Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 24, n° 3 (2012), p. 669-690.

<http://jtnb.cedram.org/item?id=JTNB_2012__24_3_669_0>

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Journal de Théorie des Nombres
de Bordeaux **24** (2012), 669-690

On classical weight one forms in Hida families

par Mladen DIMITROV et Eknath GHATE

Résumé. Nous effectuons une estimation précise du nombre de spécialisations classiques en poids un d'une famille non-CM de formes modulaires propres ordinaires cuspidales. Nous donnons aussi des exemples où plusieurs familles se spécialisent sur la même forme de poids un.

Abstract. We give precise estimates for the number of classical weight one specializations of a non-CM family of ordinary cuspidal eigenforms. We also provide examples to show how uniqueness fails with respect to membership of weight one forms in families.

1. Introduction

Let p be an odd prime and let F be a primitive p-adic Hida family of ordinary, cuspidal eigenforms. By definition F admits infinitely many classical specializations of any given weight at least two. If F is a CM family, then F contains infinitely many CM classical specialization of weight *one* as well.

However, it is known that a *non-CM* family F admits only finitely many classical weight one specializations [GV04]. The first goal of this paper is to make this finiteness result effective by giving explicit bounds on the number of such specializations. There are three types of classical weight one forms (exceptional, RM and CM) and a necessary condition for a Hida family to contain a classical weight one specialization of a given type is to be residually of the same type (see §4 for precise definitions). In the exceptional case, under some mild assumptions, we show that there is exactly one classical weight one specialization (see Theorem 5.1 and Proposition 5.2). In the RM case, we provide examples of Hida families with more than one classical weight one specializations (see §6.2) and provide an explicit bound involving the class number and fundamental unit of the underlying real quadratic number field (see Theorem 6.4). Finally, in the CM case, we provide a bound as well (see §6.3).

The second goal of this paper is to study the opposite question, which is whether there exists a unique Hida family, up to Galois conjugacy, specializing to a given ordinary classical weight one p-stabilized newform (the

Manuscrit reçu le 4 octobre 2011.
Classification math. 11F80, 11F33, 11R23.

existence of such a family is a theorem of Wiles [W88]). Recall that by a theorem of Hida, there is a unique, up to Galois conjugacy, primitive Hida family specializing to a given classical ordinary p-stabilized newform of weight two or more, and the new-quotient of Hida's ordinary Hecke algebra is étale over the Iwasawa algebra at the corresponding height one prime.

However, in §7.2, we provide examples where the new-quotient of the ordinary Hecke algebra is not étale at height one primes corresponding to classical weight one forms with RM. This phenomenon was first investigated by Cho and Vatsal [CV03, Corollary 3.3] in the context of studying universal deformation rings of residually RM Galois representations. Our arguments are also related to the existence of Λ-adic inner twists for the family, a topic of independent interest (see §7.3 and the Appendix). Finally, in §7.4, we provide an example of two non-Galois conjugate Hida families specializing to the same classical weight one form.

Acknowledgments. We thank L. Dembélé, J.-F. Mestre and G. Wiese for their advice regarding the numerical computations with Magma.

2. Hida families

Let p be a prime and fix an embedding of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_p$. Let $\Lambda = \mathbb{Z}_p[[1+p^\nu\mathbb{Z}_p]] \cong \mathbb{Z}_p[[X]]$ denote the classical Iwasawa algebra, where $\nu = 2$ if $p = 2$ and $\nu = 1$ otherwise. Fix an integer N, prime to p.

2.1. Specializations of Λ-adic cuspforms. Since there are several different normalizations in the literature concerning specializations, we fix one now which is adapted to the study of weight one forms.

Let L be the integral closure of Λ in a finite extension of its fraction field. A weight $k \geq 1$ specialization is an algebra homomorphism $L \to \bar{\mathbb{Q}}_p$ which extends the homomorphism on Λ given by $X \mapsto \zeta(1+p^\nu)^{k-1} - 1$, where ζ is a p-power root of unity.

By definition a Λ-adic ordinary cuspform of tame level N is a formal q-expansion in $L[[q]]$, such that its specializations in weights $k \geq 2$ are q-expansions of p-stabilized, ordinary, normalized cuspforms of tame level N and weight k.

If ζ is a primitive p^r-th root of unit, $r \geq 0$, then the level of the specialized form is $Np^{r+\nu}$. The normalization of the weight is slightly different from that used in earlier papers of Hida and Wiles, and also from that used in [GV04], where k was used instead of $k-1$, and so some of the formulas for the determinant etc. are slightly different from those used there. In particular, weight one specializations are obtained at $X = \zeta - 1$ when $k = 1$ (and ζ is arbitrary).

A Λ-adic cuspform is said to be a newform if all specializations in weights ≥ 2 are ordinary, cuspidal, p-stabilized newforms (see Hida [H86, Corollary 3.7] and its proof on p. 265, for $p \geq 5$).

2.2. Ordinary Hecke algebras. We define the universal ordinary Hecke algebra $\mathbb{T}_N$ of tame level N as the Λ-algebra generated by the Hecke operators U_ℓ (resp. T_ℓ, $\langle \ell \rangle$) for primes ℓ dividing Np (resp. not dividing Np) acting on the space of Λ-adic ordinary cuspforms of tame level N. Denote by $\mathbb{T}_N^{\mathrm{new}}$ its new-quotient which acts on the space of Λ-adic ordinary cuspforms of tame level N which are N-new. Hida constructed (see [H85], [EPW06]) a continuous representation:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_N^{\mathrm{new}} \otimes_\Lambda \overline{\mathrm{Frac}(\Lambda)}) \tag{2.1}$$

with several natural properties.

A height one prime of $\mathbb{T}_N$ is arithmetic if it sits above the height one prime $P_{k,\zeta}$, which is the kernel of the map $\Lambda \to \bar{\mathbb{Q}}_p$ induced by $X \mapsto \zeta(1+p^\nu)^{k-1} - 1$, for some $k \geq 2$ and some p-power root of unity ζ.

2.3. Hida families and communities. A primitive Hida family F of tame level N is by definition a Λ-adic ordinary cuspidal newform which is an eigenform, i.e., a common eigenfunction of the operators U_ℓ, T_ℓ and $\langle \ell \rangle$ as above.

The central character $\psi_F : (\mathbb{Z}/Np^\nu)^\times \to \mathbb{C}^\times$ of the family is defined by $\psi_F(\ell) =$ eigenvalue of $\langle \ell \rangle$. A primitive family determines and is uniquely determined by a Λ-algebra homomorphism $\mathbb{T}_N^{\mathrm{new}} \to K_F$, where K_F denotes the field generated by the Fourier coefficients (or equivalently the Hecke eigenvalues) of F. It follows from the control theorem of Hida that K_F is a finite field extension of the field of fractions of Λ. By (2.1) one can attach to F a continuous irreducible representation:

$$\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_F) \tag{2.2}$$

which is unramified outside Np and such that, for all primes ℓ not dividing Np, the trace of the image under ρ_F of an arithmetic Frobenius at ℓ equals the ℓ-th Fourier coefficient of F. Moreover $\det \rho_F = \psi_F \chi_{\mathrm{cyc}}$, where χ_{cyc} is the universal Λ-adic cyclotomic character given by composing the surjection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = 1 + p^\nu \mathbb{Z}_p$, where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, with the map $1 + p^\nu \mathbb{Z}_p \to \Lambda^\times$ induced by $1 + p^\nu \mapsto 1 + X$.

In fact, Hida's construction of ρ_F assumed $p \geq 5$, but it was later simplified and extended to the case of $p = 2$ and $p = 3$ by Wiles.

Specializing $\det \rho_F$ at $X = \zeta(1+p^\nu)^{k-1} - 1$, we obtain a character $\psi \chi_p^{k-1}$ with

$$\psi = \psi_F \omega_p^{1-k} \chi_\zeta \tag{2.3}$$

of finite order, where χ_p is the usual p-adic cyclotomic character, ω_p is the Teichmüller lift of χ_p mod p, if p is odd, and is the quadratic character of conductor 4 if $p = 2$, and χ_ζ is the wild Dirichlet character of p-power conductor taking $1 + p^\nu$ to ζ. Note that ψ is just the Nebentypus character of the underlying specialization. In weight one $\psi = \psi_F \chi_\zeta$ has a particularly simple form.

