

JOURNAL

de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 21, n° 1 (2009), p. 131-143.

<http://jtnb.cedram.org/item?id=JTNB_2009__21_1_131_0>

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The weight distribution of the functional codes defined by forms of degree 2 on Hermitian surfaces

par FRÉDÉRIC A. B. EDOUKOU

RÉSUMÉ. On étudie le code fonctionnel $C_2(X)$ défini sur une variété algébrique projective X , dans le cas où $X \subset \mathbb{P}^3(\mathbb{F}_q)$ est une surface Hermitienne non-dégénérée. Nous donnons d'abord des bornes pour $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q)$ meilleures que celles connues. Ensuite nous calculons le nombre de mots de code atteignant le second poids. Nous donnons aussi une estimation exacte du troisième poids, une description de la structure géométrique des mots correspondant, ainsi que leur nombre. L'article s'achève par une conjecture formulée sur les quatrième et cinquième poids du code $C_2(X)$.

ABSTRACT. We study the functional codes $C_2(X)$ defined on a projective algebraic variety X , in the case where $X \subset \mathbb{P}^3(\mathbb{F}_q)$ is a non-degenerate Hermitian surface. We first give some bounds for $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q)$, which are better than the ones known. We compute the number of codewords reaching the second weight. We also estimate the third weight, show the geometrical structure of the codewords reaching this third weight and compute their number. The paper ends with a conjecture on the fourth weight and the fifth weight of the code $C_2(X)$.

1. Introduction

The code $C_h(X)$ over \mathbb{F}_q ($q = t^2$, with t a prime power) has been studied by A. B. Sørensen [10] in his Ph. D. Thesis in the case where X is a non-singular Hermitian surface and also by G. Lachaud [8]. In fact, the code $C_2(X)$ has previously been studied over \mathbb{F}_4 by P. Spurr [11] in his master thesis; by a complete computer search he found the weight distribution of this code.

In [10], Sørensen was not able to find the minimum distance, even less the weight distribution. For f a form of degree h , he only conjectured the following result:

$$\text{for } h \leq t \quad \#X_{Z(f)}(\mathbb{F}_q) \leq h(t^3 + t^2 - t) + t + 1,$$

which should give the minimum distance. In [4] the author didn't only prove the conjecture for $h \leq 2$, but determined the second weight and three types of quadrics giving codewords of second weight.

The purpose of this paper is to express as possible the weight distribution of the code $C_2(X)$. For a $[n, k]$ linear code C , the weight distribution is the vector of A_w 's with $w = 0, 1, 2, \dots, n$, where A_w is the number of codewords of weight w . Indeed, the study of the weights is important since they measure the efficiency of the code, and their notification is useful in the procedures of decoding.

The paper has been organized as follows. First of all, we give some bounds for the number of points of the section of the Hermitian surfaces with quadrics of rank 2, quadrics cone, and the elliptic quadrics. These bounds are better than the ones obtained in [4]. Thus, codewords of second weight are obtained only from the three types of quadrics found in [4]; we compute their number. We determine the third weight, the geometrical structure of codewords reaching it and we compute their number. Finally, we conclude our work by a conjecture on the fourth and fifth weight of the code $C_2(X)$.

We now give a summary of notation and terminology. We denote by \mathbb{F}_q the field with q elements and by $\mathbb{P}^N(\mathbb{F}_q)$ the projective space of N dimension over \mathbb{F}_q . We use the term forms of degree h to describe homogeneous polynomials f of degree h ; and we denote by $\mathcal{Q} = Z(f)$, the zeros of f in $\mathbb{P}^N(\mathbb{F}_q)$, which is a quadric when $h = 2$. Let \mathcal{Q} be a quadric, the rank of \mathcal{Q} denoted $r(\mathcal{Q})$ is the smallest number of indeterminates appearing in f under any change of coordinate system.

Let \mathcal{F} be the vector space of forms of degree 2, $X \subset \mathbb{P}^3(\mathbb{F}_q)$ the non-degenerate Hermitian surface $X : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} = 0$ and $|X|$ the number of rational points of X over \mathbb{F}_q . We denote by W_i the set of points with homogeneous coordinates $(x_0 : \dots : x_3) \in \mathbb{P}^3(\mathbb{F}_q)$ such that $x_j = 0$ for $j < i$ and $x_i = 1$. The family $\{W_i\}_{0 \leq i \leq 3}$ is a partition of $\mathbb{P}^3(\mathbb{F}_q)$. The code $C_2(X)$ is the image of the linear map $c : \mathcal{F} \longrightarrow \mathbb{F}_q^{|X|}$, defined by $c(\mathcal{Q}) = (c_x(\mathcal{Q}))_{x \in X}$, where $c_x(\mathcal{Q}) = \mathcal{Q}(x_0, \dots, x_3)$ with $x = (x_0, \dots, x_3) \in W_i$.

