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## Sign changes of error terms related to arithmetical functions

par PAULO J. ALMEIDA

RÉSUMÉ. Soit  $H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$ . Motivé par une conjecture de Erdős, Lau a développé une nouvelle méthode et il a démontré que  $\#\{n \leq T : H(n)H(n+1) < 0\} \gg T$ . Nous considérons des fonctions arithmétiques  $f(n) = \sum_{d|n} \frac{b_d}{d}$  dont l'addition peut être exprimée comme  $\sum_{n \leq x} f(n) = \alpha x + P(\log(x)) + E(x)$ . Ici  $P(x)$  est un polynôme,  $E(x) = -\sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1)$  avec  $\psi(x) = x - [x] - 1/2$ . Nous généralisons la méthode de Lau et démontrons des résultats sur le nombre de changements de signe pour ces termes d'erreur.

ABSTRACT. Let  $H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$ . Motivated by a conjecture of Erdős, Lau developed a new method and proved that  $\#\{n \leq T : H(n)H(n+1) < 0\} \gg T$ . We consider arithmetical functions  $f(n) = \sum_{d|n} \frac{b_d}{d}$  whose summation can be expressed as  $\sum_{n \leq x} f(n) = \alpha x + P(\log(x)) + E(x)$ , where  $P(x)$  is a polynomial,  $E(x) = -\sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1)$  and  $\psi(x) = x - [x] - 1/2$ . We generalize Lau's method and prove results about the number of sign changes for these error terms.

### 1. Introduction

We say that an arithmetical function  $f(x)$  has a *sign change on integers* at  $x = n$ , if  $f(n)f(n+1) < 0$ . The number of sign changes on integers of  $f(x)$  on the interval  $[1, T]$  is defined as

$$N_f(T) = \#\{n \leq T, n \text{ integer} : f(n)f(n+1) < 0\}.$$

We also define  $z_f(T) = \#\{n \leq T, n \text{ integer} : f(n) = 0\}$ . Throughout this work,  $\psi(x) = x - [x] - 1/2$  and  $f(n)$  will be an arithmetical function such

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that

$$f(n) = \sum_{d|n} \frac{b_d}{d} \text{ for some sequence of real numbers } b_n.$$

The motivation for our work was a paper by Y.-K. Lau [5], where he proves that the error term,  $H(x)$ , given by

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x)$$

has a positive proportion of sign changes on integers solving a conjecture stated by P. Erdős in 1967.

An important tool that Lau used to prove his theorem, was that the error term  $H(x)$  can be expressed as

$$(1) \quad H(x) = - \sum_{n \leq \frac{x}{\log^5 x}} \frac{\mu(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^{20} x}\right), \quad (\text{S. Chowla [3]})$$

We generalize Lau's result in the following way

**Theorem 1.1.** *Suppose  $H(x)$  is a function that can be expressed as*

$$(2) \quad H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where each  $b_n$  is a real number and

(i)  $y(x)$  increasing,  $x^{\frac{1}{4}} \ll y(x) \ll \frac{x}{(\log x)^{5+\frac{D}{2}}}$ , for some  $D > 0$ , and

$$(3) \quad \sum_{n \leq x} b_n^4 \ll x \log^D x;$$

(ii)  $k(x)$  is an increasing function, satisfying  $\lim_{x \rightarrow \infty} k(x) = \infty$ .

(iii)  $H(x) = H(\lfloor x \rfloor) - \alpha\{x\} + \theta(x)$ , where  $\alpha \neq 0$  and  $\theta(x) = o(1)$ .

Let  $\prec \in \{<, =, \leq\}$ . If  $\#\{1 \leq n \leq T : \alpha H(n) \prec 0\} \gg T$  then there exists a positive constant  $c_0$  and  $c_0 T$  disjoint subintervals of  $[1, T]$ , with each of them having at least two integers,  $m$  and  $n$ , such that  $\alpha H(m) > 0$  and  $\alpha H(n) \prec 0$ . In particular,

(1)  $\#\{n \leq T : \alpha H(n) > 0\} \gg T$ ;

(2) if  $\#\{n \leq T : \alpha H(n) < 0\} \gg T$ , then  $N_H(T) \gg T$  or  $z_H(T) \gg T$ .

We consider arithmetical functions  $f(n)$  for which, the error term of the summation function satisfies the conditions of Theorem 1.1. A first class is described in the following result

**Theorem 1.2.** *Let  $f(n)$  be an arithmetical function and suppose the sequence  $b_n$  satisfies condition (3) and*

$$(4) \quad \sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right)$$

for some  $B$  real,  $D > 0$  and  $A > 6 + \frac{D}{2}$ , respectively. Let  $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$ ,

$$\gamma_b = \lim_{x \rightarrow \infty} \left( \sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

If  $\alpha \neq 0$ , then Theorem 1.1 is valid for the error term  $H(x)$ . Moreover, if  $f(n)$  is a rational function, then, except when  $\alpha = 0$ , or  $B = 0$  and  $\alpha$  is rational, we have

$$N_H(T) \gg T \quad \text{if and only if} \quad \#\{n \leq T : \alpha H(n) < 0\} \gg T.$$

Notice that this class of arithmetical functions is closed for addition, i.e., if  $f(n)$  and  $g(n)$  are members of the class then also is  $(f + g)(n)$ . In the case considered by Lau, it was known that  $H(x)$  has a positive proportion of negative values (Y.-F. S. Pétermann [6]), so the second part of Theorem 1.2 generalizes Lau's result. Another example is  $f(n) = \frac{n}{\phi(n)}$ .

Using a result of U. Balakrishnan and Y.-F. S. Pétermann [2] we are able to apply Theorem 1.1 to more general arithmetical functions:

**Theorem 1.3.** *Let  $f(n)$  be an arithmetical function and suppose the sequence  $b_n$  satisfies condition (3) and*

$$(5) \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s)g(s)$$

for some  $\beta$  real,  $D > 0$ , and a function  $g(s)$  with a Dirichlet series expansion absolutely convergent for  $\sigma > 1 - \lambda$ , for some  $\lambda > 0$ . Let  $\alpha = \zeta^\beta(2)g(2)$  and

$$H(x) = \begin{cases} \sum_{n \leq x} f(n) - \alpha x, & \text{if } \beta < 0, \\ \sum_{n \leq x} f(n) - \alpha x - \sum_{j=0}^{\lfloor \beta \rfloor} B_j (\log x)^{\beta-j} & \text{if } \beta > 0, \end{cases}$$

where the constants  $B_j$  are well defined. If  $\alpha \neq 0$ , then Theorem 1.1 is valid for the error term  $H(x)$ .

Theorem 1.3 is valid for the following examples, where  $r \neq 0$  is real:

$$\left(\frac{\phi(n)}{n}\right)^r, \quad \left(\frac{\sigma(n)}{n}\right)^r, \quad \left(\frac{\phi(n)}{\sigma(n)}\right)^r.$$

## 2. Main Lemma

The main tool used by Y.-K. Lau was his Main Lemma, where he proved that if  $H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$  then

$$\int_T^{2T} \left( \int_t^{t+h} H(u) du \right)^2 dt \ll Th,$$

for sufficiently large  $T$  and any  $1 \leq h \ll \log^4 T$ . Lau's argument depends essentially on the formula (1). In this section, we obtain a generalization of Lau's Main Lemma.

