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Newton’s method over global height fields

par XANDER FABER et ADAM TOWSLEY

RÉSUMÉ. Pour tout corps K muni d’un ensemble de valeurs absolues satisfaisant la formule du produit, nous décrivons complètement les conditions pour que la méthode de Newton, appliquée à un polynôme $f \in K[x]$ sans facteur carré, parvienne à trouver une racine dans le complété v -adique pour une infinité de valeurs absolues v de K . De plus, nous montrons que si K est un corps global, la suite d’approximation de Newton ne converge pas v -adiquement pour une partie de densité positive de v .

ABSTRACT. For any field K equipped with a set of pairwise inequivalent absolute values satisfying a product formula, we completely describe the conditions under which Newton’s method applied to a squarefree polynomial $f \in K[x]$ will succeed in finding some root of f in the v -adic topology for infinitely many places v of K . Furthermore, we show that if K is a finite extension of the rationals or of the rational function field over a finite field, then the Newton approximation sequence fails to converge v -adically for a positive density of places v .

1. Introduction

Newton’s method is one of the most efficient techniques for locating and approximating zeros of complex analytic or p -adic analytic functions in a single variable. Recall how it works: if $f(x)$ is an analytic function with a simple root α , then for any x_0 sufficiently close to α we have

$$0 = f(\alpha) \approx f(x_0) + f'(x_0)(\alpha - x_0) \implies \alpha \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Setting $N_f(x) = x - \frac{f(x)}{f'(x)}$, one can prove that the orbit (x_n) defined by $x_n = N_f(x_{n-1})$ for $n \geq 1$ converges quadratically to the root α .

Now let K be a global height field — that is, a field equipped with a set M_K of pairwise inequivalent absolute values $\{|\cdot|_v : v \in M_K\}$ satisfying an adelic property and a product formula. The basic examples are number fields and function fields of transcendence degree 1. (See Section 3.1 for more details.) Each v induces a different metric structure on K . Fix a univariate

polynomial f and an initial guess x_0 , both defined over K . Define a sequence recursively by $x_n = N_f(x_{n-1})$ for $n \geq 1$, where we set

$$N_f(x) = x - f(x)/f'(x).$$

We will say that (x_n) is the **Newton approximation sequence** associated to f and x_0 if it is not eventually periodic. Following an investigation of the adelic convergence of orbits of more general dynamical systems [10], Silverman and Voloch asked the following question. (We paraphrase. See also Remark 1.6 below.)

What can be said about the set of places v of K for which the Newton approximation sequence (x_n) converges to some root of f ?

Silverman and Voloch showed that the *complement* of the set of places of convergence is infinite. Later, the first author and Voloch showed that the set of places of convergence itself is infinite, unless there is a specific kind of dynamical obstruction. Theorem 1.1 summarizes these conclusions in the case of squarefree polynomials. Recall that if $\varphi \in K(z)$ is a rational function of degree at least 2, a point $\alpha \in \mathbb{P}^1(K)$ is called an exceptional fixed point for φ if $\varphi(\alpha) = \alpha$ and φ is totally ramified at α .

Theorem 1.1 ([7, 10]). *Let K be a number field, and let (x_n) be the Newton approximation sequence associated to a squarefree polynomial $f \in K[x]$ of degree at least 2 and an initial point $x_0 \in K$.*

- (i) *The sequence (x_n) converges v -adically to the root α of f for infinitely many places v if and only if α is not an exceptional fixed point for the dynamical system N_f . In particular, if f is not quadratic, then (x_n) converges v -adically to some root of f for infinitely many places v .*
- (ii) *The sequence (x_n) fails to converge in $\mathbb{P}^1(K_v)$ for infinitely many places v .*

The first statement says that the arithmetic conclusion about convergence to a root at infinitely many places is equivalent to a geometric conclusion about the dynamics of a certain fixed point for the Newton map. The statement itself requires only the language of global height fields, while its proof uses a Diophantine approximation technique specific to number fields. Our first goal in the present paper is to show that Theorem 1.1(i) is really a statement about global height fields by using a result of the second author [11], which in turn is a consequence of Runge's method.

