raynaud–Gruson and a dévissage argument.

Using this technical result and refining our methods from [10], we obtain the following three results:

The following theorem has been proved as [10, Lemma 4.28] for p prime to the characteristic of k; in this article, we prove it also for p equal to the characteristic of k:

Theorem (Lemma 5.1)**.** *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

In [10, Theorem 4.31], we proved:

Theorem 1.2. *Let X/k be proper, $\mathscr{A}$ and $\mathscr{A}'$ Abelian schemes a variety X over a finite field and $f : \mathscr{A}' \to \mathscr{A}$ an étale isogeny. Let $\ell \neq \operatorname{char} k$ be a prime. Then $Ш(\mathscr{A}/X)[\ell^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[\ell^\infty]$ is finite.*

In this article, we prove it also for ℓ equal to the characteristic of k:

Theorem (invariance of finiteness of Ш under isogenies, Theorem 4.1)**.** *Let X/k be a proper variety over a finite field k and $f : \mathscr{A} \to \mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p \neq \operatorname{char} k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

In [10, Theorem 4.29], we proved:

Theorem 1.3. *Let $f : X' \to X$ be a morphism of normal integral varieties over a finite field which is an alteration of degree prime to ℓ for a prime ℓ invertible on X, i.e., f is a proper, surjective, generically étale morphism of generical degree prime to ℓ. If $\mathscr{A}$ is an Abelian scheme on X such that the ℓ^∞-torsion of the Tate–Shafarevich group $Ш(\mathscr{A}'/X')$ of $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$ is finite, then the ℓ^∞-torsion of the Tate–Shafarevich group $Ш(\mathscr{A}/X)$ is finite.*

In this article, we prove it also for ℓ equal to the characteristic of k and remove the condition that the generical degree is prime to ℓ if ℓ is invertible on X:

Theorem (invariance of finiteness of Ш under alterations, Theorem 5.3 and Theorem 5.5)**.** *Let $f : X' \to X$ be a proper, surjective, generically finite morphism of generical degree d of regular, integral, separated varieties over a finite field of characteristic $p > 0$. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. Let ℓ be an arbitrary prime. Assume $(d, \ell) = 1$ if $\ell = p$. If $Ш(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[\ell^\infty]$.*

Notation. Canonical isomorphisms are often denoted by "=". We denote Pontrjagin duality by $(-)^D$ and duals of Abelian schemes and Cartier duals by $(-)^t$.

For a scheme X, we denote the set of codimension-1 points by $X^{(i)}$ and the set of closed points by $|X|$.

For an abelian group A, let A_{tors} be the torsion subgroup of A, and $A_{\text{n-tors}} = A/A_{\text{tors}}$. For A a cofinitely generated ℓ-primary group, let A_{div} be the maximal divisible subgroup of A, which equals the subgroup of divisible elements of A in this case ([11, Lemma 2.1.1 (iii)]), and $A_{\text{n-div}} = A/A_{\text{div}}$. For an integer n and an object A of an abelian category, denote the cokernel of $A \xrightarrow{n} A$ by A/n and its kernel by $A[n]$, and for a prime p the p-primary subgroup $\varinjlim_n A[p^n]$ by $A[p^\infty]$. Write $A[\text{non-}p]$ or $A[p']$ for $\varinjlim_{p \nmid n} A[n]$. For a prime ℓ, let the ℓ-adic Tate module $T_\ell A$ be $\varprojlim_n A[\ell^n]$ and the rationalized ℓ-adic Tate module $V_\ell A = T_\ell A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. The corank of $A[p^\infty]$ is the $\mathbf{Z}_p$-rank of $A[p^\infty]^D = T_p A$.

2. Vanishing of étale cohomology with supports of Abelian schemes

This is a complement to the "vanishing condition" $\mathrm{H}^i_Z(X,G)=0$ from [10, (4.4)], which is proven there only for Jacobians of curves, see [10, Lemma 4.10].

Theorem 2.1. *Let X be a regular integral Noetherian separated scheme and G/X be a finite étale commutative group scheme of order invertible on X. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X,G)=0$ for $i \leq 2$ (étale cohomology with supports in Z).*

Proof. Let $U = X \setminus Z$. One has a long exact cohomology sequence

$$\ldots \to \mathrm{H}^{i-1}(X,G) \to \mathrm{H}^{i-1}(U,G) \to$$
$$\mathrm{H}^i_Z(X,G) \to \mathrm{H}^i(X,G) \to \mathrm{H}^i(U,G) \to \ldots,$$

so one has to prove that $\mathrm{H}^i(X,G) \to \mathrm{H}^i(U,G)$ is an isomorphism for $i=0,1$ and injective for $i=2$.

For $i=0$, the claim $\mathrm{H}^i_Z(X,G)=0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G),$$

which is clear from [7, p. 105, Exercise II.4.2] since G/X is separated, X is reduced and $U \hookrightarrow X$ is dense.

For $i=1$ the claim $\mathrm{H}^i_Z(X,G)=0$ is equivalent to

$$\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G)$$

being surjective and

$$\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$$

being injective. The surjectivity of $\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G)$ follows e. g. from

Theorem 2.2. *Let S be a normal Noetherian base scheme, and let $u : T \dashrightarrow G$ be an S-rational map from a smooth S-scheme T to a smooth and separated S-group scheme G. Then, if u is defined in codimension ≤ 1, it is defined everywhere.*

Proof. See [3, p. 109, Theorem 1]. □

For the injectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$: If a principal homogeneous space P/X for G/X is trivial over U, then it is trivial over X: The trivialization over U gives a rational map from X to the principal homogeneous space and any such map (with X a regular scheme) extends to a morphism by Theorem 2.2.

