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## Variants of the Brocard-Ramanujan equation

par OMAR KIHHEL et FLORIAN LUCA

RÉSUMÉ. Dans cet article, nous étudions quelques variations sur l'équation diophantienne de Brocard-Ramanujan.

ABSTRACT. In this paper, we discuss variations on the Brocard-Ramanujan Diophantine equation.

### 1. Introduction

Brocard (see [4, 5]), and independently Ramanujan (see [15, 16]), posed the problem of finding all integral solutions to the diophantine equation

$$(1) \quad n! + 1 = x^2.$$

Although it is unlikely that equation (1) has any solution with  $n > 7$ , the fact that it has only finitely many solutions has only been conditionally proved by Overholt (see [13]). He showed that the weak form of Szpiro's conjecture implies that equation (1) has only finitely many solutions. The weak form of Szpiro's conjecture is a special case of the *ABC* conjecture and asserts that there exists a constant  $s$  such that if  $A$ ,  $B$ , and  $C$  are positive integers satisfying  $A + B = C$  with  $\gcd(A, B) = 1$ , then

$$C \leq N(ABC)^s,$$

where  $N(k)$  is the product of all primes dividing  $k$  taken without repetition. Berend and Osgood [1] showed that if  $P \in \mathbb{Z}[X]$  is a polynomial of degree  $\geq 2$ , then the density of the set of positive integers  $n$  for which there exists an integer  $x$  satisfying the more general diophantine equation

$$(2) \quad n! = P(x)$$

is zero. Erdős and Obláth [7] and Pollack and Shapiro [14] showed that if  $P(x) = x^d \pm 1$  and  $d \geq 3$ , then equation (2) has no solution with  $n > 1$ . Generalizing Overholt's result, the second author [11] showed that the full *ABC* conjecture implies that equation (2) has only finitely many solutions. A wealth of information about this equation can be found in the recent paper [2].

In this paper, we discuss some variations on the above diophantine equations. We look at the following diophantine equations:

$$(3) \quad x^p \pm y^p = \prod_{\substack{k \times n \\ k=1}}^n k,$$

and

$$(4) \quad P(x) = \prod_{\substack{k \times n \\ k=1}}^n k,$$

where  $P \in \mathbb{Q}[x]$  is a polynomial of degree  $\geq 2$ . Here and in what follows, we write  $k \times n$  to mean that  $k$  does not divide  $n$ .

### 2. New results

In what follows, we use the Vinogradov symbols  $\gg$ ,  $\ll$  and  $\asymp$  as well as the Landau symbols  $O$  and  $o$  with their regular meanings. Recall that  $A \ll B$ ,  $B \gg A$  and  $A = O(B)$  are all equivalent and mean that  $|A| \leq c|B|$  holds with some positive constant  $c$ .

**Theorem 1.** *The Diophantine equation*

$$x^p \pm y^p = \prod_{\substack{k \times n \\ k=1}}^n k$$

*admits only finitely many integer solutions  $(x, y, p, n)$  with  $p \geq 3$  a prime number and  $\gcd(x, y) = 1$ .*

*Proof.* There is no loss of generality to consider only the '+' sign and to assume that  $|x| > |y|$ . Since the right hand side is positive, we get that  $x$  is positive. Note that  $\gcd(x, y) = 1$  implies that no prime  $q \leq n$  coprime to  $n$  divides either  $x$  or  $y$ . Now either  $x \leq n$ , or  $x \geq n + 1$ . In the first case,

$$(n - 1)!^{1/2} \leq x^p + |y|^p \leq 2n^p,$$

therefore by Stirling's formula

$$\frac{(n - 1)}{2}(1 + o(1)) \log n \leq p \log n + \log 2.$$

For large  $n$ , the above inequality implies that  $p \geq n/3$ . Note however that  $x^p + y^p = (x + y)(x^p + y^p)/(x + y)$ , and, by Fermat's Little Theorem, it follows easily that  $(x^p + y^p)/(x + y) = \delta m$ , where  $\delta \in \{1, p\}$ , and every prime factor of  $m$  is  $1 \pmod p$ . Since every prime factor of  $m$  is  $\leq n \leq 3p$ , it follows that

