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Density of rational points on cyclic covers of \mathbb{P}^n

par RITABRATA MUNSHI

RÉSUMÉ. Nous obtenons une majoration de la densité des points rationnels sur les revêtements cycliques de \mathbb{P}^n . Quand $n \rightarrow \infty$ notre estimation tend vers la majoration conjecturale de Serre.

ABSTRACT. We obtain upper bound for the density of rational points on the cyclic covers of \mathbb{P}^n . As $n \rightarrow \infty$ our estimate tends to the conjectural bound of Serre.

1. Introduction

Let \mathbb{P}^n denote the n -dimensional projective space over the field of rational numbers. For a finite cover $f : X \rightarrow \mathbb{P}^n$ over \mathbb{Q} , one can define a counting function

$$N(f, B) = |\{P \in X(\mathbb{Q}) : H(f(P)) \leq B\}|,$$

where H is the standard multiplicative height function defined on \mathbb{P}^n . If the degree of f is at least two, Serre [10] proved that

$$N(f, B) \ll B^{n+\frac{1}{2}}(\log B)^\gamma$$

with $\gamma < 1$. Here the implied constant depends both on f and γ . Serre's proof is based on an analogous result proved by Cohen [3] in the affine case. The proof is a beautiful application of the large sieve inequalities. Serre [10] also conjectures that one should have

$$N(f, B) \ll B^n(\log B)^c,$$

for some c . Broberg [2] almost settled Serre's conjecture for covers of the projective line and the projective plane. More precisely, he proved:

(1) If $f : X \rightarrow \mathbb{P}^1$ is a cover of degree d , then for every $\varepsilon > 0$ we have

$$N(f, B) \ll_{\varepsilon, f} B^{\frac{2}{d}+\varepsilon}.$$

(2) If $f : X \rightarrow \mathbb{P}^2$ is a cover of degree $d > 2$, then for every $\varepsilon > 0$ we have

$$N(f, B) \ll_{\varepsilon, f} B^{2+\varepsilon}.$$

In the case $d = 2$, we have

$$N(f, B) \ll_{\varepsilon, f} B^{\frac{9}{4}+\varepsilon}$$

for every $\varepsilon > 0$.

Broberg’s method is based on that introduced by Heath-Brown in [7]. These methods, though elementary, have proved to be powerful in many situations. In fact the last result of Broberg for degree two covers of \mathbb{P}^2 , is the best known result towards Manin’s conjecture, which predicts the exponent to be $1 + \varepsilon$, for del Pezzo surfaces of degree 2. Here we will give a slightly better estimate albeit at the cost of employing heavy machinery.

To this end let $F(x_0, \dots, x_n)$ be an irreducible homogenous polynomial of degree md , with $d > 1$. Moreover suppose that the projective hypersurface defined by $F(\mathbf{x}) = 0$ is smooth. Then the equation

$$y^d = F(x_0, \dots, x_n)$$

defines a variety X in the weighted projective space $\mathbb{P}(m, 1, \dots, 1)$, where y is given weight m and each x_i is given weight 1. This variety can be viewed as a cyclic d -sheeted cover of the projective space \mathbb{P}^n , via the natural map

$$\begin{aligned} f : X &\rightarrow \mathbb{P}^n; \\ (y, x_0, \dots, x_n) &\mapsto (x_0, \dots, x_n). \end{aligned}$$

This cover is ramified above the hypersurface $F(\mathbf{x}) = 0$. Using adjunction formula one easily verifies the canonical sheaf of X to be

$$\omega_X = O_X(-m + md - n - 1).$$

Hence, if the parameters n, m and d are such that

$$md - m < n + 1$$

then the variety X is a Fano variety, i.e. the anticanonical sheaf is ample. In this case we have a more precise conjecture due to Manin et al ([1], [5]), regarding the density of rational points.

