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A contribution to infinite disjoint covering systems

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RÉSUMÉ. Supposons que la famille de suites arithmétiques $\{d_i n + b_i : n \in \mathbb{Z}\}_{i \in I}$ soit un recouvrement disjoint des nombres entiers. Nous prouvons qui si $d_i = p^k q^l$ pour des nombres premiers p, q et des entiers $k, l \geq 0$, il existe alors un $j \neq i$ tel que $d_i | d_j$. On conjecture que le résultat de divisibilité est vrai quelques soient les raisons d_i.

Un recouvrement disjoint est appelé saturé si la somme des inverses des raisons est égale à 1. La conjecture ci-dessus est vraie pour des recouvrements saturés avec des d_i dont le produit des facteurs premiers n'est pas supérieur à 1254.

ABSTRACT. Let the collection of arithmetic sequences $\{d_i n + b_i : n \in \mathbb{Z}\}_{i \in I}$ be a disjoint covering system of the integers. We prove that if $d_i = p^k q^l$ for some primes p, q and integers $k, l \geq 0$, then there is a $j \neq i$ such that $d_i | d_j$. We conjecture that the divisibility result holds for all moduli.

A disjoint covering system is called saturated if the sum of the reciprocals of the moduli is equal to 1. The above conjecture holds for saturated systems with d_i such that the product of its prime factors is at most 1254.

A *Beatty sequence* is defined by $S(\alpha, \beta) := \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$, where α is positive and β is an arbitrary real constant. A conjecture of Fraenkel asserts that if $\{S(\alpha_i, \beta_i) : i = 1 \dots m\}$ is a collection of $m \geq 3$ Beatty sequences which partitions the positive integers then α_i / α_j is an integer for some $i \neq j$. Special cases of the conjecture were verified by Fraenkel [2], Graham [4] and Simpson [9]. For more references on Beatty sequences see [1] and [11]. If α is integral, then $S(\alpha, \beta)$ is an arithmetic sequence. Mirsky and Newman, and later independently Davenport and Rado proved that if $\{\{a_i n + b_i\} : i = 1 \dots m\}$ is a partition of the positive integers, then $a_i = a_j$ for some $i \neq j$. This settles Fraenkel's conjecture for integral α's. We formulate a related conjecture for partitions to infinite number of arithmetic sequences.

We denote by $A(d, b)$ the arithmetic sequence $\{dn + b : n \in \mathbb{Z}\}$. Let a collection $\mathcal{S}$ of arithmetic sequences $\{A(d_i, b_i) : i \in I\}$ be called a *covering* *system* (CS), if the union of the sequences is $\mathbb{Z}$. The CS is finite or infinite according to the set I. The numbers d_i are called the *moduli* of the CS. A conjecture similar to Fraenkel's was posed by Schinzel, that for any finite CS, there is a pair of distinct indices i, j for which $d_i | d_j$. This was verified by Porubský [8] assuming some extra conditions.

When the sequences of a CS are disjoint, it is called a *disjoint covering system* (DCS). The structure of DCS's is a wide topic of research. We only mention here a few results about IIDCS's (such DCS's that the number of sequences are infinite, and the moduli are distinct). For further references see [7]. There is a natural method to construct DCS's, the following construction appeared in [10]:

Example 1. Let $I = \mathbb{N}$, and $d_1 | d_2 | d_3 \ldots$ be positive integers. Define the b_i's recursively to be an integer of minimal absolute value not covered by the sequences $A(d_j, b_j)$, $(j < i)$.

Indeed this gives a DCS; if $A(d_i, b_i)$ and $A(d_j, b_j)$ $(j < i)$ do intersect then $A(d_i, b_i) \subset A(d_j, b_j)$ as $d_j | d_i$, which contradicts the definition of b_i. Also the definition of b_i guarantees that an integer of absolute value n is covered by one of the first $2n + 1$ sequences.

If the sum of the reciprocals of the moduli equals 1, we call the DCS *saturated.* Apparently every finite DCS is saturated, but this property is rather "rare" for IIDCS's. The IIDCS in the example above is saturated only for $d_i = 2^i$. Stein [10] asked whether this is the unique example. Krukenberg [5] answered this in the negative, then Fraenkel and Simpson [3] characterised all IIDCS, whose moduli are of form $2^k 3^l$. Lewis [6] proved that if a prime greater than 3 divides one of the moduli, then the set of all prime divisors of the moduli is infinite.