$$\frac{x^p + y^p}{x + y} \leq p(p + 1)(2p + 1).$$

However,

$$\frac{x^p + y^p}{x + y} \geq x^{p-2} \geq 2^{p-2}.$$

Indeed, the above inequality holds for positive  $y$  because

$$x^p + y^p > x^p \geq 2x^{p-1} = (2x)x^{p-2} \geq (x + y)x^{p-2},$$

and for negative  $y$  because

$$\frac{x^p + y^p}{x + y} = x^{p-1} + x^{p-2}(-y) + \dots + (-y)^{p-1} > x^{p-1} > x^{p-2}.$$

We thus get the inequality

$$2^{p-2} \leq p(p + 1)(2p + 1),$$

which shows that  $p$  is bounded, and since  $p \geq n/3$ , we get that  $n$  is bounded as well in this case.

We now assume that  $x \geq n + 1$ . If  $y > 0$ , then  $n^n > n! > x^p \geq (n + 1)^p$ , and if  $y < 0$ , then

$$n^n > (x + y) \left( x^{p-1} + x^{p-2}(-y) + \dots + (-y)^{p-1} \right) > x^{p-1} \geq (n + 1)^{p-1}.$$

In both cases, we get  $p \leq n$ . We write again

$$x^p + y^p = (x + y) \left( \frac{x^p + y^p}{x + y} \right),$$

and we use the fact that  $(x^p + y^p)/(x + y) = \delta m$ , where  $\delta \in \{1, p\}$ , and every prime factor of  $m$  is  $1 \pmod p$ . If  $y > 0$ , we get

$$m = \frac{x^p + y^p}{\delta(x + y)} > \frac{x^p}{2xp} = \frac{x^{p-1}}{2p},$$

while if  $y < 0$ , then

$$m = \frac{1}{\delta} \left( \frac{x^p + y^p}{x + y} \right) = \frac{1}{\delta} \left( x^{p-1} + x^{p-2}(-y) + \dots + (-y)^{p-1} \right) > \frac{x^{p-1}}{2p}.$$

Thus, we always have

$$m \geq \frac{x^{p-1}}{2p} > \frac{(2x^p)^{(p-1)/p}}{4p} > \frac{\left( \prod_{k=1}^n k \right)^{(p-1)/p}}{4p},$$

where we used the fact that  $\prod_{k=1}^n k = x^p + y^p < 2x^p$ . Let  $M$  be the largest divisor of  $\prod_{k=1}^n k$  build up only from primes of the form  $q \equiv 1 \pmod p$ .

Then  $m \mid M$ , and so

$$\log M > \frac{p-1}{p} \log \left( \prod_{\substack{k=1 \\ k \times n}}^n k \right) - \log(4p) > \frac{p-1}{2p}(n-1) \log \left( \frac{n-1}{e} \right) - \log(4p).$$

In the above inequality, we used Stirling’s formula as well as the fact that  $\prod_{k=1}^n k > (n - 1)!^{1/2}$ . It is clear that for all  $q \leq n$ , the order of  $q$  in  $n!$  is

$$\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \dots < n \sum_{i \geq 1} \frac{1}{q^i} = \frac{n}{q - 1}.$$

Thus,

$$\log M < n \sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1}.$$

Comparing the above inequalities, we get

$$\frac{p - 1}{2p} \log(n - 1) - \frac{p - 1}{p} - \frac{\log 4p}{n - 1} < \frac{n}{n - 1} \sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1},$$

which together with the fact that  $p \leq n$  leads to

$$\frac{p - 1}{2p} \log(n - 1) \leq \frac{n}{n - 1} \sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1} + O(1).$$

