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# Cohomological dimension of Laumon 1-motives up to isogenies

par NICOLA MAZZARI

RÉSUMÉ. Nous prouvons que la dimension cohomologique de la catégorie des 1-motifs de Laumon à isogénie près sur un corps de caractéristique nulle est  $\leq 1$ . En conséquence, cela implique le même résultat pour la catégorie des structures de Hodge formelles de niveau  $\leq 1$  (sur  $\mathbb{Q}$ ).

ABSTRACT. We prove that the category of Laumon 1-motives up to isogenies over a field of characteristic zero is of cohomological dimension  $\leq 1$ . As a consequence this implies the same result for the category of formal Hodge structures of level  $\leq 1$  (over  $\mathbb{Q}$ ).

## 1. Introduction

In [6] P. Deligne defined a *1-motive* over a field  $k$  as  $\mathrm{Gal}(k^{\mathrm{sep}}|k)$ -equivariant morphism  $[u : \mathbf{X} \rightarrow \mathbf{G}(k^{\mathrm{sep}})]$  where  $\mathbf{X}$  is a free  $\mathrm{Gal}(k^{\mathrm{sep}}|k)$ -module and  $\mathbf{G}$  is a semi-abelian algebraic group over  $k$ . They form a category that we shall denote by  $\mathcal{M}_{1,k}$  or  $\mathcal{M}_1$ .

Deligne's definition was motivated by Hodge theory. In fact the category of 1-motives over the complex numbers is equivalent, via the so called *Hodge realization* functor, to the category  $\mathrm{MHS}_1$  of mixed Hodge structures of level  $\leq 1$ . It is known the the category  $\mathrm{MHS}_1$  is of cohomological dimension 1 (see [5]) and the same holds for  $\mathcal{M}_{1,\mathbb{C}}$ .

F. Orgogozo proved more generally that for any field  $k$ , the category  $\mathcal{M}_{1,k} \otimes \mathbb{Q}$  is of cohomological dimension  $\leq 1$  (see [14, Prop. 3.2.4]).

Over a field of characteristic 0 it is possible to define the category  $\mathcal{M}_{1,k}^a$  of Laumon 1-motives generalizing that of Deligne 1-motives (See [11]). In [3] L. Barbieri-Viale generalized the Hodge realization functor to Laumon 1-motives. He defined the category  $\mathrm{FHS}_1$  of formal Hodge structures of level  $\leq 1$  containing  $\mathrm{MHS}_1$  and proved that  $\mathrm{FHS}_1$  is equivalent to the category of Laumon 1-motives over  $\mathbb{C}$  (compatibly with the Hodge realization).

In this paper we prove that the category of Laumon 1-motives up to isogenies is of cohomological dimension 1.

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## 2. Laumon 1-motives

Let  $k$  be a (fixed) field of characteristic zero. Let  $\mathbf{Sch}_k$  be the category of schemes over  $k$  and  $\mathbf{Aff}_k$  be the full sub-category of affine schemes. According to [1, Exp. IV §6.3] the fppf topology on  $\mathbf{Sch}_k$  is the one generated by: the families of jointly surjective open immersions in  $\mathbf{Sch}_k$ ; the finite families of jointly surjective, flat, of finite presentation and quasi-finite morphisms in  $\mathbf{Aff}_k$ .

Let  $\mathbf{Ab}_k$  be the category of abelian sheaves on  $\mathbf{Aff}_k$  w.r.t. the fppf topology. We will consider both the category of commutative group schemes and that of formal group schemes (over  $k$ ) as full sub-categories of  $\mathbf{Ab}_k$ . We denote by  $\bar{k}$  the algebraic closure of  $k$ .

**Definition.** A *Laumon 1-motive* over  $k$  (or an effective free 1-motive over  $k$ , cf. [2, 1.4.1]) is the data of

- (1) a (commutative) formal group  $\mathbf{F}$  over  $k$ , such that  $\mathrm{Lie} \mathbf{F}$  is a finite dimensional  $k$ -vector space and  $\mathbf{F}(\bar{k}) = \lim_{[k':k] < \infty} \mathbf{F}(k')$  is a finitely generated and torsion-free  $\mathrm{Gal}(\bar{k}/k)$ -module;
- (2) a connected commutative algebraic group scheme  $\mathbf{G}$  over  $k$ ;
- (3) a morphism  $u : \mathbf{F} \rightarrow \mathbf{G}$  in the category  $\mathbf{Ab}_k$ .

