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On a theorem of Mestre and Schoof

par John E. CREMONA et Andrew V. SUTHERLAND

Résumé. Un théorème bien connu de Mestre et Schoof implique que la cardinalité d'une courbe elliptique E définie sur un corps premier $\mathbb{F}_q$ peut être déterminée de manière univoque en calculant les ordres de quelques points sur E et sur sa tordue quadratique, à condition que $q > 229$. Nous étendons ce résultat à tous les corps finis avec $q > 49$, et tous les corps premiers avec $q > 29$.

Abstract. A well known theorem of Mestre and Schoof implies that the order of an elliptic curve E over a prime field $\mathbb{F}_q$ can be uniquely determined by computing the orders of a few points on E and its quadratic twist, provided that $q > 229$. We extend this result to all finite fields with $q > 49$, and all prime fields with $q > 29$.

Let E be an elliptic curve defined over the finite field $\mathbb{F}_q$ with q elements. The number of points on $E/\mathbb{F}_q$, which we simply denote $\#E$, is known to lie in the Hasse interval:

$$\mathcal{H}_q = [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}].$$

Equivalently, the trace of Frobenius $t = q + 1 - \#E$ satisfies $|t| \leq 2\sqrt{q}$. A common strategy to compute $\#E$, when q is not too large, relies on the fact that the points on $E/\mathbb{F}_q$ form an abelian group $E(\mathbb{F}_q)$ of order $\#E$. For any $P \in E(\mathbb{F}_q)$, the integer $\#E$ is a multiple of the order of P, and the multiples of $|P|$ that lie in $\mathcal{H}_q$ can be efficiently determined using a baby-steps giant-steps search. If there is only one multiple in the interval, it must be $\#E$; if not, we may try other $P \in E(\mathbb{F}_q)$ in the hope of uniquely determining $\#E$. This strategy will eventually succeed if and only if the group exponent

$$\lambda(E) = \operatorname{lcm}\{|P| : P \in E(\mathbb{F}_q)\}$$

has a unique multiple in $\mathcal{H}_q$. When this condition holds we expect to determine $\#E$ quite quickly: with just two random points in $E(\mathbb{F}_q)$ we already succeed with probability greater than $6/\pi^2$ (see [2, Theorem 8.1]).

Unfortunately, $\lambda(E)$ need not have a unique multiple in $\mathcal{H}_q$. However, for prime q we have the following theorem of Mestre, as extended by

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Schoof [1, Theorem 3.2]; the result as stated in [1] refers to the order of a particular point P, but the following is an equivalent statement.

Theorem 1 (Mestre-Schoof). *Let $q > 229$ be prime and E an elliptic curve over $\mathbb{F}_q$ with quadratic twist E'. Either $\lambda(E)$ or $\lambda(E')$ has a unique multiple in $\mathcal{H}_q$.*

The quadratic twist E' is an elliptic curve defined over $\mathbb{F}_q$ that is isomorphic to E over the quadratic extension $\mathbb{F}_{q^2}$, and is easily derived from E. The orders of the groups $E(\mathbb{F}_q)$ and $E'(\mathbb{F}_q)$ satisfy $\#E + \#E' = 2(q+1)$. For prime fields with $q > 229$, Theorem 1 implies that we may determine one of $\#E$ and $\#E'$ by alternately computing the orders of points on E and E', and once we know either $\#E$ or $\#E'$, we know both.

Theorem 1 does not hold for $q = 229$. Since there are counterexamples whenever q is a square, it does not hold in general for non-prime finite fields either. The argument in the proof of [1, Theorem 3.2] does not use the primality of q, but only that q is both large enough and not a square, so that the Hasse bound on t cannot be attained. If $q = r^2$ is an even power of a prime, then there are supersingular elliptic curves E over $\mathbb{F}_q$ such that

$$E(\mathbb{F}_q) \cong (\mathbb{Z}/(r-1)\mathbb{Z})^2 \qquad \text{and} \qquad E'(\mathbb{F}_q) \cong (\mathbb{Z}/(r+1)\mathbb{Z})^2 .$$

One may easily check that there are at least 5 multiples of $r-1$, and at least 3 multiples of $r+1$, in $\mathcal{H}_q$; however for $r > 7$ ($q > 49$), the only pair that sum to $2(q+1)$ are $(r-1)^2$ and $(r+1)^2$. This resolves the ambiguity in these cases, leaving a finite number of small exceptions. For example, when $q = 49$ there is more than one pair of multiples of 6 and 8 (respectively) which sum to $2(q+1) = 100$, since $100 = 36 + 64 = 60 + 40$.

The preceding observation led to this note, whose purpose is to extend Theorem 1 to treat all finite fields (not just prime fields) $\mathbb{F}_q$ with $q > 49$, and all prime fields with $q > 29$. Specifically, we prove the following:

Theorem 2. *Let $q \notin \{3, 4, 5, 7, 9, 11, 16, 17, 23, 25, 29, 49\}$ be a prime power, and let $E/\mathbb{F}_q$ be an elliptic curve. Then there is a unique integer t with $|t| \leq 2\sqrt{q}$ such that $\lambda(E)|(q+1-t)$ and $\lambda(E')|(q+1+t)$.*

Our proof is entirely elementary, relying on just two properties of elliptic curves over finite fields:

(a) $\#E = q+1-t$ and $\#E' = q+1+t$ for some integer t with $|t| \leq 2\sqrt{q}$;
(b) $E(\mathbb{F}_q) \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ with n_1 dividing both n_2 and $q-1$.