For the surjectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$: This means that any principal homogeneous space P/U extends to a principal homogeneous space $\overline{P}/X$. By [13, p. 123, Corollary III.4.7], we have $\mathrm{PHS}(G/X) \xrightarrow{\sim} \mathrm{H}^1(X_{\mathrm{fl}},G)$ (Čech cohomology) since G/X is affine. Since G/X is smooth, [13, p. 123, Remark III.4.8 (a)] shows that we can take étale cohomology as well, and by [13, p. 101, Corollary III.2.10], one can take derived functor cohomology instead of Čech cohomology. Recall:

Theorem 2.3 (Zariski–Nagata purity)**.** *Let X be a locally Noetherian regular scheme and U an open subscheme with closed complement of codimension ≥ 2. Then the functor $X' \mapsto X' \times_X U$ is an equivalence of categories from étale coverings of X to étale coverings of U.*

Proof. See [5, Exp. X, Corollaire 3.3]. □

By Theorem 2.3, one can extend P/U uniquely to a $\overline{P}/X$, for which we have to show that it represents an element of $\mathrm{H}^1(X, G)$, i. e., that it is a G-torsor.

So we need to show that if P/U is an $G|_U$-torsor and $\overline{P}$ an extension of P to a finite étale covering of X, then $\overline{P}/X$ is also an G-torsor. For this, we use the following

Theorem 2.4. *Let X be a connected scheme, $G \to X$ a finite flat group scheme, and $\overline{P} \to X$ a scheme over X equipped with a left action $\rho : G \times_X \overline{P} \to \overline{P}$. These data define a G-torsor over X if and only if there exists a finite locally free surjective morphism $Y \to X$ such that $\overline{P} \times_X Y \to Y$ is isomorphic, as a Y-scheme with $G \times_X Y$-action, to $G \times_X Y$ acting on itself by left translations.*

Proof. See [18, p. 171, Lemma 5.3.13]. □

That P/U is an $G|_U$-torsor amounts to saying that there is an operation

$$G|_U \times_U P \to P$$

as in the previous Theorem 2.4. Since this is étale locally isomorphic to the canonical action

$$G|_U \times_U G|_U \xrightarrow{\mu} G|_U$$

which is finite étale, by faithfully flat descent the operation defines an étale covering, so extends by Zariski–Nagata purity (Theorem 2.3) uniquely to an étale covering $H \to X$, which by uniqueness has to be isomorphic to $G \times_X \overline{P} \to \overline{P}$. Now a routine check shows the condition in Theorem 2.4.

There is a finite étale Galois covering X'/X with Galois group G such that $G \times_X X'$ is isomorphic to a direct sum of μ_n with n invertible on X. The Leray spectral sequence with supports $\mathrm{H}^p(G, \mathrm{H}^q_{Z'}(X', G \times_X X')) \Rightarrow \mathrm{H}^{p+q}_Z(X, G)$ from [10, p. 228, Theorem 4.9], so it suffices to show the vanishing $\mathrm{H}^q_Z(X', G \times_X X') = 0$ for $q = 0, 1, 2$. Hence one can assume $G \cong \mu_n$ for n invertible on X.

By [13, Example III.2.22], one has an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K(X))$ with $K(X)$ the function field of X and $\mathrm{Br}(X) \to \mathrm{Br}(U) \to \mathrm{Br}(K(X))$, so

$\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. By the hypotheses on X and since the codimension of Z in X is ≥ 2, there is a restriction isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(U)$ (because of the codimension condition and [7, Proposition II.6.5 (b)], $\mathrm{Cl}\, X \xrightarrow{\sim} \mathrm{Cl}\, U$, and because of [7, Proposition II.6.16], $\mathrm{Cl}\, X \cong \mathrm{Pic}\, X$ functorial in the scheme). Hence the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)/n & \longrightarrow & \mathrm{H}^2(X, \mu_n) & \longrightarrow & \mathrm{Br}(X)[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(U)/n & \longrightarrow & \mathrm{H}^2(U, \mu_n) & \longrightarrow & \mathrm{Br}(U)[n] & \longrightarrow & 0 \end{array}$$

gives that $\mathrm{H}^2(X, \mu_n) \to \mathrm{H}^2(U, \mu_n)$ is injective, so $\mathrm{H}^2_Z(X, \mu_n) = 0$. □

Corollary 2.5. *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X, \mathscr{A})$ is torsion for all i. Furthermore, $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$, and for $i = 1, 2$, the only possible torsion is p-torsion for p not invertible on X.*

Proof. By [10, p. 224, Proposition 4.1], $\mathrm{H}^i(X, \mathscr{A})$ is torsion for $i > 0$. The Kummer exact sequence $0 \to \mathscr{A}[n] \to \mathscr{A} \to \mathscr{A} \to 0$ for n invertible on X yields a surjection

$$\mathrm{H}^i_Z(X, \mathscr{A}[n]) \twoheadrightarrow \mathrm{H}^i_Z(X, \mathscr{A})[n],$$

so it suffices to show that $\mathrm{H}^i_Z(X, \mathscr{A}[n]) = 0$ for $i = 1, 2$. But this is Theorem 2.1. The triviality $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(U, \mathscr{A}),$$

which is clear from [7, p. 105, Exercise II.4.2] since $\mathscr{A}/X$ is separated, X is reduced and $U \hookrightarrow X$ is dense. □