2. The number of intersection points of the Hermitian surface with some quadrics

The subject of this section is to find some bounds for the number of intersection points of the Hermitian surface and some quadrics which are better than the ones of [4]. In what follows we will note:

$$s(t) = 2t^3 + 2t^2 - t + 1 \quad s_2(t) = 2t^3 + t^2 + 1 \quad s_3(t) = 2t^3 + t^2 - t + 1$$

$$s_4(t) = 2t^3 + 1 \quad s_5(t) = 2t^3 - t + 1$$

2.1. Rank (\mathcal{Q}) = 2 and \mathcal{Q} is a pair of distinct non-tangent planes.

If we write $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$, then $\mathcal{P}_1 \cap X$ and $\mathcal{P}_2 \cap X$ are non-singular Hermitian curves in $\mathbb{P}^2(\mathbb{F}_{t^2})$, such that $|\mathcal{P}_1 \cap X| = |\mathcal{P}_2 \cap X| = t^3 + 1$.

Let $l = \mathcal{P}_1 \cap \mathcal{P}_2$, then l is not contained in X from theorem 9.1 [2] p. 1176. Thus, $l \cap X$ is either a single point or a set of $t + 1$ points. Hence $|\mathcal{Q} \cap X| = s_4(t)$ when $l \cap X$ is a single point or $|\mathcal{Q} \cap X| = s_5(t)$ when l is a secant.

2.2. Rank (\mathcal{Q}) = 3 (\mathcal{Q} is a cone). Here \mathcal{Q} consists of the points on $q+1$

lines passing through a vertex no three of which are coplanar. Thus, the intersection of the cone with the Hermitian surface contains at most two generators. If no line of the cone is contained in X , we get $|\mathcal{Q} \cap X| \leq t^3 + t^2 + t + 1 < s_4(t)$. From proposition 5.3 [4], we conclude that, $|\mathcal{Q} \cap X| = t^3 + t^2 + 1$ or $|\mathcal{Q} \cap X| = t^3 + 2t^2 - t + 1$ according to the cone contains exactly one or two lines of X .

2.3. Rank (\mathcal{Q}) = 4 and \mathcal{Q} is an elliptic quadric. When \mathcal{Q} is an elliptic

quadric, no line is contained in \mathcal{Q} . We get two cases: either $|X \cap \mathcal{Q}| \leq 1$ or $|X \cap \mathcal{Q}| \geq 2$.

In the case $|X \cap \mathcal{Q}| \geq 2$, we will use an analogous technique as the one used in [4] for the elliptic quadric.

Let us choose two distinct points $P_1, P_2 \in X \cap \mathcal{Q}$. Let H be a plane passing through the points P_1 and P_2 . In [6] p. 156, table 7.2, J.W.P. Hirschfeld gave the classification of quadrics in $\mathbb{P}^2(\mathbb{F}_q)$. Thus, we get that $\mathcal{Q} \cap H$ is either a line or a pair of distinct lines or a point (a pair of conjugate lines in $\mathbb{P}^2(\mathbb{F}_{q^2})$ which meet in $\mathbb{P}^2(\mathbb{F}_q)$) or a conic (a non-degenerate quadric). Since the quadric \mathcal{Q} does not contain any line and contains two points, we can say that $\mathcal{Q} \cap H$ is a conic (a non-degenerate quadric) and H is non-tangent to \mathcal{Q} .

Let \mathcal{D} be the line passing through the two points P_1 and P_2 . We consider also all the planes $(H_i)_{i \in I}$ passing through the line \mathcal{D} . Then $\mathcal{Q} \cap H_i$ as a non-degenerate quadric is a non-singular curve. On the other hand $X \cap H_i$ is a Hermitian curve (singular or not). Since H_i is non-tangent to \mathcal{Q} , as in [4] we conclude that:

$$\text{for each } i \quad |X \cap \mathcal{Q} \cap H_i| \leq 2(t + 1) \quad (2.3.1)$$

The two points P_1 and P_2 belong to each $X \cap \mathcal{Q} \cap H_i$, and therefore from (2.3.1) we get:

$$|X \cap \mathcal{Q} \cap H_i - \{P_1, P_2\}| \leq 2(t + 1) - 2 \quad (2.3.2).$$

There are exactly $q + 1$ planes passing through the line \mathcal{D} , and their union generate the whole projective space $\mathbb{P}^3(\mathbb{F}_q)$. Thus, we get:

$$\mathcal{Q} \cap X = (X \cap \mathcal{Q} \cap H_1) \cup \bigcup_{i=2}^{q+1} (X \cap \mathcal{Q} \cap H_i - \{P_1, P_2\}) \quad (2.3.3)$$

And from the relations (2.3.1), (2.3.2) and (2.3.3) we deduce that $|\mathcal{Q} \cap X| \leq 2(t+1) + q(2(t+1) - 2)$. Thus, we get finally $|\mathcal{Q} \cap X| \leq 2t^3 + 2t + 2$. We have $2t^3 + 2t + 2 < s_2(t)$ for $t \geq 3$ and for $t = 2$ we also have $|\mathcal{Q} \cap X| < |\mathcal{Q}| < s_2(2)$. Then, when \mathcal{Q} is an elliptic quadric we conclude that $|\mathcal{Q} \cap X| < s_2(t)$.