**Main Lemma.** *Suppose  $H(x)$  is a function that can be expressed as (2) and satisfies conditions (i) and (ii) of Theorem 1.1. Then, for all large  $T$  and  $h \leq \min(\log T, k^2(T))$ , we have*

$$(6) \quad \int_T^{2T} \left( \int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

For any positive integer  $N$ , define

$$(7) \quad H_N(x) = - \sum_{d \leq N} \frac{b_d}{d} \psi\left(\frac{x}{d}\right).$$

The Main Lemma will follow from the next result.

**Lemma 2.1.** *Assume the conditions of the Main Lemma and take  $D > 0$  satisfying condition (i). Let  $E = 4 + \frac{D}{2}$ , then*

(a) *For any  $\delta > 0$ , large  $T$ , any  $Y \ll T$  and  $N \leq y(T)$ , we have*

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \ll \frac{Y}{N^{1-\delta}} + \frac{Y}{k^2(T)} + y(T+Y)(\log T)^E;$$

(b) *For all large  $T$ ,  $N \leq y(T)$  and  $1 \leq h \leq \min(\log T, k^2(T))$ , we have*

$$\int_T^{2T} \left( \int_t^{t+h} H_N(u) du \right)^2 dt \ll Th^{\frac{3}{2}} + N^3(\log N)^E.$$

Now we prove the Main Lemma:

*Proof.* Take  $N = T^{\frac{1}{4}}$  and  $\delta > 0$  small. Cauchy's inequality gives us

$$\begin{aligned} \left( \int_t^{t+h} H(u) du \right)^2 &\leq 2 \left( \int_t^{t+h} H_N(u) du \right)^2 \\ &\quad + 2 \left( \int_t^{t+h} (H(u) - H_N(u)) du \right)^2. \end{aligned}$$

Since  $N = T^{\frac{1}{4}}$  then, for sufficiently large  $T$ ,  $N^3 \log^E N \ll T$ . So, using part (b) of Lemma 2.1 we have

$$\int_T^{2T} \left( \int_t^{t+h} H_N(u) du \right)^2 dt \ll Th^{\frac{3}{2}} + N^3 \log^E N \ll Th^{\frac{3}{2}}.$$

Using Cauchy's inequality and interchanging the integrals,

$$\begin{aligned} & \int_T^{2T} \left( \int_t^{t+h} (H(u) - H_N(u)) du \right)^2 dt \\ & \leq h \int_T^{2T} \left( \int_t^{t+h} (H(u) - H_N(u))^2 du \right) dt \\ & \leq h \int_T^{2T+h} \left( \int_{\max(u-h, T)}^{\min(u, 2T)} (H(u) - H_N(u))^2 dt \right) du \\ & \leq h^2 \int_T^{2T+h} (H(u) - H_N(u))^2 du \\ & \ll h^2 \left( \frac{T+h}{N^{1-\delta}} + \frac{T+h}{k^2(T)} + y(2T+h) (\log T)^E \right) \\ & \ll T + Th \ll Th^{\frac{3}{2}} \end{aligned}$$

since  $y(2T+h) \ll \frac{T}{(\log T)^{E+1}}$  and  $h \leq \min(\log T, k^2(T))$ . Hence

$$\int_T^{2T} \left( \int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

□

### 3. Step I

In this section, we will prove part (a) of Lemma 2.1. Using expression (2) and Cauchy's inequality, we obtain

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \leq 2 \int_T^{T+Y} \left( \sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi \left( \frac{u}{m} \right) \right)^2 du + O \left( \frac{Y}{k^2(T)} \right).$$

Let  $\eta(T, m, n) = \max(T, y^{-1}(m), y^{-1}(n))$ , then

$$2 \int_T^{T+Y} \left( \sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi \left( \frac{u}{m} \right) \right)^2 du = 2 \sum_{m, n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \int_{\eta(T, m, n)}^{T+Y} \psi \left( \frac{u}{m} \right) \psi \left( \frac{u}{n} \right) du.$$

The Fourier series of  $\psi(u) = u - [u] - \frac{1}{2}$ , when  $u$  is not an integer, is given by

$$(8) \quad \psi(u) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k u)}{k},$$

so we obtain

$$\frac{2}{\pi^2} \sum_{m,n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl} \int_{\eta(T,m,n)}^{T+Y} \sin\left(2\pi \frac{ku}{m}\right) \sin\left(2\pi \frac{lu}{n}\right) du.$$

Now, the integral above is equal to

$$\frac{1}{2} \int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} + \frac{l}{n}\right)\right) - \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du.$$

For the first term we get

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} + \frac{l}{n}\right)\right) du \ll \frac{1}{\left(\frac{k}{m} + \frac{l}{n}\right)}.$$

If  $\frac{k}{m} = \frac{l}{n}$ , then

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du \leq Y,$$

otherwise

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du \ll \frac{1}{\left|\frac{k}{m} - \frac{l}{n}\right|}.$$

Part (a) of Lemma 2.1 will now follow from the next three lemmas.

**Lemma 3.1.** *Let  $E = 4 + \frac{D}{2}$  as in Lemma 2.1. Then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^E.$$

**Lemma 3.2.** *If  $D > 0$  satisfies condition (3), then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl (kn + lm)} \ll X (\log X)^{1+\frac{D}{2}}.$$

**Lemma 3.3.** *For any  $\delta > 0$ ,*

$$\sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \ll \frac{1}{N^{1-\delta}}.$$

In order to finish the proof of part (a) of Lemma 2.1 we just need to take  $X = y(T + Y)$  in the previous lemmas. Hence

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \ll \frac{Y}{N^{1-\delta}} + y(T+Y) (\log T)^E + \frac{Y}{k^2(T)}.$$

□

Before we prove the three lemmas above, we need the following technical result

**Lemma 3.4.** *Let  $b_n$  be a sequence satisfying condition (3). Then*

$$\begin{aligned} \sum_{n \leq N} b_n^2 &\ll N \log^{\frac{D}{2}} N, & \sum_{n \leq N} |b_n| &\ll N \log^{\frac{D}{4}} N, & \sum_{n \leq N} \frac{b_n^2}{n} &\ll (\log N)^{1+\frac{D}{2}}, \\ \sum_{n \leq N} \frac{|b_n|}{n} &\ll (\log N)^{1+\frac{D}{4}}, & \sum_{n > N} \frac{b_n^4}{n^2} \tau(n) &\ll \frac{1}{N^{1-\delta}}, & \text{for any } \delta > 0. \end{aligned}$$

*Proof.* Follows from Cauchy's inequality, partial summations and the fact that, for any  $\epsilon > 0$ ,  $\tau(n) = O(n^\epsilon)$ . □

**Remark.** If  $H(x)$  can be expressed in the form (2), then

$$(9) \quad |H(x)| \leq \sum_{n \leq y(x)} \frac{|b_n|}{n} + O\left(\frac{1}{k(x)}\right) \ll (\log x)^{1+\frac{D}{4}}.$$