With vanishing condition (4.4) in [10, Theorem 4.5] satisfied for $\mathscr{A}/X$ by Corollary 2.5, the statement there generalizes from $\mathscr{A}$ a Jacobian to $\mathscr{A}$ a general Abelian scheme:

Theorem 2.6. *Let X be regular, Noetherian, integral and separated and let $\mathscr{A}$ be an Abelian scheme over X. For $x \in X$, denote the function field of X by K, the quotient field of the strict Henselization of $\mathscr{O}_{X,x}$ by K_x^{nr}, the inclusion of the generic point by $j : \{\eta\} \hookrightarrow X$ and let $j_x : \mathrm{Spec}(K_x^{nr}) \hookrightarrow \mathrm{Spec}(\mathscr{O}_{X,x}^{sh}) \hookrightarrow X$ be the composition. Then we have*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x \in X} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big).$$

One can replace the product over all points by the following:

(a) *the closed points $x \in |X|$: One has isomorphisms*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in|X|} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

and

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in|X|} \mathrm{H}^1(K_x^{h}, j_x^*\mathscr{A})\Big)$$

with $K_x^h = \mathrm{Quot}(\mathscr{O}_{X,x}^h)$ the quotient field of the Henselization if $\kappa(x)$ is finite. Or,

(b) *the codimension-1 points $x \in X^{(1)}$: One has an isomorphism*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

if one disregards the p-torsion ($p = \mathrm{char}\, k$) and X/k is smooth projective over k finitely generated. For $\dim X \le 2$, this also holds for the p-torsion.

For $x \in X^{(1)}$, one can also replace K_x^{nr} and K_x^h by the quotient field of the completions $\widehat{\mathscr{O}}_{X,x}^{sh}$ and $\widehat{\mathscr{O}}_{X,x}^{h}$, respectively.

3. Finiteness theorems for $\mathbf{H}^1_{\mathrm{fppf}}$ over finite fields

The aim of this section is to show that $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite for X a normal proper variety over a finite field of characteristic p and G/X a finite flat group scheme.

The proof is by reduction to the case of a finite flat *simple* group scheme over an algebraically closed field, which is isomorphic to $\mathbf{Z}/\ell$ (étale-étale), $\mathbf{Z}/p$ (étale-local), μ_p (local-étale) or α_p (local-local).

We use the interpretation of $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ as G-torsors on X [13, Proposition III.4.7] since G/X is affine. We also exploit de Jong's alteration theorem [9, Theorem 4.1].

Let us first recall some well-known facts on flat cohomology.

Definition 3.1. An *isogeny* of commutative group schemes G, H of finite type over an arbitrary base scheme X is a group scheme homomorphism $f : G \to H$ such that for all $x \in X$, the induced homomorphism $f_x : G_x \to H_x$ on the fibers over x is finite and surjective on identity components.

Remark 3.2. See [3, p. 180, Definition 4]. We will usually consider isogenies between abelian schemes, for example the finite flat n-multiplication, which is étale iff n is invertible on the base scheme or the abelian schemes are trivial.

Lemma 3.3. *Let G, G' be commutative group schemes over a scheme X which are smooth and of finite type over X with connected fibers and $\dim G = \dim G'$ and let $f : G' \to G$ be a morphism of commutative group schemes over X.*

If f is flat (respectively, étale) then $\ker(f)$ *is a flat (respectively, étale) group scheme over X, f is quasi-finite, surjective and defines an epimorphism in the category of flat (respectively, étale) sheaves over X.*

Proof. See [11, Lemma 2.3.3]. ☐

Lemma 3.4 (Kummer sequence)**.** *Let $f : G \to G'$ be a faithfully flat isogeny between smooth commutative group schemes over a base scheme X. Then the sequence*

$$0 \to \ker(f) \to G \xrightarrow{f} G' \to 0$$

is exact on X_{fppf}. This applies in particular to $G = \mathbf{G}_m$ and $G = \mathscr{A}$ an abelian scheme and the n-multiplication morphism for arbitrary $n \neq 0$.

Proof. Since f is faithfully flat, in particular surjective, it is an epimorphism of sheaves by Lemma 3.3. An isogeny of abelian schemes is faithfully flat by [14, Proposition 8.1]. ☐

Lemma 3.5. *Let G/X be a* smooth *commutative group scheme. Then there are comparison isomorphisms*

$$\mathrm{H}^i_{\mathrm{fppf}}(X, G) = \mathrm{H}^i_{\text{ét}}(X, G).$$

In particular, $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite if X is proper over a finite field and G is a commutative finite étale group scheme.

Proof. See [13, Remark III.3.11 (b)] and note that the proof given there gives a comparison isomorphism for any topologies between the étale and the flat site. ☐

Lemma 3.6. *Let X be a Noetherian integral scheme with function field $K(X)$ and $U \subseteq X$ dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0.$$

Proof. The assumptions imply that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{O}_X(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \twoheaddownarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{O}_X(U)^\times & \longrightarrow & K(U)^\times & \longrightarrow & \mathrm{Div}(U) & \longrightarrow & \mathrm{Cl}(U) & \longrightarrow & 0. \end{array}$$

A diagram chase yields the result. ☐

Corollary 3.7. *Let X be a Noetherian integral regular scheme and let $U \subseteq X$ be dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0.$$

Proof. By the assumptions, $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ and $\mathrm{Cl}(U) = \mathrm{Pic}(U)$. □

Corollary 3.8. *Let $X/\mathbf{F}_q$ be an integral Noetherian regular proper variety and let $j : U \hookrightarrow X$ be the inclusion of an open subscheme of X. Then $\mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n})$ is finite for all n and any prime p.*

