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Computation of 2-groups of positive classes of exceptional number fields

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Résumé. Nous développons un algorithme pour déterminer le 2-groupe $\mathcal{C}\ell_F^{pos}$ des classes positives dans le cas où le corps de nombres considéré F possède des places paires exceptionnelles. Cela donne en particulier le 2-rang du noyau sauvage $WK_2(F)$.

Abstract. We present an algorithm for computing the 2-group $\mathcal{C}\ell_F^{pos}$ of the positive divisor classes in case the number field F has exceptional dyadic places. As an application, we compute the 2-rank of the wild kernel $WK_2(F)$ in $K_2(F)$.

1. Introduction

The logarithmic ℓ-class group $\widetilde{\mathcal{C}\ell}_F$ was introduced in [10] by J.-F. Jaulent who used it to study the ℓ-part $WK_2(F)$ of the wild kernel in number fields: if F contains a primitive $2\ell^t$-th root of unity ($t > 0$), there is a natural isomorphism

$$\mu_{\ell^t} \otimes_{\mathbb{Z}} \widetilde{\mathcal{C}\ell}_F \simeq WK_2(F)/WK_2(F)^{\ell^t},$$

so the ℓ-rank of $WK_2(F)$ coincides with the ℓ-rank of the logarithmic group $\widetilde{\mathcal{C}\ell}_F$. An algorithm for computing $\widetilde{\mathcal{C}\ell}_F$ for Galois extensions F was developed in [4] and later generalized and improved for arbitrary number fields in [3].

In case the prime ℓ is odd, the assumption $\mu_\ell \subset F$ may be easily passed if one considers the cyclotomic extension $F(\mu_\ell)$ and gets back to F via the so-called transfer (see [12], [15] and [17]). However for $\ell = 2$ the connection between symbols and logarithmic classes is more intricate: in the non-exceptional situation (*i.e.* when the cyclotomic $\mathbb{Z}_2$-extension F^c contains the fourth root of unity i) the 2-rank of $WK_2(F)$ still coincides with the 2-rank of $\widetilde{\mathcal{C}\ell}_F$. Even more if the number field F has no exceptional dyadic place (*i.e.* if one has $i \in F^c_{\mathfrak{q}}$ for any $\mathfrak{q}|2$), the same result holds if one replaces the ordinary logarithmic class group $\widetilde{\mathcal{C}\ell}_F$ by a narrow version $\widetilde{\mathcal{C}\ell}_F^{res}$. The algorithmic aspect of this is treated in [11].

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Last in [13] the authors pass the difficulty in the remaining case by introducing a new 2-class group $\mathcal{C}\ell_F^{pos}$, the *2-group of positive divisor classes*, which satisfies the rank identity: $\mathrm{rk}_2\, \mathcal{C}\ell_F^{pos} = \mathrm{rk}_2\, WK_2(F)$.

In this paper we develop an algorithm for computing both $\mathcal{C}\ell_F^{pos}$ and $\widetilde{\mathcal{C}\ell}_F^{pos}$ in case the number field F does contain exceptional dyadic places.

We conclude with several examples. Combining our algorithm with the work of Belabas and Gangl [1] on the computation of the tame kernel of K_2 we obtain the complete structure of the wild kernel in some cases.

2. Positive divisor classes of degree zero

2.1. The group of logarithmic divisor classes of degree zero.

Throughout this paper the prime number ℓ equals 2 and we let i be a primitive fourth root of unity. Let F be a number field of degree $n = r + 2c$. According to [9], for every place $\mathfrak{p}$ of F there exists a 2-adic valuation $\widetilde{v}_{\mathfrak{p}}$ which is related to the wild 2-symbol in case the cyclotomic $\mathbb{Z}_2$-extension of $F_{\mathfrak{p}}$ contains i. The degree $\deg \mathfrak{p}$ of $\mathfrak{p}$ is a 2-adic integer such that the image of the map $\mathrm{Log}\,|\ |_{\mathfrak{p}}$ is the $\mathbb{Z}_2$-module $\deg(\mathfrak{p})\,\mathbb{Z}_2$ (see [10]). (By Log we mean the usual 2-adic logarithm.) The construction of the 2-adic logarithmic valuations $\widetilde{v}_{\mathfrak{p}}$ yields

$$\forall \alpha \in \mathcal{R}_F := \mathbb{Z}_2 \otimes_{\mathbb{Z}} F^{\times} \ : \ \sum_{\mathfrak{p} \in Pl_F^0} \widetilde{v}_{\mathfrak{p}}(\alpha) \deg(\mathfrak{p}) = 0 \ , \tag{1}$$

where Pl_F^0 denotes the set of finite places of the number field F. Setting

$$\widetilde{\mathrm{div}}(\alpha) := \sum_{\mathfrak{p} \in Pl_F^0} \widetilde{v}_{\mathfrak{p}}(\alpha) \mathfrak{p}$$

we obtain by $\mathbb{Z}_2$-linearity:

$$\deg(\widetilde{\mathrm{div}}(\alpha)) = 0 \ . \tag{2}$$

We define the 2-*group of logarithmic divisors of degree 0* as the kernel of the degree map deg in the direct sum $\mathcal{D}\ell_F = \sum_{\mathfrak{p} \in Pl_F^0} \mathbb{Z}_2\, \mathfrak{p}$:

$$\widetilde{\mathcal{D}\ell}_F := \left\{ \textstyle\sum_{\mathfrak{p} \in Pl_F^0} a_{\mathfrak{p}} \mathfrak{p} \in \mathcal{D}\ell_F \mid \sum_{\mathfrak{p} \in Pl_F^0} a_{\mathfrak{p}} \deg(\mathfrak{p}) = 0 \right\} ;$$

and the *subgroup of principal logarithmic divisors* as the image of the logarithmical map $\widetilde{\mathrm{div}}$:

$$\widetilde{\mathcal{P}\ell}_F := \{ \widetilde{\mathrm{div}}(\alpha) \mid \alpha \in \mathcal{R}_F \} \ .$$

Because of (2) $\widetilde{\mathcal{P}\ell}_F$ is clearly a subgroup of $\widetilde{\mathcal{D}\ell}_F$. Moreover by the so-called generalised Gross conjecture, the factorgroup

$$\widetilde{\mathcal{C}\ell}_F := \widetilde{\mathcal{D}\ell}_F / \widetilde{\mathcal{P}\ell}_F$$

is a finite 2-group, the *2-group of logarithmic divisor classes*. So, under this conjecture, $\widetilde{\mathcal{C}\ell}_F$ is just the torsion subgroup of the group

$$\mathcal{C}\ell_F := \mathcal{D}\ell_F / \widetilde{\mathcal{P}\ell}_F$$

of logarithmic classes (without any assumption of degree).