For later use we coin some terminology.

Definition 2.1. A Hida community, denoted by $\{F\}$, is the (finite) union of all families F of the same tame level having the same residual representation $\bar{\rho}_F$.

In terms of Hida's ordinary Hecke algebra $\mathbb{T}_N^{\mathrm{new}}$:

• minimal primes correspond to $\mathrm{Aut}_\Lambda(\overline{\mathrm{Frac}(\Lambda)})$-orbits of Hida families,

• maximal primes correspond to $\mathrm{Aut}_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$-orbits of Hida communities,

• height one arithmetic primes correspond $\mathrm{Aut}_{\mathbb{Q}_p}(\overline{\mathbb{Q}_p})$-orbits of classical weight $k \geq 2$ specializations.

An important theorem due to Hida states that for all $k \geq 2$ and all ζ a p-power root of unity, the $\Lambda_{P_{k,\zeta}}$-algebra $\mathbb{T}_N^{\mathrm{new}} \otimes_\Lambda \Lambda_{P_{k,\zeta}}$ is étale, at least if p is odd. The proof uses his exact control theorem for the ordinary Hecke algebra in weights $k \geq 2$.

3. Finiteness results for non-CM families

A Hida eigenfamily is said to be CM if its Λ-adic q-expansion is the theta series attached to a Λ-adic Hecke character over a quadratic extension $K/\mathbb{Q}$, which is necessarily imaginary. Thus if F is CM, then $\rho_F \cong \rho_F \otimes \varepsilon_{K|\mathbb{Q}}$, where $\varepsilon_{K|\mathbb{Q}}$ is the quadratic character attached to $K/\mathbb{Q}$. While there is no classicity condition on the weight one specializations in the notion of a Hida family, a CM family has infinitely many classical (CM) specializations in all weights, including in weight one. Following the classical case, one says that a Hida eigenfamily is non-CM, if it is not a CM family. This definition is reasonable since it turns out that the notion of CM-ness is pure with respect to families in weight strictly bigger than one (at least if p is odd; for the case of $p = 2$, see [GK12]). That is, each weight two or more classical specializations of a non-CM (respectively CM) family is always a non-CM (CM) form. However, in weight one this is, somewhat surprisingly, false: there are examples of non-CM families containing classical CM forms of weight one.

The following finiteness result for the number of classical weight one specializations in a non-CM family was proved in the course of the proof of the main result of [GV04] (see in particular the proof of the implications (ii) $\implies$ (iii) $\implies$ (iv) of Prop. 14 in that paper). It was stated there for odd primes p, under some conditions on the residual representation attached to F: an absolute irreducibility hypothesis over the quadratic field corresponding to the (odd) prime p, and a p-distinguished hypothesis as well, though these conditions were needed elsewhere in the proof. We state the result again, without these conditions, and shorten considerably a key step in the proof.

Proposition 3.1. *A non-CM eigenfamily admits only finitely many classical weight one specializations.*

Proof. Assume that F has infinitely many classical weight one specializations. By [GV04, Prop. 14, (ii) $\implies$ (iii)], F then contains infinitely many CM forms with CM by the same imaginary quadratic field, call it K. A quick check shows that the argument also works for $p = 2$. In [GV04, Prop. 14, (iii) $\implies$ (iv)], a somewhat lengthy argument (for odd primes p) was given to show that this forces F to be a CM family. While this argument had the advantage of being explicit, in that it explicitly constructed a Λ-adic Hecke character Φ which interpolates the finite order characters φ corresponding to the weight one forms above, it is possible to give a much shorter proof of this part, as follows.

One first notes that since the Fourier coefficients at the inert primes in K vanish for all the CM weight one specializations, so do the corresponding Fourier coefficients in the q-expansion of the corresponding Λ-adic form F. Indeed in the coefficient ring of F, which is the integral closure of Λ in a finite extension of the quotient field of Λ, the intersection of infinitely many height one primes is $\{0\}$. Hence, $\mathrm{Tr}(\rho_F)(\mathrm{Frob}_\ell) = \mathrm{Tr}(\rho_F \otimes \varepsilon_{K|\mathbb{Q}})(\mathrm{Frob}_\ell)$ for all but finitely many primes ℓ, since this is vacuous for split primes, and for inert primes both sides vanish. It follows that $\rho_F \cong \rho_F \otimes \varepsilon_{K|\mathbb{Q}}$ (a priori, only up to semi-simplification, but it is well known that ρ_F is irreducible). It follows that $\rho_F \cong \mathrm{Ind}_K^{\mathbb{Q}} \Phi$ for some Λ-adic Hecke character Φ, which shows that F has CM. □

P. Sarnak once asked whether the above result can be made effective. The first goal of this paper is to show that this is indeed possible, and several results to this end will be described below. Our approach is Galois theoretic, since it uses ρ_F rather then F itself, and we believe that it can be adapted for Hilbert modular forms.

4. Residual type of a family

For the rest of this paper we mostly assume p is odd, though occasionally we point out some interesting phenomena that could occur if $p = 2$.

Our results will be decomposed according to the residual type of the Hida family F, which we define now.

The projective image of the residual Galois representation $\bar{\rho}_F$ attached to F lies in $\mathrm{PGL}_2(\bar{\mathbb{F}}_p)$. We recall the following well-known theorem.

Theorem 4.1 (Dickson). *Assume $p \geq 3$. Let $\mathbb{F} \supset \mathbb{F}_p$ be a finite field and let H be a subgroup of* $\mathrm{PGL}_2(\mathbb{F})$. *Then either*

(1) *H is a subgroup of a Borel subgroup in* $\mathrm{PGL}_2(\mathbb{F}')$ *with $\mathbb{F}'/\mathbb{F}$ quadratic, or,*
(2) *H is conjugate to* $\mathrm{PGL}_2(\mathbb{F}')$ *or* $\mathrm{PSL}_2(\mathbb{F}')$, *for $\mathbb{F}'$ a subfield of $\mathbb{F}$, or,*
(3) *H is isomorphic to D_{2m} with m coprime to p, or A_4, S_4, A_5.*

While, in general, $\bar{\rho}_F$ may have image as described in (1) and (2) of Dickson's theorem, we have:

Lemma 4.2. *Let $p \geq 3$. If F is an eigenfamily with a classical weight one specialization, then $\bar{\rho}_F$ must have projective image as in part* (3) *of* Theorem 4.1.

Proof. Indeed, it is well known that the Galois representation associated to a classical weight one form has projective image that is either D_{2n}, for some n, or A_4, S_4, or A_5. The corresponding form is said to be of dihedral (respectively tetrahedral, octahedral, icosahedral) type. The image of the Galois representation in $\mathrm{GL}_2(\mathbb{C})$ can be taken to lie in $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in a p-adic field. Since the reduction map from $\mathrm{PGL}_2(\mathcal{O})$ to $\mathrm{PGL}_2(\mathbb{F})$ has kernel a pro-p group, the projective image of such a representation in $\mathrm{PGL}_2(\mathcal{O})$ injects into $\mathrm{PGL}_2(\mathbb{F})$, if it has order prime to p or if, at least, it does not contain a non-trivial normal subgroup of p-power order. When p is odd, A_4, S_4, and A_5 do not contain a normal subgroup of p-power order, so these groups inject under reduction. On the other hand D_{2n} contains a normal subgroup of each order $d \geq 3$ dividing n and the corresponding subgroup of p-power order must die under the reduction map. Thus the reductions of these groups are as in part (3) of Dickson's theorem. Since each specialization of a family F gives rise to the same residual representation $\bar{\rho}_F$, we see that if F has a classical weight one specialization, then $\bar{\rho}_F$ must also have projective image as described in part (3) of Dickson's theorem. $\square$

Thus, in view of our goal to estimate the number of classical weight one forms in a non-CM family, by Lemma 4.2, we may as well restrict to families for which the projective image of $\bar{\rho}_F$ is as in Theorem 4.1, part (3). This motivates the following definition.

Definition 4.3. Say that a Hida family F is of residually dihedral type (respectively residually tetrahedral, octahedral or icosahedral type) if the projective image of $\bar{\rho}_F$ is dihedral (respectively tetrahedral, octahedral, or icosahedral). In the last three cases, we say that F is residually of exceptional type.