*Proof of Lemma 3.2 :* Since the arithmetical mean is greater or equal to the geometrical mean, we have

$$\begin{aligned} \sum_{m, n \leq X} |b_m b_n| \sum_{k, l=1}^{\infty} \frac{1}{kl(kn + lm)} &\leq 2 \sum_{m, n \leq X} |b_m b_n| \sum_{k, l=1}^{\infty} \frac{1}{kl\sqrt{knlm}} \\ &\ll \left( \sum_{m \leq X} \frac{|b_m|}{\sqrt{m}} \right)^2 \\ &\ll \left( \sum_{m \leq X} 1 \right) \left( \sum_{M \leq X} \frac{b_M^2}{M} \right) \\ &\ll X (\log X)^{1+\frac{D}{2}}. \end{aligned}$$

□

*Proof of Lemma 3.3 :* For the second sum, take  $d = (m, n)$ ,  $m = d\alpha$  and  $n = d\beta$ . Since  $kn = lm$ , then  $\alpha|k$  and  $\beta|l$ . Taking  $k = \alpha\gamma$ , we also have  $l = \beta\gamma$ . As

$$(10) \quad \sum_{\substack{k, l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} = \frac{1}{\alpha\beta} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^2} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn}.$$



Then,

$$\begin{aligned} \sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k, l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} &= \frac{\pi^2}{6} \sum_{N < m, n \leq X} \frac{|b_m b_n|(m, n)^2}{m^2 n^2} \\ &\leq \frac{\pi^2}{6} \sum_{d \leq X} \left( d \sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2. \end{aligned}$$

The next step is to estimate the inner sum using Hölder inequality

$$\left( \sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \leq \left( \sum_{\substack{N < m \leq X \\ d|m}} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \left( \sum_{\substack{N < M \leq X \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}}.$$

Set  $M = \beta d$ , then

$$(11) \quad \left( \sum_{\substack{N < M \leq X \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}} = \frac{1}{d^3} \left( \sum_{\substack{N < \beta d \leq X \\ d|\beta d}} \frac{1}{\beta^2} \right)^{\frac{3}{2}} \ll \frac{1}{d^3} \left( \min \left\{ 1, \frac{d^3}{N^3} \right\} \right)^{\frac{1}{2}}.$$

To complete the proof of Lemma 3.3 we use Cauchy's inequality and Lemma 3.4. For any  $\delta > 0$ ,

$$\sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k, l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \ll \sum_{d \leq X} \left( \frac{1}{d} \left( \min \left\{ 1, \frac{d^3}{N^3} \right\} \right)^{\frac{1}{2}} \left( \sum_{\substack{N < m \leq X \\ d|m}} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \right),$$

therefore

$$\begin{aligned} \left( \sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k, l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \right)^2 &\ll \sum_{d \leq X} \frac{1}{d^2} \min \left\{ 1, \frac{d^3}{N^3} \right\} \sum_{D \leq X} \left( \sum_{\substack{N < m \leq X \\ D|m}} \frac{b_m^4}{m^2} \right) \\ &\ll \left( \frac{1}{N^3} \sum_{d \leq N} d + \sum_{d > N} \frac{1}{d^2} \right) \sum_{N < m \leq X} \left( \frac{b_m^4}{m^2} \sum_{D|m} 1 \right) \\ &\ll \frac{1}{N} \sum_{N < m \leq X} \frac{b_m^4}{m^2} \tau(m) \ll \frac{1}{N^{2-\delta}}. \end{aligned}$$

□

*Proof of Lemma 3.1 :* This lemma is a generalization of *Hilfssatz 6* in [9] of A. Walfisz. Notice first that

$$\sum_{m, n \leq X} \left( |b_m b_n| \sum_{\substack{k, l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \right) \leq 2 \sum_{m \leq n \leq X} \left( |b_m b_n| \sum_{\substack{k, l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \right).$$

Like in [9] we begin by separating the interior sum into four terms:

$$\begin{aligned} \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} &= \sum_{\substack{k,l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} \left( \frac{1}{kl |kn - lm|} \right) + \sum_{\substack{k,l=1 \\ \frac{kn}{2} < lm < kn}}^{\infty} \left( \frac{1}{kl |kn - lm|} \right) \\ &+ \sum_{\substack{k,l=1 \\ kn < lm < 2kn}}^{\infty} \left( \frac{1}{kl |kn - lm|} \right) + \sum_{\substack{k,l=1 \\ lm \geq 2kn}}^{\infty} \left( \frac{1}{kl |kn - lm|} \right). \end{aligned}$$

For the first term, we use Lemma 3.4,

$$\begin{aligned} \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} \frac{1}{kl |kn - lm|} &\leq 2 \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} \frac{1}{k^2 ln} \\ &= 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{2m}} \frac{1}{l} \\ &\ll \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \left( \frac{\log k}{k^2} + \frac{\log X}{k^2} \right) \\ &\ll \log X \sum_{n \leq X} \frac{|b_n|}{n} \sum_{m \leq n} |b_m| \ll X (\log X)^{1+\frac{D}{2}}. \end{aligned}$$

From the fourth inequality of Lemma 3.4, we get

$$\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \leq \frac{lm}{2}}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^{1+\frac{D}{2}}.$$

The estimation of the third term is more complicated and we have to use a different approach. In this case,  $\frac{1}{l} < \frac{2m}{kn}$ , so that

$$\begin{aligned} \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ \frac{kn}{2} < lm < kn}}^{\infty} \frac{1}{kl |kn - lm|} \\ &< 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{2m} < l < \frac{kn}{m}}} \frac{m}{kn - lm} \end{aligned}$$

$$\begin{aligned}
&< 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{m} - 1} \left( \frac{1}{\frac{kn}{m} - l} \right) \\
&\quad + 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\frac{kn}{m} - 1 < l < \frac{kn}{m}} \frac{m}{kn - lm}.
\end{aligned}$$

Now, taking  $L = \left\lfloor \frac{kn}{m} - l \right\rfloor$ ,

$$\begin{aligned}
\sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{m} - 1} \left( \frac{1}{\frac{kn}{m} - l} \right) &\leq \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{L \leq \frac{kn}{m} - 1} \frac{1}{L} \\
&\ll X (\log X)^{1 + \frac{D}{2}},
\end{aligned}$$

as in the first term. If there exists an integer  $l$  with  $\frac{kn}{m} - 1 < l < \frac{kn}{m}$ , then  $m \nmid kn$ . In this case,  $kn - lm = m \left\{ \frac{kn}{m} \right\}$  and  $m < n$ . So, we have to estimate

$$(12) \quad \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}}.$$

Notice that the fractional part of  $\frac{kn}{m}$  is at least  $\frac{1}{m}$ . So, when  $k \geq m$ ,

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=m}^{\infty} \frac{m}{k^2} \ll \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \ll X (\log X)^{\frac{D}{2}}.$$