Remark 1. Let F^+ be the set of all totally positive elements of $F^\times$ (*i.e.* the subgroup $F^+ := \{x \in F^\times | \, x_\mathfrak{p} > 0 \quad \text{for all real } \mathfrak{p}\}$). For

$$\widetilde{\mathcal{P}\ell}_F^+ := \{\widetilde{\operatorname{div}}(\alpha) \mid \alpha \in \mathcal{R}_F^+ := \mathbb{Z}_2 \otimes_\mathbb{Z} F^+\}$$

the factor group

$$\mathcal{C}\ell_F^{res} := \mathcal{D}\ell_F / \widetilde{\mathcal{P}\ell}_F^+ \qquad (\text{resp.} \quad \widetilde{\mathcal{C}\ell}_F^{res} := \widetilde{\mathcal{D}\ell}_F / \widetilde{\mathcal{P}\ell}_F^+)$$

is the *2-group of narrow logarithmic divisor classes* of the number field F (resp. the *2-group of narrow logarithmic divisor classes of degree 0*) introduced in [16] and computed in [11].

2.2. Signs and places. For a field F we denote by F^c, (respectively $F^c[i]$) the cyclotomic $\mathbb{Z}_2$-extension (resp. the maximal cyclotomic pro-2-extension) of F.

We adopt the notations and definitions in this section from [13].

Definition 1 (signed places). Let F be a number field. We say that a non-complex place $\mathfrak{p}$ of F is *signed* if and only if $F_\mathfrak{p}$ does not contain the fourth root of unity i. These are the places which do not decompose in the extension $F[i]/F$.

We say that $\mathfrak{p}$ is *logarithmically signed* if and only if the cyclotomic $\mathbb{Z}_2$-extension $F_\mathfrak{p}^c$ does not contain i. These are the places which do not decompose in $F^c[i]/F^c$.

Definition 2 (sets of signed places). By *PS*, respectively *PLS*, we denote the sets of signed, respectively logarithmically signed, places:

$$PS := \{\mathfrak{p} \mid i \notin F_\mathfrak{p}\} \,,$$
$$PLS := \{\mathfrak{p} \mid i \notin F_\mathfrak{p}^c\} \,.$$

A finite place $\mathfrak{p} \in PLS$ is called *exceptional*. The set of exceptional places is denoted by *PE*. Exceptional places are even (i.e. finite places dividing 2).

These sets satisfy the following inclusions:

$$PS \subset PLS = PE \cup PR \subset Pl(2) \cup Pl(\infty)$$

where $Pl(2)$, $Pl(\infty)$, *PR* denote the sets of even, infinite and real places of F, respectively. From this the finiteness of *PLS* is obvious.

We recall the canonical decomposition $\mathbb{Q}_2^\times = 2^\mathbb{Z} \times (1 + 4\mathbb{Z}_2) \times \langle -1 \rangle$ and we denote by ϵ the projection from $\mathbb{Q}_2^\times$ onto $\langle -1 \rangle$.

Definition 3 (**sign function**)**.** For all places $\mathfrak{p}$ we define a sign function via

$$\mathrm{sg}_{\mathfrak{p}} \;:\; F_{\mathfrak{p}}^{\times} \to \langle -1 \rangle \;:\; x \mapsto \begin{cases} 1 & \text{for} \quad \mathfrak{p} \text{ complex} \\ \mathrm{sign}(x) & \text{for} \quad \mathfrak{p} \text{ real} \\ \epsilon(N\mathfrak{p}^{-\nu_{\mathfrak{p}}(x)}) & \text{for} \quad \mathfrak{p} \nmid 2\infty \\ \epsilon(N_{F_{\mathfrak{p}}/\mathbb{Q}_2}(x) N\mathfrak{p}^{-\nu_{\mathfrak{p}}(x)}) & \text{for} \quad \mathfrak{p} \mid 2\,. \end{cases}$$

These sign functions satisfy the product formula:

$$\forall x \in F^{\times} \qquad \prod_{\mathfrak{p} \in Pl_F} \mathrm{sg}_{\mathfrak{p}}(x) = 1.$$

In addition we have:

Proposition 1. *The places $\mathfrak{p}$ of F satisfy the following properties:*

(i) if $\mathfrak{p} \in PLS$ then $(\mathrm{sg}_{\mathfrak{p}}, \widetilde{v}_{\mathfrak{p}})$ is surjective;
(ii) if $\mathfrak{p} \in PS \setminus PLS$ then $\mathrm{sg}_{\mathfrak{p}}(\ ) = (-1)^{\widetilde{v}_{\mathfrak{p}}(\ )}$ and $\widetilde{v}_{\mathfrak{p}}$ is surjective;
(iii) if $\mathfrak{p} \notin PS$ then $\mathrm{sg}_{\mathfrak{p}}(F_{\mathfrak{p}}^{\times}) = 1$ and $\widetilde{v}_{\mathfrak{p}}$ is surjective.

Remark 2. The logarithmic valuation $\widetilde{v}_{\mathfrak{p}}$ is surjective in all three cases. Part 2 of the preceding result is often used for testing $\mathfrak{p} \in PLS$.

2.3. The group of positive divisor classes. For the introduction of that group we modify several notations from [13] in order to make them suitable for actual computations.

Since *PLS* is finite we can fix the order of the logarithmically signed places, say $PLS = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$, with $PE = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}$ and $PR = \{\mathfrak{p}_{e+1}, \ldots, \mathfrak{p}_m\}$. Accordingly we define vectors $\mathbf{e} = (e_1, \ldots, e_m) \in \{\pm 1\}^m$.

For each divisor $\mathfrak{a} = \sum_{\mathfrak{p} \in Pl_F^0} a_{\mathfrak{p}} \mathfrak{p}$, we form pairs $(\mathfrak{a}, \mathbf{e})$ and put

$$\mathrm{sg}(\mathfrak{a}, \mathbf{e}) := \prod_{\mathfrak{p} \in PS \setminus PLS} (-1)^{a_{\mathfrak{p}}} \times \prod_{i=1}^{m} e_i\,. \tag{3}$$

Let $\mathcal{D}\ell_F(PE) := \left\{ \mathfrak{a} \in \mathcal{D}\ell_F \,\middle|\, \mathfrak{a} = \sum_{\mathfrak{p} \in PE} a_{\mathfrak{p}} \mathfrak{p} \right\}$ be the $\mathbb{Z}_2$-submodule of $\mathcal{D}\ell_F$ generated by the exceptional dyadic places. And let $\mathcal{D}\ell_F^{PE}$ be the factor group $\mathcal{D}\ell_F / \mathcal{D}\ell_F(PE)$. Thus the *group of positive divisors* is the $\mathbb{Z}_2$-module:

$$\mathcal{D}\ell_F^{pos} := \left\{ (\mathfrak{a}, \mathbf{e}) \in \mathcal{D}\ell_F^{PE} \times \{\pm 1\}^m \,\middle|\, \mathrm{sg}(\mathfrak{a}, \mathbf{e}) = 1 \right\}\,. \tag{4}$$