Proof. The Kummer sequence Lemma 3.4 on U_{fppf} together with $\mathrm{Pic}(U) = \mathrm{H}^1_{\mathrm{fppf}}(U, \mathbf{G}_{m,U})$ by Lemma 3.5 yields the exact sequence

$$1 \to \mathbf{G}_m(U)/p^n \to \mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n}) \to \mathrm{Pic}(U)[p^n] \to 0.$$

Since $\mathbf{G}_m(X) = \Gamma(X, \mathbf{G}_m)^\times$ is finite by the coherence theorem [6, Théorème (3.2.1)], since $X/\mathbf{F}_q$ is proper and $\mathbf{F}_q$ is finite, and since $\mathrm{Pic}(X)$ is finitely generated since its sits in a short exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$ and $\mathrm{Pic}^0(X)$ is finite since it is the group of rational points of an Abelian variety over a finite field and $\mathrm{NS}(X)$ is always finitely generated by [2, Exp. XIII, §5], by Corollary 3.7 and the finiteness of $(X \setminus U)^{(1)}$, this exact sequence gives the finiteness of $\mathbf{G}_m(U)/p^n$ and of $\mathrm{Pic}(U)[p^n]$. □

The following statements and proofs in this section are an extended version of the sketch of Theorem 3.14 given by "darx" in [19].

Lemma 3.9. *Let X be a normal integral scheme and G/X be a finite flat group scheme. If T is a G-torsor on X trivial over the generic point of X, then T is trivial. Hence, $\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(K(X), G)$ is injective, and if $f : Y \to X$ is birational, $f^* : \mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G)$ is injective.*

Proof. Since T is trivial over the generic point of X, generically, there is a section of $\pi : T \to X$. This extends to a rational map $\sigma : X \dashrightarrow T$. Take the schematic closure $i : X' \hookrightarrow T$ of σ. The composition $\pi \circ i : X' \to T \to X$ is birational and finite (as a composition of a closed immersion and a finite morphism). By [4, Corollary 12.88], since X is normal, $X' \to X$ is an isomorphism. Hence σ is a section of π, so T/X is trivial. □

Lemma 3.10. *Let X be a proper variety over a finite field and Y/X be a finite flat scheme. Let Z/X be proper. Then $Y(Z)$ is finite.*

Proof. Since $\mathrm{Mor}_X(Z, Y) = \mathrm{Mor}_Z(Z, Y \times_X Z)$, one can assume $Z = X$. So we have to show that there are only finitely many sections to $\pi : Y \to X$. Such a section corresponds to an $\mathscr{O}_X$-algebra map $\pi_*\mathscr{O}_Y \to \mathscr{O}_X$. But $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}\mathrm{om}_X(\pi_*\mathscr{O}_Y, \mathscr{O}_X))$ is finite by the coherence theorem [6, Théorème (3.2.1)] as it is a finite dimensional vector space over a finite field. □

Lemma 3.11. *Let $Y \to X$ be an alteration of proper integral varieties with X normal, and G/X be a finite flat commutative group scheme. Then* $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y,G))$ *is finite. Hence* $\mathrm{H}^1_{\mathrm{fppf}}(X,G)$ *is finite if* $\mathrm{H}^1_{\mathrm{fppf}}(Y,G)$ *is.*

Proof. If $Y \to X$ is a blow-up, the kernel is trivial by Lemma 3.9 since a blow-up is birational. Hence the statement holds for blow-ups.

By [16, Théorème 5.2.2], there is a blow-up $f : X' \to X$ such that $Y' := Y \times_X X'$ is flat over X'. Since a normalization morphism of integral schemes is birational [12, Proposition 4.1.22], one can assume X' normal. There is a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y,G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X,G) \\ & & \downarrow & & \downarrow{\scriptstyle f^*} \\ 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X',G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y',G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X',G) \end{array}$$

By the snake lemma, since $\ker f^*$ is finite as f is a blow-up,

$$\ker(\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y,G))$$

is finite if we can show that

$$\ker(\mathrm{H}^1_{\mathrm{fppf}}(X',G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y',G))$$

is finite. Hence, we can assume $Y \to X$ finite flat.

Let $T \to X$ be in the kernel, i. e., it is a G-torsor on X trivial when pulled back to Y. Choose a section $\sigma : Y \to T \times_X Y$; there are only finitely many of them by Lemma 3.10. Two such sections differ by an element of $G(Y)$. Since the base change $T \times_X (Y \times_X Y) \to Y \times_X Y$ is a G-torsor, one can take the 1-cocycle

$$\tau := d^0(\sigma) = \mathrm{pr}_0^*(\sigma) - \mathrm{pr}_1^*(\sigma) \in G(Y \times_X Y).$$

The section τ corresponds to the isomorphism class of the G-torsor T by the descent theory for the fppf covering $\{Y \to X\}$: As $\mathrm{H}^1_{\mathrm{fppf}}(-,G)$ can be computed by Čech cohomology and as the class of T in $\mathrm{H}^1_{\mathrm{fppf}}(X,G) = \check{H}^1_{\mathrm{fppf}}(X,G) = \varinjlim_{\mathscr{U}} \check{H}^1_{\mathrm{fppf}}(\mathscr{U},G)$ (the colimit taken over the coverings of X; the natural morphism from the first Čech cohomology to the first derived functor cohomology is always an isomorphism) is trivialized by the covering