We are left with the estimation of

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}}.$$

Since  $m \nmid kn$ , given  $k$  and  $n$ , we can take  $a_{k,n}$ , such that  $1 \leq a_{k,n} < m$  and  $a_{k,n} \equiv kn \pmod{m}$ . Then,

$$\begin{aligned}
\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}} &\leq \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k < m} \frac{m}{k^2 a_{k,n}} \\
&\leq \sum_{a, k \leq X} \frac{1}{ak^2} \sum_{\max(a, k) < m \leq X} m |b_m| \sum_{\substack{m < n \leq X \\ kn \equiv a \\ \pmod{m}}} \frac{|b_n|}{n}.
\end{aligned}$$

We need to estimate the inner sums. In order to do that, we partition the interval  $[1, X]$  in intervals of the form  $[M, 2M)$  and apply Cauchy's

inequality. Take  $1 \leq P \leq Q \leq X$ , then,

$$\sum_{P \leq m < 2P} m |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} \ll \frac{P}{Q} \sum_{P \leq m < 2P} |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} |b_n|.$$

Next, we apply Cauchy's inequality twice, first to the first sum on the right and afterwards to the second sum:

$$\begin{aligned} & \left( \sum_{P \leq m < 2P} \left( |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} |b_n| \right) \right)^2 \\ & \leq \sum_{P \leq M < 2P} b_M^2 \sum_{P \leq m < 2P} \left( \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} |b_n| \right)^2 \\ & \ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left( \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} b_n^2 \sum_{\substack{Q \leq N < 2Q \\ kN \equiv a \pmod{m}}} 1 \right) \\ & \ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left( \left( 1 + \frac{Q}{(k,m)} \right) \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} b_n^2 \right). \end{aligned}$$

Since  $m \leq 2P \leq 2Q$ , we have  $\frac{Q}{m} \geq \frac{1}{2}$ . Using also  $(k, m) \leq k$ , we obtain

$$1 + \frac{Q}{(k,m)} \leq 1 + \frac{Qk}{m} \leq \frac{Q}{m}(k+2) \leq 3 \frac{Qk}{m}.$$

Therefore,

$$\begin{aligned} & \left( \sum_{P \leq m < 2P} \left( |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} |b_n| \right) \right)^2 \ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left( 3 \frac{Qk}{m} \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} b_n^2 \right) \\ & \ll P \frac{3kQ}{P} \log^{\frac{D}{2}} P \sum_{Q \leq n < 2Q} \left( b_n^2 \sum_{\substack{P \leq m < 2P \\ m | kn - a}} 1 \right) \\ & \ll kQ (\log P)^{\frac{D}{2}} \sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a). \end{aligned}$$

By a theorem of S. Ramanujan [7],  $\sum_{n \leq X} \tau^2(n) \sim X \log^3 X$  and by another application of Cauchy inequality and condition (3), we get

$$\begin{aligned} \left( \sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a) \right)^2 &\leq \left( \sum_{Q \leq n < 2Q} b_n^4 \right) \left( \sum_{Q \leq n < 2Q} \tau^2(kn - a) \right) \\ &\ll Q \log^D Q \sum_{kQ-a \leq N < 2kQ-a} \tau^2(N) \\ &\ll kQ^2 \log^{D+3} X. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{P \leq m < 2P} \left( m |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} \right) &\ll \frac{P}{Q} \left( kQ (\log P)^{\frac{D}{2}} \sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a) \right)^{\frac{1}{2}} \\ &\ll \frac{P}{Q} \left( kQ (\log X)^{\frac{D}{2}} (kQ^2 \log^{D+3} X)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\ll P k^{\frac{3}{4}} (\log X)^{1 + \frac{D}{2}}. \end{aligned}$$

The number of pairs of intervals of the form  $([P, 2P], [Q, 2Q])$  to be considered is at most  $\ll \log^2 X$ , hence

$$\begin{aligned} \sum_{a, k \leq X} \frac{1}{ak^2} \sum_{a < m \leq X} m |b_m| \sum_{\substack{m < n \leq X \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} &\ll \sum_{a, k \leq X} \frac{k^{\frac{3}{4}}}{ak^2} \sum_{P, Q} P (\log X)^{1 + \frac{D}{2}} \\ &\ll X (\log X)^{4 + \frac{D}{2}}. \end{aligned}$$

The fourth term is treated as the third, hence

$$\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn < lm < 2kn}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^{4 + \frac{D}{2}}.$$

This completes the proof of Lemma 3.1.  $\square$

#### 4. Step II

In this section we prove part (b) of Lemma 2.1. From equation (8), we get

$$\int_a^b \psi(u) du = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \left( \frac{\cos(2\pi kb)}{k^2} - \frac{\cos(2\pi ka)}{k^2} \right).$$

Using the definition of  $H_N$  stated in (7), we obtain

$$\begin{aligned} \int_t^{t+h} H_N(u) du &= - \sum_{m \leq N} \frac{b_m}{m} \int_t^{t+h} \psi\left(\frac{u}{m}\right) du \\ &= - \frac{1}{2\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\cos\left(2\pi \frac{k(t+h)}{m}\right) - \cos\left(2\pi \frac{kt}{m}\right)}{k^2}. \end{aligned}$$

As usual, let us write  $e(t)$  for  $e^{2\pi it}$ , then

$$\int_t^{t+h} H_N(u) du = \frac{1}{4\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) e\left(\frac{kt}{m}\right) \left(e\left(-k\frac{(2t+h)}{m}\right) - 1\right)}{k^2}.$$

Therefore,

$$\begin{aligned} &16\pi^4 \int_T^{2T} \left| \int_t^{t+h} H_N(u) du \right|^2 dt \\ &= \int_T^{2T} \left| \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) e\left(\frac{kt}{m}\right) \left(e\left(-k\frac{(2t+h)}{m}\right) - 1\right)}{k^2} \right|^2 dt \\ &= \sum_{m, n \leq N} b_m b_n \sum_{k, l=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) \left(e\left(-\frac{lh}{n}\right) - 1\right)}{(kl)^2} \\ &\quad \times \int_T^{2T} e\left(\frac{kt}{m}\right) e\left(-\frac{lt}{n}\right) \left(e\left(-k\frac{(2t+h)}{m}\right) - 1\right) \left(e\left(l\frac{(2t+h)}{n}\right) - 1\right) dt. \end{aligned}$$

After multiplying the terms inside the integral above, we obtain the following four terms that we will estimate below:

$$\begin{aligned} &\int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \\ &\quad - e\left(-\frac{kh}{m}\right) \int_T^{2T} e\left(-\frac{lt}{n} - \frac{kt}{m}\right) dt - e\left(\frac{lh}{n}\right) \int_T^{2T} e\left(\frac{lt}{n} + \frac{kt}{m}\right) dt. \end{aligned}$$

Notice that,  $\left| \int_T^{2T} e^{2\pi i r t} dt \right| \leq \frac{1}{\pi|r|}$ , for any  $r \neq 0$ . We begin with the last term and use  $|e(t) - 1| \leq 2$  and Lemma 3.2. Then

$$\begin{aligned} & \left| \sum_{m,n \leq N} b_m b_n \sum_{k,l=1}^{\infty} \frac{\left( e\left(\frac{kh}{m}\right) - 1 \right) \left( e\left(-\frac{lh}{n}\right) - 1 \right) e\left(\frac{lh}{n}\right)}{(kl)^2} \int_T^{2T} e\left(\frac{lt}{n} + \frac{kt}{m}\right) dt \right| \\ & \leq \frac{4}{\pi} \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{(kl)^2 \left(\frac{l}{n} + \frac{k}{m}\right)} \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \binom{m}{k} \binom{n}{l} \frac{1}{kl(lm + kn)} \ll N^3 (\log N)^{1+\frac{D}{2}}. \end{aligned}$$