For $\alpha \in \mathcal{R}_F := \mathbb{Z}_2 \otimes_{\mathbb{Z}} F^{\times}$, let $\widetilde{\mathrm{div}}'(\alpha)$ denote the image of $\widetilde{\mathrm{div}}(\alpha)$ in $\mathcal{D}\ell_F^{PE}$ and $\mathrm{sg}(\alpha)$ the vector of signs $(\mathrm{sg}_{\mathfrak{p}_1}(\alpha), \ldots, \mathrm{sg}_{\mathfrak{p}_m}(\alpha))$ in $\{\pm 1\}^m$. Then

$$\widetilde{\mathcal{P}\ell}_F^{pos} := \left\{ (\widetilde{\mathrm{div}}'(\alpha), \mathrm{sg}(\alpha)) \in \mathcal{D}\ell_F^{PE} \times \{\pm 1\}^m \,\middle|\, \alpha \in \mathcal{R}_F \right\} \tag{5}$$

is obviously a submodule of $\mathcal{D}\ell_F^{pos}$ which is called the *principal submodule.*

Definition 4 (positive divisor classes). With the notations above:

(i) The group of *positive logarithmic divisor classes* is the factor group

$$\mathcal{C}\ell_F^{pos} = \mathcal{D}\ell_F^{pos} / \widetilde{\mathcal{P}\ell}_F^{pos} .$$

(ii) The subgroup of *positive logarithmic divisor classes of degree zero* is the kernel $\widetilde{\mathcal{C}\ell}_F^{pos}$ of the degree map deg in $\mathcal{C}\ell_F^{pos}$:

$$\widetilde{\mathcal{C}\ell}_F^{pos} := \{(\mathfrak{a}, \mathbf{e}) + \widetilde{\mathcal{P}\ell}_F^{pos} \mid \deg(\mathfrak{a}) \in \deg(\mathcal{D}\ell_F(PE))\} .$$

Remark 3. The group $\mathcal{C}\ell_F^{pos}$ is infinite whenever the number field F has no exceptional places, since in this case $\deg(\mathcal{C}\ell_F^{pos})$ is isomorphic to $\mathbb{Z}_2$. The finiteness of $\mathcal{C}\ell_F^{pos}$ in case $PE \neq \emptyset$ follows from the so-called generalized Gross conjecture.

For the computation of $\widetilde{\mathcal{C}\ell}_F^{pos}$ we need to introduce *primitive divisors*.

Definition 5. A divisor $\mathfrak{b}$ of F is called a *primitive* divisor if $\deg(\mathfrak{b})$ generates the $\mathbb{Z}_2$-module $\deg(\mathcal{D}\ell_F) = 4[F \cap \mathbb{Q}^c : \mathbb{Q}]\mathbb{Z}_2$.

We close this section by presenting a method for exhibiting such a divisor:

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be all dyadic primes and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be a finite set of non-dyadic primes which generates the 2-group of 2-ideal-classes $\mathcal{C}\ell'_F$ (*i.e.* the quotient of the usual 2-class group by the subgroup generated by ideals above 2).

Then every $\mathfrak{p} \in \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s, \mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ with minimal 2-valuation $\nu_2(\deg \mathfrak{p})$ is primitive.

2.4. Galois interpretations and applications to *K*-theory. Let F^{lc} be the locally cyclotomic 2-extension of F (*i.e.* the maximal abelian pro-2-extension of F which is completely split at every place over the cyclotomic $\mathbb{Z}_2$-extension F^c). Then by ℓ-adic class field theory (*cf.* [9]), one has the following interpretations of the logarithmic class groups:

$$\mathrm{Gal}(F^{lc}/F) \simeq \mathcal{C}\ell_F \qquad \text{and} \qquad \mathrm{Gal}(F^{lc}/F^c) \simeq \widetilde{\mathcal{C}\ell}_F .$$

Remark 4. Let us assume $i \notin F^c$. Thus we may list the following special cases:

(i) In case $PLS = \emptyset$, the group $\mathcal{C}\ell_F^{pos} \simeq \mathbb{Z}_2 \oplus \widetilde{\mathcal{C}\ell}_F^{pos}$ of positive divisor classes has index 2 in the group $\mathcal{C}\ell_F \simeq \mathbb{Z}_2 \oplus \widetilde{\mathcal{C}\ell}_F$ of logarithmic classes of arbitrary degree; as a consequence its torsion subgroup $\widetilde{\mathcal{C}\ell}_F^{pos}$ has index 2 in the finite group $\widetilde{\mathcal{C}\ell}_F$ of logarithmic classes of degree 0 which was already computed in [3].

(*ii*) In case $PE = \emptyset$, the group $\mathcal{C}\ell_F^{pos} \simeq \mathbb{Z}_2 \oplus \widetilde{\mathcal{C}\ell}_F^{pos}$ has index 2 in the group $\mathcal{C}\ell_F^{res} \simeq \mathbb{Z}_2 \oplus \widetilde{\mathcal{C}\ell}_F^{res}$ of narrow logarithmic classes of arbitrary degree; and its torsion subgroup $\widetilde{\mathcal{C}\ell}_F^{pos}$ has index 2 in the finite group $\widetilde{\mathcal{C}\ell}_F^{res}$ of narrow logarithmic classes of degree 0 which was introduced in [16] and computed in [11].

Definition 6. We adopt the following conventions from [6, 7, 13, 14]:

(*i*) F is *exceptional* whenever one has $i \notin F^c$ (*i.e.* $[F^c[i] : F^c] = 2$);
(*ii*) F is *logarithmically signed* whenever one has $i \notin F^{lc}$ (*i.e.* $PLS \neq \emptyset$);
(*iii*) F is *primitive* whenever at least one of the exceptional places does not split in (the first step of the cyclotomic $\mathbb{Z}_2$-extension) F^c/F.