$\{Y \to X\}$, it can be represented as the 1-cocycle $\tau = d^0(\sigma)$, which is a 1-coboundary:

$$\check{H}^1(\{Y \to X\}, G) = \frac{\ker\left(G(Y \times_X Y) \xrightarrow{d^1} G(Y \times_X Y \times_X Y)\right)}{\operatorname{im}\left(G(Y) \xhookrightarrow{d^0} G(Y \times_X Y)\right)}$$

But by Lemma 3.10, $G(Y \times_X Y)$ is finite. □

Lemma 3.12. *Let X be an integral scheme with function field K and G/X be a finite flat group scheme. Let $H_K \hookrightarrow G_K$ be a finite flat group scheme. Then there is a blow-up $\widetilde{X}/X$ such that H_K extends to a finite flat subgroup scheme of $G \times_X \widetilde{X}$.*

Proof. Let $H \hookrightarrow G$ be the schematic closure of $H_K \hookrightarrow G$. The morphism $H \to G \to X$ is finite as a composition of a closed immersion and a finite morphism. By [16, Théorème 5.2.2], there is a blow-up $X' \to X$ such that $H' := H \times_X X' \to X'$ is flat. Then, H' is the schematic closure of $H_K \hookrightarrow G' := G \times_X X'$. So one can assume H/X finite flat.

Let $Y \to X$ be finite flat. Since the morphism is affine, locally, one has the diagram

$$\begin{array}{ccc} A & \hookrightarrow & A \otimes_R \operatorname{Quot}(R) \\ \uparrow & & \uparrow \\ R & \hookrightarrow & \operatorname{Quot}(R). \end{array}$$

Here, the upper horizontal arrow is injective by flatness of $R \to A$. Hence Y is the schematic closure of Y_K in Y.

By flatness, the schematic closure of $H_K \times_K H_K$ in $G \times_X G$ is $H \times_X H$. By the universal property of the schematic closure [4, (10.8)], one has the factorization

$$\begin{array}{ccc} H_K \times_K H_K & \xrightarrow{\mu} & H_K \\ \downarrow & & \downarrow \\ H \times_X H & \overset{\mu}{\dashrightarrow} & H \\ \downarrow & & \downarrow \\ G \times_X G & \xrightarrow{\mu} & G, \end{array}$$

for the multiplication μ, and similar for the inverse and unit section. □

Lemma 3.13. *Let X be a proper integral variety over a field and G/X be a finite flat commutative group scheme. After an alteration $X' \to X$, there exists a filtration of G by finite flat group schemes with subquotients of prime order.*

Proof. Over the algebraic closure of the function field of X, there is such an filtration since the only simple objects in the category of finite flat group schemes of p-power order are μ_p, $\mathbf{Z}/p$ and α_p. Since everything is of finite presentation, these are defined over a finite extension of the function field [4, Corollary 10.79]. Now take the normalization in this finite extension of function fields and use Lemma 3.12. □

Theorem 3.14. *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite.*

Proof. By Lemma 3.13, Lemma 3.11 and the long exact cohomology sequence one can assume G of prime order p (since the case of G/X étale is easily dealt with). Since then G is simple by [17, p. 38] and since $F \circ V = [p] = 0$ by [17, p. 62] and [15, p. 141], either $V = 0$ or $F = 0$ on G.

If $V = 0$, by [8, Proposition 2.2], there is a short exact sequence

$$0 \to G \to \mathscr{L} \to \mathscr{M} \to 0$$

with vector bundles $\mathscr{L}, \mathscr{M}$. By the coherence theorem [6, Théorème (3.2.1)], as X is proper and lives over a finite ground field, and by comparison of Zariski and fppf cohomology [13, Proposition III.3.7], the long exact cohomology sequence shows that $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite.

If $F = 0$, after replacing X by an alteration by Lemma 3.11 as in the proof of Lemma 3.13, one can assume that G is isomorphic to μ_p over the generic point. Since for $Y, Z/X$ of finite presentation such that $Y_K \cong Z_K$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $Y_U \cong Z_U$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $G_U \cong \mu_{p,U}$. By [9], there is an alteration $f : X' \to X$ such that X' is regular. By Corollary 3.8, $\mathrm{H}^1_{\mathrm{fppf}}(f^{-1}(U), \mu_p)$ is finite. By Lemma 3.9, $\mathrm{H}^1_{\mathrm{fppf}}(X', G \times_X X')$ is finite, so by Lemma 3.11, $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite. □

4. Isogeny invariance of finiteness of Ш, the *p*-part

In this section, we extend [10, p. 240, Theorem 4.31] to p^∞-torsion.

Theorem 4.1. *Let X/k be a proper variety over a finite field k and $f : \mathscr{A} \to \mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p \neq \operatorname{char} k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

Proof. In the case where ℓ is invertible on X and f is étale (i. e., of degree invertible on X), this is [10, p. 240, Theorem 4.31].

Now assume $p = \operatorname{char} k$. Then the short exact sequence of flat sheaves Lemma 3.4 yields an exact sequence in cohomology

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f)) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \xrightarrow{f} \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}')$$

and note that $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A}) = Ш(\mathscr{A}/X)$ by Lemma 3.5 since $\mathscr{A}/X$ is smooth, and that $\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f))$ is finite by Theorem 3.14. Note that all groups are torsion (the Tate–Shafarevich groups by [10, p. 224, Proposition 4.1]), hence the sequence stays exact after taking p^∞-torsion. So $Ш(\mathscr{A}/X)[p^\infty]$ is finite if $Ш(\mathscr{A}'/X)[p^\infty]$ is.