In this paper, we employ the power sieve together with Poisson summation formula, and deep results of Deligne [4], and of Katz [9] about cancellation in mixed character sums over finite fields, to estimate the number of d -powers in the set

$$\{F(x_0, \dots, x_n) : -B < x_i < B\}$$

with some smooth weight W . This will give us our main result.

Theorem 1.1. *Let $f : X \rightarrow \mathbb{P}^n$ be a finite cyclic cover of \mathbb{P}^n over \mathbb{Q} . Then we have*

$$N(f, B) \ll_f B^{n + \frac{1}{n+2}} (\log B)^{\frac{n+1}{n+2}}.$$

Remark. In the special case of cyclic covers our result is better than that of Serre [10]. Also it is exciting to note that as the dimension n gets larger our bound comes closer to the conjectured bound of Serre.

Remark. For $n = 2$ and $d = 2$ our result gives

$$N(f, B) \ll B^{\frac{9}{4}} (\log B)^{\frac{3}{4}},$$

which improves the result of Broberg by an ε . However, for $n = 2, d > 2$ our result is worse.

2. Power sieve

In this section we briefly recall the concept of d -power sieve (see [6] for $d = 2$). Let p be a prime such that $p \equiv 1 \pmod{d}$. Then for every non-zero element a in the finite field \mathbb{F}_p , we can define a d -th root of unity by $a^{\frac{p-1}{d}}$. Now since \mathbb{F}_p^* is cyclic, we can define a non-canonical isomorphism

$$\theta_p : \mathbb{F}_p^* \rightarrow \mu_{p-1},$$

where μ_{p-1} denotes the group of $(p - 1)$ -th roots of unity in \mathbb{C}^* . Using this we can define a primitive Dirichlet character modulo p ,

$$\chi_p(n) = \theta_p\left(\bar{n}^{\frac{p-1}{d}}\right)$$

for $(p, n) = 1$, and $\chi_p(n) = 0$ for $p|n$. The crucial property of this character is the following:

$$\chi_p(n) = 1 \quad \text{if } (p, n) = 1, \text{ and } n = m^d \text{ for some } m.$$

Now let $\mathcal{A} = (a(n))$ be a finite sequence of non-negative quantities. We are interested in estimating the sum over the d -powers

$$S(\mathcal{A}) = \sum_n a(n^d).$$

Let \mathcal{P} be a set of P primes each having residue 1 modulo d . Suppose that $a(n) = 0$ if $|n| \geq e^P$, and consider the expression

$$S = \sum_n a(n) \left| \sum_{p \in \mathcal{P}} \chi_p(n) \right|^2$$

Each n is counted with a non-negative weight, and if $n = m^d$ and $0 < |n| < e^P$ then

$$\sum_{p \in \mathcal{P}} \chi_p(n) = \sum_{\substack{p \in \mathcal{P} \\ (p,n)=1}} 1 \gg P.$$

Hence $P^2 S(\mathcal{A}) \ll S$. Moreover

$$\begin{aligned} S &= \sum_{p, q \in \mathcal{P}} \sum_n a(n) \chi_p(n) \bar{\chi}_q(n) \\ &\leq P \sum_n a(n) + \sum_{p \neq q \in \mathcal{P}} \left| \sum_n a(n) \chi_p(n) \bar{\chi}_q(n) \right|. \end{aligned}$$

Hence we obtain the following:

Lemma 2.1. *Let \mathcal{P} be a set of P primes each having residue 1 modulo d . Suppose that $a(n) = 0$ for $n = 0$ and for $|n| \geq e^P$. Then*

$$S(\mathcal{A}) \ll P^{-1} \sum_n a(n) + P^{-2} \sum_{p \neq q \in \mathcal{P}} \left| \sum_n a(n) \chi_p(n) \bar{\chi}_q(n) \right|.$$