The following theorem is a consequence of the results in [6, 7, 9, 10, 13, 14]:

Theorem 1. *Let $WK_2(F)$ (resp. $K_2^{\infty}(F) := \cap_{n\geq 1} K_2^{2^n}(F)$) be the 2-part of the wild kernel (resp. the 2-subgroup of infinite height elements) in $K_2(F)$.*

(*i*) *In case $i \in F^{lc}$ (i.e. in case $PLS = \emptyset$), we have both:*

$$\mathrm{rk}_2\, WK_2(F) = \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}_F = \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}_F^{res}.$$

(*ii*) *In case $i \notin F^{lc}$ but F has no exceptional places (i.e. $PE = \emptyset$), we have:*

$$\mathrm{rk}_2\, WK_2(F) = \mathrm{rk}_2\, \widetilde{\mathcal{C}\ell}_F^{res}.$$

(*iii*) *In case $PE \neq \emptyset$, then we have*

$$\mathrm{rk}_2\, WK_2(F) = \mathrm{rk}_2\, \mathcal{C}\ell_F^{pos}.$$

And in this last situation there are two subcases:
(*a*) *If F is primitive, i.e. if the set PE of exceptional dyadic places contains a primitive place, we have:*

$$K_2^{\infty}(F) = WK_2(F)\ .$$

(*b*) *If F is imprimitive and $K_2^{\infty}(F) = \oplus_{i=1}^{n} \mathbb{Z}/2^{n_i}\mathbb{Z}$, we get:*
(*i*) $WK_2(F) = \mathbb{Z}/2^{n_1+1}\mathbb{Z} \oplus (\oplus_{i=2}^{n} \mathbb{Z}/2^{n_i}\mathbb{Z})$ *if* $\mathrm{rk}_2(\widetilde{\mathcal{C}\ell}_F^{pos}) = \mathrm{rk}_2(\mathcal{C}\ell_F^{pos})$;
(*ii*) $WK_2(F) = \mathbb{Z}/2\mathbb{Z} \oplus (\oplus_{i=1}^{n} \mathbb{Z}/2^{n_i}\mathbb{Z})$ *if* $\mathrm{rk}_2(\widetilde{\mathcal{C}\ell}_F^{pos}) < \mathrm{rk}_2(\mathcal{C}\ell_F^{pos})$.

3. Computation of positive divisor classes

We assume in the following that the set PE of exceptional places is not empty.

3.1. Computation of exceptional units. Classically the group of logarithmic units is the kernel in $\mathcal{R}_F$ of the logarithmic valuations (see [9]):

$$\widetilde{\mathcal{E}}_F = \{x \in \mathcal{R}_F \mid \forall \mathfrak{p} \quad \widetilde{v}_{\mathfrak{p}}(x) = 0\} .$$

In order to compute positive divisor classes in case PE is not empty, we introduce a new group of units:

Definition 7. We define the group of *logarithmic exceptional units* as the kernel of the non-exceptional logarithmic valuations:

$$\widetilde{\mathcal{E}}_F^{exc} = \{x \in \mathcal{R}_F \mid \forall \mathfrak{p} \notin PE \quad \widetilde{v}_{\mathfrak{p}}(x) = 0\} . \tag{6}$$

We only know that the group of logarithmic exceptional units is a subgroup of the 2-group of 2-units $\mathcal{E}'_F = \mathbb{Z}_2 \otimes E'_F$. If we assume that there are exactly s places in F containing 2 we have, say:

$$\mathcal{E}'_F = \mu_F \times \langle \varepsilon_1, \ldots, \varepsilon_{r+c-1+s} \rangle .$$

For the calculation of $\widetilde{\mathcal{E}}_F^{exc}$ we use the same precision η as for our 2-adic approximations used in the course of the calculation of $\widetilde{\mathcal{C}\ell}_F$. We obtain a system of generators of $\widetilde{\mathcal{E}}_F^{exc}$ by computing the nullspace of the matrix

$$B = \left(\begin{array}{c|ccc} & 2^\eta & \ldots & 0 \\ \widetilde{v}_{\mathfrak{p}_i}(\varepsilon_j) & \cdot & \ldots & \cdot \\ & 0 & \ldots & 2^\eta \end{array} \right)$$

with $r + c - 1 + s + e$ columns and e rows, where e is the cardinality of PE and the precision η is determined as explained in [3].

We assume that the nullspace of B is generated by the columns of the matrix

$$B' = \begin{pmatrix} C \\ - \quad - \quad - \\ D \end{pmatrix}$$

where C has $r+c-1+s$ and D exactly e rows. It suffices to consider C. Each column $(n_1,\ldots,n_{r+c-1+s})^{tr}$ of C corresponds to a unit

$$\prod_{i=1}^{r+c-1+s} \varepsilon_i^{n_i} \in \widetilde{\mathcal{E}}_F^{exc}\mathcal{R}_F^{2^\eta}$$

so that we can choose

$$\widetilde{\varepsilon} := \prod_{i=1}^{r+c-1+s} \varepsilon_i^{n_i}$$

as an approximation for an exceptional unit. This procedure yields $k \geq r+c+e$ exceptional units, say $\widetilde{\varepsilon}_1,\ldots,\widetilde{\varepsilon}_k$. By the so-called generalized conjecture of Gross we would have exactly $r+c+e$ such units. So we assume in the following that the procedure does give $k = r+c+e$ (otherwise we would refute the conjecture). Hence, from now on we may assume that we have determined exactly $r+c+e$ generators $\widetilde{\varepsilon}_1,\ldots,\widetilde{\varepsilon}_{r+c+e}$ of $\widetilde{\mathcal{E}}_F^{exc}$, and we write:

$$\widetilde{\mathcal{E}}_F^{exc} = \langle -1\rangle \times \langle \widetilde{\varepsilon}_1,\ldots,\widetilde{\varepsilon}_{r+c-1+e}\rangle \ .$$

Definition 8. The kernel of the canonical map $\mathcal{R}_F \to \mathcal{D}\ell_F^{pos}$ is the subgroup of *positive logarithmic units*:

$$\widetilde{\mathcal{E}}_F^{pos} = \{\widetilde{\varepsilon} \in \widetilde{\mathcal{E}}_F^{exc} \mid \forall \mathfrak{p} \in PLS \quad \mathrm{sg}_{\mathfrak{p}}(\widetilde{\varepsilon}) = +1\} \ .$$

The subgroup $\widetilde{\mathcal{E}}_F^{pos}$ has finite index in the group $\widetilde{\mathcal{E}}_F^{exc}$ of exceptional units.