For the converse, note that by [11, Proposition 2.19], there is a polarization $\lambda : \mathscr{A}^t \to \mathscr{A}$. Hence, the argument above for λ and λ^t implies that $Ш(\mathscr{A}^t/X)[p^\infty]$ is finite iff $Ш(\mathscr{A}/X)[p^\infty]$ is, and analogously for $Ш(\mathscr{A}'/X)[p^\infty]$. Taking the dual Kummer sequence $0 \to \ker(f^t) \to \mathscr{A}'^t \to \mathscr{A}^t \to 0$ yields an exact sequence

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f^t)) \to Ш(\mathscr{A}'^t/X) \to Ш(\mathscr{A}^t/X).$$

By the same argument as above, $Ш(\mathscr{A}'^t/X)[p^\infty]$ is finite if $Ш(\mathscr{A}^t/X)[p^\infty]$ is if $Ш(\mathscr{A}/X)[p^\infty]$ is. So $Ш(\mathscr{A}'/X)[p^\infty]$ is finite. □

5. Descent of finiteness of Ш, the *p*-part

In this section, we extend [10, p. 238, Theorem 4.29] to p^∞-torsion.

Lemma 5.1. *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

Recall that $Ш(\mathscr{A}/X)$ was defined as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$ in [10, p. 225, Definition 4.2].

Proof. The long exact cohomology sequence associated to the Kummer sequence Lemma 3.4 gives us a surjection

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n]) \twoheadrightarrow \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A})[p^n] \to 0$$

Now, since $\mathscr{A}/X$ is a smooth group scheme, Lemma 3.5 gives us an isomorphism $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$, which by definition equals $Ш(\mathscr{A}/X)$. By Theorem 3.14, $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n])$ is finite since $X/\mathbf{F}_q$ is proper. From this, one sees that $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})[p]$ is finite. Hence $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated by [11, Lemma 2.38]. □

Lemma 5.2 (existence of trace morphism)**.** *Let $f : X' \to X$ be a finite étale morphism of constant degree d and let $\mathscr{F}$ be an fppf sheaf on X. Then there is a trace map $\mathrm{Tr}_f : f_* f^* \mathscr{F} \to \mathscr{F}$, functorial in $\mathscr{F}$, such that $\varphi \mapsto \mathrm{Tr}_f \circ f_*(\varphi)$ is an isomorphism $\mathrm{Hom}_{X'}(\mathscr{F}', f^*\mathscr{F}) \to \mathrm{Hom}_X(\pi_*\mathscr{F}', \mathscr{F})$ for any fppf sheaf $\mathscr{F}'$ on X'. Thus, $f_* = f_!$, that is, f_* is left adjoint to f^*, and Tr_f is the adjunction map. The composites*

$$\mathscr{F} \to f_* f^* \mathscr{F} \overset{\mathrm{Tr}_f}{\to} \mathscr{F}$$

and

$$\mathrm{H}^r_{\mathrm{fppf}}(X,\mathscr{F}) \xrightarrow{f^*} \mathrm{H}^r_{\mathrm{fppf}}(X',f^*\mathscr{F}) \xrightarrow{\mathrm{can}} \mathrm{H}^r_{\mathrm{fppf}}(X,f_*f^*\mathscr{F}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^r_{\mathrm{fppf}}(X,\mathscr{F})$$

are multiplication by d.

Proof. On may copy the proof of [13, p. 168, Lemma V.1.12] almost verbatim: Let $\mathscr{F}$ be a fppf sheaf on X. Let $X'' \to X$ be finite Galois with Galois group G factoring as $X'' \to X' \to X$; $X'' \to X'$ is Galois with Galois group $H \le G$. For any U/X flat, we have $\Gamma(U,\mathscr{F}) \hookrightarrow \Gamma(U',\mathscr{F}) \hookrightarrow \Gamma(U'',\mathscr{F})$ and $\Gamma(U,\mathscr{F}) \xrightarrow{\sim} \Gamma(U'',\mathscr{F})^G$, where $U' = U \times_X X'$ and $U'' = U \times_X X''$. For a section $s \in \Gamma(U, f_*f^*\mathscr{F}) := \Gamma(U',\mathscr{F})$, we define

$$\mathrm{Tr}_f(s) := \sum_{\sigma\in G/H} \sigma(s|_{U''});$$

as this is fixed by G, it may be regarded as an element of $\Gamma(U,\mathscr{F}) \xrightarrow{\sim} \Gamma(U'',\mathscr{F})^G$. Clearly, Tr_f defines a morphism $f_*f^*\mathscr{F} \to \mathscr{F}$ such that its composite with $\mathscr{F} \to f_*f^*\mathscr{F}$ is multiplication by the degree d of f.

If X' is a disjoint union of d copies of X, obviously $\mathrm{Hom}_{X'}(\mathscr{F}', f^*\mathscr{F}) \to \mathrm{Hom}_X(f_*\mathscr{F}',\mathscr{F})$, and one may reduce the question to this split case by passing to a finite étale covering of X, for example to $X'' \to X$, and using the fact that Hom is a sheaf.