3. Proof of Theorem 1

Let $W : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a non-negative smooth function supported in the dyadic box $[-B, B]^{n+1}$, and such that the partial derivatives satisfy the following bound:

$$\left| \frac{d^{i_0 + \dots + i_n} W(x_0, \dots, x_n)}{dx_0^{i_0} \dots dx_n^{i_n}} \right| \ll B^{-(i_0 + \dots + i_n)}.$$

Then via integration-by-parts we note the following bound for the Fourier transform

$$\hat{W}(\mathbf{u}) = \int_{\mathbb{R}^{n+1}} W(\mathbf{x}) e(-\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbf{x} \ll B^{n+1} \prod_{i=0}^n (1 + |u_i|B)^{-2}$$

where $\mathbf{u} = (u_0, \dots, u_n)$.

Then we define a sequence of non-negative numbers by

$$a(k) = \sum_{\mathbf{x} \in \mathbb{Z}^{n+1}, F(\mathbf{x})=k} W(\mathbf{x}).$$

We wish to obtain an upper bound for

$$S(\mathcal{A}) = \sum_k a(k^d)$$

using the d -power sieve. Now clearly we have

$$\sum_k a(k) \ll B^{n+1}.$$

Hence to apply the lemma we need to estimate the sum

$$S(p, q) = \sum_k a(k) \chi_p(k) \bar{\chi}_q(k) = \sum_{\mathbf{x} \in \mathbb{Z}^{n+1}} W(\mathbf{x}) \chi_p(F(\mathbf{x})) \bar{\chi}_q(F(\mathbf{x})),$$

where p and q are two distinct primes. Using Poisson summation formula we obtain

$$\begin{aligned} S(p, q) &= \sum_{\mathbf{a} \pmod{pq}} \chi_p(F(\mathbf{a})) \bar{\chi}_q(F(\mathbf{a})) \sum_{\mathbf{x} \equiv \mathbf{a}} W(\mathbf{x}) \\ &= (pq)^{-(n+1)} \sum_{\mathbf{u} \in \mathbb{Z}^{n+1}} g(\mathbf{u}; p, q) \hat{W}\left(\frac{\mathbf{u}}{pq}\right), \end{aligned}$$

where

$$g(\mathbf{u}; p, q) = \sum_{\mathbf{a} \pmod{pq}} \chi_p(F(\mathbf{a})) \bar{\chi}_q(F(\mathbf{a})) e\left(\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{pq}\right)$$

is a mixed character sum. Then we observe that the above sum satisfies the following multiplicative property

$$g(\mathbf{u}; p, q) = g(\bar{q}\mathbf{u}; \chi_p) g(\bar{p}\mathbf{u}; \bar{\chi}_q),$$

where $p\bar{p} \equiv 1 \pmod{q}$, $q\bar{q} \equiv 1 \pmod{p}$ and

$$g(\mathbf{u}; \chi_p) = \sum_{\mathbf{a} \pmod{p}} \chi_p(F(\mathbf{a})) e\left(\frac{\langle \mathbf{a}, \mathbf{u} \rangle}{p}\right).$$

So our job is reduced to estimating the mixed character sum over the finite field \mathbb{F}_p . Let V be the hypersurface defined by $F(\mathbf{x}) = 0$ over the field of rationals. Let V^* be its dual variety, which in this case is again a hypersurface. Assuming that the prime p is such that the reduction $V_p = V \pmod{p}$ is smooth, we have three situations:

(1) First suppose that the vector $\mathbf{u} \in \mathbb{Z}^{n+1}$ is such that $\mathbf{u} \equiv \mathbf{0} \pmod{p}$, then

$$g(\mathbf{u}; \chi_p) = \sum_{\mathbf{a} \pmod{p}} \chi_p(F(\mathbf{a})) \ll p^{\frac{n+1}{2}},$$

where the implied constant depends only on n and the degree of F . This follows from the multiplicative analogue of Deligne’s bound established by Katz [8].