3. The second weight of the code $C_2(X)$

Theorem 3.1 ([4]). *The second weight of the code $C_2(X)$ is: $w_2 = t^5 - t^3$.*

Theorem 3.2. *The codewords of second weight correspond to:*

- hyperbolic quadrics containing three skew lines of X ,
- quadrics which are union of two tangent planes meeting at a line contained in X ,
- quadrics which are pair of two planes, one tangent to X and the second not tangent to X and the line of intersection of the two planes meeting X at a single point.

Proof. From paragraph 2.3, we deduce that the codewords of second weight can not correspond to elliptic quadrics. Therefore the three types of quadrics found in [4] are the only ones which give codewords of second weight. \square

Lemma 3.1. *Let \mathcal{P}_1 and \mathcal{P}'_1 two tangent planes to X , \mathcal{P}_2 and \mathcal{P}'_2 two non-tangent planes with $\mathcal{P}_1 \cup \mathcal{P}_2 \neq \mathcal{P}'_1 \cup \mathcal{P}'_2$ and such that $|\mathcal{P}_1 \cap \mathcal{P}_2 \cap X| = |\mathcal{P}'_1 \cap \mathcal{P}'_2 \cap X| = 1$. Then $(\mathcal{P}_1 \cap X) \cup (\mathcal{P}_2 \cap X) \neq (\mathcal{P}'_1 \cap X) \cup (\mathcal{P}'_2 \cap X)$.*

Lemma 3.2. *Let \mathcal{Q}' be a hyperbolic quadric containing three skew lines of X . Let $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$ be a pair of distinct planes, one tangent to X . Then $\mathcal{Q}' \cap X \neq (\mathcal{P}_1 \cap X) \cup (\mathcal{P}_2 \cap X)$.*

Lemma 3.3. *Let $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$ be a pair of distinct tangent planes and $l = \mathcal{P}_1 \cap \mathcal{P}_2 \subset X$. Let $\mathcal{Q}' = \mathcal{P}'_1 \cup \mathcal{P}'_2$ be two planes, \mathcal{P}'_1 tangent to X , \mathcal{P}'_2 non-tangent to X and $l' = \mathcal{P}'_1 \cap \mathcal{P}'_2$ intersecting X at a single point. Then $(\mathcal{P}_1 \cap X) \cup (\mathcal{P}_2 \cap X) \neq (\mathcal{P}'_1 \cap X) \cup (\mathcal{P}'_2 \cap X)$.*

The proofs of these three lemmas are obvious. There are analogous to the proofs of lemma 6.3 and lemma 6.4 [4].

We come now to the computation of the codewords of second weight.

Theorem 3.3. *The number of codewords of second weight is:*

$$A_{w_2} = \frac{1}{2}(t^2 - 1)(t^5 + t^3 + t^2 + 1)(3t^2 - t + 1)t^2.$$

Proof. Since there are three types of codewords of second weight we will find the number of each of them.

(i) From J.W.P. Hirschfeld [7] p. 124, there are exactly $n_q = \frac{1}{2}q^2(q\sqrt{q} + 1)(q + 1)$ hyperbolic quadrics containing three skew lines of the Hermitian surface. Let \mathcal{Q} and \mathcal{Q}' be two distinct hyperbolic quadrics giving codewords of second weight, we have $\mathcal{Q} \cap X \neq \mathcal{Q}' \cap X$. Thus, there are exactly $(q - 1)[\frac{1}{2}q^2(q\sqrt{q} + 1)(q + 1)]$ codewords of second weight obtained from the hyperbolic quadrics.