3.2. The algorithm for computing $\mathcal{C}\ell_F^{pos}$. We assume $PE \neq \emptyset$ and that the logarithmic 2-class group $\widetilde{\mathcal{C}\ell}_F$ is isomorphic to the direct sum

$$\widetilde{\mathcal{C}\ell}_F \;\cong\; \oplus_{i=1}^{\nu}\, \mathbb{Z}/2^{n_i}\mathbb{Z}$$

subject to $1 \leq n_1 \leq \ldots \leq n_\nu$. Let $\mathfrak{a}_i$ $(1 \leq i \leq \nu)$ be fixed representatives of the ν generating divisor classes. Then any divisor $\mathfrak{a}$ of $\mathcal{D}\ell_F$ can be written as

$$\mathfrak{a} = \sum_{i=1}^{\nu} a_i\mathfrak{a}_i + \lambda\mathfrak{b} + \widetilde{\mathrm{div}}(\alpha)$$

with suitable integers $a_i \in \mathbb{Z}_2$, a primitive divisor $\mathfrak{b}$, $\lambda = \frac{\deg(\mathfrak{a})}{\deg(\mathfrak{b})}$ and an appropriate element α of $\mathcal{R}_F$. With each divisor $\mathfrak{a}_i$ we associate a vector

$$\mathbf{e}_i := (\mathrm{sg}(\mathfrak{a}_i,\mathbf{1}),1,\ldots,1) \in \{\pm 1\}^m \ ,$$

where m again denotes the number of divisors in PLS. Clearly, that representation then satisfies $\mathrm{sg}(\mathfrak{a}_i,\mathbf{e}_i) = 1$, hence the element $(\mathfrak{a}_i,\mathbf{e}_i)$ belongs to $\mathcal{D}\ell_F^{pos}$. Setting $\mathbf{e}_{\mathfrak{b}} = (\mathrm{sg}(\mathfrak{b},\mathbf{1}),1,\ldots,1)$ as above and writing

$$\mathbf{e}' := \mathrm{sg}(\alpha) \times \prod_{i=1}^{\nu} \mathbf{e}_i^{a_i} \times \mathbf{e} \times \mathbf{e}_{\mathfrak{b}}^{\lambda}$$

for abbreviation, any element $(\mathfrak{a}, \mathbf{e})$ of $\mathcal{D}\ell_F^{pos}$ can then be written in the form

$$(\mathfrak{a}, \mathbf{e}) = \left(\sum_{i=1}^{\nu} a_i \mathfrak{a}_i + \lambda \mathfrak{b} + \widetilde{\operatorname{div}}(\alpha), \mathbf{e}' \times \prod_{i=1}^{\nu} \mathbf{e}_i^{a_i} \times \operatorname{sg}(\alpha) \times \mathbf{e}_{\mathfrak{b}}^{\lambda} \right)$$
$$= \sum_{i=1}^{\nu} a_i (\mathfrak{a}_i, \mathbf{e}_i) + \lambda(\mathfrak{b}, \mathbf{e}_{\mathfrak{b}}) + (\mathbf{0}, \mathbf{e}') + (\widetilde{\operatorname{div}}(\alpha), \operatorname{sg}(\alpha)) \; .$$

The multiplications are carried out coordinatewise. The vector $\mathbf{e}'$ is therefore contained in the $\mathbb{Z}_2$-module generated by $\mathbf{g}_i \in \mathbb{Z}^m \quad (1 \le i \le m)$ with $\mathbf{g}_1 = (1, \ldots, 1)$, whereas $\mathbf{g}_i$ has first and i-th coordinate -1, all other coordinates 1 for $i > 1$.

As a consequence, the set

$$\{(\mathfrak{a}_j, \mathbf{e}_j) \mid 1 \le j \le \nu\} \cup \{(0, \mathbf{g}_i) \mid 2 \le i \le m\} \cup \{(\mathfrak{b}, \mathbf{e}\}$$

contains a system of generators of $\mathcal{C}\ell_F^{pos}$ (note that $(0, \mathbf{g}_1)$ is trivial in $\mathcal{C}\ell_F^{pos}$).
We still need to expose the relations among those. But the latter are easy to characterize. We must have

$$\sum_{j=1}^{\nu} a_j (\mathfrak{a}_j, \mathbf{e}_j) + \sum_{i=2}^{m} b_i (\mathbf{0}, \mathbf{g}_i) + \lambda(\mathfrak{b}, \mathbf{e}_{\mathfrak{b}}) \equiv 0 \bmod \widetilde{\mathcal{P}\ell}_F^{pos} \; ,$$
$$\sum_{j=1}^{\nu} a_j (\mathfrak{a}_j, \mathbf{e}_j) + \sum_{i=2}^{m} b_i (\mathbf{0}, \mathbf{g}_i) + \lambda(\mathfrak{b}, \mathbf{e}_{\mathfrak{b}}) = (\widetilde{\operatorname{div}}(\alpha), \operatorname{sg}(\alpha)) + \sum_{\mathfrak{p} \in PE} (d_{\mathfrak{p}} \mathfrak{p}, \mathbf{1})$$

with indeterminates $a_j, b_i, d_{\mathfrak{p}}$ from $\mathbb{Z}_2$. Considering the two components separately, we obtain the conditions

$$\sum_{j=1}^{\nu} a_j \mathfrak{a}_j + \lambda \mathfrak{b} \equiv \sum_{\mathfrak{p} \in PE} d_{\mathfrak{p}} \mathfrak{p} \bmod \widetilde{\mathcal{P}\ell}_F \tag{7}$$

and

$$\prod_{j=1}^{\nu} \mathbf{e}_j^{a_j} \times \prod_{i=2}^{m} \mathbf{g}_i^{b_i} \times \mathbf{e}_{\mathfrak{b}}^{\lambda} = \operatorname{sg}(\alpha) \; . \tag{8}$$

Let us recall that we have already ordered *PLS* so that exactly the first e elements $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$ belong to *PE*. Then the first one of the conditions above is tantamount to

$$\sum_{j=1}^{\nu} a_j \mathfrak{a}_j \equiv \sum_{i=1}^{e} d_{\mathfrak{p}_i} \left(\mathfrak{p}_i - \frac{\deg \mathfrak{p}_i}{\deg \mathfrak{b}} \mathfrak{b} \right) \bmod \widetilde{\mathcal{P}\ell}_F \; .$$

The divisors

$$\mathfrak{p}_i - \frac{\deg \mathfrak{p}_i}{\deg \mathfrak{b}} \mathfrak{b}$$

on the right-hand side can again be expressed by the $\mathfrak{a}_j$. For $1 \le i \le e$ we let

$$\widetilde{\operatorname{div}}(\alpha_i) \;+\; \mathfrak{p}_i - \frac{\deg \mathfrak{p}_i}{\deg \mathfrak{b}}\mathfrak{b} = \sum_{j=1}^{\nu} c_{ij}\mathfrak{a}_j \ .$$

The calculation of the α_i, c_{ij} is described in [15].