CM families are clearly residually of dihedral type, but non-CM families may also be residually of dihedral type. We also point out that being of residually dihedral or exceptional type is a necessary condition for a Hida family to have a classical weight one specialization, but not a sufficent condition. We prove this following some remarks of the referee. First note the following easily checked fact which we learned from Greenberg and Vatsal.

Lemma 4.4. *If G is a primitive Hida family of tame level N which is of Steinberg type at some prime ℓ dividing N (that is, $\ell || N$ and ψ_G is trivial at ℓ), then G has no classical weight one specializations.*

Now, start with any p-ordinary newform f of level M, with $p \nmid M$, and weight 2, which is residually of exceptional or dihedral type. Choose an auxiliary prime $\ell \equiv 1 \pmod p$ such that the projective image of $\bar{\rho}_f(\mathrm{Frob}_\ell)$ is trivial, so that

$$\frac{(\mathrm{Tr}\,\bar{\rho}_f(\mathrm{Frob}_\ell))^2}{\det \bar{\rho}_f(\mathrm{Frob}_\ell)} = 4 \equiv \ell^{-1}(1+\ell)^2 \pmod p.$$

By Ribet's level raising theorem, there exists a newform g of weight 2, level $N = M\ell$ and trivial Nebentypus at ℓ, which is congruent to f modulo a prime ideal sitting above p. Let G be the (non-CM) primitive Hida family specializing to (the p-stabilization) of g. It is residually of the same type as f, i.e., exceptional or dihedral. G also has tame level $N = M\ell$ and trivial character at ℓ, hence is of Steinberg type at ℓ. But, by Lemma 4.4, G has no classical weight one specializations.

However, we have the following lemma, whose proof is similar to that of Lemma 4.2.

Lemma 4.5. *Let $p \geq 3$. If a Hida family F is residually of dihedral or exceptional type, then the Galois representation of every classical weight one specialization f (if any) must be of the same kind (all dihedral, or all tetrahedral, or all octahedral, or all icosahedral). Moreover, in the exceptional case the projective images of ρ_f and $\bar{\rho}_F$ are isomorphic, while in the dihedral case, if the projective image of ρ_f is $D_{2n} \subset \mathrm{PGL}_2(\mathcal{O})$, then the projective image of $\bar{\rho}_F$ is $D_{2m} \subset \mathrm{PGL}_2(\mathbb{F})$, where m is the prime-to-p part of n.*

Remark 4.6. Families of dihedral type with $m = 1$ have residually reducible Galois representation, whereas those with $m \geq 2$ have residually absolutely irreducible Galois representation.

Remark 4.7. The lemma may no longer be true for $p = 2$. The group S_4 (resp. A_4) contains the Klein four group D_4 as a normal subgroup and the quotient is isomorphic to S_3 (resp. $\mathbb{Z}/3\mathbb{Z}$). Since $S_3 \simeq \mathrm{PGL}_2(\mathbb{F}_2)$, there could be an octahedral (resp. tetrahedral) weight one form whose Galois representation has projective residual image contained in S_3. On the other hand, A_5 is a simple group isomorphic to $\mathrm{PGL}_2(\mathbb{F}_4)$, hence an icosahedral weight one form has projective residual image of the same type.

5. Families of exceptional type

We now give a bound on the number of classical weight one forms in a family F which is residually of exceptional type. We assume p is odd from now on. Recall that a p-ordinary representation (Λ-adic, classical, or residual) is said to p-distinguished if the semisimplification of the underlying local residual representation obtained by restricting to the decomposition subgroup G_p at p is a direct sum of two distinct characters.

Theorem 5.1. *If F is residually of exceptional type, then F has at most four classical weight one specializations. More precisely, F has at most $a \cdot b$ classical weight one specializations, where*

- *$a = 1$, except if $p = 5$ and the type of F is icosahedral, in which case $a = 2$, and*
- *$b = 1$, except if $p = 3$ or 5 and $\bar\rho_F$ is p-ramified but not p-distinguished, in which case $b = 2$.*

Thus, if $p \geq 7$, then F has at most one classical weight one specialization.

Proof. It is enough to show that there are at most a choices for the projective trace $\frac{(\mathrm{Tr}\,\rho_f)^2}{\det \rho_f}$ and for each of them at most b choices for the determinant of the Galois representation ρ_f of a classical weight one specialization f of F. In fact, knowing $(\mathrm{Tr}\,\rho_f)^2$ and the fact that $\mathrm{Tr}\,\rho_f$ is congruent to $\mathrm{Tr}\,\bar\rho_F$ modulo p uniquely determines $\mathrm{Tr}\,\rho_f$ since p is odd, hence f.

By Lemma 4.5, the fields cut out by the projectivizations of ρ_f and $\bar\rho_F$ are the same, and by assumption have Galois group A_4, S_4 or A_5. It is well known that A_4 and S_4 have a unique (up to conjugacy) embedding in $\mathrm{PGL}_2(\mathbb{C})$, whereas A_5 has two such embeddings [S77, p. 247]. It follows immediately that there is at most one choice for the projective trace in the A_4 and S_4 cases. In the A_5 case we note that the two embeddings of A_5 can be congruent modulo p only for $p = 5$. In fact, since $\bar\rho_F(g)$ has projective order ≤ 5, a standard computation shows that its projective trace

$$\frac{(\mathrm{Tr}\,\bar\rho_F(g))^2}{\det \bar\rho_F(g)} \text{ belongs to the set } \{4, 0, 1, 2, \text{ roots of } X^2 - 3X + 1\}.$$

Hence, the projective trace of $\bar\rho_F(g)$ together with the projective order of $\rho_f(g)$ uniquely determine the projective trace of $\rho_f(g)$, unless $p = 5$ and $\rho_f(g)$ has projective order 5, in which case there are two choices. Hence in all cases there are at most a choices for the projective trace.

It remains to show that there are at most b choices for the determinant of a classical weight one specializations f of F with a given projective trace.

Let I_p be the inertia subgroup at p. Then by ordinariness $\rho_F|_{I_p}$ has the shape $\left(\begin{smallmatrix} \det\rho_F & * \\ 0 & 1 \end{smallmatrix}\right)$. Hence $\rho_f|_{I_p}$ has the shape $\left(\begin{smallmatrix} \psi_F\chi_\zeta & * \\ 0 & 1 \end{smallmatrix}\right)$ and therefore the image of I_p injects into $\mathrm{PGL}_2(\mathbb{C})$. This immediately implies that for $p \geq 7$, we have $\zeta = 1$, hence $\det\rho_f = \psi_F$.

Assume now that f and f' are two distinct classical weight one specializations of F sharing the same projective trace. Then, there exists a p-power order character $\varepsilon : G_{\mathbb{Q}} \to \mathcal{O}^\times$ such that $\rho_f \cong \varepsilon \otimes \rho_{f'}$. For the remaining part of the proof, we assume that $\varepsilon \neq 1$, in particular $p = 3$ or 5. By using the determinant, we see that ε is unramified outside p. Indeed the relation $\psi_F\chi_\zeta = \psi_F\chi'_\zeta\varepsilon^2$ shows that $\varepsilon^2 = 1$ at primes away from p, and since ε has p-power order, with p odd, the claim follows. Finally, by ordinariness

$$\begin{pmatrix} \psi_F\chi_\zeta & * \\ 0 & 1 \end{pmatrix} \cong \rho_f|_{I_p} \cong \varepsilon \otimes \rho_{f'}|_{I_p} \cong \begin{pmatrix} \psi_F\chi_{\zeta'}\varepsilon & * \\ 0 & \varepsilon \end{pmatrix}.$$

Comparing cross terms on the diagonal, it follows that ψ_F is unramified at p and $\varepsilon = \chi_\zeta = \chi_{\zeta'}^{-1}$. In particular $\zeta' = \zeta^{-1}$, hence there are at most two choices for the determinant.