Theorem 1.2. *Let K be a global height field, and let (x_n) be the Newton approximation sequence associated to a point $x_0 \in K$ and a squarefree polynomial $f \in K[x]$ with $\deg(f) \geq 2$. Then the sequence (x_n) converges*

v-adically to a root α of f for infinitely many places v of K if and only if α is not an exceptional fixed point for the dynamical system N_f .

Remark 1.3. The condition that f be squarefree implies that f' is not identically zero, and hence that N_f is well defined. Moreover, if K has characteristic $p > 0$ and g is a function of x^p , say $g = h(x^p)$, then $N_{f \cdot g} = N_f$, which means the Newton map ignores inseparable factors. For purposes of root finding, it is also reasonable to assume that f is squarefree by replacing f with $f/\gcd(f, f')$ if necessary.

The conclusion in Theorem 1.1(i) that (x_n) converges for infinitely many places v if $\deg(f) > 2$ is a consequence of a classification of polynomials f that have a dynamically exceptional root α — i.e., a root that is an exceptional fixed point for N_f . For global height fields of characteristic zero, this classification carries over verbatim; with some modification of the argument in positive characteristic, one can still give a classification of polynomials f for which *all* of its roots are exceptional fixed points for N_f (Proposition 2.9). This immediately yields the following corollary.

Corollary 1.4. *Let K^a be a fixed algebraic closure of K . With the notation of Theorem 1.2, the sequence (x_n) converges *v*-adically to some root of f for infinitely many places v of K unless one of the following is true:*

- *f is quadratic;*
- *K has characteristic $p > 0$ and there exist $A, B, C \in K^a$ and an integer $r \geq 1$ such that $Af(Bx + C) = x^{p^r} - x$; or*
- *K has characteristic $p > 0$ and there exist $\lambda, A, B, C \in K^a$ and an integer $r \geq 1$ such that $Af(Bx + C) = x^{p^r+1} - (\lambda + 1)x^{p^r} + \lambda x$.*

We now discuss quantitative extensions of Theorem 1.1(ii). The first author and Voloch conjectured that something much stronger should be true in the case of number fields: the set of places v of K for which (x_n) does not converge *v*-adically to any root of f has density one when ordered by norm. They were only able to provide numerical and heuristic evidence to this effect. A clever technique using the Chebotarev density theorem was later developed by several authors to address a related question on collisions of orbits modulo primes [2]. It implies that Newton's method fails to converge to some root for a *positive* density of places of a number field [2, Thm. 4.6]. (It does not give an effective lower bound on the density.) We are able to give a unified proof of positive density for any global field.

Theorem 1.5. *Let K be a finite extension of the rational field \mathbb{Q} or of the rational function field $\mathbb{F}_p(T)$. Assume the setup of Theorem 1.2. The set of places v of K for which the Newton approximation sequence (x_n) diverges *v*-adically has positive lower density when ordered by norm.*

Remark 1.6. Silverman and Voloch proved a preliminary form of the above theorem for a number field K or an arbitrary function field K/k of transcendence degree 1. They showed that the set of places v of K for which the Newton approximation sequence diverges is infinite [10, Remark 10]. Surprisingly, they were able to show that the result continues to hold in the function field setting *even when the map is purely inseparable*, and we model the purely inseparable case of our proof on theirs.