Consequently, the coefficient vectors $(a_1, \ldots, a_\nu, \lambda)$ can be chosen as $\mathbb{Z}_2$-linear combinations of the rows of the following matrix $A \in \mathbb{Z}_2^{(\nu+e)\times(\nu+1)}$:

$$A = \left(\begin{array}{cccccc|c} 2^{n_1} & 0 & \ldots & 0 & 0 & & 0 \\ 0 & 2^{n_2} & \ldots & 0 & 0 & & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \ldots & \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & 2^{n_{\nu-1}} & 0 & & 0 \\ 0 & 0 & \ldots & 0 & 2^{n_\nu} & & 0 \\ -- & -- & --- & -- & -- & & --- \\ & & & & & & \frac{\deg(\mathfrak{p}_1)}{\deg(\mathfrak{b})} \\ & & c_{ij} & & & & \vdots \\ & & & & & & \frac{\deg(\mathfrak{p}_e)}{\deg(\mathfrak{b})} \end{array}\right) .$$

Each row $(a_1, \ldots, a_\nu, \lambda)$ of A corresponds to a linear combination satisfying

$$\sum_{j=1}^{\nu} a_j\mathfrak{a}_j + \lambda\mathfrak{b} \equiv \widetilde{\operatorname{div}}(\alpha) \ \operatorname{mod}\ \mathcal{D}\ell_F(PE) \ . \tag{9}$$

Condition (8) gives

$$\prod_{i=2}^{m} \mathbf{g}_i^{b_i} = \operatorname{sg}(\alpha) \times \prod_{j=1}^{\nu} \mathbf{e}_j^{a_j} \times \mathbf{e}_{\mathfrak{b}}^{\lambda} \ . \tag{10}$$

Obviously, the family $(\mathbf{g}_i)_{2\le i\le m}$ is free over $\mathbb{F}_2$ implying that the exponents b_i are uniquely defined. Consequently, if the k-th coordinate of the product $\operatorname{sg}(\alpha) \times \prod_{j=1}^{\nu} \mathbf{e}_j^{a_j} \times \mathbf{e}_{\mathfrak{b}}^{\lambda}$ is -1 we must have $b_k = 1$, otherwise $b_k = 0$ for $2 \le k \le m$. (We note that the product over all coordinates is always 1.) Therefore, we denote by $b_{2,j}, \ldots, b_{m,j}$ the exponents of the relation belonging to the j-th column of A for $j = 1, \ldots, \nu + e$.

Unfortunately, the elements α are only given up to exceptional units. Hence, we must additionally consider the signs of the exceptional units of F. For

$$\widetilde{\mathcal{E}}_F^{exc} = \langle -1 \rangle \times \langle \widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_{r+c-1+e} \rangle \tag{11}$$

we put:

$$\operatorname{sg}(\widetilde{\varepsilon}_j) = \prod_{i=1}^{m} \mathbf{g}_i^{b_{i,j+v+e}} \ . \tag{12}$$

Using the notations of (11) and (12) the rows of the following matrix $A' \in \mathbb{Z}_2^{(\nu+2e+r+c-1)\times(\nu+m)}$ generate all relations for the $(\mathfrak{a}_j, \mathbf{e}_j),\ (\mathfrak{b}, \mathbf{e}_\mathfrak{b}),\ (\mathbf{0}, \mathbf{g}_i)$.

$$A' = \left(\begin{array}{c|ccc} & b_{2,1} & \dots & b_{m,1} \\ & \cdot & \dots & \cdot \\ A & \cdot & \dots & \cdot \\ & \cdot & \dots & \cdot \\ & b_{2,\nu+e} & \dots & b_{m,\nu+e} \\ - \ \ - - - \ \ - & - & - - - & - \\ & b_{2,\nu+e+1} & \dots & b_{m,\nu+e+1} \\ & \cdot & \dots & \cdot \\ \mathbf{O} & \cdot & \dots & \cdot \\ & \cdot & \dots & \cdot \\ & b_{2,\nu+2e+r+c-1} & \dots & b_{m,\nu+2e+r+c-1} \end{array} \right) .$$

3.3. The algorithm for computing $\widetilde{\mathcal{C}\ell}_F^{pos}$. We assume that $PE = \{\mathfrak{p}_1, \dots, \mathfrak{p}_e\} \neq \emptyset$ is ordered by increasing 2-valuations $v_2(\deg \mathfrak{p}_i)$; that the group $\mathcal{C}\ell_F^{pos}$ of positive divisor classes is isomorphic to the direct sum

$$\mathcal{C}\ell_F^{pos} \ \cong \ \oplus_{i=1}^{w} \mathbb{Z}/2^{m_i}\mathbb{Z}\,;$$

and that we know a full set of representatives $(\mathfrak{b}_i, \mathbf{f}_i)$ $(1 \leq i \leq w)$ for all classes.

Then each $(\mathfrak{b}, \mathbf{f}) \in \widetilde{\mathcal{D}\ell}_F^{pos}$ satisfies $\deg(\mathfrak{b}) \in \deg(\mathcal{D}\ell_F(PE))$ and

$$\mathfrak{b} \equiv \sum_{i=1}^{w} b_i \mathfrak{b}_i \mod (\mathcal{D}\ell_F(PE) + \widetilde{\mathcal{P}\ell}_F) \ .$$

Obviously, we obtain

$$0 \equiv \deg(\mathfrak{b}) \equiv \sum_{i=1}^{w} b_i \deg(\mathfrak{b}_i) \mod \deg(\mathcal{D}\ell_F(PE)) \ .$$

We reorder the $\mathfrak{b}_i$ if necessary so that

$$v_2(\deg(\mathfrak{b}_1)) \ \leq \ v_2(\deg(\mathfrak{b}_i)) \ \ (2 \leq i \leq w)$$

is fulfilled. We put

$$\begin{aligned} t &:= \max(\min(\{v_2(\deg(\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{D}\ell_F(PE)\}) - v_2(\deg(\mathfrak{b}_1)), 0) \\ &= \max(v_2(\deg(\mathfrak{p}_1)) - v_2(\deg(\mathfrak{b}_1), 0) \end{aligned}$$

and

$$\delta := b_1 + \sum_{i=2}^{w} \frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)} b_i \ .$$

Then we get:

$$\mathfrak{b} \equiv \sum_{i=2}^{w} b_i \left(\mathfrak{b}_i - \frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)} \mathfrak{b}_1 \right) + \delta \mathfrak{b}_1 \mod (\mathcal{D}\ell_F(PE) + \widetilde{\mathcal{P}\ell}_F)$$

and so

$$\deg \mathfrak{b} \equiv 0 \equiv \sum b_i \times 0 + \delta \deg \mathfrak{b}_1 \bmod \deg \mathcal{D}\ell_F(PE).$$

From this it is immediate that a full set of representatives of the elements of $\widetilde{\mathcal{C}\ell}_F^{pos}$ is given by

$$\left(\mathfrak{b}_i - \frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)} \mathfrak{b}_1, \mathbf{f}_i \times \mathbf{f}_1^{-\deg(\mathfrak{b}_i)/\deg(\mathfrak{b}_1)} \right) \quad \text{for} \ \ 2 \le i \le w$$

and

$$(\mathfrak{b}_1' := 2^t \mathfrak{b}_1 - 2^t \frac{\deg \mathfrak{b}_1}{\deg \mathfrak{p}_1} \mathfrak{p}_1, \mathbf{f}_1^{2^t}) \ .$$

Let us denote the class of $(\mathbf{c}, \mathbf{f})$ in $\widetilde{\mathcal{C}\ell}_F^{pos}$ by $[\mathbf{c}, \mathbf{f}]$.