(ii) We also know from [2], theorem 7.3 p. 1172, that there are exactly $\#X(\mathbb{F}_q) = t^5 + t^3 + t^2 + 1$ tangent planes to the Hermitian surface X . We want now to determine the number of quadrics $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$ where \mathcal{P}_1 and \mathcal{P}_2 are two distinct tangent planes meeting at a line contained in X . Given a tangent plane \mathcal{P}_1 , there are exactly $t^2(t + 1) = t^3 + t^2$ possibilities to choose the plane \mathcal{P}_2 ; so we get $(t^5 + t^3 + t^2 + 1)(t^3 + t^2)$ couples $(\mathcal{P}_1, \mathcal{P}_2)$. Since the two couples $(\mathcal{P}_1, \mathcal{P}_2)$ and $(\mathcal{P}_2, \mathcal{P}_1)$ are the same (they give the same quadric), we get from these tangent planes $\frac{1}{2}(t^5 + t^3 + t^2 + 1)(t^3 + t^2)$ quadrics giving codewords of second weight. And from lemma 6.3 and 6.4 [4], the codewords obtained from these quadrics are all distinct. Thus, we get $(t^2 - 1)[\frac{1}{2}(t^5 + t^3 + t^2 + 1)(t^3 + t^2)]$ codewords of second weight.

(iii) Let $Q = \mathcal{P}_1 \cup \mathcal{P}_2$ be a quadric where \mathcal{P}_1 and \mathcal{P}_2 are respectively a tangent plane to X , a non-tangent plane to X , where $l = \mathcal{P}_1 \cap \mathcal{P}_2$ with $l \cap X = \{P_1\}$ ($l \not\subset X$). Here, we need to find first of all the number of planes $(H_i)_{i \in I}$ for which we have $H_i \cap \mathcal{P}_1 = l$ where l is a line intersecting X in a single point P_1 . We know that there are $(t^2 - t)$ lines of such type (lines l).

Given a line l , there pass exactly $q + 1$ planes through l (\mathcal{P}_1 is one of these planes). Thus, we get at maximum $q(t^2 - t)$ planes H_i , i.e. $t^4 - t^3$ planes H_i . In fact, there are exactly $t^4 - t^3$ planes H_i , since if H_1 and H_2 are respectively planes through l_1 and l_2 two distinct lines, then $H_1 \neq H_2$.

We also know that the number of tangent planes to X is $\#X(\mathbb{F}_q) = t^5 + t^3 + t^2 + 1$, so we deduce that there are exactly $(t^5 + t^3 + t^2 + 1)(t^4 - t^3)$ quadrics $Q = \mathcal{P}_1 \cup \mathcal{P}_2$ (where \mathcal{P}_1 is a tangent plane to X , \mathcal{P}_2 is a non-tangent plane to X , and $l = \mathcal{P}_1 \cap \mathcal{P}_2$ intersecting X at a single point). And from lemma 3.3, the codewords obtained from these quadrics are all distinct. Thus, we get exactly $(t^2 - 1)(t^5 + t^3 + t^2 + 1)(t^4 - t^3)$ codewords of second weight from these quadrics.

From lemmas 3.4 and 3.5, we deduce that the codewords obtained from (i), (ii) and (iii) are all distinct.

Therefore we have exactly: $(t^2 - 1)[\frac{1}{2}t^4(t^3 + 1)(t^2 + 1)] + (t^2 - 1)[\frac{1}{2}(t^5 + t^3 + t^2 + 1)(t^3 + t^2)] + (t^2 - 1)(t^5 + t^3 + t^2 + 1)(t^4 - t^3) = \frac{1}{2}(t^2 - 1)(t^5 + t^3 + t^2 + 1)(3t^2 - t + 1)t^2$ codewords of second weight. □

Observe that, in the case $t = 2$, there are exactly $\frac{1}{2} \times 3 \times 45 \times 11 \times 4 = 2970$ codewords of second weight. Thus, we recover the result of Paul P. Spurr [3] p. 120 calculated by computer.

4. The weight distribution of the code $C_2(X)$

In what follows $s(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$ and $s_5(t)$ have the same values as in section 2. It has been shown in [4] with the help of the relations (3.2.2) and (3.4.3) that $s(t)$ and $s_2(t)$ give respectively the first weight (minimum distance) and the second weight.

For $t > 3$, $s_3(t) > 2t^3 + 2t + 2$; and for $t = 2$, the elliptic quadric \mathcal{Q} over \mathbb{F}_4 has $|\mathcal{Q}| = 17$ points, and we have $17 < s_3(2) = 19$. Therefore, from table 4.2, the third weight is given by $s_3(t)$ for $t \neq 3$.

Theorem 4.1. *Let l be a line in $\mathbb{P}^3(\mathbb{F}_q)$ and X the Hermitian surface then l meets X at a single point or $t + 1$ points or is contained in X . This line is respectively called a tangent, a secant, or a generator.*

Proof. See [2], p. 1179. □

Definition. A regulus (notion used by Hirschfeld [7] p. 4) is the set of transversals of three skew lines. It consists of $q + 1$ skew lines.