Writing  $q = 1 + pt$  for some  $t \leq n/p$  and using the trivial inequality

$$\sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1} \leq \frac{\log n}{p} \sum_{t \leq n/p} \frac{1}{t} \ll \frac{\log^2 n}{p},$$

we get

$$\frac{1}{3} \log(n - 1) \leq \frac{p - 1}{2p} \log(n - 1) \ll \frac{\log^2 n}{p} + O(1),$$

therefore  $p \ll \log n$ . Using the Montgomery-Vaughan Theorem concerning primes in arithmetic progressions (see [12]) as well as partial summation, we deduce that

$$\sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1} \ll \frac{\log n}{p},$$

and therefore get

$$\frac{1}{3} \log(n - 1) \leq \frac{p - 1}{2p} \log(n - 1) \ll \frac{\log n}{p} + O(1),$$

which leads to  $p \ll 1$ . Since now  $p$  may be assumed fixed, we may apply Dirichlet’s theorem on primes in arithmetical progressions, to get that

$$\sum_{\substack{q \leq n \\ q \equiv 1 \pmod{p}}} \frac{\log q}{q - 1} = \frac{\log(n - 1)}{p - 1} + O(1),$$

and now we are led to

$$\frac{p-1}{2p} \log(n-1) \leq \frac{\log(n-1)}{p-1} + O(1),$$

which tells us that

$$\left(\frac{p-1}{2p} - \frac{1}{p-1}\right) = O\left(\frac{1}{\log(n-1)}\right),$$

which admits only finitely many solutions  $(p, n)$  with  $p \geq 5$  because

$$\frac{p-1}{2p} - \frac{1}{p-1} = \frac{p^2 - 4p + 1}{2p(p-1)} \geq \frac{6}{40}$$

for  $p \geq 5$ . Finally, since  $n^n > x^{p-1}$  and  $n$  and  $p$  are bounded, we get that  $x$  is also bounded. The statement with  $p = 3$  follows from the same arguments by strengthening the inequality

$$\prod_{\substack{k \times n \\ k=1}}^n k > (n-1)!^{1/2},$$

to say

$$\prod_{\substack{k \times n \\ k=1}}^n k > n!^{2/3}$$

for  $n$  sufficiently large. To see that this last inequality holds, note that

$$\prod_{\substack{k \times n \\ k=1}}^n k \geq \frac{n!}{n^{\tau(n)}},$$

where  $\tau(n)$  is the number of divisors of  $n$ . Thus, it suffices to show that the inequality

$$n^{\tau(n)} < n!^{1/3}$$

holds for large  $n$ , and this inequality is implied by

$$\tau(n) \log n < \frac{n}{3} \log(n/e).$$

Since  $\log(n/e) > (\log n)/2$  if  $n$  is large, it follows that it is enough that the inequality

$$\tau(n) < \frac{n}{6}$$

holds for large  $n$ , and this last inequality is obvious. □

Before stating and proving Theorem 2, we restate the *ABC* conjecture mentioned already in the introduction. The *ABC* conjecture asserts that for any  $\varepsilon > 0$  there exists a constant  $C(\varepsilon)$  depending only on  $\varepsilon$ , such that

if  $A, B$  and  $C$  are three nonzero coprime integers satisfying  $A + B = C$ , then

$$\max(|A|, |B|, |C|) < C(\varepsilon)N(ABC)^{1+\varepsilon}.$$

**Theorem 2.** *Let  $P \in \mathbb{Q}[x]$  be a polynomial of degree  $\geq 2$ . Then the ABC conjecture implies that the equation*

$$P(x) = \prod_{\substack{k \times n \\ k=1}}^n k$$

has only finitely many solutions  $(x, n)$ , where  $x$  is a rational number and  $n$  is a positive integer.