In

$$\mathrm{H}^r_{\mathrm{fppf}}(X,\mathscr{F}) \xrightarrow{f^*} \mathrm{H}^r_{\mathrm{fppf}}(X',f^*\mathscr{F}) \xrightarrow{\mathrm{can}} \mathrm{H}^r_{\mathrm{fppf}}(X,f_*f^*\mathscr{F}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^r_{\mathrm{fppf}}(X,\mathscr{F})$$

the composite of the first two maps is induced by $\mathscr{F} \to f_*f^*\mathscr{F}$, and the composite of all three is induced by $(\mathscr{F} \to f_*f^*\mathscr{F} \xrightarrow{\mathrm{Tr}_f} \mathscr{F})$, which is multiplication by d. ☐

Theorem 5.3. *Let p be a prime and X be a scheme of characteristic p. Let $f : X' \to X$ be a proper, surjective, generically étale morphism of generical degree prime to p of regular, integral, separated varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. If $Ш(\mathscr{A}'/X')[p^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[p^\infty]$.*

Proof. The same proof as in [10, Theorem 4.29] works, one only needs $Ш(\mathscr{A}/X)[p^\infty]$ to be cofinitely generated in Step 2, which is Lemma 5.1. The trace morphism in Step 3 for fppf cohomology comes from Lemma 5.2. Note that the proof given there does not need the regularity of X, X' and that varieties over a field are excellent by [12, Corollary 2.40 (a)]. For the convenience of the reader, we reproduce the proof of [10, Theorem 4.29] adapted to our situation here:

Step 1: $\mathbf{H}^1_{\mathbf{fppf}}(\boldsymbol{X}, \boldsymbol{f_*}\mathscr{A}')[\boldsymbol{p^\infty}]$ *is finite.* This follows from the low terms exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{fppf}}(X, f_*\mathscr{A}') \to \mathrm{H}^1_{\mathrm{fppf}}(X',\mathscr{A}')$$

associated to the Leray spectral sequence

$$\mathrm{H}^p_{\mathrm{fppf}}(X, \mathrm{R}^q f_* \mathscr{A}') \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(X', \mathscr{A}')$$

and the finiteness of

$$\mathrm{H}^1_{\mathrm{fppf}}(X', \mathscr{A}')[p^\infty] = Ш(\mathscr{A}'/X')[p^\infty].$$

Step 2: The theorem holds if there is a trace morphism. Since by Lemma 5.2 there is a trace morphism $f_* f^* \mathscr{A} \to \mathscr{A}$ such that the composition with the adjunction morphism

$$\mathscr{A} \to f_* f^* \mathscr{A} \to \mathscr{A}$$

is multiplication by $\deg f \neq 0$, the finiteness of $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A})[p^\infty]$ follows from that of $\mathrm{H}^1_{\mathrm{fppf}}(X, f_* \mathscr{A}')[p^\infty]$ because both groups are cofinitely generated by Lemma 5.1.

Step 3: Proof of the theorem in the general case. Let η be the generic point of X. Define X'_η by the commutativity of the cartesian diagram

(5.1)
$$\begin{array}{ccc} X'_\eta & \xhookrightarrow{g'} & X' \\ \big\downarrow{\scriptstyle f_\eta} & & \big\downarrow{\scriptstyle f} \\ \{\eta\} & \xhookrightarrow{g} & X. \end{array}$$

Since f is generically étale, we can apply Lemma 5.2 to f_η in this commutative diagram. From the commutativity of that diagram, the kernel of $f^* : \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \to \mathrm{H}^1_{\mathrm{fppf}}(X', \mathscr{A}')$ is contained in the kernel of the composition

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \stackrel{g^*}{\to} \mathrm{H}^1_{\mathrm{fppf}}(\{\eta\}, \mathscr{A}_\eta) \stackrel{f_\eta^*}{\to} \mathrm{H}^1_{\mathrm{fppf}}(X'_\eta, \mathscr{A}'_{X'_\eta}),$$

so it suffices to show that the first arrow g^* is injective. But by the Néron mapping property $\mathscr{A} \xrightarrow{\sim} g_* g^* \mathscr{A}$ [10, Theorem 3.3] (for the *étale* topology!), $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X, g_* \mathscr{A}_\eta)$. However, the Leray spectral sequence $\mathrm{H}^p_{\text{ét}}(X, \mathrm{R}^q g_* \mathscr{A}_\eta) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\{\eta\}, \mathscr{A}_\eta)$ gives an injection

$$0 \to \mathrm{H}^1_{\text{ét}}(X, g_* \mathscr{A}_\eta) \to \mathrm{H}^1_{\text{ét}}(\{\eta\}, \mathscr{A}_\eta).$$

But because $\mathscr{A}/X$ and $\mathscr{A}_\eta/\{\eta\}$ are smooth commutative group schemes, their étale cohomology agrees with their flat cohomology, see Lemma 3.5, and the comparison of topology morphisms are functorial. □

Theorem 5.4 (Stein factorization, alteration = finite ∘ modification)**.** *Let $f : X' \to X$ be a proper morphism of Noetherian schemes. Then one can factor f into $g \circ f'$, where $f' : X' \to Y := \mathbf{Spec}_X f_* \mathscr{O}_{X'}$ is a proper morphism with connected fibers, and $g : Y \to X$ is a finite morphism. If f is an alteration, f' is birational and proper (a* modification*).*

Proof. See [6, Théorème 4.3.1] for the statement on the existence of the factorization, which includes $Y = \mathbf{Spec}_X f_* \mathscr{O}_{X'}$.