Theorem 4.2 ([7] p. 23). *A hyperbolic quadric \mathcal{Q} consists of $(q+1)^2$ points, which are all on a pair of complementary reguli. The two reguli are the two systems of generators of \mathcal{Q} . A hyperbolic quadric whose complementary reguli are \mathcal{R} and \mathcal{R}' is denoted by $\mathcal{H}(\mathcal{R}, \mathcal{R}')$.*

Remark. When \mathcal{Q} is a hyperbolic quadric containing exactly two skew lines on the Hermitian surface X (i.e. type 12 in table 4.2), we get:

- for $t \geq 3$, $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q) \leq t^3 + 3t^2 - t + 1 < s_4(t)$,
- for $t = 2$, we get $\#X_{Z(\mathcal{Q})}(\mathbb{F}_4) \leq 2^3 + 3 \times 2^2 - 2 + 1 = 19 = s_3(2)$. Let us suppose in the case ($t = 2$) that, $\#X_{Z(\mathcal{Q})}(\mathbb{F}_4) < 19$; therefore it exists a line l on the considered regulus of \mathcal{Q} such that $l \cap X$ is a single point. Thus we get $\#X_{Z(\mathcal{Q})}(\mathbb{F}_4) \leq 2(t^2 + 1) + (t^2 - 2)(t + 1) + 1 = 17 = s_4(2)$. Therefore we can assert that

$$\#X_{Z(\mathcal{Q})}(\mathbb{F}_4) = s_3(2) \text{ or } \#X_{Z(\mathcal{Q})}(\mathbb{F}_q) \leq s_4(t) \quad (4.4.1)$$

Theorem 4.3. *Let \mathcal{Q} be a hyperbolic quadric containing exactly two skew lines on the Hermitian surface X . Then we have $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q) \leq s_4(t)$.*

Proof. See appendix. □

4.1. Table of the intersection of quadrics with the Hermitian surface. Let us show in a table the geometrical structure of the intersection of quadrics with the Hermitian surface.

Rank (\mathcal{Q})	Description	Types
1	repeated plane	1, 2
2	line	3, 4, 5
2	pair of distinct planes	6, 7, 8
3	quadric cone	9, 10
4	hyperbolic quadric	11, 12, 13, 14
4	elliptic quadric	15

Types	Description
1	a tangent plane to X
2	a non-tangent plane to X
3	tangent to X
4	secant
5	a generator of X
6	two non-tangent planes to X
7	\mathcal{P}_1 is non-tangent to X and \mathcal{P}_2 is tangent to X
8	two tangent planes to X
9	no generartor in the cone
10	1 or 2 generators in the cone
11	3 skew generators in \mathcal{R}
12	2 skew generators \mathcal{R}
13	1 generator in \mathcal{R}
14	no generator in \mathcal{R}
15	elliptic quadric

4.2. Table of the weight distribution. Let us show in a table the quadrics (geometrical structure) with the corresponding weights associated to the codewords obtained from them. From the results of section 2 and those of section 5 [4], we have the following table: Table 3.

Note that, for the code $C_2(X)$ over \mathbb{F}_4 , the codewords corresponding to quadrics of type.1 to type.14 have even weights. It has also been proved by Spurr (by a computer program) that $C_2(X)$ over \mathbb{F}_4 is an even weight code.

5. The third weight of the code $C_2(X)$

Theorem 5.1. *The third weight (for $t \neq 3$) of the code $C_2(X)$ is: $w_3 = t^5 - t^3 + t$.*

Proof. In fact when \mathcal{Q} is a quadric not corresponding to the codewords of minimum weight and second weight, we get $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q) \leq s_3(t)$. \square

Rank	Types	$\#X_{Z(\mathcal{Q})}(\mathbb{F}_q)$	$\#X_{Z(\mathcal{Q})}(\mathbb{F}_4)$	Weight over \mathbb{F}_q	Weight over \mathbb{F}_4
1 (repeated plane)	1	$t^3 + t^2 + 1$	13	t^5	32
	2	$t^3 + 1$	9	$t^5 + t^2$	36
2 (line)	3	1	1	$t^5 + t^3 + t^2$	44
	4	$t + 1$	3	$t^5 + t^3 + t^2 - t$	42
	5	$t^2 + 1$	5	$t^5 + t^3$	40
2 (pair of distinct planes) $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$	6	$s_4(t)$	17	$t^5 - t^3 + t^2$	28
		$s_5(t)$	15	$t^5 - t^3 + t^2 + t$	30
	7	$s_3(t)$	19	$t^5 - t^3 + t$	26
		$s_2(t)$	21	$t^5 - t^3$	24
	8	$s(t)$	23	$t^5 - t^3 - t^2 + t$	22
		$s_2(t)$	21	$t^5 - t^3$	24
3 (cone)	9	$\leq t^3 + t^2 + t$ $+1 < s_4(t)$	≤ 15	$\geq t^5 - t$	$30 \leq w$ ≤ 44
		$t^3 + t^2 + 1$	13	t^5	32
	10	$t^3 + 2t^2 - t + 1$	15	$t^5 - t^2 + t$	30
4 (hyperbolic) $\mathcal{H}(\mathcal{R}, \mathcal{R}')$	11	$s_2(t)$	21	$t^5 - t^3$	24
	12	$\leq t^3 + 3t^2 - t$ $+1 \leq s_3(t)$	≤ 19	$\geq t^5 - t^3$	$26 \leq w$ ≤ 32
		$\leq t^3 + 2t^2$ $+1 \leq s_4(t)$	≤ 17	$\geq t^5 - t^2$	$28 \leq w$ ≤ 36
	14	$\leq t^3 + t^2 + t + 1 < s_4(t)$	≤ 15	$\geq t^5 - t$	$30 \leq w$ < 40
4 (elliptic)	15	$\leq 2t^3 + 2t$ $+2 < s_2(t)$	≤ 17	$\geq t^5 - t^3 + t^2$ $-2t - 1$	$28 \leq w$ ≤ 45