*Proof.* We write the equation as

$$a_0x^d + a_1x^{d-1} + \dots + a_d = a_{d+1} \prod_{\substack{k \times n \\ k=1}}^n k,$$

where  $a_0, \dots, a_{d+1}$  are integers with  $a_0a_{d+1} \neq 0$ . Here, one may choose  $a_{d+1}$  to be the least common denominator of all the coefficients of  $P(x)$ . Multiplying both sides of the above equation by  $a_0^{d-1}$  and letting  $y = a_0x$ , it follows that we arrive at the equation

$$y^d + b_1y^{d-1} + \dots + b_d = b_{d+1} \prod_{\substack{k \times n \\ k=1}}^n k,$$

where  $b_i = a_i a_0^{i-1}$  for  $i = 1, \dots, d$ , and  $b_{d+1} = a_{d+1} a_0^{d-1}$ . Note now that  $y$  is an integer because for every fixed positive integer  $n$  the above equation shows that the rational number  $y$  is a root of a monic polynomial with integer coefficients; hence, an algebraic integer, and therefore a rational integer. With the substitution  $z = y + b_1/d$ , we may rewrite the above equation as

$$z^d + c_2z^{d-2} + \dots + c_d = c_{d+1} \prod_{\substack{k \times n \\ k=1}}^n k,$$

where  $c_i$  are rational numbers whose denominator divides  $d^d$ . Finally, we multiply the above equation by  $d^d$  and use the substitution  $t = dz$  to arrive at

$$t^d + e_2t^{d-2} + \dots + e_d = e_{d+1} \prod_{\substack{k \times n \\ k=1}}^n k,$$

where  $t$  and  $e_i$  are integers for  $i = 2, \dots, d + 1$ . Let  $j \leq d$  be the largest index in  $\{2, \dots, d\}$  such that  $e_j \neq 0$ . If this index does not exist, then the

above equation is

$$t^d = e_{d+1} \prod_{\substack{k \times n \\ k=1}}^n k.$$

By the Prime Number Theorem, for large  $n$ , the interval  $(n/2, n)$  contains  $\approx n/(2 \log n)$  prime numbers  $p$  and none of those divides  $n$ . Since  $d > 1$  and  $t$  is an integer, it follows that every such prime number divides  $e_{d+1}$ . In particular,  $n/2 < e_{d+1}$ , which shows that  $n$  is bounded.

Assume now that  $j$  exists and rewrite the equation as

$$(5) \quad t^j + (e_2 t^{j-2} + \dots + e_j) = \frac{e_{d+1}}{t^{d-j}} \prod_{\substack{k \times n \\ k=1}}^n k.$$

Since for large  $t$  we have that  $|t^d + e_2 t^{d-2} + \dots + e_j| \asymp |t|^d$ , it follows that

$$|t|^d \gg |t^d + e_2 t^{d-2} + \dots + e_d| \gg \prod_{\substack{k \times n \\ k=1}}^n k \gg (n-1)!^{1/2},$$

therefore, by taking logarithms and invoking Stirling's formula, we get

$$(6) \quad |t| \geq \exp\left(\frac{1}{2d}(1 + o(1))n \log n + O(1)\right).$$

We now set  $A = t^j$ ,  $B = (e_2 t^{j-2} + \dots + e_j)$ ,  $C = \frac{e_{d+1}}{t^{d-j}} \prod_{\substack{k \times n \\ k=1}}^n k$ , and we apply the  $ABC$  conjecture to equation (5). We note that our  $A$ ,  $B$ ,  $C$  are not necessarily coprime, but their greatest common divisor is  $O(1)$ . Indeed, let  $D_1 = \gcd(t, e_j)$ . Clearly,  $D_1 \leq |e_j|$ , and

$$\gcd(A, B) = \gcd(t^j, e_2 t^{j-2} + \dots + e_j) \mid \left(\gcd(t, e_2 t^{j-2} + \dots + e_j)\right)^j \mid D_1^d.$$