Proofs of (a) and (b) may be found in most standard references, including [3]. We note that $n_2 = \lambda(E)$, and $n_1 = 1$ when $E(\mathbb{F}_q)$ is cyclic.

Proof of Theorem 2. Let E be an elliptic curve over $\mathbb{F}_q$, and put $\#E = mM$ with $M = \lambda(E)$, and $\#E' = nN$ with $N = \lambda(E')$. Without loss of generality, we assume $a = q + 1 - \#E \geq 0$. Taking $t = a$ shows existence, by (a) and (b) above, so we need only prove that $t = a$ is the unique t satisfying the conditions stated in the theorem. For any such t we have $t \equiv q+1 \bmod M$ and $t \equiv -(q+1) \bmod N$; hence t lies in an arithmetic sequence with difference $\operatorname{lcm}(M, N)$. We also have $|t| \leq 2\sqrt{q}$; thus if $\operatorname{lcm}(M, N) > 4\sqrt{q}$, then $t = a$ is certainly unique.

We now show that $\operatorname{lcm}(M, N) \leq 4\sqrt{q}$ implies $q \leq 1024$. We start from

$$mMnN = (q+1-a)(q+1+a) = (q+1)^2 - a^2 \geq (q+1)^2 - 4q = (q-1)^2,$$

which yields

$$mn \geq \frac{(q-1)^2}{MN} = \frac{(q-1)^2}{\gcd(M, N)\operatorname{lcm}(M, N)}. \tag{0.1}$$

Let $d = \gcd(m, n)$. Then d^2 divides $mM + nN = 2(q+1)$, so $d|(q+1)$, but also $d|(q-1)$, hence $d \leq 2$. This implies $2\operatorname{lcm}(M, N) \geq 2\operatorname{lcm}(m, n) \geq mn$. We also have $\gcd(M, N) \leq \gcd(m, n)\gcd(M/m, N/n) \leq 2\gcd(M/m, N/n)$. Applying these inequalities to (0.1) we obtain

$$\operatorname{lcm}(M, N)^2 \geq \frac{(q-1)^2}{4\gcd(M/m, N/n)}. \tag{0.2}$$

We now suppose $\operatorname{lcm}(M, N) \leq 4\sqrt{q}$, for otherwise the theorem holds. We have $nN = q + 1 + a > q$, since we assumed $a \geq 0$, and $N \leq 4\sqrt{q}$ implies that $n > \sqrt{q}/4$, so $N/n < 16$. Applying $\gcd(M/m, N/n) \leq N/n < 16$ to (0.2) yields

$$4\sqrt{q} \geq \operatorname{lcm}(M, N) > (q-1)/8,$$

which implies that the prime power q is at most 1024.

The cases for $q \leq 1024$ are addressed by a program listed in the appendix that outputs the values of q, $M = \lambda(E)$, and $N = \lambda(E')$ for which exceptions can arise. This yields the set of excluded q and completes the proof. ☐

Application. The proof of Theorem 2 suggests an algorithm to compute $\#E$, provided that q is small enough for the orders of randomly chosen points in $E(\mathbb{F}_q)$ to be easily computed. It suffices to determine integers a and m for which the set $S = \{x : x \equiv a \bmod m\}$ contains $t = q+1-\#E$ but no $t' \neq t$ with $|t'| \leq 2\sqrt{q}$. Beginning with $m = 1$ and $a = 0$, we compute $|P|$ for random points P in $E(\mathbb{F}_q)$ or $E'(\mathbb{F}_q)$, and update a and m to reflect the fact that $t \equiv q+1 \bmod |P|$ when $P \in E(\mathbb{F}_q)$, and $t \equiv -(q+1) \bmod |P|$ when $P \in E'(\mathbb{F}_q)$. The new values of a and m may be determined via the extended Euclidean algorithm. When the set S contains a unique t with $|t| \leq 2\sqrt{q}$, we can conclude that $\#E = q + 1 - t$ (and also that $\#E' = q + 1 + t$).

The probabilistic algorithm we have described is a *Las Vegas* algorithm, that is, its output is always correct and its expected running time is finite. The correctness of the algorithm follows from property (a). Theorem 2 ensures that the algorithm can terminate (provided that q is not in the excluded set), and [2, Theorem 8.2] bounds its expected running time.

An examination of Table 1 reveals that in many cases an ambiguous t' could be ruled out if $\lambda(E)$ or $\lambda(E')$ were known. For example, when $q = 49$, the trace $t' = -10$ yields $\#E = 60$ and $\#E' = 40$, so both $\lambda(E)$ and $\lambda(E')$ are divisible by 5 (and are not 6 or 8). If E has trace -10, the algorithm above will likely discover this and terminate within a few iterations. But when the trace of E is 14 (and $\lambda(E) = 6$ and $\lambda(E') = 8$), we can never be completely certain that we have ruled out -10 as a possibility. Thus when an unconditional result is required, we must avoid $q \in \{3, 4, 5, 7, 9, 11, 16, 17, 23, 25, 29, 49\}$.