Assume now that f is an alteration. If $U \subseteq X$ is an open subscheme such that $f|_U$ is finite (in particular affine), one may shrink U such that it is affine, so by finiteness of g, $g : g^{-1}(U) \to U$ is finite and can be written as $\operatorname{Spec} B \to \operatorname{Spec} A$. From the statement of Stein factorization, $g^{-1}(U) = \mathbf{Spec}_U f_* \mathscr{O}_{f^{-1}(U)}$, but f' has geometrically connected fibers, so $f'_* \mathscr{O}_{X'} = \mathscr{O}_Y$, so $f'|_{g^{-1}(U)}$ is an isomorphism because it is affine. □

We also remove the hypotheses that f is generically étale and has degree prime to ℓ if ℓ is invertible on the base scheme in [10, Theorem 4.29]:

Theorem 5.5. *Let $f : X' \to X$ be a proper, surjective, generically finite morphism of regular, integral, separated varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. Let ℓ be invertible on X. If $\text{Ш}(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$.*

Proof. By the Stein factorization Theorem 5.4, f factors as a proper, surjective, birational morphism followed by a finite morphism. In particular, it is a generically étale alteration. The finite morphism factors as a finite purely inseparable morphism followed by a finite generically étale morphism. We prove the finiteness assertion of the theorem for all such morphisms separately:

If f is generically étale, this is Theorem 5.3. If f is a proper, surjective, birational morphism, it is generically an isomorphism, i.e., generically étale of degree 1.

If f is a universal homeomorphism, the étale sites of X and X' are equivalent by f^* and f_* by [1, VIII.1.1]. In particular, the étale cohomology groups $\text{Ш}(\mathscr{A}/X) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$ and $\text{Ш}(\mathscr{A}'/X') = \mathrm{H}^1_{\text{ét}}(X', f^*\mathscr{A})$ are isomorphic via f^*. □

Acknowledgments. I thank Jean-Louis Colliot-Thélène, Aise Johan de Jong, Ulrich Görtz, Uwe Jannsen and Moritz Kerz, and Ariyan Javanpeykar, darx and David Loeffler from mathoverflow. I also thank the anonymous referee for improving the exposition.

References

[1] M. ARTIN, A. GROTHENDIECK & J.-L. VERDIER, *Séminaire de Géométrie Algébrique du Bois Marie – 1963–64 – Théorie des topos et cohomologie étale des schémas – (SGA 4) – Vol. 2*, Lecture Notes in Mathematics, vol. 270, Springer, 1972, iv+418 pages.

[2] P. BERTHELOT, A. GROTHENDIECK & L. ILLUSIE, *Séminaire de Géométrie Algébrique du Bois Marie – 1966–67 – Théorie des intersections et théorème de Riemann-Roch – (SGA 6)*, Lecture Notes in Mathematics, vol. 225, Springer, 1971, xii+700 pages.

[3] S. BOSCH, W. LÜTKEBOHMERT & M. RAYNAUD, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 21, Springer, 1990, x+325 pages.

[4] U. GÖRTZ & T. WEDHORN, *Algebraic geometry I. Schemes. With examples and exercises*, Advanced Lectures in Mathematics, Vieweg+Teubner, 2010, vii+615 pages.

[5] A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie – 1960–61 – Revêtements étales et groupe fondamental – (SGA 1)*, Lecture Notes in Mathematics, vol. 224, Springer, 1971, xxii+447 pages.

[6] A. Grothendieck & J. Dieudonné, "Eléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie.", *Publ. Math., Inst. Hautes Étud. Sci.* **11** (1961), p. 5-167.

[7] R. Hartshorne, *Algebraic geometry. Corr. 3rd printing.*, Graduate Texts in Mathematics, vol. 52, Springer, 1983, xvi+496 pages.

[8] A. J. de Jong, "Finite locally free group schemes in characteristic p and Dieudonné modules.", *Invent. Math.* **114** (1993), no. 1, p. 89-137.

[9] ———, "Smoothness, semi-stability and alterations.", *Publ. Math., Inst. Hautes Étud. Sci.* **83** (1996), p. 51-93.

[10] T. Keller, "On the Tate–Shafarevich group of Abelian schemes over higher dimensional bases over finite fields", *Manuscr. Math.* **150** (2016), p. 211-245.

[11] ———, "On an analogue of the conjecture of Birch and Swinnerton–Dyer for Abelian schemes over higherdimensional bases over finite fields", *Doc. Math.* **24** (2019), p. 915-993.

[12] Q. Liu, *Algebraic geometry and arithmetic curves. Transl. by Reinie Erné.*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2006, xv+577 pages.

[13] J. S. Milne, *Étale cohomology.*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980, xiii+323 pages.

[14] ———, "Abelian varieties", in *Arithmetic geometry (Pap. Conf., Storrs/Conn., 1984)*, Springer, 1986, p. 103-150.

[15] D. Mumford, *Abelian varieties*, Oxford University Press, 1970, viii+242 pages.

[16] M. Raynaud & L. Gruson, "Critères de platitude et de projectivité. Techniques de « platification » d'un module", *Invent. Math.* **13** (1971), p. 1-89.

[17] S. S. Shatz, "Group schemes, formal groups, and p-divisible groups", in *Arithmetic geometry (Pap. Conf., Storrs/Conn., 1984)*, Springer, 1986, p. 29-78.

[18] T. Szamuely, *Galois groups and fundamental groups.*, Cambridge Studies in Advanced Mathematics, vol. 117, Cambridge University Press, 2009, ix+270 pages.

[19] user19475, "Finitness of syntomic/fppf cohomology with coefficients in a finite flat group scheme", MathOverflow, `https://mathoverflow.net/q/257441` (version: 2016-12-19).

Timo Keller
Universität Bayreuth
Lehrstuhl Mathematik II (Computeralgebra)
Universitätsstraße 30
95440 Bayreuth, Germany
E-mail: `Timo.Keller@uni-bayreuth.de`
URL: `https://www.timo-keller.de`