By examining further $\rho_f|_{I_p} \cong \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we see that $\rho_f(I_p)$ is a cyclic group of order p, since A_4, S_4, A_5 have no elements of order p^2. By Lemma 4.5 the same is true for $\bar\rho_F(I_p)$ showing that $\bar\rho_F|_{I_p} \cong \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is (wildly) ramified at p. Also by ordinariness, the restriction to the decomposition group G_p is:

$$\rho_f|_{G_p} \cong \begin{pmatrix} \mathrm{unr}(a_p)^{-1}\psi_F\chi_\zeta & 0 \\ 0 & \mathrm{unr}(a_p) \end{pmatrix},$$

where $\mathrm{unr}(a_p)$ denotes the unramified character of G_p sending a Frobenius to $a_p(f)$. In particular, the image of G_p is abelian. It follows that the reduction also has abelian image. So the semi-simplification of the residual representation on G_p must be a direct sum of the same character. Hence $\bar\rho_F$ is not p-distinguished as required. □

As mentioned in Section 4, there are Hida families that are residually of exceptional type which have no classical weight one specializations. However, we now show that under some mild conditions, a Hida community which is residually of exceptional type has at least one classical weight one specialization. Note that it makes sense to speak of the residual type of a Hida community, since each family in the community shares a common residual representation. For the same reason, it makes sense to speak of a community being p-distinguished. We note that some authors, e.g. [EPW06], use the terminology Hida family for Hida community. With our definitions, we have:

Proposition 5.2. *Let $p \geq 7$ and $\{F\}$ be a Hida community of exceptional type which is p-distinguished and such that the tame level N of $\{F\}$ is the same as the Artin conductor of $\bar{\rho}_F$. Then $\{F\}$ has at least one classical weight one specialization f. Moreover, any other classical weight one specialization of $\{F\}$ can be written as $f \otimes \varepsilon$, where ε is a p-power Dirichlet character of conductor dividing N. In particular, if p does not divide $\phi(N)$, where ϕ is Euler's totient function, then $\{F\}$ has a unique classical weight one specialization.*

Proof. Since the image of $\bar{\rho}_F$ in $\mathrm{GL}_2(\mathbb{F})$ has order prime to p, we can consider its Teichmüller lift $\widetilde{\rho}$ to $\mathrm{GL}_2(W(\mathbb{F}))$ where $W(\mathbb{F})$ are the Witt vectors of $\mathbb{F}$. Such a representation is necessarily split on G_p (it is ordinary, hence reducible on G_p, and since it is a finite image characteristic zero representation, it is semisimple on G_p). By hypothesis it is also p-distinguished. By a modularity lifting theorem of Buzzard [B03] and Buzzard-Taylor [BT99] it arises from a classical weight one cusp form f, and by a theorem of Wiles [W88] there exists a Hida family G specializing to f. This family $G \in \{F\}$ since the tame level of f, hence G, is the same as $\{F\}$ by assumption (*a priori* it could have been of smaller tame level).

For the second claim, let f and g be two classical weight one specializations of $\{F\}$. By the proof of Theorem 5.1, it follows that $g = f \otimes \varepsilon$ for some p-power order Dirichlet character of conductor dividing Np^r, for some $r \geq 1$, and an inspection of the ordinariness condition, as in the proof of Theorem 5.1, shows that ε is unramified at p. This completes the proof of the proposition. $\square$

6. Families of dihedral type

Let F be a non-CM Hida family residually of dihedral type with coefficients in a finite extension L of Λ. We continue to assume that $p \geq 3$.

Since $\bar{\rho}_F$ is projectively dihedral there is a quadratic extension K of $\mathbb{Q}$ such that $\bar{\rho}_F \cong \bar{\rho}_F \otimes \varepsilon_{K|\mathbb{Q}}$, where $\varepsilon_{K|\mathbb{Q}}$ is the quadratic character associated to $K/\mathbb{Q}$. Alternatively, $\bar{\rho}_F \cong \mathrm{Ind}_K^{\mathbb{Q}} \bar{\varphi}$ for a finite order character $\bar{\varphi} : G_K \to \bar{\mathbb{F}}^\times$. Note that there is not always a unique choice for K. However, we have the following easily proved lemma.

Lemma 6.1. *If the projective image of $\bar{\rho}_F$ has order more than four, then K is unique. If the image is the Klein four group D_4, then there are three possibilities for K.*

In the D_4 case, the three quadratic fields K cannot all be real, by the oddness of $\bar{\rho}_F$, and so one must be real and the other two imaginary. Such residually dihedral families will play an important role in §7.4.

Definition 6.2. If the field K is imaginary, we shall say that F is residually of CM type and if the field K is real, we say that F is residually of RM type.

Thus, in the D_4 case, a family is residually both of RM and CM type.

We wish to estimate the number of weight one forms in residually dihedral families. We start with the following easy but useful lemma.

Lemma 6.3.

(1) *If F has a classical weight one specialization f such that $\rho_f(I_p)$ has order at least 3, then p splits in K.*

(2) *If the number of classical weight one specializations of F, up to Galois conjugacy, is greater than the degree $[K_F : \mathrm{Frac}(\Lambda)]$, then p splits in K.*

Proof. (1) Let us first prove that $K/\mathbb{Q}$ is unramified at p. By ordinariness $\rho_f(I_p)$ injects in its projectivization. A cyclic subgroup of order ≥ 3 of a dihedral group is contained in its unique maximal cyclic subgroup. Our assumption on the order of $\rho_f(I_p)$ implies that I_p is contained in G_K, that is, $K/\mathbb{Q}$ is unramified at p. Now $\rho_f(G_K) = \bar{\varphi} \oplus \bar{\varphi}'$. Let $\mathfrak{p}$ be a prime of K lying over p. Exactly one of the characters $\bar{\varphi}$, $\bar{\varphi}'$ is ramified at $\mathfrak{p}$, by ordinariness. Thus p splits in K.

(2) Indeed, since the number of height one primes of L lying above a particular height one prime of Λ is bounded by the given degree, at least one of the specializations in weight one corresponds to an algebra homomorphism $L \to \bar{\mathbb{Q}}_p$ sending $X \in \Lambda$ to $\zeta - 1$, with $\zeta \neq 1$. Then, as in the proof of Theorem 5.1, one sees that $\rho_f(I_p)$ would have order divisible by $p \geq 3$, and so p would split in K by (1). $\square$

Lemma 6.3 is false when the inertial order is 2. For example, the 23-adic family F specializing to the Ramanujan Δ function in weight 12, has a CM form f of weight 1 with $\rho_f|_{I_{23}} = \varepsilon_{K|\mathbb{Q}} \oplus 1$, for $K = \mathbb{Q}(\sqrt{-23})$, and 23 ramifies in K. Note that when p ramifies in K, the family F is necessarily a non-CM family (otherwise F would have classical weight one specializations with corresponding Galois representation having arbitrarily large inertial order, thus contradicting Lemma 6.3; alternatively, F would have weight 2 members which are non-ordinary at p). In particular, $F \otimes \varepsilon_{K|\mathbb{Q}}$ is another family specializing to f. However, $F \otimes \varepsilon_{K|\mathbb{Q}}$ it is not ordinary since p ramifies in K, and so does not fit into the framework of classical Hida theory, but it is *nearly* ordinary in the sense that, after specialization, the corresponding Galois representation is locally reducible at p. This discussion shows that étaleness of the nearly ordinary Hecke algebra fails at weight one classical specializations. We shall soon give examples where étaleness fails at primes in the *ordinary* Hecke algebra, corresponding to classical weight one forms.

6.1. Families residually of RM type. The following proposition bounds the number or classical weight one forms in the RM case in terms of invariants attached to the underlying quadratic field.

Theorem 6.4. *Let F be a non-CM family of tame level N, and K be real quadratic such that $\bar{\rho}_F \cong \bar{\rho}_F \otimes \varepsilon_{K|\mathbb{Q}}$. Then the number of classical weight one specializations of F is bounded by the p-part of*

$$|\mathrm{Cl}_K| \cdot \mathrm{N}_{K/\mathbb{Q}}(\epsilon_K^{p-1} - 1) \cdot \prod_{\substack{\ell \mid N \\ \ell \text{ split in } K}} (\ell - 1) \cdot \prod_{\substack{\ell \mid N \\ \ell \text{ inert in } K}} (\ell + 1),$$

where Cl_K denotes the class group of K and ϵ_K is a fundamental unit of K.