However, when $\lambda(E)$ and $\lambda(E')$ are known we have the following corollary, which extends Proposition 4.19 of [3].

Corollary 1. *Let $E/\mathbb{F}_q$ be an elliptic curve. Up to isomorphism, the integers $\lambda(E)$ and $\lambda(E')$ uniquely determine the groups $E(\mathbb{F}_q)$ and $E'(\mathbb{F}_q)$, provided that $q \notin \{5, 7, 9, 11, 17, 23, 29\}$. In every case, $\lambda(E)$ and $\lambda(E')$ uniquely determine the set $\{E(\mathbb{F}_q), E'(\mathbb{F}_q)\}$.*

Note that $\lambda(E)$ and $\#E$ together determine $E(\mathbb{F}_q)$, by property (b). To prove the corollary, apply Theorem 1 with a modified version of the algorithm in the appendix that also requires $(q + 1 - t')/M$ to divide M and $(q + 1 + t')/N$ to divide N.

As a final remark, we note that all the exceptional cases listed in Table 0.1 can be eliminated if the orders of the 2-torsion and 3-torsion subgroups of $E(\mathbb{F}_q)$ are known (these orders may be computed using the division polynomials). Alternatively, one can simply enumerate the points on $E/\mathbb{F}_q$ to determine $\#E$ when $q \leq 49$.

Appendix

For a prime power q, we wish to enumerate all M, N, and t such that:

(i) M divides $q + 1 - t$ and N divides $q + 1 + t$, with $0 \leq t \leq 2\sqrt{q}$.

(ii) $(q + 1 - t)/M$ divides M and $q - 1$, and $(q + 1 + t)/N$ divides N and $q - 1$.

(iii) M divides $q + 1 - t'$ and N divides $q + 1 + t'$ for some $t' \neq t$ with $|t'| \leq 2\sqrt{q}$.

Any exception to Theorem 2 must arise from an elliptic curve $E/\mathbb{F}_q$ with $\lambda(E) = M$, $\lambda(E') = N$, and $\#E = q + 1 - t$ (or from its twist, but the cases are symmetric, so we restrict to $t \geq 0$). Properties (i) and (ii) follow from (a) and (b) above, and (iii) implies that t does not uniquely satisfy the requirements of the theorem.

Algorithm 1 below finds all M, N, and t satisfying (i), (ii), and (iii). For $q \leq 1024$, exceptional cases are found only for the twelve values of q listed in Theorem 2. Not every case output by Algorithm 1 is actually realized by an elliptic curve (in fact, all but one of the exceptions fail the condition that $(q+1-t)/M \equiv (q+1+t)/N \pmod 2$), but for each combination of q and t at least one is. An example of each such case is listed in Table 0.1, where we only list cases with $t \geq 0$: for the symmetric cases with $t < 0$, change the sign of t and swap M and N.

Algorithm 1. Given a prime power q, output all quadruples of integers (M, N, t, t') satisfying (i), (ii), and (iii) above:

```
for all pairs of integers $(M, N)$ with $\sqrt{q} - 1 \leq M, N \leq 4\sqrt{q}$ do
  for all integers $t \in [0, 2\sqrt{q}]$ with $M|(q+1-t)$ and $N|(q+1+t)$ do
    Let $m = (q+1-t)/M$ and $n = (q+1+t)/N$.
    if $m|M$ and $m|(q-1)$ and $n|N$ and $n|(q-1)$ then
      for all integers $t' \in [-2\sqrt{q}, 2\sqrt{q}]$ do
        if $M|(q+1-t')$ and $N|(q+1+t')$ then
          print $M, N, t, t'$.
        end if
      end for
    end if
  end for
end for
```

q	M	N	t	E	t'
3	2	2	0	$y^2 = x^3 - x$	-2,2
4	1	3	4	$y^2 + y = x^3 + \alpha^2$	-2,1
5	2	4	2	$y^2 = x^3 + x$	-2
7	2	6	4	$y^2 = x^3 - 1$	-2
7	4	4	0	$y^2 = x^3 + 3x$	-4,4
9	2	4	6	$y^2 = x^3 + \alpha^2 x$	-6,-2,2
11	4	8	4	$y^2 = x^3 + x + 9$	-4
11	6	6	0	$y^2 = x^3 + 2x$	-6,6
16	3	5	8	$y^2 + y = x^3$	-7
17	6	12	6	$y^2 = x^3 + x + 7$	-6
23	8	16	8	$y^2 = x^3 + 5x + 15$	-8
25	4	6	10	$y^2 + y = x^3 + \alpha^7$	-2
29	10	20	10	$y^2 = x^3 + x$	-10
49	6	8	14	$y^2 = x^3 + \alpha^2 x$	-10

TABLE 0.1. Exceptional Cases with $t \geq 0$.

The coefficient α denotes a primitive element of $\mathbb{F}_q$.

References

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