

JOURNAL

de Théorie des Nombres

de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 22, n° 3 (2010), p. 741-745.

<http://jtnb.cedram.org/item?id=JTNB_2010__22_3_741_0>

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A family of varieties with exactly one pointless rational fiber

par BIANCA VIRAY

RÉSUMÉ. On construit un exemple concret d'une famille à un paramètre de variétés lisses, projectives, et géométriquement intègres sur un sous-schéma ouvert de $\mathbb{P}_{\mathbb{Q}}^1$, de sorte qu'il y ait précisément une fibre rationnelle sans point rationnel. Ceci rend explicite une construction de Poonen.

ABSTRACT. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}_{\mathbb{Q}}^1$ such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

1. Introduction

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of $\mathbb{P}_{\mathbb{Q}}^1$ with the following curious arithmetic property: there is exactly one \mathbb{Q} -fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [6, Prop. 7.2], thus giving “an extreme example of geometry *not* controlling arithmetic” [6, p.2]. We believe that this is the first example of its kind.

Theorem 1.1. *Define $P_0(x) := (x^2 - 2)(3 - x^2)$ and $P_{\infty}(x) := 2x^4 + 3x^2 - 1$. Let $\pi: X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ be the Châtelet surface bundle over $\mathbb{P}_{\mathbb{Q}}^1$ given by*

$$y^2 + z^2 = (6u^2 - v^2)^2 P_0(x) + (12v^2)^2 P_{\infty}(x),$$

where π is projection onto $(u : v)$. Then $\pi(X(\mathbb{Q})) = \mathbb{A}_{\mathbb{Q}}^1(\mathbb{Q})$.

Note that the degenerate fibers of π do not lie over $\mathbb{P}^1(\mathbb{Q})$ so the family of smooth projective geometrically integral surfaces mentioned above contains all \mathbb{Q} -fibers.

The non-effectivity in [6, Prop. 7.2] stems from the use of higher genus curves and Faltings' theorem. (This is described in more detail in [6, §9]).

Manuscrit reçu le 4 septembre 2009.

This research was supported by the Mentored Research Award from University of California, Berkeley.

We circumvent the use of higher genus curves by an appropriate choice of $P_\infty(x)$.

2. Background

This information can be found in [6, §3,5, and 6]. We review it here for the reader’s convenience.

Let \mathcal{E} be a rank 3 vector sheaf on a k -variety B . A conic bundle C over B is the zero locus in $\mathbb{P}\mathcal{E}$ of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}\mathcal{E}, \text{Sym}^2(\mathcal{E}))$. A diagonal conic bundle is a conic bundle where $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $s = s_1 + s_2 + s_3, s_i \in \Gamma(\mathbb{P}\mathcal{E}, \mathcal{L}_i^{\otimes 2})$.

Now let $\alpha \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle X given by $B = \mathbb{P}^1, \mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(2), s_1 = 1, s_2 = -\alpha, s_3 = -w^4 P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - \alpha z^2 = P(x) \subset \mathbb{A}^3$ as an open subscheme. We say that X is the Châtelet surface given by

$$y^2 - \alpha z^2 = P(x).$$

Note that since $P(x)$ is not identically zero, X is an integral surface.

A Châtelet surface bundle over \mathbb{P}^1 is a flat proper morphism $V \rightarrow \mathbb{P}^1$ such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let $P, Q \in k[x, w]$ be linearly independent homogeneous polynomials of degree 4 and let $\alpha \in k^\times$. Let V be the diagonal conic bundle over $\mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(w:x)}$ given by $\mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(1, 2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2 P + b^2 Q)$. By composing $V \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with the projection onto the first factor, we realize V as a Châtelet surface bundle. We say that V is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2 P(x) + b^2 Q(x),$$

where $P(x) = P(x, 1)$ and $Q(x) = Q(x, 1)$. We can also view a, b as relatively prime, homogeneous, degree d polynomials in u, v by pulling back by a suitable degree d map $\phi: \mathbb{P}^1_{(u:v)} \rightarrow \mathbb{P}^1_{(a:b)}$.

3. Proof of Theorem 1.1

By [5], we know that the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle, i.e. it has \mathbb{Q}_v -rational points for all completions v , but no \mathbb{Q} -rational points. Thus, $\pi(X(\mathbb{Q})) \subseteq \mathbb{A}_{\mathbb{Q}}^1(\mathbb{Q})$. Therefore, it remains to show that $X_{(u:1)}$, the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x),$$

has a rational point for all $u \in \mathbb{Q}$.

If $P_{(u:1)} := (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x)$ is irreducible, then by [3], [4] we know that $X_{(u:1)}$ satisfies the Hasse principle. Thus it suffices to show that $P_{(u:1)}$ is irreducible and $X_{(u:1)}(\mathbb{Q}_v) \neq \emptyset$ for all $u \in \mathbb{Q}$ and all places v of \mathbb{Q} .

3.1. Irreducibility. We prove that for any $u \in \mathbb{Q}$, the polynomial $P_{(u:1)}(x)$ is irreducible in $\mathbb{Q}[x]$ by proving the slightly more general statement, that for all $t \in \mathbb{Q}$

$$\begin{aligned} P_t(x) &:= (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) \\ &= x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1) \end{aligned}$$

is irreducible in $\mathbb{Q}[x]$. We will use the fact that if $a, b, c \in \mathbb{Q}$ are such that $b^2 - 4ac$ and ac are not squares in \mathbb{Q} then $p(x) := ax^4 + bx^2 + c$ is irreducible in $\mathbb{Q}[x]$.