Note that in the case $t = 2$, we recover the third weight of the code $C_2(X)$ over \mathbb{F}_4 in agreement with [3] which is $2^5 - 2^3 + 2 = 26$.

Theorem 5.2. *The codewords of third weight (for $t \neq 3$) correspond to quadrics which are union of two planes, one tangent to the Hermitian surface X , the second not tangent to X , and the line of intersection of the two planes intersects the Hermitian surface in $t + 1$ points.*

Proof. The codewords of third weight are not given by hyperbolic quadrics of type 12 in table 4.2 from theorem 4.5, nor by elliptic quadrics if $t \neq 3$. \square

We come now to the computation of the codewords of third weight.

Theorem 5.3. *The number of codewords of third weight (for $t \neq 3$) is:*

$$A_{w_3} = (t^2 - 1)(t^5 + t^3 + t^2 + 1)(t^6 - t^5).$$

Proof. Let us write $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$, where:

- \mathcal{P}_1 is a tangent plane to the Hermitian surface X at a point P_1 ,
- \mathcal{P}_2 is a non-tangent plane to the Hermitian surface X ,
- and $l = \mathcal{P}_1 \cap \mathcal{P}_2$ meets the Hermitian surface in $t + 1$ points.

Given a fixed point P_1 in \mathcal{P}_1 , there are exactly $q + 1$ lines passing through P_1 and contained in the same plane. We also know that in a plane, there pass exactly $q^2 + q + 1$ lines. We wish that the line l of intersection is not a generator, nor a tangent (the number of generators and tangent lines is $q + 1$). Thus, the remaining lines which number is $q^2 + q + 1 - (q + 1) = q^2$, is the number of possibilities we can choose the line l such that l intersects X at $t + 1$ points. For each line l , it pass exactly $q + 1$ planes $(\mathcal{P}_i)_{1 \leq i \leq q+1}$ (counting \mathcal{P}_1). Let us consider the q planes $(\mathcal{P}_i)_{2 \leq i \leq q+1}$; they are not all non-tangent to X (some of them are tangent to X). For a fixed point P_1 , there are $q^2 \cdot q = q^3$ planes \mathcal{P}_i constructed and these q^3 planes are all distinct.

In total there are exactly $(t^5 + t^3 + t^2 + 1)q^3$ couples $(\mathcal{P}_1, \mathcal{P}_2)$, where \mathcal{P}_1 is a tangent plane to the Hermitian surface X , \mathcal{P}_2 is a non-tangent plane to the Hermitian surface X with $l = \mathcal{P}_1 \cap \mathcal{P}_2$ intersecting the Hermitian surface at $t + 1$ points. We also know from the proof of theorem 6.5 [4] that $(t^5 + t^3 + t^2 + 1)t^5$ is the number of couples $(\mathcal{P}_1, \mathcal{P}_2)$ with \mathcal{P}_1 and \mathcal{P}_2 two tangent planes to the Hermitian surface X , with the intersection line meeting X at $t + 1$ points. Thus, one deduces that $(t^5 + t^3 + t^2 + 1)(t^6 - t^5)$ is the number of couples $(\mathcal{P}_1, \mathcal{P}_2)$ giving codewords of third weight. Therefore, there are $(t^2 - 1)(t^5 + t^3 + t^2 + 1)(t^6 - t^5)$ codewords of third weight. \square

Observe that, for $t = 2$, there are exactly $3 \times 45 \times 32 = 4320$ codewords of third weight in agreement with Spurr [3] p. 120.

Remark. Conjecture on the fourth and fifth weight.