Proof. By hypothesis $\bar{\rho}_F \cong \mathrm{Ind}_K^{\mathbb{Q}} \bar{\varphi}$ for some finite order character $\bar{\varphi} : G_K \to \bar{\mathbb{F}}^\times$. If F has a classical weight one specialization, its Galois representation must be of the form $\rho \cong \mathrm{Ind}_K^{\mathbb{Q}} \varphi$, for a finite order character φ which lifts $\bar{\varphi}$. Then any other classical weight one specialization with Galois representation $\rho' \cong \mathrm{Ind}_K^{\mathbb{Q}} \varphi'$ is uniquely determined by the p-power order character $\xi = \varphi/\varphi' : G_K \to \mathcal{O}^\times$. Hence counting classical weight one specialization of F amounts to counting such characters ξ.

If $\xi = 1$, then $\rho \cong \rho'$ and we are done. Otherwise, by ordinariness, it follows from the proof of Lemma 6.3 that p splits in K and that ξ is ramified only at one of the primes of K dividing p, say $\mathfrak{p}$.

Let $\mathrm{Ver} : G_{\mathbb{Q}}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ be the transfer map. By taking the determinants we see that the Dirichlet character

$$\xi \circ \mathrm{Ver} = \frac{\det \rho}{\det \rho'} = \frac{\chi_\zeta}{\chi_{\zeta'}}$$

has p-power order and conductor.

By class field theory, we can see ξ as a character of the finite group $\mathrm{Cl}_K^+(N\mathfrak{p}^\infty)$ sitting in the following exact sequence :

$$1 \to \overline{\langle \epsilon_K \rangle} \to \mathbb{Z}_p^\times \times (\mathcal{O}_K/N)^\times \to \mathrm{Cl}_K^+(N\mathfrak{p}^\infty) \to \mathrm{Cl}_K^+ \to 1,$$

where Cl_K^+ denotes the narrow class group of K. Since we are only interested in characters ξ having p-power order and conductor such that $\xi \circ \mathrm{Ver}$ is unramified outside p, it follows that the restriction of ξ to $\mathbb{Z}_p^\times \times (\mathcal{O}_K/N)^\times$ factors through its finite quotient:

$$(1 + p\mathbb{Z}_p)/\overline{\langle \epsilon_K^{p-1} \rangle} \prod_{\substack{\ell \mid N \\ \ell \text{ split in } K}} \mathbb{F}_\ell^\times \prod_{\substack{\ell \mid N \\ \ell \text{ inert in } K}} \mathbb{F}_{\ell^2}^\times / \mathbb{F}_\ell^\times.$$

Since p is odd, the p-parts of $|\mathrm{Cl}_K^+|$ and $|\mathrm{Cl}_K|$ are equal, hence the bound given in the statement of the proposition. $\square$

6.2. An example of a family with more than one weight one specializations. By inspecting the proof above, one can exhibit non-CM Hida families having more than one classical weight one specialization.

It is not difficult to find a real quadratic field K and a prime p which splits in K as $\mathfrak{p}\mathfrak{p}'$, and such that

$$[\mathrm{Cl}_K(\mathfrak{p}^2) : \mathrm{Cl}_K(\mathfrak{p})] = p. \tag{6.1}$$

For example, one can take $K = \mathbb{Q}(\sqrt{23}) = \mathbb{Q}(\sqrt{4 \cdot 23})$ and $p = 7$, in which case $\mathrm{Cl}_K(\mathfrak{p})$ is trivial, whereas $\mathrm{Cl}_K(\mathfrak{p}^2)$ has order 7. Let ε_D be the quadratic character corresponding to the quadratic field $\mathbb{Q}(\sqrt{D})$ of discriminant D. The space of weight one forms of level $4 \cdot 7 \cdot 23$ and Nebentypus $\varepsilon_{-4}\varepsilon_{-7}\varepsilon_{-23}$ has dimension 6 and contains two newforms with rational coefficients and one with coefficients in $\mathbb{Q}(\zeta_8)$. Denote by f one of the forms with rational coefficients. It has RM by K and its existence can also be seen using the construction described in §7.1. Now f is ordinary at $p = 7$, and we let F be a 7-adic family of tame level $4 \cdot 23$ specializing to f. The projective image of ρ_f is the Klein four group since f also has CM by $\mathbb{Q}(\sqrt{-7})$ and by $\mathbb{Q}(\sqrt{-4 \cdot 7 \cdot 23})$. In particular, F has to be a non-CM family, since 7 ramifies in these fields.

Now, by (6.1), there exists a 7-ordinary weight one newform f' with RM by K of level $4 \cdot 7^2 \cdot 23$ and Nebentypus $\varepsilon_{-4}\psi_7\varepsilon_{-23}$, where ψ_7 denotes a character of order 14 of $(\mathbb{Z}/49\mathbb{Z})^\times$ (see the last row in Table 7.1). It is easy to see using Lemma 4.5 that f and f' share the same residual Galois representation $\bar{\rho}$, that is to say, occur as specializations of the same Hida community $\{F\}$.

It remains to prove that f and f' are specializations of the same Hida family. The family F occurs in the last entry of Table 7.2. In particular, by (7.1), one has that $\mathrm{rk}_\Lambda \mathbb{T}^{\mathrm{new}}_{92,\bar{\rho}} = 2$, hence F and $F \otimes \varepsilon_{-4}\varepsilon_{-23}$ are the only two Hida families in $\{F\}$. By Wiles' theorem alluded to in the introduction each of f and f' is the specialization of at least one of these families. Since $f = f \otimes \varepsilon_{-4}\varepsilon_{-23}$ and $f' = f' \otimes \varepsilon_{-4}\varepsilon_{-23}$, they are both specializations of both F and $F \otimes \varepsilon_{-4}\varepsilon_{-23}$.

6.3. Families residually of CM type. In this section we assume that K is imaginary quadratic and that p splits in K, and that F is a non-CM Hida family, residually of CM type by K. (While the level raising argument in Section 4 gives examples of such families, we do not know of an example of such a family with a classical weight one specialization.)

We give a weak bound for the number of classical weight one specializations in F. Say that the Fourier coefficients of F lie in L. Since F is non-CM, there exists a prime ℓ inert in K, such that the Fourier coefficient $a_\ell(F) \in L$ is non-zero (since, as already proved in the course of proving Proposition 3.1, otherwise F would have CM by K). Then $a_\ell(F)$ lies in only finitely many height one primes of L, and in particular $a_\ell(F)$ lies only in a finite number of primes of L lying above of primes of Λ induced by $X \mapsto \zeta - 1$, for ζ a p-power root of unity. Denote by $\lambda_{F,\ell}$ this number. When $L = \Lambda$ then $\lambda_{F,\ell}$ is bounded by the the degree of the distinguished polynomial part of $a_\ell(F)$ under the Weierstrass preparation theorem. Put

$$\lambda_F = \min_{\ell \text{ inert in } K} \lambda_{F,\ell}.$$

Lemma 6.5. *Let F be a non-CM Hida family, residually of CM type, with CM by K. Then the number of weight one specializations of F is bounded by λ_F.*

Proof. Say f is a classical weight one specialization of F at a point of L lying over $X \mapsto \zeta - 1$. Then f must have CM by K as well, and so $a_\ell(f) = 0$ for all primes ℓ inert in K. Thus for all ℓ inert in K, $a_\ell(F)$ lies in the height one prime which is the kernel of the above specialization map. If F has more than λ_F weight one specializations, then each $a_\ell(F)$ vanishes for at least $\lambda_F + 1$ specializations, a contradiction. □

7. Uniqueness and étaleness at weight one points

The aim of this section is to provide concrete examples where the new-quotient $\mathbb{T}_N^{\text{new}}$ of Hida's ordinary Hecke algebra is not étale at a classical weight one RM point. Even more, we show that it is possible for two non-Galois conjugate Hida families to specialize to the same RM weight one form. Recall that in weight two or more, étaleness implies uniqueness (up to Galois conjugacy), and it is a classical result of Hida that étaleness holds, at least for odd primes p.

7.1. Constructing weight one RM forms. We first recall the definition and construction of a p-ordinary weight one RM form.