The third term is treated similarly to obtain

$$\begin{aligned} & \left| \sum_{m,n \leq N} b_m b_n \sum_{k,l=1}^{\infty} \frac{\left( e\left(\frac{kh}{m}\right) - 1 \right) \left( e\left(-\frac{lh}{n}\right) - 1 \right) e\left(-\frac{kh}{m}\right)}{(kl)^2} \int_T^{2T} e\left(-\frac{lt}{n} - \frac{kt}{m}\right) dt \right| \\ & \ll N^3 (\log N)^{1+\frac{D}{2}}. \end{aligned}$$

Now, if  $kn = lm$ , then

$$\int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt = 2T.$$

If  $kn \neq lm$  then,

$$\int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \ll \frac{1}{\left|\frac{k}{m} - \frac{l}{n}\right|}.$$

Let us study first the case when  $kn \neq lm$ ,

$$\begin{aligned} & \left| \sum_{m,n \leq N} b_m b_n \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{\left( e\left(\frac{kh}{m}\right) - 1 \right) \left( e\left(-\frac{lh}{n}\right) - 1 \right)}{(kl)^2} \int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt \right| \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{(kl)^2 \left|\frac{k}{m} - \frac{l}{n}\right|} \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \binom{m}{k} \binom{n}{l} \frac{1}{kl|kn - lm|} \ll N^3 \log^E N, \end{aligned}$$

by Lemma 3.1. Similarly,

$$\left| \sum_{m,n \leq N} b_m b_n \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) \left(e\left(-\frac{lh}{n}\right) - 1\right) e\left(\frac{lh}{n} - \frac{kh}{m}\right)}{(kl)^2} \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \right| \ll N^3 \log^E N.$$

If  $kn = ml$ , we will use  $|e(t) - 1| \ll \min(1, |t|)$  instead. The expression obtained has some similarities with Lemma 3.3. We are going to use the same argument to prove:

$$(13) \quad \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) \ll h^{\frac{3}{2}}.$$

As in Lemma 3.3, take  $d = (m, n)$ ,  $\alpha = \frac{m}{d}$ ,  $\beta = \frac{n}{d}$  and  $\gamma = \frac{k}{\alpha}$ . So  $l = \beta\gamma$  and then

$$\sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) = \frac{1}{\alpha^2 \beta^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right)\right)^2.$$

If  $d \leq h$ , we obtain  $\sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right)\right)^2 = \frac{\pi^4}{90}$ , and if  $h < d \leq N$ ,

$$\begin{aligned} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right)\right)^2 &= \left(\frac{h}{d}\right)^2 \sum_{\gamma \leq \frac{d}{h}} \frac{1}{\gamma^2} + \sum_{\gamma > \frac{d}{h}} \frac{1}{\gamma^4} \\ &\ll \left(\frac{h}{d}\right)^2 + \left(\frac{h}{d}\right)^3 \ll \left(\frac{h}{d}\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) \\ &= \sum_{m,n \leq N} |b_m b_n| \frac{(m, n)^4}{m^2 n^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{(m, n)}\right)\right)^2 \\ &\leq \sum_{d \leq N} d^4 \sum_{\substack{m,n \leq N \\ d=(m,n)}} \frac{|b_m b_n|}{m^2 n^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left[\min\left(1, \frac{h\gamma}{d}\right)\right]^2 \\ &\ll \sum_{d \leq h} \left(d^2 \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2}\right)^2 + h^2 \sum_{h < d \leq N} \left(d \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2}\right)^2. \end{aligned}$$



Since  $d|m$  we have  $m > h$  and so,  $h^2 \sum_{h < d \leq N} \left( d \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \ll h^{1+\delta}$ . To estimate the first term we begin with Hölder inequality:

$$\sum_{d \leq h} d^4 \left( \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \leq \sum_{d \leq h} d^4 \left( \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \left( \sum_{\substack{M \leq N \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}}.$$

The third sum is  $O\left(\frac{1}{d^3}\right)$  (similar to (11)). Then

$$\begin{aligned} \sum_{d \leq h} d^4 \left( \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \left( \sum_{\substack{M \leq N \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}} &\ll \sum_{d \leq h} d \left( \sum_{\substack{d \leq m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \\ &\ll \left( \sum_{d \leq h} d^2 \right)^{\frac{1}{2}} \left[ \sum_{D \leq h} \left( \sum_{\substack{D \leq m \leq N \\ D|m}} \frac{|b_m|^4}{m^2} \right) \right]^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \left( \sum_{m \leq N} \frac{|b_m|^4}{m^2} \sum_{\substack{D \leq h \\ D|m}} 1 \right)^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \left( \sum_{m \leq N} \frac{|b_m|^4}{m^2} \tau(m) \right)^{\frac{1}{2}}. \end{aligned}$$

The last part of Lemma 3.4 implies  $\sum_{m \leq N} \frac{|b_m|^4}{m^2} \tau(m) = O(1)$ , where the underlying constant doesn't depend on  $N$ . Therefore, we obtain inequality (13) and part (b) of Lemma 2.1, follows.  $\square$

## 5. A general Theorem

In this section, we prove Theorem 1.1, from which the main Theorems 1.2 and 1.3, will be deduced.

*Proof of Theorem 1.1 :* From the Main Lemma, we have, for all large  $T$  and  $h \leq \min(\log T, k^2(T))$ ,

$$\int_T^{2T} \left( \int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

Assume  $\#\{n \leq T : \alpha H(n) < 0\} \gg T$ . Let  $c > 0$  be a constant and  $T$  be sufficiently large, such that  $\#\{n \leq 2T : \alpha H(n) < 0\} > cT$ . Divide the interval  $[1, 2T]$  into subintervals of length  $h$ , where  $h$  is a sufficiently large integer satisfying  $h \leq \log T$ . Then more than  $cT/h$  of those subintervals must have at least one integer  $n$  with  $\alpha H(n) < 0$ . Let  $\mathcal{C}$  be the set of the subintervals which satisfy this property. Write  $\mathcal{C} = \{J_r \mid 1 \leq r \leq R\}$ , where the subintervals are indexed by their positions in the interval  $[1, 2T]$  and

where  $R > cT/h$ . Define  $K_s = J_{3s-2}$ , for  $1 \leq s \leq R/3$ , and let  $\mathcal{D}$  be the set of these subintervals. We have  $\#(\mathcal{D}) > cT/3h$ . Notice that any two members of  $\mathcal{D}$  are separated by a distance of at least  $2h$ .

Let  $M$  be the number of subintervals  $K$  in  $\mathcal{D}$  for which there exists an integer  $n$  in  $K$  such that  $\alpha H(n) < 0$  and  $\alpha H(m) \leq 0$  for every integer  $m \in (n, n + 2h)$ , and let  $\mathcal{S}$  be the set of the corresponding values of  $n$ .