Let us first check that for all $t \in \mathbb{Q}$, $(3 + 5t^2)^2 - 4(2 - t^2)(-6t^2 - 1)$ is not a square in \mathbb{Q} . This is equivalent to proving that the affine curve $C: w^2 = t^4 + 74t^2 + 17$ has no rational points. The smooth projective model, $\overline{C}: w^2 = t^4 + 74t^2s^2 + 17s^4$ in weighted projective space $\mathbb{P}(1, 1, 2)$, has 2 rational points at infinity. Therefore \overline{C} is isomorphic to its Jacobian. A computation in **Magma** shows that $\text{Jac}(C)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}[1]$. Therefore, the points at infinity are the only 2 rational points of \overline{C} and thus C has no rational points.

Now we will show that $(-6t^2 - 1)(2 - t^2)$ is not a square in \mathbb{Q} for any $t \in \mathbb{Q}$. As above, this is equivalent to determining whether $C': w^2 = (-6t^2 - 1)(2 - t^2)$ has a rational point. Since 6 is not a square in \mathbb{Q} , this is equivalent to determining whether the smooth projective model, \overline{C}' , has a rational point. The curve \overline{C}' is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in **Magma** shows that $\text{Jac}(C')(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}[1]$. Thus $\#C'(\mathbb{Q}) = 0$ or 2. If (t, w) is a rational point of C' , then $(\pm t, \pm w)$ is also a rational point. Therefore, $\#C(\mathbb{Q}) = 2$ if and only if there is a point with $t = 0$ or $w = 0$ and one can easily check that this is not the case.

3.2. Local solvability.

Lemma 3.1. *For any point $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$, the Châtelet surface $X_{(u:v)}$ has \mathbb{R} -points and \mathbb{Q}_p -points for every prime p .*

Proof. Let $a = 6u^2 - v^2$ and let $b = 12v^2$. We will refer to $a^2P_0(x) + b^2P_\infty(x)$ both as $P_{(a:b)}$ and $P_{(u:v)}$.

\mathbb{R} -points: It suffices to show that given $(u : v)$ there exists an x such that

$$P_{(a:b)} = x^4(2b^2 - a^2) + x^2(3b^2 + 5a^2) + (-6a^2 - b^2)$$

is positive. If $2b^2 - a^2$ is positive, then any x sufficiently large will work. So assume $2b^2 - a^2$ is negative. Then $\alpha = \frac{-(3b^2 + 5a^2)}{2(2b^2 - a^2)}$ is positive. We claim

$P_{(a;b)}(\sqrt{\alpha})$ is positive.

$$\begin{aligned} P_{(a;b)}(\sqrt{\alpha}) &= \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2) \\ &= \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-(3b^2 + 5a^2)^2}{2(2b^2 - a^2)} + (-6a^2 - b^2) \\ &= \frac{1}{4(2b^2 - a^2)} \left(4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2 \right) \\ &= \frac{1}{4(2b^2 - a^2)} \left(-17b^4 - 74a^2b^2 - a^4 \right) \end{aligned}$$

Since $2b^2 - a^2$ is negative by assumption and $-17b^4 - 74a^2b^2 - a^4$ is always negative, we have our result.

\mathbb{Q}_p -points:

$p \geq 5$: Without loss of generality, let a and b be relatively prime integers. Let $\bar{X}_{(a;b)}$ denote the reduction of $X_{(a;b)}$ modulo p . We claim that there exists a smooth \mathbb{F}_p -point of $\bar{X}_{(a;b)}$ that, by Hensel's lemma, we can lift to a \mathbb{Q}_p -point of $X_{(a;b)}$.

Since $P_{(a;b)}$ has degree at most 4 and is not identically zero modulo p , there is some $x \in \mathbb{F}_p$ such that $P_{(a;b)}(x)$ is nonzero. Now let y, z run over all values in \mathbb{F}_p . Then the polynomials $y^2, P_{(a;b)}(x) - z^2$ each take $(p+1)/2$ distinct values. By the pigeonhole principle, y^2 and $P_{(a;b)}(x) - z^2$ must agree for at least one pair $(y, z) \in \mathbb{F}_p^2$ and one can check that this pair is not $(0, 0)$. Thus, this tuple (x, y, z) gives a smooth \mathbb{F}_p -point of $\bar{X}_{(a;b)}$. (The proof above that the quadratic form $y^2 + z^2$ represents any element in \mathbb{F}_p is not new. For example, it can be found in [2, Prop 5.2.1].)

$p = 3$: From the equations for a and b , one can check that for any $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$, $v_3(b/a)$ is positive. Since $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$ is an unramified extension, it suffices to show that given a, b as above, there exists an x such that $P_{(a;b)}(x)$ has even valuation. Since $v_3(b/a)$ is positive, $v_3(2b^2 - a^2) = 2v_3(a)$. Therefore, if $x = 3^{-n}$, for n sufficiently large, the valuation of $P_{(a;b)}(x)$ is $-4n + 2v_3(a)$ which is even.

$p = 2$: From the equations for a and b , one can check that for any $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$, $v_2(b/a)$ is at least 2. Let $x = 0$ and $y = a$. Then we need to find a solution to $z^2 = a^2(-7 + (b/a)^2)$. Since $v_2(b/a) > 1$, $-7 + (b/a)^2 \equiv 1^2 \pmod{8}$. By Hensel's lemma, we can lift this to a solution in \mathbb{Q}_2 .

□

Acknowledgements

I thank my advisor, Bjorn Poonen, for suggesting the problem and many helpful conversations. I thank the referee for suggesting the pigeonhole principle argument in Lemma 3.1. I also thank Daniel Erman and the referee for comments improving the exposition.

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