We formulate the following conjecture:

Conjecture 2. *If* $\{A(d_i, b_i) : i \in I\}$ *is a DCS, then for all* i *there exists an index* $j \neq i$ *such that* $d_i | d_j$.

This conjecture is valid for the above examples, and also valid for those appearing in [3]. It is also known for finite DCS's, being a consequence of Corollary 2 of [8].

We prove the following special case of Conjecture 2.

Theorem 3. *If* $\{A(d_i, b_i) : i \in I\}$ *is a DCS, and* $d_i = p^k q^l$ *for some primes* p, q *and integers* $k, l \geq 0$, *then* $d_i | d_j$ *for some* $j \neq i$.

Proof. Let (a, b) denote the greatest common divisor of the integers a, b, and $[a, b]$ their least common multiple. We will use the fact, that the sequences $A(d_1, b_1)$ and $A(d_2, b_2)$ are disjoint if and only if $(d_1, d_2) \nmid x_1 - x_2$, where x_i is an arbitrary number covered by $A(d_i, b_i)$ for $i = 1, 2$.

If $l = 0$, consider the sequence $A(d_j, b_j)$ that covers $b_i + p^{k-1}$. Then $(d_j, d_i) \nmid b_i + p^{k-1} - b_i = p^{k-1}$. Thus $(d_j, d_i) = p^k = d_i$, so $d_i | d_j$ which was to be proved. Now we may assume that $k, l > 0$.

Assume to the contrary, that the theorem is false. Defining $b'_j = b_j - b_i$, we get another DCS $\{A(d_j, b'_j) : j \in I\}$, where $b'_i = 0$. Hence we may assume that $b_i = 0$.

Let $A_j = A(d_j, b_j) \cap A(p^{k-1}q^{l-1}, 0)$. Either A_j is empty or an arithmetic sequence, whose modulus is $[p^{k-1}q^{l-1}, d_j]$. Let $B_j = \left\{ \frac{x}{p^{k-1}q^{l-1}} \middle| x \in A_j \right\}$. The nonempty sequences among the B_j's form a DCS. Notice that pq divides the modulus of B_j if and only if $d_i = p^k q^l | d_j$. Since the modulus of B_i is pq, it remains to prove the theorem for $k = l = 1$.

Assume $d_i = pq$, and $pq \nmid d_j$ for $i \neq j$. Assume that $p + q$ is covered by the sequence $A(d_t, b_t)$ of the DCS. We prove that $p \nmid d_t$.

Assume to the contrary that $p | d_t$. Let $d_t = d \cdot p^m$, where $p \nmid d$. Then $(pq, d) | q$, and there exist a pair of positive integers u, v such that $q = pq \cdot u - d \cdot v$. Let $a = p + q + dv = p + pqu$. Assume that a is covered by $A(d_s, b_s)$. If $A(d_s, b_s)$ and $A(d_t, b_t)$ are the same sequences, then $d_s = d_t$, and $p | d_s$. Otherwise $(d_s, d_t) \nmid p + q + dv - p - q = dv$, which yields $p | d_s$. Since $pq \nmid a$, $s \neq i$, and $(d_s, d_i) \nmid p + pqu - pqu = p$, thus $q | d_s$. This contradicts $d_i \nmid d_s$.

Similar argument shows $q \nmid d_t$, which contradicts $(d_i, d_t) \nmid b_i - b_t$. So the proof is complete. □

As a byproduct of the previous proof, we got the following lemma:

Lemma 4. *Suppose there is a DCS $\{A(d_i, b_i) : i \in I\}$ and an index $i \in I$ such, that $d_i = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where the p's are distinct primes, and $d_i \nmid d_j$ for all $j \neq i$. Then there exists another DCS $\{A(\hat{d}_i, \hat{b}_i) : i \in \hat{I}\}$ and an index $\hat{\imath} \in \hat{I}$ such that $\hat{d}_{\hat{\imath}} = p_1 p_2 \dots p_k$ and $\hat{d}_{\hat{\imath}} \nmid \hat{d}_j$ for all $j \neq \hat{\imath}$.*