The author has tried to study again $\#X_{Z(\mathcal{Q})}(\mathbb{F}_q)$ for \mathcal{Q} an elliptic quadric and conjecture that $s_4(t)$, $s_5(t)$ should give respectively the fourth and the fifth weight.

The codewords of fourth weight correspond (for $t \geq 3$) now to two types of quadrics:

- quadrics which are union of two non-tangent planes to X , and the line of intersection of the two planes meets X at a single point.
- some particular elliptic quadrics.

The codewords of fifth weight correspond (for $t > 3$) to two types of quadrics:

- quadrics which are union of two non-tangent planes to X , and the line of intersection of the two planes meets X in $t + 1$ points.
- and some particular elliptic quadrics.

Unfortunately no proof is found yet.

6. Appendix

Lemma 6.1. *Let \bar{l} be a line defined over $\overline{\mathbb{F}}_q$ in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$ and $l = \bar{l} \cap \mathbb{P}^3(\mathbb{F}_q)$. Then l is empty, or a single point or a line defined over \mathbb{F}_q .*

Proof. It is a direct consequence of the equation of a line $\mathbb{P}^3(\mathbb{F}_q)$. \square

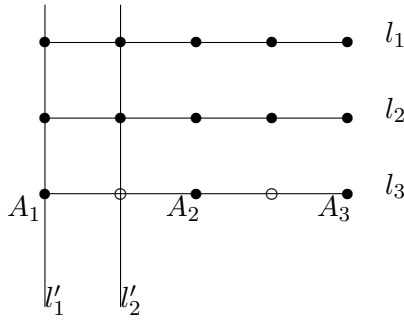
Let \mathcal{Q} be a non-degenerate quadric over $\overline{\mathbb{F}}_q$. Note $d_i =$ degree of the curve \mathcal{C}_i defined on \mathcal{Q} . From [9] (chap. 4 example 2 p. 241), we deduce an interesting relation between the arithmetical genus of a curve and its degree. Thus, we get the following result.

Proposition 6.1. *Let $\mathcal{Q} \subset \mathbb{P}^3(\overline{\mathbb{F}}_q)$ be a non-singular quadric surface in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$ and \mathcal{C} an irreducible curve on \mathcal{Q} .*
 - If $\text{deg } \mathcal{C} = 2l + 2$ then, $p_a(\mathcal{C}) \leq l^2$ (*)
 - If $\text{deg } \mathcal{C} = 2l + 1$ then, $p_a(\mathcal{C}) \leq l(l - 1)$ (**)

Let us recall the result of Yves Aubry and Marc Perret on the Weil theorem for singular curves, see [1]. With this result we get a bound for the number of rational points of any irreducible curve not necessary smooth.

Proposition 6.2. *Let C be a reduced connected projective algebraic curve over a finite field \mathbb{F}_q , with r irreducible components and of arithmetical genus $p_a(C)$. Then $|\#C(\mathbb{F}_q) - (rq + 1)| \leq 2p_a(C)\sqrt{q}$.*

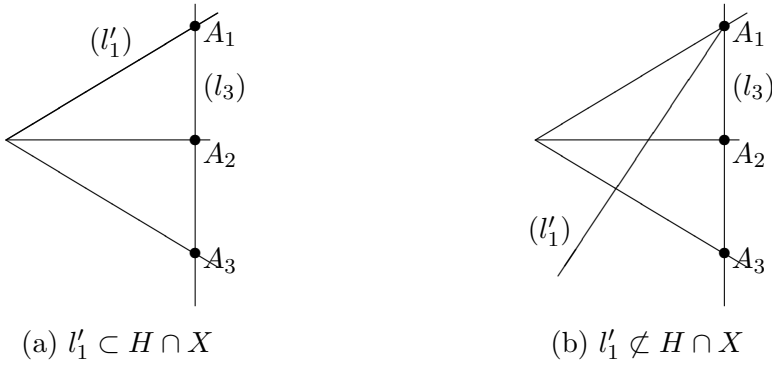
Proof of theorem 4.5. From the relation 4.4.1 of remark 4.4, we need to prove that $\#X_{Z(\mathcal{Q})}(\mathbb{F}_4) \neq s_3(2)$. A hyperbolic quadric over \mathbb{F}_4 is a pair of two reguli, each regulus containing five skew lines. Let \mathcal{R} be the regulus of \mathcal{Q} containing the two skew lines l_1 and l_2 of the Hermitian surface.