Let $K = \mathbb{Q}(\sqrt{D})$ be the real quadratic extension of $\mathbb{Q}$ of discriminant $D > 0$ and let p be an odd prime splitting in K as $\mathfrak{p}\mathfrak{p}'$. Let $\mathfrak{c}$ be an integral ideal of K not divisible by $\mathfrak{p}'$ (though we allow $\mathfrak{c}$ to be divisible by $\mathfrak{p}$). Assume that $C = \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c})$ and D are relatively prime and that at least one of the mixed sign ray class groups $\mathrm{Cl}_K^{+-}(\mathfrak{c})$ or $\mathrm{Cl}_K^{-+}(\mathfrak{c})$ is not isomorphic to the usual ray class group $\mathrm{Cl}_K(\mathfrak{c})$ (so the kernel of the canonical surjection has order two). One can then consider a finite order Hecke character φ on K of conductor dividing $\mathfrak{c}$ and such that for $x_\infty = (x_1, x_2) \in (K \otimes \mathbb{R})^\times = \mathbb{R}^\times \times \mathbb{R}^\times$ one has $\varphi_\infty(x_\infty) = \mathrm{sgn}(x_1)$ or $\mathrm{sgn}(x_2)$ (in fact it is enough to consider a character of $\mathrm{Cl}_K^{+-}(\mathfrak{c})$ or $\mathrm{Cl}_K^{-+}(\mathfrak{c})$ that does not factor through $\mathrm{Cl}_K(\mathfrak{c})$). By changing $\mathfrak{c}$, one can assume then the conductor of φ is exactly $\mathfrak{c}$. It follows from the shape of φ_∞, that φ does not factor through the norm and that its transfer $\varphi \circ \mathrm{Ver}$ is an odd Dirichlet character. Hence the representation $\mathrm{Ind}_K^{\mathbb{Q}}\varphi$ is irreducible and odd, and by a theorem of Weil it corresponds to a weight one cuspform f of level CD.

The projective image of the p-adic Galois representation ρ_f is isomorphic to a dihedral group D_{2m}, where m denotes the order of the character φ/φ^σ, with σ the non-trivial automorphism of $K/\mathbb{Q}$. Since $\varphi_\infty \neq \varphi_\infty^\sigma$ have order 2 it follows that m is even. By Remark 4.6, the residual Galois representation attached to f is absolutely irreducible.

Table 7.1 lists several weight one RM forms of small level. It has been obtained using Pari, although it has been cross-checked using Magma.

Table 7.1. Weight one newforms with RM

N	ψ	D	CM	m
39	$\varepsilon_{13}\varepsilon_{-3}$	13	$-3, -39$	2
55	$\varepsilon_5\varepsilon_{-11}$	5	$-11, -55$	2
56	$\varepsilon_8\varepsilon_{-7}$	8	$-7, -56$	2
95	$\varepsilon_5\varepsilon_{-19}$	5	$-19, -95$	2
120	$\varepsilon_5\varepsilon_8\varepsilon_{-3}$	40	$-15, -24$	4
145	$\varepsilon_5\omega_{29}^7$	5	–	4
145	$\varepsilon_{29}\omega_5$	29	–	4
155	$\varepsilon_5\varepsilon_{-31}$	5	$-31, -155$	2
183	$\varepsilon_{61}\varepsilon_{-3}$	61	$-3, -183$	2
184	$\varepsilon_8\varepsilon_{-23}$	8	$-23, -184$	2
255	$\varepsilon_5\varepsilon_{17}\varepsilon_{-3}$	85	$-15, -51$	4
259	$\varepsilon_{37}\varepsilon_{-7}$	37	$-7, -259$	2
328	$\varepsilon_8\omega_{41}^5$	8	–	8
371	$\varepsilon_{53}\omega_7$	53	–	6
644	$\varepsilon_{-4}\varepsilon_{-7}\varepsilon_{-23}$	92	$-7, -644$	2
4508	$\varepsilon_{-4}\psi_7\varepsilon_{-23}$	92	–	14

Every line in the table corresponds to (the Galois conjugacy class of) a weight one newform f with RM by $K = \mathbb{Q}(\sqrt{D})$ of level CD and Nebentypus $\psi = \varepsilon_D \cdot \varphi \circ \mathrm{Ver}$. In the second column, ω_p is the Teichmüller lift of the mod p cyclotomic character, and ψ_7 is the character defined in Section 6.2. The last two columns show respectively the discriminants of quadratic fields by which f has CM and the order of the character φ/φ^σ.

7.2. Etaleness fails at RM forms. We can now provide examples of classical weight one height one primes in $\mathbb{T}_N^{\mathrm{new}}$ for which étaleness fails.

Consider a weight one form f as in §7.1 with RM by K and a prime p split in K. One first needs to choose a p-stabilization of f. If p divides C, by the above assumptions f is an eigenform for U_p with eigenvalue $\varphi(\mathfrak{p}')$, therefore is a p-stabilized ordinary form. Otherwise, if p is prime to C, choosing a p-stabilization of f amounts to choosing one of the roots $\{\varphi(\mathfrak{p}), \varphi(\mathfrak{p}')\}$ of the Hecke polynomial at p. Note that in weight one both roots are units and the corresponding stabilizations are p-ordinary. Also note that it might happen that $\varphi(\mathfrak{p}) = \varphi(\mathfrak{p}')$ in which case the Galois representation associated to f is not p-distinguished.

Let F be a Hida family containing the p-ordinary stabilizations of f fixed above. Then F is residually of dihedral type with RM by K. The tame level N of F satisfies $N|CD$ and equals CD if $p \nmid C$.

Proposition 7.1. *Let F be a Hida family as above and denote by G the (primitive) Hida family underlying $F \otimes \varepsilon_{K|\mathbb{Q}}$. Then F and G are two different Hida families of the same tame level and containing the classical weight one form f with RM by K. In particular, the Hecke algebra acting on the space of primitive Λ-adic cuspforms of tame level N is not étale over Λ at the point defined by f.*

Proof. Write ε_D for $\varepsilon_{K|\mathbb{Q}}$. By assumption, p splits in K, so $a_p(G) = a_p(F)\varepsilon_D(p) = a_p(F)$ and G is a p-ordinary family.

Since $\rho_f \cong \rho_f \otimes \varepsilon_D$ both F and $F \otimes \varepsilon_D$ specialize to f outside D.

Moreover $F \otimes \varepsilon_D$ is new at the primes $\ell \neq p$ not dividing D, since F is primitive and ε_D is unramified at those primes (one can see this by considering any weight bigger than two specialization).

At an odd prime ℓ dividing D, $F \otimes \varepsilon_D$ has level ℓ^2 (C and D are relatively prime) and we have to show that it is ℓ-old, of level ℓ. We leave the case of $\ell = 2$ to the reader. By (2.3) the ℓ-part of ψ_F equals the ℓ-part of the central character of f, which is nothing but (the ℓ-part of) ε_D. Since N and D are divisible by ℓ and not by ℓ^2, it follows that every classical specializations f_k of F in weight $k \geq 2$ has to be a minimally ramified principal series at ℓ. More precisely, the restriction of the associated Galois representation to the inertia group I_ℓ equals (the restriction of) $1 \oplus \varepsilon_D$. Since ε_D is quadratic, the twisted form $f \otimes \varepsilon_D$ is ℓ-old, of level ℓ. It follows that $F \otimes \varepsilon_D$ is ℓ-old as well. This proves that G also has tame level N as desired.

It remains to see that $a_\ell(G)$ specializes to $a_\ell(f)$, for primes $\ell|D$. By the above discussion the restriction of ρ_f to the decomposition group G_ℓ can be written as

$$\mathrm{unr}(a_\ell) \oplus \varepsilon_D \mathrm{unr}(b_\ell),$$

where a_ℓ and b_ℓ are some scalars. Using the Local Langlands correspondence for f at ℓ one can easily check that $a_\ell = a_\ell(f)$. Similar reasoning for the form $f \otimes \varepsilon_D$ shows that b_ℓ is the specialization of $a_\ell(G)$. Since $\rho_f \cong \rho_f \otimes \epsilon_D$, comparing the unramified characters in the decompositions of the restrictions to G_ℓ of the corresponding Galois representations, we see that $a_\ell = b_\ell$, as desired. ☐

7.3. Inner twists in the quadratic case. In the previous section we showed that $\mathbb{T}_N^{\mathrm{new}}$ is not étale at a height one prime corresponding to a classical RM weight one form f, by constructing two different Hida families F and G specializing to f. We remark that this does not immediately imply that uniqueness fails as well, since it could very well happen that F and G are Galois conjugates. In this section, we analyze this in more detail.