**Lemma 5.1.** *For some absolute constant  $c_1$ , we have  $M \leq c_1 \frac{T}{h^{\frac{3}{2}}}$ .*

*Proof.* Since  $H(x) = H(\lfloor x \rfloor) - \alpha\{x\} + \theta(x)$ , then

$$\alpha H(x) - \alpha H(\lfloor x \rfloor) = -\alpha^2\{x\} + \alpha\theta(x).$$

So, if  $x$  is sufficiently large and not an integer then

$$(14) \quad -\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H(\lfloor x \rfloor) < -\frac{3}{4}\alpha^2\{x\}.$$

Let  $n_1$  be the smallest integer such that any non integer  $x > n_1$  satisfies condition (14). If  $\#\{n \in \mathcal{S} : n \geq n_1\} = 0$  then  $M \leq n_1$ , so, the lemma is clearly true for sufficiently large  $T$ . Otherwise,

$$\#\{n \in \mathcal{S} : n \geq n_1\} \geq M - n_1 \gg M.$$

Take  $n \in \mathcal{S}$  with  $n \geq n_1$  and  $t \in [n, n + h]$ . For any integer  $m \in [t, t + h]$ ,  $\alpha H(m) \leq 0$ . Now, for any  $1 \leq j < h$ ,

$$\int_{\lfloor t \rfloor + j}^{\lfloor t \rfloor + j + 1} H(u) du = \int_{\lfloor t \rfloor + j}^{\lfloor t \rfloor + j + 1} (H(u) - H(\lfloor t \rfloor + j)) du + \int_{\lfloor t \rfloor + j}^{\lfloor t \rfloor + j + 1} H(\lfloor t \rfloor + j) du.$$

Therefore, by (14),

$$\begin{aligned} \int_{\lfloor t \rfloor + j}^{\lfloor t \rfloor + j + 1} \alpha H(u) du &< \int_0^1 \left( -\frac{3}{4}\alpha^2 x \right) dx + \alpha H(\lfloor t \rfloor + j) \\ &< -\frac{3}{8}\alpha^2, \end{aligned}$$

because  $\alpha H(\lfloor t \rfloor + j) \leq 0$ . Since  $\lfloor t \rfloor \geq n$ , we also have

$$\int_t^{\lfloor t \rfloor + 1} \alpha H(u) du < \int_{\{t\}}^1 \left( -\frac{3}{4}\alpha^2 x \right) dx + \alpha H(\lfloor t \rfloor) (1 - \{t\}) < 0$$

and

$$\int_{\lfloor t \rfloor + h}^{t+h} \alpha H(u) du < \int_0^{\{t\}} \left( -\frac{3}{4}\alpha^2 x \right) dx + \alpha H(\lfloor t \rfloor + h) \{t\} \leq 0.$$

Hence,

$$\left| \int_t^{t+h} H(u) du \right| \geq \frac{3}{8} |\alpha| (h - 1).$$

Take an integer  $r = r(T)$  such that  $2^r > (\log T)^{3+\frac{D}{2}}$ . Using (6) and (9), we obtain

$$\begin{aligned} \int_0^{2T} \left( \int_t^{t+h} H(u) \, du \right)^2 dt &= \int_0^{\frac{T}{2^r}} \left( \int_t^{t+h} H(u) \, du \right)^2 dt \\ &\quad + \sum_{j=0}^r \int_{\frac{T}{2^j}}^{\frac{T}{2^{j-1}}} \left( \int_t^{t+h} H(u) \, du \right)^2 dt \\ &\ll \frac{T}{2^r} h^2 (\log T)^{2+\frac{D}{2}} + h^{\frac{3}{2}} \sum_{j=0}^r \frac{T}{2^j} \\ &\ll Th^{\frac{3}{2}}, \end{aligned}$$

due to  $h \leq \log T$ . On the other hand,

$$\begin{aligned} \int_0^{2T} \left( \int_t^{t+h} H(u) \, du \right)^2 dt &\geq \sum_{n \in \mathcal{S}} \int_n^{n+h} \left( \int_t^{t+h} H(u) \, du \right)^2 dt \\ &\geq \sum_{\substack{n \in \mathcal{S} \\ n \geq n_1}} \int_n^{n+h} \left( \frac{3}{8} |\alpha|(h-1) \right)^2 dt \\ &\gg Mh^3. \end{aligned}$$

Hence  $M \leq c_1 \frac{T}{h^{\frac{3}{2}}}$  for some absolute constant  $c_1$ .  $\square$

Take  $c_0 = c/6h$ . If  $h$  is a suitably large integer such that  $c_1 T < c_0 T h^{\frac{3}{2}}$ , then there are at least  $c_0 T$  intervals  $K$  in  $\mathcal{D}$  such that  $\alpha H(n) < 0$  for some integer  $n \in K$  and  $\alpha H(m) > 0$  for some integer  $m$  lying in  $(n, n+2h)$ . Now, suppose  $\#\{n \leq T : \alpha H(n) \leq 0\} \gg T$ . Take  $T$  sufficiently large and take the order relation ' $\prec$ ' to be ' $\leq$ '. Therefore, we have  $c_0 T$  integers  $m$  in the interval  $[1, 2T]$ , for which  $\alpha H(m)$  is positive. In this case,

$$\#\{n \leq T : \alpha H(n) > 0\} \gg T.$$

If we don't have  $\#\{n \leq T : \alpha H(n) \leq 0\} \gg T$ , then

$$\#\{n \leq T : \alpha H(n) > 0\} = T(1 + o(1)).$$

Hence, part 1 of Theorem 1.1 is proved. Next, we prove part 2. Take ' $\prec$ ' to be ' $<$ '. Then, there exists a positive constant  $c_0$  and  $c_0 T$  disjoint subintervals of  $[1, T]$ , with each of them having at least two integers,  $m$  and  $n$ , such that  $H(m) > 0$  and  $H(n) < 0$ . Therefore, in each of those intervals we have at least one  $l$  with either  $H(l) = 0$  or  $H(l)H(l+1) < 0$ . Whence,  $z_H(T) > \frac{c_0}{2} T$  or  $N_H(T) > \frac{c_0}{2} T$ .  $\square$

## 6. A class of arithmetical functions

In this section, we consider arithmetical functions  $f(n)$ , such that the sequence  $b_n$  satisfies conditions (3) and (4). We begin with some elementary results about this class of arithmetical functions. Using condition (4), we immediately obtain the following lemma:

**Lemma 6.1.** *Let  $b_n$  be a sequence of real numbers satisfying (4), for some constants  $B$  and  $A > 1$ , then there exist constants  $\gamma_b$  and  $\alpha$  such that*

$$(15) \quad \sum_{n \leq x} \frac{b_n}{n} = B \log x + \gamma_b + O\left(\frac{1}{\log^{A-1} x}\right),$$

$$(16) \quad \sum_{n=1}^{\infty} \frac{b_n}{n^2} = \alpha,$$

$$(17) \quad \sum_{n > x} \frac{b_n}{n^2} = \frac{B}{x} + O\left(\frac{1}{x \log^{A-1} x}\right).$$

Next, we calculate the sum of  $f(n)$  and describe the error term  $H(x)$ .