Note that we can consider a Laumon 1-motive (over  $k$ )  $M = [u : \mathbf{F} \rightarrow \mathbf{G}]$  as a complex of sheaves in  $\mathbf{Ab}_k$  concentrated in degree 0, 1.

It is known that any formal  $k$ -group  $\mathbf{F}$  splits canonically as product  $\mathbf{F}^o \times \mathbf{F}_{\acute{e}t}$  where  $\mathbf{F}^o$  is the identity component of  $\mathbf{F}$  and is a connected formal  $k$ -group, and  $\mathbf{F}_{\acute{e}t} = \mathbf{F}/\mathbf{F}^o$  is étale. Moreover,  $\mathbf{F}_{\acute{e}t}$  admits a maximal subgroup scheme  $\mathbf{F}_{\mathrm{tor}}$ , étale and finite, such that the quotient  $\mathbf{F}_{\acute{e}t}/\mathbf{F}_{\mathrm{tor}} = \mathbf{F}_{\mathrm{fr}}$  is constant of the type  $\mathbb{Z}^r$  over  $\bar{k}$ . One says that  $\mathbf{F}$  is torsion-free if  $\mathbf{F}_{\mathrm{tor}} = 0$ .

By a theorem of Chevalley any connected algebraic group scheme  $\mathbf{G}$  is the extension of an abelian variety  $\mathbf{A}$  by a linear  $k$ -group scheme  $\mathbf{L}$  that is product of its maximal sub-torus  $\mathbf{T}$  with a vector  $k$ -group scheme  $\mathbf{V}$ . See [7] for more details on algebraic and formal groups.

**Definition.** A *morphism* of Laumon 1-motives is a commutative square in the category  $\mathbf{Ab}_k$ . We denote by  $\mathcal{M}_1^a = \mathcal{M}_{1,k}^a$  the category of Laumon  $k$ -1-motives, i.e. the full sub-category of  $C^b(\mathbf{Ab}_k)$  whose objects are Laumon 1-motives. We denote by  $\mathcal{M}_1^a$  the full sub-category of  $\mathcal{M}_1^a$  whose objects are Deligne 1-motives (over  $k$ ) [6, §10.1.2].

**Proposition 2.1.** *The category  $\mathcal{M}_1^a$  of Laumon 1-motives (over  $k$ ) is an additive category with kernels and co-kernels. In particular let  $(f, g)$  be a*

morphism from  $M = [u : \mathbf{F} \rightarrow \mathbf{G}]$  to  $M' = [u' : \mathbf{F}' \rightarrow \mathbf{G}']$  (i.e.  $u'f = gu$ ), then

$$(2.1) \quad \text{Ker}(f, g) = [u^* \text{Ker}(g)^o \rightarrow \text{Ker}(g)^o]$$

and

$$(2.2) \quad \text{Coker}(f, g) = [\text{Coker}(f)_{fr} \rightarrow \text{Coker}(g)]$$

*Proof.* See [11, Prop. 5.1.3]. □

**Remark.** The category of connected algebraic groups is fully embedded in  $\mathcal{M}_1^a$  and it is not abelian. So the category of Laumon 1-motives is not abelian too. In fact consider a surjective morphism of connected algebraic groups  $g : \mathbf{G} \rightarrow \mathbf{G}'$ . Then  $\text{Ker}(g)$  is not necessarily connected and the canonical map (in the category of connected algebraic groups)

$$\text{Coim}(g) = \mathbf{G} / \text{Ker}(g)^o \rightarrow \text{Im}(g) = \mathbf{G}'$$

is not an isomorphism in general.

According to [14] we define the category  $\mathcal{M}_1^a \otimes \mathbb{Q}$  of Laumon 1-motives up to isogenies: the objects are the same of  $\mathcal{M}_1^a$ ; the Hom groups are  $\text{Hom}_{\mathcal{M}_1^a}(M, M') \otimes_{\mathbb{Z}} \mathbb{Q}$ .

**Remark.** Note that a morphism  $(f, g) : M \rightarrow M'$  is an isogeny (i.e. an isomorphism in  $\mathcal{M}_1^a \otimes \mathbb{Q}$ ) if and only if  $f$  is injective with finite co-kernel and  $g$  is surjective with finite kernel.