If each one of the remaining three lines of \mathcal{R} is not secant to X (i.e. meets X at three points), then the problem is solved. Suppose now that there is a secant line. Let l_3 be a line of \mathcal{R} secant to X ; l_3 meets X at three points A_1, A_2 and A_3 . Let us consider the five planes passing through l_3 ; each of them is tangent to the quadric \mathcal{Q} (since $l_3 \cap X = \{A_1, A_2, A_3\} \subset H \cap \mathcal{Q}$). In fact, $H \cap \mathcal{Q}$ as a degenerate quadric of rank 2 in $\mathbb{P}^2(\mathbb{F}_q)$, is a set of 2 lines from J.W.P. Hirschfeld [6] p. 156. Thus, we can distinguish two cases: $H \cap \mathcal{Q} = \{l_3, l\}$ where l is a line of \mathcal{Q} passing through one of the three points of $\{A_1, A_2, A_3\}$ or l does not meet any of the points A_1, A_2 and A_3 . Without loss of generality, we may assume that $H \cap \mathcal{Q} = \{l_3, l'_1\}$ or $H \cap \mathcal{Q} = \{l_3, l'_2\}$.

First case: Suppose that $H \cap \mathcal{Q} = \{l_3, l'_1\}$

If H is tangent to X : In this case $H \cap X$ is a degenerate Hermitian curve of rank 2 consisting of 13 points (a set of three lines meeting at a common point). But, two of the three points A_1, A_2 and A_3 can not be on a the same line of $H \cap X$, otherwise l_3 should be contained in X . So we get two eventualities:

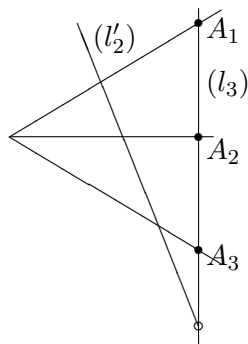


(a) $l'_1 \subset H \cap X$: In this case, \mathcal{Q} contains three lines of X among which two lines are in the same regulus and the third is in the complementary regulus. We can now write: $\mathcal{C} = \mathcal{Q} \cap X = l_1 \cup l_2 \cup l'_1 \cup \overline{\mathcal{C}}$ (where $\overline{\mathcal{C}}$ is a curve of degree 3 over \mathbb{F}_4). If $\overline{\mathcal{C}}$ is irreducible, it is defined on \mathbb{F}_4 and from proposition 6.2, we have $p_a(\overline{\mathcal{C}}) = 0$ and therefore from proposition 6.3, we get $|\mathcal{Q} \cap X| \leq 18$. If $\overline{\mathcal{C}}$ is not irreducible and since there is no plane conic on X , we deduce that $\overline{\mathcal{C}} = \overline{\mathcal{C}}_1 \cup \overline{\mathcal{C}}_2 \cup \overline{\mathcal{C}}_3$ (union of three lines); at most one line among them should be defined over \mathbb{F}_q . Thus, from lemma 6.1 we deduce that $|\mathcal{Q} \cap X| \leq 18$.

(b) $l'_1 \not\subset H \cap X$: Here $l'_1 \cap H \cap X = 3$ points. So that $|H \cap X \cap \mathcal{Q}| \leq 5$.
 If H is non-tangent to X : In this case, $H \cap X$ is a non-degenerate Hermitian curve with $|H \cap X| = 9$. Here we need to distinguish two cases:
 - If $l'_1 \subset H \cap X$, we recover the case which has been done previously.
 - If $l'_1 \not\subset H \cap X$, then $|l \cap (H \cap X)| = 3$; so $|\mathcal{Q} \cap H \cap X| \leq 3 + 3 = 6$.

Second case: Suppose that $H \cap \mathcal{Q} = \{l_3, l'_2\}$

Similary, if H is tangent to X , then $H \cap X$ is a set of three lines meeting at a common point. And we get the following configuration:



Thus $|\mathcal{Q} \cap H \cap X| \leq 3 + 3 = 6$.

If H is non-tangent to X , then $H \cap X$ is a non-singular plane curve and therefore irreducible. Here $H \cap X$ does not contain the line l'_2 , so l'_2 meets $H \cap X$ in exactly three points. Then, $|\mathcal{Q} \cap H \cap X| \leq 6$.

Summary: When \mathcal{Q} is a hyperbolic quadric which regulus \mathcal{R} contains exactly two skew lines of the Hermitian surface X , \mathcal{Q} should contain an other line l of X , in this case l is contained in the complementary regulus of \mathcal{R} and then $|\mathcal{Q} \cap X| \leq 18$; or $|\mathcal{Q} \cap H_i \cap X| \leq 6$ for each one of the five planes $(H_i)_{1 \leq i \leq 5}$ passing through the line l_3 , which leads to $|\mathcal{Q} \cap X| \leq 18$, therefore $|\mathcal{Q} \cap X| \in \{13, 15, 17\}$. This concludes the proof of theorem 4.5. \square

Acknowledgment: The author would like to thank Professor F. Rodier whose remarks and patience encouraged him to work on the problem.

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