Let $\mathbb{F}$ be the field over which the residual representation $\bar{\rho} = \bar{\rho}_F$ is defined and let $W = W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$.

Note that, since $W[[X]]$ is unramified over Λ at $X = 0$, a Hida family with coefficient field $\mathrm{Frac}(W[[X]])$ would never specialize to a classical weight one form with RM. Therefore, if a Hida family F admits a weight one classical specialization with RM, then K_F should be at least a non-scalar quadratic extension of $\mathrm{Frac}(\Lambda)$.

Let $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}}$ be the local component of the semi-local algebra $\mathbb{T}_N^{\mathrm{new}}$ defined by F (in fact it only depends on the community $\{F\}$). For the rest of this section we place ourselves in second simplest case after the case $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}} = W[[X]]$, namely we assume that

$$\mathrm{rk}_{W[[X]]}\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}} = 2. \tag{7.1}$$

There are two cases:

Case 1: $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}}$ has a unique minimal prime, that is $K_F = \mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}} \otimes_\Lambda \mathrm{Frac}\Lambda$ is a field. Then K_F is obtained by adjoining to $\mathrm{Frac}(W[[X]])$ a square-root of an element in $W[[X]]$, and $F^\gamma = F \otimes \varepsilon_D$, where γ denotes the non-trivial $W[[X]]$-linear automorphism of K_F.

Case 2: $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}} \otimes_\Lambda \mathrm{Frac}\Lambda \cong \mathrm{Frac}(W[[X]])^2$, in which case $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}}$ is the set of tuples in $W[[X]] \oplus W[[X]]$ which are congruent modulo some ideal in $W[[X]]$. Then F and $F \otimes \varepsilon_D$ are two non-Galois conjugate families.

In Table 7.2 we provide several examples of Hida families containing a classical weight one cuspform with RM for which the condition (7.1) is fulfilled. The method of computation consists in studying specializations in weights two or more, where efficient algorithms using modular symbols have been implemented in Magma. Note that the rank of $\mathbb{T}_{N,\psi_F}^{\mathrm{new}}$ (the quotient of $\mathbb{T}_N^{\mathrm{new}}$ by the ideal generated by $\langle \ell \rangle - \psi_F(\ell)$, ℓ not dividing Np) is over Λ, whereas the rank of $\mathbb{T}_{N,\bar{\rho}}^{\mathrm{new}}$ is over $W[[X]]$. All families in Table 7.2 are non-CM and specialize to the weight one form with RM listed on the same line in Table 7.1.

All the examples wind up in Case 1. This is in accordance with [BD12, Theorem 1.1] which says that the eigencurve is smooth (but not étale over the weight space) at weight one forms f which are p-distinguished and admit RM by a field in which p splits (using some techniques of Cho and

TABLE 7.2. Hida families specializing to weight one forms from Table 7.1

N	p	ψ_F	$\mathrm{rk}_\Lambda \mathbb{T}^{\mathrm{new}}_{N,\psi_F}$	inner twists	$\lvert\mathbb{F}\rvert$	proj.im$(\bar\rho_F)$
13	3	$\varepsilon_{13}\varepsilon_{-3}$	2	ε_{13}	3	D_4
5	11	$\varepsilon_5\varepsilon_{-11}$	2	ε_5	11	D_4
8	7	$\varepsilon_8\varepsilon_{-7}$	2	ε_8	7	D_4
5	19	$\varepsilon_5\varepsilon_{-19}$	2	ε_5	19	D_4
$5\cdot 8$	3	$\varepsilon_5\varepsilon_8\varepsilon_{-3}$	16	$\varepsilon_5\varepsilon_8$	3^2	D_8
5	29	$\varepsilon_5\omega_{29}^7$	2	ε_5	29	D_8
29	5	$\varepsilon_{29}\omega_5$	12	ε_{29}	5	D_8
5	31	$\varepsilon_5\varepsilon_{-31}$	6	ε_5	31	D_4
61	3	$\varepsilon_{61}\varepsilon_{-3}$	14	ε_{61}	3	D_4
8	23	$\varepsilon_8\varepsilon_{-23}$	8	ε_8	23	D_4
$5\cdot 17$	3	$\varepsilon_5\varepsilon_{17}\varepsilon_{-3}$	24	$\varepsilon_5\varepsilon_{17}$	3^2	D_8
37	7	$\varepsilon_{37}\varepsilon_{-7}$	8	ε_{37}	7	D_4
8	41	$\varepsilon_8\omega_{41}^5$	4	ε_8	41	D_{16}
53	7	$\varepsilon_{53}\omega_7$	30	ε_{53}	7	D_{12}
$4\cdot 23$	7	$\varepsilon_{-4}\varepsilon_{-23}\varepsilon_{-7}$	34	$\varepsilon_{-4}\varepsilon_{-23}$	7	D_4

Vatsal [CV03], one can further show in this case that the ramification index is two).

Let us describe in greater detail a typical example, which corresponds to the first entry of the Table 7.2. We start with a quadratic Hecke character on $\mathbb{Q}(\sqrt{13})$ of conductor having finite part one of the two primes lying above 3, and infinite part the sign character at exactly one of the two infinite places. The corresponding theta series f is a RM form in $S_1(39, \varepsilon_{-3}\varepsilon_{13})$. We take $p = 3$ which is ordinary for f. The weight k specialization f_k of the resulting 3-adic Hida family lie in $S_k(39, \varepsilon_{-3}^{-k}\varepsilon_{13})$. A brief check using Pari shows that the p-adic completions $K_{f_k,p}$ of the Hecke fields of f_k for the first few weights k are all quadratic extensions of $\mathbb{Q}_3$ as follows:

k	1	2	3	4
$K_{f_k,p}$	$\mathbb{Q}_3[Y]/Y^2$	$\mathbb{Q}_3(\sqrt{-3})$	$\mathbb{Q}_3(\sqrt{3})$	$\mathbb{Q}_9$
k	5	6	7	$\cdots$
$K_{f_k,p}$	$\mathbb{Q}_3(\sqrt{-3})$	$\mathbb{Q}_3(\sqrt{3})$	$\mathbb{Q}_3\times\mathbb{Q}_3$	$\cdots$

Based on this (and more) data, we see $F \in L[[q]]$ with L most likely given by:

$$L = \frac{\mathbb{Z}_3[[X]][Y]}{(Y^2 + X)}$$

since specialization of L at $X = (1+p)^{k-1} - 1 = 4^{k-1} - 1$ matches well with the above data. We see that $F^{\gamma} = F \otimes \varepsilon_{13}$ where γ is the automorphism that takes Y to $-Y$, so we are in Case 1. In particular, while étaleness fails for L/Λ at $X = 0$, the two Hida families F and $F \otimes \varepsilon_{13}$ are Galois conjugate families specializing to f, and there is no violation of uniqueness.

Remark 7.2. Computing the structure of the Hida fields K_F (or, even more, L) is a very interesting problem. There seems to be almost no examples in the literature of rings L which are not scalar extensions of the canonical power series ring Λ, and the computation above should be considered as a step in this direction.

7.4. Uniqueness fails for RM-CM weight one forms. In view of the previous section, slightly new ideas are required to give an example of two non-conjugate Hida families specializing to the same weight one form. We describe these now.

Before we do this, it will help to briefly recall the two approaches discussed above, which failed numerically to produce examples:

(1) Find a non-CM family F, with a CM weight one specialization, with p splitting in the corresponding imaginary quadratic field K. Then f would also live in a CM family G by explicit interpolation, violating uniqueness. However, we know of no such families F (the few examples of non-CM families with CM weight one forms we know all have p ramified in K, and such forms cannot live in CM families). In fact, such families should be difficult to find, since by [BD12, Theorem 1.1] the eigencurve is étale over the weight space at classical weight one forms f which are p-distinguished and admit CM by a field in which p splits.