**Proposition 2.2.** *The category of Laumon 1-motives up to isogenies is abelian.*

*Proof.* By construction  $\mathcal{M}_1^a \otimes \mathbb{Q}$  is an additive category. Let  $(f, g) : M \rightarrow M'$  be a morphism of Laumon 1-motives. We know that the group  $\pi_0(\text{Ker}(g)) = \text{Ker}(g) / \text{Ker}(g)^o$  is a finite group scheme, hence there exists an integer  $n$  such that the following diagram commutes in  $\mathbf{Ab}_k$

$$\begin{array}{ccccc} & & \text{Ker}(f) & & \\ & & \downarrow n \cdot u & \searrow 0 & \\ & & \text{Ker}(g)^o & \longrightarrow & \text{Ker}(g) & \longrightarrow & \pi_0(\text{Ker}(g)) \end{array}$$

Then  $n \cdot u$  factors through  $\text{Ker}(g)^o$  and it is easy to check that  $\text{Ker}((f, g)) = [(u^* \text{Ker}(g)^o) \rightarrow \text{Ker}(g)^o]$  is isogenous to  $[\text{Ker}(f) \rightarrow \text{Ker}(g)^0]$ .

It follows that  $\text{Coim}(f, g)$  is isogenous to  $[(\mathbf{F} / \text{Ker}(f))_{fr} \rightarrow \mathbf{G} / \text{Ker}(g)]$ . As  $\mathbf{G} / \text{Ker}(g)^o \rightarrow \mathbf{G} / \text{Ker}(g)$  is an isogeny we get that the canonical map  $\text{Coim}(f, g) \rightarrow \text{Im}(f, g)$  is an isogeny too.

This is enough to prove that the category  $\mathcal{M}_1^a \otimes \mathbb{Q}$  is abelian. □

**Remark.** One can also define the category  ${}^t\mathcal{M}_1^a$  of *1-motives with torsion* over  $k$  (See [2, Def. 1.4.4]). We note that using the same arguments as in [4, C.7.3] it is easy to show that there is an equivalence of categories

$$\mathcal{M}_1^a \otimes \mathbb{Q} \xrightarrow{\sim} {}^t\mathcal{M}_1^a \otimes \mathbb{Q} .$$

**2.1. Weights.** A Deligne 1-motive is endowed with an increasing filtration (of sub-1-motives) called the weight filtration ([6, §10.1.4]) defined as follows

$$W_i = W_i M := \begin{cases} [\mathbf{X} \rightarrow \mathbf{G}] & i \geq 0 \\ [0 \rightarrow \mathbf{G}] & i = -1 \\ [0 \rightarrow \mathbf{T}] & i = -2 \\ [0 \rightarrow 0] & i \leq -3 \end{cases} ; \quad \text{gr}_i^W M = \begin{cases} [\mathbf{X} \rightarrow 0] & i = 0 \\ [0 \rightarrow \mathbf{A}] & i = -1 \\ [0 \rightarrow \mathbf{T}] & i = -2 \\ [0 \rightarrow 0] & \text{otherwise} \end{cases} .$$

According to [4, C.11.1] we extend the weight filtration to Laumon 1-motives.

**Definition.** Let  $M = [u : \mathbf{F} \rightarrow \mathbf{G}]$  be a Laumon 1-motive. The *weight filtration* of  $M$  is

$$W_{-3} = 0 \subset W_{-2} = [0 \rightarrow \mathbf{L}] \subset W_{-1} = [0 \rightarrow \mathbf{G}] \subset W_0 = M .$$

**Remark.**

- (1) The morphisms of Laumon 1-motives are compatible w.r.t. the weight filtration. Also the weight filtration extends to a filtration on the objects of  $\mathcal{M}_1^a \otimes \mathbb{Q}$ .
- (2) Let  $\text{Mod}_k^f$  be the category of finite dimensional  $k$ -vector spaces. The full sub-category of  $\mathcal{M}_1^a \otimes \mathbb{Q}$  of Laumon 1-motives of weight 0 is equivalent to the category  $\text{Mod}_k^f \times \text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{k}/k))$  via the functor  $\mathbf{F} \mapsto (\text{Lie}(\mathbf{F}), \mathbf{F}(k) \otimes \mathbb{Q})$ .