We update the main conjecture of [7] to include function fields:

Conjecture 1.7 (Newton Approximation Fails for 100% of Places). *Let K be a finite extension of the rational field \mathbb{Q} or of the rational function field $\mathbb{F}_p(T)$. Let (x_n) be the Newton approximation sequence associated to a squarefree polynomial $f \in K[x]$ of degree at least 2 and a point $x_0 \in K$. Define $C(K, f, x_0)$ to be the set of places v of K for which (x_n) fails to converge v -adically to some root of f . Then the set of places $C(K, f, x_0)$ has natural density one when ordered by norm.*

For other global height fields K , we expect the set of places of divergence to be “large,” but there is some issue in clarifying what “large” should mean. We are able to give a satisfying answer in the case where K/k is the function field of a curve Y defined over a global height field k . Here “large” may be quantified in terms of a height function on Y .

Theorem 1.8. *Let k be a global height field, and let K/k be a function field of transcendence degree one. Fix a nonsingular connected curve Y/k such that $K = k(Y)$, and fix a height function h_Y on $Y(k^a)$ associated to an ample divisor. Let (x_n) be the Newton approximation sequence associated to a squarefree polynomial $f \in K[x]$ of degree at least 2 and a point $x_0 \in K$. Then there exists a positive real number $M = M(f, x_0, h_Y)$ such that (x_n) fails to converge in K_y for all $y \in Y(k^a)$ satisfying $h_Y(y) \geq M$.*

More generally, one can place a fundamental cardinality restriction on the set of places at which Newton approximation may detect a root.

Proposition 1.9. *Let K be a global height field, and let (x_n) be the Newton approximation sequence associated to a squarefree polynomial $f \in K[x]$ of degree at least 2 and a point $x_0 \in K$. The sequence (x_n) converges v -adically to a root of f for at most a countable number of places v of K .*

The proof is trivial. Pass to a finite extension of K if necessary so that f splits completely. For each of the countably many pairs $(n, \alpha) \in \mathbb{N} \times \text{Zeros}(f)$, the inequality $|x_n - \alpha|_v < 1$ may only be satisfied for finitely many places v of K .

Morally, the above proposition holds because there is a finitely generated subfield of K over which f and x_0 are defined, and a finitely generated field

admits only countably many places. Note that, unlike \mathbb{Q} or $\mathbb{F}_p(T)$, the places of an arbitrary finitely generated field do not come with any sort of canonical ordering with which to measure density. With these observations in mind, we ask the following:

Question 1.10. Is there a meaningful way to assert that Newton approximation fails for a “large” set of places of an arbitrary finitely generated global height field?

To close the introduction, we describe the contents of this paper. Section 2 describes a number of geometric properties of the correspondence $f \mapsto N_f$ between polynomials and Newton maps. This includes the following dynamical characterization: Up to conjugation, Newton maps are precisely the rational functions of degree $d \geq 2$ that have at least d critical fixed points. The results in Section 2 are well known in characteristic zero, so our goal is to compare and contrast the behavior of this correspondence in characteristic zero with that in positive characteristic. We prove Theorem 1.2 in Section 3, and the proof of Theorems 1.5 and 1.8 will occupy Section 4.

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2. Geometry of the Newton map

Throughout this section, we fix a separably closed field L of characteristic $p \geq 0$. We begin by describing separability properties of the Newton map N_f in terms of the polynomial f . Then we discuss the fixed points of the Newton map and use them to give a dynamical characterization of rational functions that are conjugate to a Newton map. In the final subsection we characterize those squarefree polynomials f for which every root is a totally ramified fixed point of the Newton map.

2.1. Inseparability.

Definition 2.1. A rational function $\varphi \in L(x)$ is **inseparable** if L has positive characteristic and φ can be written as $\varphi(x) = \psi(x^p)$ for some $\psi \in L(x)$; φ is said to be **separable** otherwise. By the Hurwitz formula, φ is inseparable if and only if its formal derivative is identically zero. We say that φ is **purely inseparable** if $\varphi(x) = \eta(x^{p^r})$ with $r > 0$ and η fractional linear.

It will be important for our arithmetic questions to know which polynomials f give rise to purely inseparable Newton maps N_f .