Thus, we may apply the  $ABC$ -conjecture and get that

$$|t|^j \ll \left( |t| |B| \prod_{p \leq n} p \right)^{1+\varepsilon} \ll |t|^{(j-1)(1+\varepsilon)} \cdot 4^{(1+\varepsilon)n}.$$

Choosing  $\varepsilon = 1/j$ , we get that  $(j-1)(1+\varepsilon) = j - 1/j$ , and that the inequality

$$|t|^{1/j} \ll 4^{(1+\varepsilon)n} < 4^{2n}$$

holds. This last inequality leads to

$$(7) \quad |t| \leq \exp(2jn \log 4 + O(1)) \leq \exp(2dn \log 4 + O(1)).$$

Comparing (6) and (7), we get

$$2dn \log 4 - \frac{1}{2d}(1 + o(1))n \log n = O(1),$$



which certainly implies that  $n$  is bounded. □

Dabrowski (see [6]), showed that if  $P(x) = x^2 - A$ , where  $A$  is an integer which is not a perfect square, then equation (2) has only finitely many solutions. We consider the diophantine equation

$$(8) \quad x^2 - A = \prod_{\substack{k \times n \\ k=1}}^n k,$$

and prove the following result.

**Theorem 3.** *If the integer  $A$  is not a perfect square, and  $n$  and  $x$  are positive integers satisfying equation (8), then either*

$$n \leq p \text{ or } n = 2p,$$

where  $p$  is the smallest prime such that  $\left(\frac{A}{p}\right) = -1$ . Here,  $\left(\frac{\bullet}{p}\right)$  stands for the Legendre symbol. In particular, equation (8) has only finitely many positive integer solutions.

Before proving Theorem 3, we need the following Lemma.

**Lemma 4.** *Every prime  $p \leq n$  divides  $\prod_{k=1}^n k$ , except when  $n = p, 2p$ , cases in which  $p$  is the only prime  $\leq n$  which does not divide  $\prod_{k=1}^n k$ .*

*Proof.* Suppose that  $p \leq n$  and that  $n \neq p, 2p$ . If  $p \in (n/2, n)$ , then  $p$  does not divide  $n$ , therefore it divides  $\prod_{k=1}^n k$ . We now assume that  $p < \frac{n}{2}$ .

Hence, there exists a positive integer  $i$  such that

$$\frac{n}{2} \leq 2^i p < n.$$

- (i) If  $\frac{n}{2} < 2^i p < n$ , then  $2^i p$  does not divide  $n$ , and so it divides  $\prod_{k=1}^n k$ .
- (ii) If  $\frac{n}{2} = 2^i p$ , then  $3 \cdot 2^{i-1} p < n$ , and does not divide  $n$ , therefore it divides  $\prod_{k=1}^n k$ .

□

*Proof of Theorem 3.* Since  $A$  is not a perfect square, there exists a prime  $p$  such that  $\left(\frac{A}{p}\right) = -1$ . Then  $p$  does not divide  $\prod_{k=1}^n k$ . Lemma 4 now shows that either  $n \leq p$  or  $n = 2p$ . □

In the general case in which  $A$  is any integer in equation (8), we have the conditional result given in the following theorem.

**Theorem 5.** *If the weak form of Hall's conjecture is true, then equation (8) has only finitely many solutions.*

The weak form of Hall’s conjecture is a special case of the *ABC*-conjecture and asserts that for every  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon)$  depending only on  $\varepsilon > 0$ , such that if  $x, y$ , and  $k$  are nonzero integers satisfying  $x^2 = y^3 + k$ , then

$$\max(|x^2|, |y^3|) \leq C(\varepsilon)|k|^{6+\varepsilon}.$$

*Proof of Theorem 5.* We assume that  $n \geq 3$ . Let  $d$  and  $y$  be the two integers with  $d$  cubefree such that  $\prod_{k=1}^n k = dy^3$ . Then, from Chebyshev’s bound we obtain that

$$d \leq \left( \prod_{\substack{p < n \\ p \text{ prime}}} p \right)^2 < 4^{2(n-1)}.$$