**Lemma 6.2.** *Let  $b_n$  be a sequence of real numbers as in Lemma 6.1, then*

$$(18) \quad \sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} \frac{b_d}{d} = \alpha x - \frac{B \log 2\pi x}{2} - \frac{\gamma_b}{2} + H(x),$$

where,

$$(19) \quad H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right),$$

for any  $0 < C < A - 1$ .

*Proof.* We have

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} \frac{b_d}{d} = \sum_{d \leq x} \frac{b_d}{d} \sum_{\substack{n \leq x \\ d|n}} 1 = \sum_{m \leq x} \sum_{d \leq \frac{x}{m}} \frac{b_d}{d}.$$

We separate the double sum above in two parts. Let  $0 < C < A - 1$  and  $y = \log^C x$ . Then

$$(20) \quad \sum_{m \leq x} \sum_{d \leq \frac{x}{m}} \frac{b_d}{d} = \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} + \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \sum_{y < n \leq \frac{x}{d}} 1.$$

In order to evaluate the first term on the right, we start with an application of formula (15):

$$\begin{aligned} \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} &= \sum_{n \leq [y]} \left( B \log x - B \log n + \gamma_b + O\left(\frac{1}{\log^{A-1}\left(\frac{x}{n}\right)}\right) \right) \\ &= B[y] \log x - B \sum_{n \leq [y]} \log n + \gamma_b [y] + O\left(\frac{y}{\log^{A-1}\left(\frac{x}{y}\right)}\right). \end{aligned}$$

Recall that by Stirling formula,  $\sum_{n \leq [y]} \log n = [y] \log [y] - [y] + \frac{\log [y]}{2} + \frac{\log 2\pi}{2} + O\left(\frac{1}{y}\right)$ . Hence,

$$\begin{aligned} \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} &= B[y] (\log x - \log [y] + 1) + \gamma_b [y] - \frac{B(\log 2\pi y)}{2} \\ &\quad + O\left(\frac{y}{\log^{A-1} x}\right) + O\left(\frac{1}{y}\right). \end{aligned}$$

For the second term, we get

$$\begin{aligned} \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \sum_{y < n \leq \frac{x}{d}} 1 &= \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \left( \left\lfloor \frac{x}{d} \right\rfloor - [y] \right) \\ &= x \sum_{d=1}^{\infty} \frac{b_d}{d^2} - x \sum_{d > \frac{x}{y}} \frac{b_d}{d^2} - \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) - \left(\frac{1}{2} + [y]\right) \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \\ &= \alpha x - B y - \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) - \frac{\gamma_b + B \log x}{2} \\ &\quad - B[y] (\log x - \log y) + \frac{B \log y}{2} - \gamma_b [y] + O\left(\frac{y}{\log^{A-1} x}\right). \end{aligned}$$

Notice also that  $B[y] (\log y - \log [y]) = B[y] \left(-\log\left(1 - \frac{\{y\}}{y}\right)\right) = B\{y\} + O\left(\frac{1}{y}\right)$ . Joining everything together, we obtain

$$\begin{aligned} H(x) &= \sum_{n \leq x} f(n) - \left( \alpha x - \frac{B \log 2\pi x}{2} - \frac{\gamma_b}{2} \right) \\ &= - \sum_{d \leq \frac{x}{\log^C x}} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right) + O\left(\frac{1}{\log^C x}\right). \end{aligned}$$

□

*Proof of Theorem 1.2 :* We just have to show that  $H(x)$  satisfies the conditions of Theorem 1.1. From Lemma 6.2, for any  $x$

$$\begin{aligned} H(x) - H(\lfloor x \rfloor) &= -\alpha\{x\} - \frac{B}{2} (\log 2\pi\lfloor x \rfloor - \log 2\pi x) \\ &= -\alpha\{x\} + \frac{B}{2} \frac{\{x\}}{x} + O\left(\frac{1}{x^2}\right). \end{aligned}$$

In Lemma 6.2, we also obtained

$$H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right),$$

for any  $0 < C < A - 1$ . Take  $C = 5 + \frac{D}{2}$ ,  $y(x) = \frac{x}{\log^C x}$  and  $k(x) = \min(\log^C x, \log^{A-C-1} x)$ . Since  $A > 6 + D/2$ , then  $C < A - 1$  and  $A - C - 1 > 0$ . The first part of Theorem 1.2 now follows from Theorem 1.1.

Suppose that  $f(n)$  takes only rational values. In order to prove the second part of Theorem 1.2, we use the following result of A. Baker [1].

**Proposition.** *Let  $\alpha_1, \dots, \alpha_n$  and  $\beta_0, \dots, \beta_n$  denote nonzero algebraic numbers. Then  $\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n \neq 0$ .*

Using the result above, we obtain the next lemma.

**Lemma 6.3.** *Let  $f(n)$  be a rational valued arithmetical function and suppose the sequence  $b_n$  satisfies condition (4) for some real  $B$  and  $A > 1$ . Let  $r$  be a real number and suppose  $H(x)$  is given by (18). Then*

- (1) *If  $B = 0$  and  $\alpha$  is irrational then  $\#\{n \text{ integer} : H(n) = r\} \leq 1$ ;*
- (2) *If  $B$  is a nonzero algebraic number then  $\#\{n \text{ integer} : H(n) = r\} \leq 2$ ;*
- (3) *If  $B$  is transcendental then there exists a constant  $C$  that depends on  $r$  and on the function  $f(n)$ , such that*

$$\#\{n \leq T, n \text{ integer} : H(n) = r\} < (\log T)^C.$$

*Proof.* Suppose that  $B = 0$  and  $\alpha$  is irrational. Suppose also that there are two integers, say  $M \neq N$ , such that  $H(M) = H(N)$ . Then

$$\sum_{n \leq M} f(n) - \alpha M + \frac{\gamma_b}{2} = \sum_{n \leq N} f(n) - \alpha N + \frac{\gamma_b}{2}.$$

But this implies that  $\alpha$  is rational, a contradiction.