### 3. Cohomological dimension

**3.1. Extensions.** Let  $\mathbf{A}$  be any abelian category (we don't suppose it has enough injective objects), then we can define its derived category  $D(\mathbf{A})$  and the group of  $n$ -fold extension classes  $\text{Ext}_{\mathbf{A}}^n(A, B) := \text{Hom}_{D(\mathbf{A})}(A, B[n])$ ,  $A, B \in \mathbf{A}$ . As usual we identify this group with the group of classes of *Yoneda extensions*, i.e. the set of exact sequences

$$0 \rightarrow B \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow A \rightarrow 0$$

modulo congruences (See [10] or [9]).

We will use the two following well-known facts about extensions and filtrations.

- (1) Let  $W_{-2} \subset W_{-1} \subset W_0 = W$  be a filtration of  $W \in \mathbf{A}$ . We have the following exact sequences

$$\gamma : \quad 0 \rightarrow W_{-2} \rightarrow W_{-1} \rightarrow W_0/W_{-2} \rightarrow W_0/W_{-1} \rightarrow 0$$

$$\gamma_1 : \quad 0 \rightarrow W_{-2} \rightarrow W_{-1} \rightarrow W_{-1}/W_{-2} \rightarrow 0$$

$$\gamma_2 : \quad 0 \rightarrow W_{-1}/W_{-2} \rightarrow W_0/W_{-2} \rightarrow W_0/W_{-1} \rightarrow 0$$

and  $\gamma = \gamma_1 \cdot \gamma_2 \in \text{Ext}_{\mathbb{A}}^2(W_0/W_{-1}, W_{-2})$  by definition. This class is trivial, *i.e.*  $\gamma = 0$  in  $\text{Ext}_{\mathbb{A}}^2(W_0/W_{-1}, W_{-2})$ . (See [14, Lemma 3.2.5], or [9, p. 184])

- (2) Assume that the objects of  $\mathbb{A}$  are filtered (by a separated and exhaustive filtration  $W$ ) and that the morphisms in  $\mathbb{A}$  are compatible w.r.t. this filtration. If  $\text{Ext}_{\mathbb{A}}^n(\text{gr}_i^W A, \text{gr}_j^W B) = 0$  for any  $i, j$ , then  $\text{Ext}_{\mathbb{A}}^n(A, B) = 0$ . In fact assume for instance that  $B$  has a 3 steps filtration  $0 \subset W_{-2} \subset W_{-1} \subset W_0 = B$ : then we have the canonical exact sequences

$$0 \rightarrow W_{-1}M' \rightarrow M' \rightarrow \text{gr}_0^W M' \rightarrow 0$$

$$0 \rightarrow W_{-2}M' \rightarrow W_{-1}M' \rightarrow \text{gr}_{-1}^W M' \rightarrow 0$$

By applying  $\text{Hom}(A, -)$  we get two long exact sequences

$$\dots \text{Ext}^2(A, W_{-1}B') \rightarrow \text{Ext}^2(A, B) \rightarrow \text{Ext}^2(M, \text{gr}_0^W B) \dots$$

$$\dots \text{Ext}^2(A, \text{gr}_{-2}^W B) \rightarrow \text{Ext}^2(A, W_{-1}B) \rightarrow \text{Ext}^2(A, \text{gr}_{-1}^W B) \dots$$

from this follows that we can reduce to prove  $\text{Ext}^2(A, \text{gr}_i^W B) = 0$ . This process can be easily adapted to the general case.

Now we can give a sketch of the proof of the main theorem: one first checks that  $\text{Ext}_{\mathbb{Q}}^1(M, M') = 0$  if  $M, M'$  are pure of weights  $w, w'$  and  $w \leq w'$  (a 1-motive is *pure* if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). By point (2) above this formally reduces the problem to checking that if  $M, M', M''$  are pure respectively of weights  $0, -1, -2$ , then the Yoneda product of two classes  $(\gamma_1, \gamma_2) \in \text{Ext}_{\mathbb{Q}}^1(M', M'') \times \text{Ext}_{\mathbb{Q}}^1(M, M')$  is 0. Of course we may assume  $\gamma_1$  and  $\gamma_2$  integral. Then the point is that  $\gamma_1$  and  $\gamma_2$  *glue* into a 1-motive and we can conclude by (1) above.

**3.2. Main result.** From now on we call 1-motive a Laumon 1-motive (over  $k$ ) and  $\text{Ext}_{\mathbb{Q}}^i(M, M')$  is the group of classes of  $i$ -fold extensions in  $\mathcal{M}_1^a \otimes \mathbb{Q}$ .