Equation (8) gives

$$dy^3 + A = x^2,$$

so

$$Y^3 + d^2A = X^2,$$

where  $X = dx$  and  $Y = dy$ . Taking  $\varepsilon = 1$  in the weak form of Hall’s conjecture we get that

$$d^2 \prod_{\substack{k \times n \\ k=1}}^n k = d^3y^3 = Y^3 \leq C(1)|d^2A|^7.$$

Since

$$(n-1)!^{1/2} \leq \prod_{\substack{k \times n \\ k=1}}^n k,$$

it follows, from Stirling’s formula, that

$$(4(n-1)^{(n-1)}e^{-(n-1)})^{1/2} \leq (n-1)!^{1/2} \leq \prod_{\substack{k \times n \\ k=1}}^n k.$$

Hence,

$$(4(n-1)^{(n-1)}e^{-(n-1)})^{1/2} \leq \prod_{\substack{k \times n \\ k=1}}^n k \ll |d^2|^6|A|^7 \ll 4^{24(n-1)}|A|^7.$$

Thus,

$$\left( \frac{n-1}{4^{48}} \right)^{\frac{n-1}{2}} \ll |A|^7.$$

This proves that  $n$  is bounded. □

We considered equation (8) with  $A = 1$ , namely

$$(9) \quad \prod_{\substack{k \times n \\ k=1}}^n k + 1 = y^2,$$

and did some computations. Except from the obvious solutions  $n = 4$  and  $n = 5$ , we didn't find any other solution for equation (9) up to  $n = 10^5$ .

Finally, we look at yet another variant of the Brocard-Ramanujan diophantine equation, namely

$$(10) \quad 1 + \prod_{\substack{k \leq n \\ \gcd(k,n)=1}} k = y^2.$$

**Theorem 6.** *Suppose that there exist integers  $n > 4$  and  $y$  satisfying equation (10). Then either  $n$  is equal to  $p^\alpha$  or  $2p^\alpha$  for some prime  $p$  and positive integer  $\alpha$ , or all odd primes dividing  $n$  are  $\pm 1 \pmod{8}$ .*

*Proof.* We know from Gauss generalization to Wilson's Theorem (see [17]) that

$$\prod_{\substack{k \leq n \\ \gcd(k,n)=1}} k = \begin{cases} -1 \pmod{n}, & \text{if } n = 4, p^\alpha, 2p^\alpha, \\ 1 \pmod{n}, & \text{otherwise.} \end{cases}$$

Thus, if  $n \neq 4, p^\alpha, \text{ or } 2p^\alpha$ , then  $\prod_{\substack{k \leq n \\ \gcd(k,n)=1}} k + 1 \equiv 2 \pmod{n}$ . This implies

that  $y^2 \equiv 2 \pmod{n}$ . In particular,  $y^2 \equiv 2 \pmod{q}$  holds for all odd prime factors  $q$  of  $n$ . Hence,  $\left(\frac{2}{q}\right) = 1$ , leading to the conclusion that  $q \equiv \pm 1 \pmod{8}$ .  $\square$

We remark that results of Landau (see pages 668–669 in [10]), together with the Prime Number Theorem, imply that if  $x$  is any positive real number, then the number of positive integers  $n \leq x$  such that  $n = p^\alpha, 2p^\alpha$ , or  $n$  is free of prime factors  $\equiv \pm 3 \pmod{8}$  is  $\ll x/\sqrt{\log x}$ . In particular, the set of  $n$  for which equation (10) can have a positive integer solution  $y$  is of asymptotic density zero, which is an analogue of the result of Berend and Osgood from [1] for the particular polynomial  $P(X) = X^2 - 1$  and our variant of the Brocard-Ramanujan equation.

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