Now we establish a matrix of relations for the generating classes. For this we consider relations:

$$\sum_{i=2}^{w} a_i \left[\mathfrak{b}_i - \frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)} \mathfrak{b}_1, \mathbf{f}_i \times \mathbf{f}_1^{-\frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)}} \right] + a_1 \left[2^t \mathfrak{b}_1', \mathbf{f}_1^{2^t} \right] = 0 \ ,$$

hence

$$\sum_{i=2}^{w} a_i [\mathfrak{b}_i, \mathbf{f}_i] + \left(2^t a_1 - \sum_{i=2}^{w} \frac{\deg(\mathfrak{b}_i)}{\deg(\mathfrak{b}_1)} a_i \right) [\mathfrak{b}_1, \mathbf{f}_1] = 0 \ .$$

A system of generators for all relations can then be computed analogously to the previous section. We calculate a basis of the nullspace of the matrix $A'' = (a''_{ij}) \in \mathbb{Z}^{w \times 2w}$ with first row

$$\left(2^t, -\frac{\deg(\mathfrak{b}_2)}{\deg(\mathfrak{b}_1)}, \ldots, -\frac{\deg(\mathfrak{b}_w)}{\deg(\mathfrak{b}_1)}, 2^{m_1}, 0, \ldots, 0 \right)$$

and in rows $i = 2, \ldots, w$ all entries are zero except for $a''_{ii} = 1$ and $a''_{i,w+i} = 2^{m_i}$. We note that we are only interested in the first w coordinates of the obtained vectors of that nullspace.

4. Examples

The methods described here are implemented in the computer algebra system Magma [2]. Many of the fields used in the examples were results of queries to the QaoS number field database [5, section 6]. More extensive tables of examples can be found at:

`http://www.math.tu-berlin.de/~pauli/K`

In the tables abelian groups are given as a list of the orders of their cyclic factors.

[:] denotes the index $(K_2(O_F) : WK_2(F))$ (see [1, equation (6)]);
d_F denotes the discriminant for a number field F;
$\mathcal{C}\ell_F$ denotes the class group, P the set of dyadic places;
$\mathcal{C}\ell'_F$ denotes the 2-part of $\mathcal{C}\ell/\langle P\rangle$;
$\widetilde{\mathcal{C}\ell}_F$ denotes the logarithmic classgroup;
$\mathcal{C}\ell_F^{pos}$ denotes the group of positive divisor classes;
$\widetilde{\mathcal{C}\ell}_F^{pos}$ denotes the group of positive divisor classes of degree 0;
rk_2 denotes the 2-rank of the wild kernel WK_2.

K. Belabas and H. Gangl have developed an algorithm for the computation of the tame kernel $K_2\mathcal{O}_F$ [1]. The following table contains the structure of $K_2\mathcal{O}_F$ as computed by Belabas and Gangl and the 2-rank of the wild kernel WK_2 calculated with our methods for some imaginary quadratic fields. We also give the structure of the wild kernel if it can be deduced from the structure of $K_2\mathcal{O}_F$ and of the rank of the wild kernel computed here or in [15].

4.1. Imaginary quadratic fields.

d_F	$\mathcal{C}\ell_F$	$K_2\mathcal{O}_F$	[:]	$\|P\|$	$\|PE\|$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\mathcal{C}\ell_F^{pos}$	$\widetilde{\mathcal{C}\ell}_F^{pos}$	rk_2	WK_2
-184	[4]	[2]	1	1	1	[2]	[1]	[2]	[ ]	1	[2]
-248	[8]	[2]	1	1	1	[4]	[2]	[4]	[2]	1	[2]
-399	[2,8]	[2,12]	2	2	2	[2]	[4]	[2]	[2]	1	[4]
-632	[8]	[2]	1	1	1	[4]	[2]	[4]	[2]	1	[2]
-759	[2,12]	[2,18]	6	2	2	[2]	[2]	[2]	[2]	1	[6]
-799	[16]	[2,4]	2	2	2	[2]	[2,4]	[2]	[2]	1	[2]
-959	[36]	[2,4]	2	2	2	[4]	[4,8]	[4]	[4]	1	[4]

4.2. Real quadratic fields.

d_F	$\mathcal{C}\ell_F$	[:]	$\|P\|$	$\|PE\|$	$\mathcal{C}\ell'$	$\widetilde{\mathcal{C}\ell}_F$	$\mathcal{C}\ell_F^{pos}$	$\widetilde{\mathcal{C}\ell}_F^{pos}$	rk_2
776	[2]	4	1	1	[2]	[ ]	[2,2]	[2]	2
904	[8]	4	1	1	[4]	[2]	[4]	[2]	1
29665	[2,16]	8	2	2	[2]	[2]	[2,2]	[2,2]	2
34689	[32]	8	2	2	[ ]	[ ]	[2]	[2]	1
69064	[4,8]	4	1	1	[2,8]	[8]	[2,8]	[8]	2
90321	[2,2,8]	24	2	2	[2,2]	[2,4]	[2,2,2,2]	[2,2,2,2]	4
104584	[4,8]	4	1	1	[2,8]	[2,4]	[2,8]	[2,4]	2
248584	[4,8]	4	1	1	[2,8]	[2,4]	[2,2,8]	[2,2,4]	3
300040	[2,2,8]	4	1	1	[2,8]	[8]	[2,8]	[8]	2
374105	[32]	8	2	2	[ ]	[ ]	[2]	[2]	1
171865	[2,32]	8	2	2	[4]	[4]	[2,4]	[2,4]	2
285160	[2,32]	4	1	1	[32]	[32]	[32]	[32]	1
318097	[64]	8	2	2	[ ]	[ ]	[2]	[2]	1
469221	[64]	12	1	1	[64]	[64]	[2,64]	[2,64]	2
651784	[2,32]	4	1	1	[2,16]	[2,8]	[2,2,16]	[2,2,8]	3

4.3. Examples of degree 3. The studied fields are given by a generating polynomial f and have Galois group of their normal closure isomorphic to C_3 (cyclic) or $\mathfrak{S}_3$ (dihedral); r denotes the number of real places.