**Theorem 3.1.** *The category  $\mathcal{M}_1^a \otimes \mathbb{Q}$  (and in particular  $\text{FHS}_1 \otimes \mathbb{Q}$ ) is of cohomological dimension 1.*

*Proof.* By the general facts on extensions (§3.1 (2)) we can restrict to consider only pure motives  $M = \text{gr}_w^W M$  and  $M' = \text{gr}_{w'}^W M'$  of weight  $w$  and  $w'$ , respectively.

(*Equal weights*) If  $w = w'$  we can show that  $\text{Ext}_{\mathbb{Q}}^1(M, M') = 0$ . Let  $0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0$  be an exact sequence in  $\mathcal{M}_1^a \otimes \mathbb{Q}$ , then also  $E$  is pure of weight  $w$ . We have to consider 3 cases: first note the category  $\text{Mod}_k^f \times \text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{k}/k))$  is semi-simple by Maschke's lemma [15, p. 47] and

so the claim holds for the weight zero case by point (2) of the remark in §2.1: also the category of abelian varieties up to isogenies (*i.e.*  $w = -1$ ) is semi-simple by [13, p. 173]; the third case (weight  $-2$ ) can be reduced to the first by Cartier duality (See [11, §5]) or proved explicitly.

(*Different weights*) Fix a 2-fold extension  $\gamma \in \text{Ext}_{\mathbb{Q}}^2(M, M')$  represented by

$$0 \rightarrow M' \rightarrow E_1 \rightarrow E_2 \rightarrow M \rightarrow 0$$

and let  $E = \text{Ker}(E_2 \rightarrow M)$ . Then we can write  $\gamma = \gamma_1 \cdot \gamma_2$ , where  $\gamma_2 \in \text{Ext}_{\mathbb{Q}}^1(M, E)$ ,  $\gamma_1 \in \text{Ext}_{\mathbb{Q}}^1(E, M')$ . Using the canonical exact sequence induced by weights and the first part of the proof it is easy to reduce to the case  $E = \text{gr}_{-1} E$ , *i.e.*  $E$  is an abelian variety.

If  $w < w'$  then  $\gamma_1$  is an extension of an abelian variety  $E$  by  $M'$  which is a formal group or an abelian variety. Then  $\gamma_1 = 0$  (if  $M'$  is a formal group we refer to [2, Lemma A.4.5]).

It remains to study what happens if  $w > w'$ . If  $w$  or  $w'$  is equal to  $-1$  there is nothing to prove because  $E$  is an abelian variety too. So the only case left is when  $w = 0$  and  $w' = -2$ , *i.e.*  $M = \mathbf{F}[1]$ ,  $M' = \mathbf{L}[0]$ . We want to reduce to the situation considered in § 3.1 (1). Thus we have to show that there exists a 1-motive  $N$  such that  $\gamma_1 \in \text{Ext}_{\mathbb{Q}}^1(E, \mathbf{L})$  is represented by  $0 \rightarrow W_{-2}N \rightarrow W_{-1}N \rightarrow \text{gr}_{-1}N \rightarrow 0$ ;  $\gamma_2 \in \text{Ext}_{\mathbb{Q}}^1(\mathbf{F}[1], E)$  is represented by  $0 \rightarrow \text{gr}_{-1}N \rightarrow W_0N/W_{-2} \rightarrow \text{gr}_0N \rightarrow 0$ .

We claim that  $\gamma_1$  and  $\gamma_2$  can be represented by extensions in the category Laumon-1-motives. In fact let

$$\gamma_1 : 0 \rightarrow \mathbf{L} \xrightarrow{f \otimes n^{-1}} \mathbf{G} \xrightarrow{g \otimes m^{-1}} E \rightarrow 0$$

be an extension in the category of 1-motives modulo isogenies:  $f, g$  are morphism of algebraic groups,  $n, m \in \mathbb{Z}$ . Then consider the push-forward by  $n^{-1}$  and the pull-back by  $m^{-1}$ , we get the following commutative diagram with exact rows in  $\mathcal{M}_1^{\text{a,fr}} \otimes \mathbb{Q}$