Next, suppose  $B \neq 0$  is algebraic number and that there are  $M > N > Q$  integers, satisfying  $H(M) = H(N) = H(Q)$ . We have

$$\sum_{n \leq M} f(n) - \alpha M + \frac{B \log 2\pi M}{2} + \frac{\gamma_b}{2} = \sum_{n \leq N} f(n) - \alpha N + \frac{B \log 2\pi N}{2} + \frac{\gamma_b}{2}$$

which implies

$$\alpha = \frac{B}{M-N} \log \left( \frac{M}{N} \right) + \frac{1}{M-N} \sum_{N < n \leq M} f(n).$$

Consequently

$$B \log \left( \frac{\left( \frac{M}{N} \right)^{\frac{1}{M-N}}}{\left( \frac{M}{Q} \right)^{\frac{1}{M-Q}}} \right) = \frac{1}{M-Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M-N} \sum_{N < n \leq M} f(n).$$

We are going to prove that

$$(21) \quad \left( \frac{M}{N} \right)^{\frac{1}{M-N}} \neq \left( \frac{M}{Q} \right)^{\frac{1}{M-Q}}.$$

Since  $B$  is a nonzero algebraic number and the values of  $f(n)$  are rational, for any integer  $n$ , the proposition implies

$$B \log \left( \frac{\left( \frac{M}{N} \right)^{\frac{1}{M-N}}}{\left( \frac{M}{Q} \right)^{\frac{1}{M-Q}}} \right) \neq \frac{1}{M-Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M-N} \sum_{N < n \leq M} f(n),$$

and so we get a contradiction, which implies  $\#\{n \text{ integer} : H(n) = r\} \leq 2$ , for any real number  $r$ . In fact, instead of proving (21), we are going to prove that

$$(22) \quad M^{N-Q} Q^{M-N} < N^{M-Q},$$

for any positive integers  $M > N > Q$ . Clearly, this implies (21). The inequality (22) is just a particular case of the geometric mean-analytic mean inequality

$$(23) \quad \left( \prod_{i=1}^n u_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^n u_i,$$

where equality happens only if  $u_1 = u_2 = \dots = u_n$ . In fact, if we take  $n = M - Q$ ,  $u_i = M$  for  $1 \leq i \leq N - Q$  and  $u_i = Q$  for  $N - Q < i \leq M - Q$ , we derive

$$\left( M^{N-Q} Q^{M-N} \right)^{\frac{1}{M-Q}} < \frac{1}{M-Q} ((N-Q)M + (M-N)Q) = N.$$

Hence, we obtain (22) and part 2 of the Lemma.

Finally we prove part 3. Suppose  $r$  is a real number such that

$$\#\{n \leq T : H(n) = r\} \geq 4.$$

Let  $Q < N < M$  be the three smallest positive integers in the above set, then

$$0 \neq B \log \left( \frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = \frac{1}{M-Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M-N} \sum_{N < n \leq M} f(n).$$

Suppose  $L$  is such that  $H(L) = r$ . Then  $L > N > Q$ , and as in part 2:

$$0 \neq B \log \left( \frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right) = \frac{1}{L-Q} \sum_{Q < n \leq L} f(n) - \frac{1}{L-N} \sum_{N < n \leq L} f(n).$$

After we cross multiply the two expressions above, we obtain

$$\log \left( \frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = r_1 \log \left( \frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right),$$

for some rational  $r_1$ . Therefore, there are four rational numbers  $r_2, r_3, r_4$  and  $r_5$ , such that

$$L^{r_2} = M^{r_3} N^{r_4} Q^{r_5}.$$

Now, any prime dividing  $L$  must divide  $MNQ$ . Notice that, if  $p$  is a prime,  $k$  is an integer and  $p^k \leq x$  then  $k \leq \frac{\log x}{\log p}$ . Therefore, the number of integers smaller than  $x$ , which have all prime divisors smaller than  $M$  is smaller than  $(\log x)^{\pi(M)}$ . This finishes the proof.  $\square$

Except when  $\alpha = 0$ , or  $B = 0$  and  $\alpha$  is rational, we cannot have  $z_H(T) \gg T$ . Hence, we obtain the second part of Theorem 1.2.  $\square$

**Example.** We finish this section by proving that Theorem 1.2 is valid for the arithmetical function  $\frac{n}{\phi(n)}$ .

Notice that

$$\frac{n}{\phi(n)} = \prod_{p|n} \left(1 + \frac{1}{p-1}\right) = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

Let  $b_n = \frac{\mu^2(n)n}{\phi(n)}$ , then  $f(n) = \sum_{d|n} \frac{b_d}{d}$ . In [8], R. Sitaramachandrarao proved that

$$\sum_{n \leq x} \frac{\mu^2(n)n}{\phi(n)} = x + O\left(x^{\frac{1}{2}}\right),$$



so condition (4) is satisfied for any  $A$  and with  $B = 1$ . By Merten's Theorem  $\prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{p \leq n} \left(1 - \frac{1}{p}\right)^{-1} \sim e^\gamma \log n$ , hence

$$\sum_{n \leq x} b_n^4 = \sum_{n \leq x} \mu^2(n) \frac{n^4}{\phi^4(n)} = \sum_{n \leq x} O(\log^4 n) = O(x \log^4 x),$$

and condition (3) is satisfied for  $D \geq 4$ . In this case,

$$\alpha = \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \quad \gamma_b = \gamma + \sum_p \frac{\log p}{p(p-1)}$$

and

$$H(x) = \sum_{n \leq x} \frac{n}{\phi(n)} - \frac{\zeta(2)\zeta(3)}{\zeta(6)}x + \frac{\log x}{2} + \frac{\log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)}}{2}.$$

Since  $B = 1$  we can apply Theorem 6.3, and so  $z(T) \leq 2$ . Therefore, if  $\#\{n \leq T : \alpha H(n) < 0\} \gg T$ , then  $N_H(T) \gg T$ .

## 7. Second class of arithmetical functions

Given a sequence of real numbers  $b_n$ , and a complex number  $s$ , we define the Dirichlet series  $B(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ . In this section, we consider arithmetical functions  $f(n)$ , such that the sequence  $b_n$  satisfies conditions (3) and (5) for some  $D > 0$ ,  $\beta$  real and a function  $g(s)$  with a Dirichlet series expansion absolutely convergent for  $\sigma > 1 - \lambda$ , for some  $\lambda > 0$ .

U. Balakrishnan and Y.-F. S. Pétermann [2] proved that:

**Proposition.** *Let  $f(n)$  be a complex valued arithmetical function satisfying*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^\beta(s+1)g(s+1),$$

for a complex number  $\beta$ , and  $g(s)$  having a Dirichlet series expansion

$$g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

which is absolutely convergent in the half plane  $\sigma > 1 - \lambda$  for some  $\lambda > 0$ . Let  $\beta_0$  be the real part of  $\beta$ . If

$$\zeta^\beta(s)g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

then there is a real number  $b$ ,  $0 < b < 1/2$ , and constants  $B_j$ , such that, taking  $y(x) = x \exp(-(\log x)^b)$  and  $\alpha = \zeta^\beta(2)g(2)$ ,

$$\sum_{n \leq x} f(n) = \begin{cases} \alpha x - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1) & \text{if } \beta_0 < 0, \\ \alpha x + \sum_{j=0}^{[\beta_0]} B_j (\log x)^{\beta-j} - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1) & \text{if } \beta_0 \geq 0, \end{cases}$$

The real version of the previous proposition allows us to prove Theorem 1.3:

*Proof.* Notice that, for any  $c > 0$ ,  $\log^c \lfloor x \rfloor = \log^c x - c \frac{\{x\}}{x} \log^{c-1} x + O\left(\frac{1}{x}\right)$ . So,  $H(x) = H(\lfloor x \rfloor) - \alpha \{x\} + o(1)$ . From the previous proposition, there is an increasing function  $k(x)$ , with  $\lim_{x \rightarrow \infty} k(x) = \infty$ , such that

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where  $y(x) = x \exp(-(\log x)^b)$ , for some  $0 < b < 1/2$ . Hence, the result follows from Theorem 1.1.  $\square$

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