f	d_F	r	Gal	C_F	[:]	$\lvert P\rvert$	$\lvert PE\rvert$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\mathcal{C}\ell^{pos}$	$\widetilde{\mathcal{C}\ell}_F^{pos}$	rk_2
$x^3+x^2-10x-8$	961	3	C_3	[]	32	3	3	[]	[]	[]	[]	0
x^3+x^2-6x-1	985	3	$\mathfrak{S}_3$	[]	8	1	1	[]	[]	[2]	[2]	1
$x^3+x^2-14x-23$	2777	3	$\mathfrak{S}_3$	[2]	8	1	1	[2]	[2]	[2, 2]	[2, 2]	2
x^3+x^2-9x+1	2804	3	$\mathfrak{S}_3$	[]	8	1	1	[]	[]	[]	[]	0
x^3-9x+2	2808	3	$\mathfrak{S}_3$	[]	16	2	1	[]	[]	[2]	[2]	1
x^3+x^2-9x-7	2836	3	$\mathfrak{S}_3$	[]	8	1	1	[]	[]	[2]	[2]	1
$x^3-40x+1349$	-997523	1	$\mathfrak{S}_3$	[16]	4	2	2	[ ]	[ ]	[2]	[2]	1
$x^3-25x+198$	-996008	1	$\mathfrak{S}_3$	[2,8]	4	2	2	[4]	[4]	[4]	[4]	1
$x^3+x^2-47x-1365$	-994476	1	$\mathfrak{S}_3$	[16]	6	1	1	[16]	[16]	[16]	[16]	1
$x^3+x^2+126x+234$	-992696	1	$\mathfrak{S}_3$	[2,8]	4	2	2	[2]	[2]	[2,2]	[2,2]	2
$x^3+x^2+39x-155$	-992620	1	$\mathfrak{S}_3$	[2,8]	2	1	1	[2,8]	[2,8]	[2,8]	[2,8]	2
$x^3+x^2+59x-63$	-991852	1	$\mathfrak{S}_3$	[16]	2	1	1	[16]	[16]	[16]	[16]	1
$x^3+x^2-108x+2304$	-991516	12	$\mathfrak{S}_3$	[16]	8	3	3	[ ]	[ ]	[2]	[2]	1
$x^3+x^2-49x-48$	453317	3	$\mathfrak{S}_3$	[16]	16	2	2	[ ]	[ ]	[2]	[2]	1
$x^3-148x+673$	738085	3	$\mathfrak{S}_3$	[16]	16	2	2	[2]	[2]	[2,2,2]	[2,2,2]	3
$x^3-203x+548$	1014140	3	$\mathfrak{S}_3$	[16]	32	2	1	[ ]	[ ]	[2]	[2]	1
$x^3+x^2-164x+64$	1085681	3	$\mathfrak{S}_3$	[16]	32	3	3	[2]	[2]	[2,2]	[2,2]	2
$x^3+x^2-232x-1840$	-526836	1	$\mathfrak{S}_3$	[2,32]	12	2	2	[2]	[2]	[2,2]	[2,2]	2
$x^3+70x+236$	-718948	1	$\mathfrak{S}_3$	[64]	8	2	1	[2]	[2]	[2]	[2]	1
$x^3+x^2+45x+154$	-878683	1	$\mathfrak{S}_3$	[2,32]	8	2	2	[8]	[8]	[8]	[8]	1

4.4. Examples of higher degree.

f	d_F	r	Gal	$\mathcal{C}\ell_F$	[:]	$\|P\|$	$\|PE\|$	$\mathcal{C}\ell'_F$	$\widetilde{\mathcal{C}\ell}_F$	$\mathcal{C}\ell_F^{pos}$	$\widetilde{\mathcal{C}\ell}_F^{pos}$	rk_2
$x^4 - 59x^2 - 120x - 416$	-860400	2	D_4	[16]	8	2	2	[ ]	[ ]	[2]	[2]	1
$x^4 - x^3 - 2x^2 + 5x + 1$	-3967	2	$\mathfrak{S}_4$	[ ]	8	2	2	[ ]	[ ]	[2]	[2]	1
$x^4 - x^3 + 86x^2 - 66x + 1791$	701125	0	D_4	[2,8]	1	1	1	[2,8]	[2,8]	[2,8]	[2,8]	2
$x^4 + 14$	702464	0	D_4	[4,4]	1	1	1	[4]	[2]	[4]	[2]	1
$x^4 + 58x^2 + 1$	705600	0	E_4	[4,8]	2	2	2	[4]	[2]	[4]	[2]	1
$x^4 - 2x^3 + 59x^2 - 24x + 738$	728128	0	D_4	[32]	2	2	2	[2]	[4]	[2]	[2]	1
$x^4 + 21x^2 + 120$	730080	0	D_4	[4,8]	6	2	2	[2]	[4]	[2,2]	[2,2]	2
$x^4 - 5x + 30$	766125	0	$\mathfrak{S}_4$	[2,16]	20	3	3	[ ]	[ ]	[ ]	[ ]	0
$x^4 + 58x^2 + 1$	705600	0	E_4	[4,8]	2	2	2	[4]	[2]	[4]	[2]	1
$x^4 - x^3 + 96x^2 - 96 * x + 1901$	741125	0	C_4	[2,2,4]	1	1	1	[2,2,4]	[2,2,4]	[2,2,4]	[2,2,4]	3
$x^4 - x^3 + 99x^2 - 80x + 2320$	910025	0	D_4	[32]	2	2	2	[4]	[2,4]	[4]	[2,4]	1
$x^5 + x^4 + x^3 - 8x^2 - 12x + 16$	-4424116	3	$\mathfrak{S}_5$	[4]	64	3	2	[ ]	[ ]	[2]	[2]	1
$x^5 + x^4 - 13x^3 - 26x^2 - 8x - 1$	-3504168	3	$\mathfrak{S}_5$	[4]	16	2	2	[ ]	[ ]	[2]	[2]	1
$x^5 - 10x^3 + 9x^2 + 7x - 1$	-3477048	3	$\mathfrak{S}_5$	[4]	16	2	2	[ ]	[ ]	[ ]	[ ]	0
$x^5 + 2x^4 + 6x^3 + 11x^2 - 2x - 9$	-3420711	3	$\mathfrak{S}_5$	[4]	8	1	1	[4]	[4]	[4]	[4]	1
$x^5 - 14x^3 + 26x^2 - 11x - 1$	-3356683	3	$\mathfrak{S}_5$	[4]	16	2	2	[ ]	[ ]	[2]	[2]	1
$x^5 + 2x^4 + 9x^3 + 3x^2 + 10x - 24$	2761273	1	$\mathfrak{S}_5$	[10]	8	3	3	[ ]	[ ]	[ ]	[ ]	0
$x^5 + x^4 - 3x^3 + 15x^2 + 36x - 18$	3825936	1	D_5	[11]	288	3	1	[ ]	[ ]	[ ]	[ ]	0
$x^5 + 2x^4 - 8x^3 - 4x^2 + 7x + 1$	13664837	5	$\mathfrak{S}_5$	[4]	64	2	2	[ ]	[ ]	[2]	[2]	1
$x^5 + 2x^4 - 11x^3 - 27x^2 - 10x + 1$	17371748	5	$\mathfrak{S}_5$	[2]	64	2	2	[ ]	[ ]	[2]	[2]	1

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