Remark 2.2. A nonconstant rational function $\varphi \in L(x)$ is separable (resp. inseparable, purely inseparable) if and only if the field extension $L(x)/L(\varphi)$ is separable (resp. inseparable, purely inseparable) in the sense of classical field theory.

Proposition 2.3. *Suppose that L has positive characteristic, and let $f \in L[x]$ be a squarefree polynomial of degree at least 2.*

(i) *The Newton Map N_f is inseparable if and only if*

$$f(x) = xg(x^p) + h(x^p)$$

for polynomials $g, h \in L[x]$.

(ii) *The Newton map N_f is purely inseparable if and only if*

$$f(x) = xg(x^{p^r}) + h(x^{p^r})$$

for polynomials $g, h \in L[x]$ of degree at most 1 and a positive integer r .

Proof. Since f is squarefree, its derivative cannot vanish identically, and N_f is well defined.

If $f(x) = xg(x^p) + h(x^p)$, then $N_f(x) = -\frac{h(x^p)}{g(x^p)}$, which is inseparable.

Conversely, assume that N_f is inseparable. By definition, $N_f(x) = \frac{xf'(x)-f(x)}{f'(x)}$, and the numerator and denominator have no factor in common since f is squarefree. It follows that $f'(x) = g(x^p)$ for some $g \in L[x]$. Integration yields that $f(x) = xg(x^p) + h(x^p)$ for some h . This completes the proof of (i).

For (ii), an immediate calculation shows that N_f is purely inseparable if f has the given form. Conversely, if N_f is purely inseparable, then part (i) shows that $f(x) = xg(x^{p^r}) + h(x^{p^s})$ for some polynomials g, h and some integers $r, s > 0$. We may assume that r and s are chosen to be maximal. Then $N_f(x) = -\frac{h(x^{p^s})}{g(x^{p^r})}$. As N_f is purely inseparable, we must have $r = s$ and h/g is fractional linear. \square

2.2. Degree and fixed points. Typically, a polynomial of degree d will give rise to a Newton map of degree d . However, in positive characteristic, the Newton map will have degree $d-1$ when $d \equiv 1 \pmod{p}$. This difference has only a slight effect on the basic properties of the correspondence $f \mapsto N_f$.

Proposition 2.4. *Let $f \in L[x]$ be a squarefree polynomial of degree $d \geq 2$.*

(i) The Newton map has degree

$$\deg(N_f) = \begin{cases} d & \text{if } d \neq 1 \text{ in } L \\ d - 1 & \text{if } d = 1 \text{ in } L. \end{cases}$$

(ii) Let $\alpha_1, \dots, \alpha_d$ be the roots of f . Then the fixed points of the Newton map are

$$\text{Fix}(N_f) = \begin{cases} \{\alpha_1, \dots, \alpha_d, \infty\} & \text{if } d \neq 1 \text{ in } L \\ \{\alpha_1, \dots, \alpha_d\} & \text{if } d = 1 \text{ in } L. \end{cases}$$

(iii) Each of the roots α_i of f is a critical fixed point for N_f . If $d \neq 1$ in L , then ∞ has fixed point multiplier $\frac{d}{d-1}$. In particular, if $d = 0$ or 1 in L , then all of the fixed points of N_f are critical.

Proof. The squarefree hypothesis shows there is no common factor shared between the numerator and denominator of $N_f(x) = (xf' - f) / f'$.

For part (i), observe that the leading terms of the numerator and denominator of N_f are $(d - 1)x^d$ and dx^{d-1} , respectively. We are finished unless $d = 1$ in L and $N_f = 0$. But then we find that $f(x) = xg(x^p) + h(x^p)$ for some polynomials g, h (Proposition 2.3). Computing N_f shows that $h = 0$, and the squarefree hypothesis on f forces g to be constant. As we assumed $\deg(f) \geq 2$, this is a contradiction.