(2) Find a non-CM family F with a weight one specialization with RM by K and p split in K, and consider the Hida family G underlying $F \otimes \varepsilon_{K|\mathbb{Q}}$. In all numerical cases we looked at the families F and G were Galois conjugate, so this method failed to produce an example.

We now describe a third method to violate uniqueness which is a bit more subtle, but is able to produce examples.

We start with a weight one form f whose Galois representation has projective image D_4 (the Klein four group), has RM by $\mathbb{Q}(\sqrt{D})$, for $D > 0$, and has CM by $\mathbb{Q}(\sqrt{D'})$, $D' < 0$, and by $\mathbb{Q}(\sqrt{D''})$ where $D'' = DD' < 0$. We choose a prime p which splits in all these three fields. Let F (resp. G) be a Hida family with CM by $\mathbb{Q}(\sqrt{D'})$ (resp. by $\mathbb{Q}(\sqrt{D''})$) specializing to f. The existence of CM families is guaranteed by [H86, Theorem 7.1], since p is split in these fields. Now these two families cannot possibly be the same. Indeed, since $F = F \otimes \varepsilon_{D'}$ and $G = G \otimes \varepsilon_{D''}$, if $F = G$, then $F = F \otimes \varepsilon_{D}$, a contradiction, since there are no 'RM Hida families' (such families would produce RM forms in higher weights, a contradiction). Moreover, F and G are not even Galois conjugate, since if $F = G^\gamma$, then since the action of γ commutes with twisting by quadratic characters, we still get the contradiction $F = F \otimes \varepsilon_D$.

Thus F and G are non-conjugate CM families specializing to the same RM weight one form!

An example of such a phenomenon is not too hard to find, following the construction in §7.1. For instance, one can check that there exists a weight one form f of level 111, central character $\varepsilon_{-3}\varepsilon_{37}$ having RM by $\mathbb{Q}(\sqrt{37})$ and CM by $\mathbb{Q}(\sqrt{-3})$ and by $\mathbb{Q}(\sqrt{-111})$. The prime $p = 7$ splits in all these fields and the Hecke polynomial of f at 7 is $(X+1)^2$. Hence f has a unique 7-stabilization, which is a 7-old ordinary weight one eigenform of level 777 with U_7 eigenvalue equals to -1. A quick numerical check gives two non-Galois conjugate 7-adic Hida families F (resp. G) with CM by $\mathbb{Q}(\sqrt{-3})$ (resp. $\mathbb{Q}(\sqrt{-111})$), of tame level 111, central character $\varepsilon_{-3}\varepsilon_{37}$ and specializing to the 7-stabilization of f.

A. Appendix: Inner twists for Hida families

Let us recall that a classical newform $f = \sum a_n q^n \in S_k(N, \psi)$ with coefficients in K_f has an inner twist by an embedding $\gamma : K_f \hookrightarrow \mathbb{C}$, if there exists a Dirichlet character χ such that $f^\gamma = f \otimes \chi$ (meaning that $\gamma(a_p) = a_p\chi(p)$ for almost all primes p). Assume now that f has no CM, so that γ uniquely determines χ. Then it is a theorem of Momose and Ribet that such a γ necessarily induces an automorphism of K_f and that $\Gamma_f = \{\gamma \in \mathrm{Aut}_{\mathbb{Q}}(K_f) \mid \exists\chi \text{ such that } f^\gamma = f \otimes \chi\}$ is an abelian group.

Knowing the inner twists of a newform is an essential ingredient in determining the image of its p-adic Galois representation. It is therefore natural to try to develop a theory of inner twists for Hida families. We start with a definition.

Definition A.1. A Hida family $F = \sum a_n q^n$ with coefficients field K_F (a finite extension of $\mathrm{Frac}(\Lambda)$) has inner twist by $\gamma : K_F \hookrightarrow \overline{\mathrm{Frac}(\Lambda)}$ if there exists a Dirichlet character χ such that $F^\gamma = F \otimes \chi$ (that is, $\gamma(a_p) = a_p\chi(p)$ for almost all primes p).

In other terms, F has an inner twist by a Dirichlet character χ if both F and $F \otimes \chi$ correspond to the same minimal prime in $\mathbb{T}_N^{\mathrm{new}}$.

Note that, if F has no CM, then χ is uniquely determined by γ and will be denoted by χ_γ. Assume henceforth that F has no CM.

Lemma A.2. *The set of inner twists γ of F is an abelian subgroup $\Gamma_F \subset \mathrm{Aut}_\Lambda(K_F)$.*

Proof. The proof is similar to the classical case. Since F is a primitive family, ψ_F has values in $\mu_n(K_F) \subset K_F^\times$ for some $n \geq 1$. By comparing determinants, one obtains $\psi_F^\gamma = \psi_F\chi_\gamma^2$, hence $\chi_\gamma^2 = \psi_F^{\gamma-1}$ takes values in $\mu_n(K_F)^2$. Therefore χ_γ takes values in $\mu_n(K_F)$ and γ is a Λ-linear automorphism of K_F, as desired. The group law is given by $\chi_{\gamma\delta} = \chi_\gamma\chi_\delta^\gamma$ and Γ_F is abelian if, and only if, $\chi_\delta^{\gamma-1} = \chi_\gamma^{\delta-1}$. The last is true since they are both equal to $\psi_F^{(\gamma-1)(\delta-1)/2}$. □

By the lemma, γ is an automorphism of the local algebra L, hence induces an automorphism of the residue field $\mathbb{F}$ that we denote by $\bar\gamma$. It follows that for all $\gamma \in \Gamma_F$, we have:

$$\bar\rho_F^{\bar\gamma} \cong \bar\rho_F \otimes \chi_\gamma.$$

It follows that if $\bar\gamma = 1$ for some $\gamma \in \Gamma_F$, then F is residually of dihedral type (or reducible). In particular, if $\Gamma_F \neq \emptyset$ and $\mathbb{F} = \mathbb{F}_p$, then F is necessarily of dihedral type (or reducible).

Finally, as in the classical situation $H_F := \cap_{\gamma\in\Gamma_F}\ker(\chi_\gamma)$ is an abelian extension of $\mathbb{Q}$ and the traces of elements of $\rho_F(\mathrm{Gal}(\bar{\mathbb{Q}}/H_F))$ lie in $K_F^{\Gamma_F}$.

We study next the behavior of inner twists under specialization. Recall that we have fixed an embedding of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_p$, hence we can see $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Let $F \in L[[q]]$ be a Hida family, where L is the integral closure of Λ in its field of coefficients K_F (a finite extension of the fraction field of Λ).

Let $P_k = ((1+X)-(1+p^\nu)^{k-1})$. For every $\gamma \in \mathrm{Aut}_\Lambda(L)$ and every $k \geq 2$, there exists $\gamma_k \in \mathrm{Aut}_{\mathbb{Z}_p}(L/P_kL)$ making the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "\gamma", "\sim"'] \arrow[d] & L \arrow[d] \\ L/P_kL \arrow[r, "\gamma_k", "\sim"'] & L/P_kL \end{tikzcd}$$

The next proposition extends a result of A. Fischman [F02].

Proposition A.3. *Let $k \geq 2$ be such that $(L/P_kL)[\frac{1}{p}]$ is a field, and let f_k be the specialization of F at the unique prime ideal of L above P_k. Then, there is a natural bijection between Γ_F and $\Gamma_{f_k} \cap \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.*

Proof. Since $(L/P_kL)[\frac{1}{p}] \cong K_{f_k,\mathcal{P}}$, where $\mathcal{P}$ denotes the prime of K_{f_k} uniquely determined by the fixed embedding of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_p$, every $\gamma \in \Gamma_F$ induces $\gamma_k \in \Gamma_{f_k} \cap \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Conversely, for $\gamma_k \in \Gamma_{f_k} \cap \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, let χ denote the Dirichlet character for which $f_k^{\gamma_k} = f_k \otimes \chi$. Then F and the primitive Hida family underlying $F \otimes \chi$ specialize both to f_k (in fact $F \otimes \chi$ specializes to $f_k^{\gamma_k}$, hence to f_k, by further composing the specialization homomorphism with γ_k^{-1}). By a theorem of Hida, F and $F \otimes \chi$ should be Galois conjugates, as claimed. □

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