So it is sufficient to verify the conjecture for square-free moduli. When d_i has more than two different prime factors, the situation seems to be much more complicated. We can still say something for saturated DCS's. We need the following concepts:

Suppose $A \subseteq \mathbb{Z}$. Let $S_n(A) = |\{x \in A : -n < x < n\}|$ be the number of elements of A with absolute value less than n. We define the *density* of A to be $d(A) = \lim_{n \to \infty} \frac{S_n(A)}{2n-1}$ if the limit exists, and in that case we say, that the density of A exists. We will use following facts. The density is finitely additive, and the density of arithmetic sequences exist, and $d(A(d, b)) = \frac{1}{d}$. Let $\{A(d_i, b_i) : i \in I\}$ be a saturated DCS, and $J \subseteq I$. Lemma 2.2 of [6] states, that the density of $X := \bigcup_{j \in J} A(d_j, b_j)$ exists, and $d(X) = \sum_{j \in J} \frac{1}{d_j}$.

Let a be an arbitrary and b a positive integer. Denote by $a \bmod b$ the unique integer $0 \le c < b$, that $b|a-c$.

Lemma 5. *Suppose there is a saturated DCS $\{A(d_i,b_i) : i \in I\}$, and an index $i \in I$ such that for all $j \neq i$ $d_i \nmid d_j$. Let D denote the set of positive divisors of d_i different from 1 and d_i. Then there exist nonnegative real numbers $x_{s,t}$, where $s \in D$ and $0 \le t < s$, such that for all $0 < u < d_i$*

$$\sum_{s\in D} x_{s,u \bmod s} = 1,$$

and $x_{s,0} = 0$ for all $s \in D$.

Proof. Assume $b_i = 0$. Denote by $I_{s,t}$ the set of indices $j \neq i$ such that $(d_i,d_j) = s$, and $b_j \bmod s = t$. Notice that $I_{s,0} = \emptyset$ and $I = \{i\} \cup \bigcup_{s\in D;\, 0\le t<s} I_{s,t}$. Let $y_{s,t} = d(\bigcup_{j\in I_{s,t}} A(d_j,b_j))$. Then $y_{s,0} = 0$ for all s. Since the DCS is saturated, $y_{s,t} = \sum_{j\in I_{s,t}} \frac{1}{d_j}$.

Let $0 < u < d_i$ and

$$Y_{s,u} = \bigcup_{j\in I_{s,u \bmod s}} (A(d_j,b_j) \cap A(d_i,u)).$$

If some element of $A(d_i,u)$ is covered by the sequence $A(d_j,b_j)$, and $(d_i,d_j) = s$, then $s|u-b_j$. Thus $A(d_i,u) = \bigcup_{s\in D} Y_{s,u}$. Notice that $Y_{s,u}$ is the union of sequences of form $A(d_j \cdot \frac{d_i}{s}, u + kd_i)$ for some k, depending on j. Consider the saturated DCS, that consists of the sequences $A(d_j,b_j) \cap A(d_i,u)$, for $j \in I$, $0 \le u < d_i$. As $Y_{s,u}$ is the union of some sequences of this DCS:

$$d(Y_{s,u}) = \frac{s}{d_i} \sum_{j\in I_{s,u \bmod s}} \frac{1}{d_j} = \frac{s}{d_i} y_{s,u \bmod s}.$$

Since $A(d_i,u) = \bigcup_{s\in D} Y_{s,u}$, $d(d_i,u) = \sum_{s\in D} d(Y_{s,u})$, we get $\frac{1}{d_i} = \sum_{s\in D} \frac{s}{d_i} y_{s,u \bmod s}$. We finish the proof by setting $x_{s,t} = \frac{y_{s,t}}{s}$. □

We conjecture, that the system of linear equations in Lemma 5 has no solutions, which would prove Conjecture 2 for saturated DCS's. We have checked this with a computer program for square-free numbers $d_i \le 1254 = 2 \cdot 3 \cdot 11 \cdot 19$.

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