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{f/n} & \mathbf{G} & \xrightarrow{g/m} & E & \longrightarrow & 0 \\
 & & \downarrow n^{-1} & & \downarrow \text{id} & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{f} & \mathbf{G} & \xrightarrow{g/m} & E & \longrightarrow & 0 \\
 & & \uparrow \text{id} & & \uparrow \text{id} & & \uparrow m^{-1} & & \\
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{f} & \mathbf{G} & \xrightarrow{g} & E & \longrightarrow & 0
 \end{array}$$

The exactness of the last row is equivalent to the following:  $\text{Ker } f$  is finite; let  $(\text{Ker } g)^0$  be the connected component of  $\text{Ker } g$ , then  $\text{Im } f \rightarrow (\text{Ker } g)^0$  is surjective with finite kernel  $K$ ;  $g$  is surjective. So after replacing  $\mathbf{L}, E$  with isogenous groups we have an exact sequence in  $\mathcal{M}_1^{\text{a,fr}}$

$$0 \rightarrow \mathbf{L} \rightarrow \mathbf{G} \rightarrow E \rightarrow 0$$

Explicitly

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{f} & \mathbf{G} & \xrightarrow{g} & \mathbf{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \mathbf{L}/\text{Ker } f & \longrightarrow & \mathbf{G} & \xrightarrow{g} & \mathbf{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Im } f/K & \longrightarrow & \mathbf{G} & \xrightarrow{g} & \mathbf{E} & \longrightarrow & 0 \\
 & & \uparrow \text{id} & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \text{Im } f/K & \longrightarrow & \mathbf{G} & \longrightarrow & \mathbf{G}/(\text{Ker } g)^0 & \longrightarrow & 0
 \end{array}$$

With similar arguments we can prove that  $\gamma_2$  is represented by an extension in the category  $\mathcal{M}_1^{\text{a,fr}}$

$$0 \rightarrow E \rightarrow N \rightarrow \mathbf{F}[1] \rightarrow 0$$

with  $N = [u : \mathbf{F} \rightarrow E]$ .

To conclude we have to prove that there is lifting  $u' : \mathbf{F} \rightarrow \mathbf{G}$ . First suppose  $\mathbf{F} = \mathbf{F}_{\text{ét}}$ : consider the long exact sequence

$$\text{Hom}_{\text{Ab}_k}(\mathbf{F}, \mathbf{G}) \rightarrow \text{Hom}_{\text{Ab}_k}(\mathbf{F}, E) \xrightarrow{\partial} \text{Ext}_{\text{Ab}_k}^1(\mathbf{F}, \mathbf{L}) .$$

We can consider a (Galois) extension  $k'/k$  of finite degree  $d$  trivializing  $\mathbf{F}$ . By [12, Theorem 3.9] we get the vanishing of  $\text{Ext}_{\text{Ab}_{k'}}^1(\mathbf{F}_{k'}, \mathbf{L}_{k'})$ . Recall that the multiplication by  $d$  on  $\mathbf{F}$  can be written as the composition

$$\mathbf{F} \xrightarrow{\text{can.}} \Pi_{k'/k} \mathbf{F}_{k'} \xrightarrow{\text{tr}} \mathbf{F}$$

where  $\Pi_{k'/k}$  is Weil restriction functor and  $\text{tr}$  is the trace map. This implies that  $\text{Ext}_{\text{Ab}_k}^1(\mathbf{F}, \mathbf{L})$  is torsion, hence  $\partial u = 0$  and the lift exists (up to isogeny).

In case  $\mathbf{F} = \mathbf{F}^o$  is a connected formal group we have a commutative diagram in  $\text{Ab}_k$

$$\begin{array}{ccc}
 & & \mathbf{F} \\
 & & \swarrow \downarrow u \\
 \widehat{\mathbf{G}} & \xrightarrow{\widehat{\pi}} & \widehat{\mathbf{E}} \\
 & \searrow & \swarrow \downarrow \pi \\
 & \mathbf{G} & \longrightarrow \mathbf{E}
 \end{array}$$

where  $\widehat{?}$  is the formal completion at the origin of  $? = \mathbf{G}, E$ . The formal completion is an exact functor so  $\widehat{\pi}$  is an epimorphism. The category of



connected formal groups is equivalent to  $\text{Mod}_k^f$ , thus we can choose a section  $\sigma$  of  $\hat{\pi}$ . Then we can easily construct a (non canonical) lifting of  $u$ .  $\square$

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