(2) If $\mathbf{u} \pmod{p}$ is non-zero and the associated hyperplane $\langle \mathbf{u}, \mathbf{x} \rangle = 0$ is not a tangent to the hypersurface V_p , then by Theorem 1.1 in [9] we have square-root cancellation i.e.

$$g(\mathbf{u}; \chi_p) \ll p^{\frac{n+1}{2}}.$$

Here again the implied constant depends only on n and the degree of F . In particular, it does not depend on \mathbf{u} .

(3) If $\mathbf{u} \pmod{p}$ is non-zero and the associated hyperplane $\langle \mathbf{u}, \mathbf{x} \rangle = 0$ is a tangent to the hypersurface V_p , then using the primitivity of the character we get

$$g(\mathbf{u}; \chi_p) = \frac{\chi_p(-1)\tau(\chi_p)}{p} \sum_{\mathbf{b} \pmod{p}} \bar{\chi}_p(\mathbf{b}) \sum_{\mathbf{a} \pmod{p}} e\left(\frac{\mathbf{b}F(\mathbf{a}) + \langle \mathbf{a}, \mathbf{u} \rangle}{p}\right),$$

where $\tau(\chi)$ stands for the Gauss sum of the character χ . Then we apply Deligne’s bound [4] to the inner sum to obtain

$$g(\mathbf{u}; \chi_p) \ll \sqrt{pp}^{\frac{n+1}{2}}.$$

Again the implied constant depends only on n and the degree of F .

In the first two cases we say that the vector $\mathbf{u} \in \mathbb{Z}^{n+1}$ is ‘good’ modulo p . In the third case we say that \mathbf{u} is ‘bad’ modulo p .

We first compute the contribution to $S(p, q)$ from those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are ‘good’ modulo both p and q . Using square-root cancellation we see that this is bounded by

$$\ll B^{n+1} \frac{(pq)^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{u} \in \mathbb{Z}^{n+1}} \prod_{i=0}^n \left(1 + \frac{|u_i|B}{pq}\right)^{-2} \ll (pq)^{\frac{n+1}{2}}.$$

In the last inequality we are assuming that $pq \geq B$, which follows from our choice of the set \mathcal{P} . Now the contribution of those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are ‘bad’ for the prime p but ‘good’ for the prime q is given by

$$(pq)^{-(n+1)} \sum_{\mathbf{v} \in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^p \sum_{\mathbf{u} \equiv \lambda \mathbf{v} \pmod{p}}^* g(\mathbf{u}; p, q) \hat{W}\left(\frac{\mathbf{u}}{pq}\right).$$

The $*$ above the sum indicates that the sum is restricted over those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are ‘good’ modulo q . The above expression is bounded by

$$\begin{aligned} & B^{n+1} \frac{p^{\frac{n+2}{2}} q^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{v} \in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^p \sum_{\mathbf{u} \equiv \lambda \mathbf{v} \pmod{p}}^* \prod_{i=0}^n \left(1 + \frac{|u_i|B}{pq}\right)^{-2} \\ & \ll B^{n+1} \frac{p^{\frac{n+2}{2}} q^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{v} \in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^p \left(\frac{q}{B}\right)^{n+1} \ll p^{\frac{n}{2}} q^{\frac{n+1}{2}}. \end{aligned}$$

In the last inequality we have used the fact that $|V_p^*(\mathbb{F}_p)| \ll p^{n-1}$. Similarly one can show that the contribution of those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are ‘bad’ for both p and q , is bounded by $(pq)^{\frac{n}{2}}$. Hence we conclude that

$$S(p, q) \ll (pq)^{\frac{n+1}{2}}.$$

Now for the set \mathcal{P} we pick P consecutive primes in the progression $(1 + dk)_k$ of size $\sim P \log P$. Then using the d -power sieve inequality, we get

$$S(\mathcal{A}) \ll \frac{B^{n+1}}{P} + (P \log P)^{n+1}.$$

Then choosing

$$P = \left(\frac{B}{\log B}\right)^{\frac{n+1}{n+2}}$$

the theorem follows.

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