For part (ii), it is clear that the α_i are fixed points of N_f . If $d \neq 1$ in L , then the previous paragraph shows that the numerator of N_f has larger degree than the denominator, so infinity is a fixed point. A rational function of degree δ has $\delta + 1$ fixed points, so we have found them all.

For (iii), we pick a root α of f and write $f(x) = (x - \alpha)g(x)$ for some polynomial g . Now

$$N_f(x) - \alpha = (x - \alpha) - \frac{f(x)}{f'(x)} = (x - \alpha)^2 \frac{g'(x)}{g(x) + (x - \alpha)g'(x)}.$$

Since f is squarefree, we find $g(\alpha) \neq 0$, and $N_f(x)$ is ramified at α .

If $d \neq 1$ in L , we can write

$$N_f(x) = \frac{(d - 1)x^d + \dots}{dx^{d-1} + \dots},$$

where we are ignoring lower order terms. We compute the fixed point multiplier at ∞ by a change of coordinate and ignoring higher order terms:

$$1/N_f(1/x) = \frac{dx + \dots}{(d - 1) + \dots} = \frac{d}{d - 1}x + \dots \quad \square$$

Corollary 2.5. *If $d = 1$ in L , then there is no Newton map of degree d .*

The following result gives a dynamical description of Newton maps. It is certainly well known in the case of fields of characteristic zero; see for

example [9]. The only difference that arises in positive characteristic is that it is possible to have rational functions all of whose fixed points are critical. Such a function is necessarily conjugate to a Newton map.

Proposition 2.6. *Fix an integer $d \geq 2$, and let $\varphi \in L(x)$ be a rational function of degree d . The following are equivalent:*

- (i) φ is conjugate to a Newton map N_f for some squarefree polynomial $f \in L[x]$;
- (ii) φ has at least d critical fixed points; and
- (iii) If $d \neq 0$ in L (resp. $d = 0$ in L), then φ has exactly d (resp. exactly $d + 1$) critical fixed points.

Proof. (i) \Rightarrow (ii). Evident from Proposition 2.4.

(ii) \Rightarrow (i). Suppose that φ has at least d critical fixed points, and write $\varphi = A/B$ with A and B coprime polynomials. If one of the fixed points of φ is not critical, we may conjugate it to ∞ without loss of generality. Otherwise, let us conjugate any of the critical fixed points to ∞ . In particular, we may assume that $\deg(B) < \deg(A)$.

Set $f(x) = xB(x) - A(x)$; the roots of f are exactly the finite fixed points of φ . We claim that $N_f = \varphi$. First, we show that $A' = xB'$. The finite critical points of φ are the roots of

$$BA' - AB' = B(A' - xB') + B'(xB - A).$$

Since every finite fixed point is also a critical point, $xB - A \mid B(A' - xB')$. As A and B are coprime, we conclude that $xB - A \mid A' - xB'$. But now $\deg(xB - A) = d$, while $\deg(A' - xB') \leq d - 1$. Hence, $A' = xB'$, as desired. This relation implies that $f' = xB' - A' + B = B$, and

$$\gcd(f, f') = \gcd(xB - A, B) = \gcd(A, B) = 1.$$

That is, f is squarefree. A direct computation shows that $N_f = A/B$, as desired.

(iii) \Rightarrow (ii). Evident.

(ii) \Rightarrow (iii). Write $\lambda_0, \dots, \lambda_d$ for the multipliers at the $d + 1$ (distinct) fixed points of φ , and let us suppose that $\lambda_1 = \dots = \lambda_d = 0$. The fixed point index formula shows that

$$1 = \sum \frac{1}{1 - \lambda_i} = d + \frac{1}{1 - \lambda_0}.$$

We see that $d = 0$ in L if and only if $\lambda_0 = 0$. □

2.3. Dynamically exceptional roots.

Definition 2.7. A root α of f will be called **dynamically exceptional** if it is exceptional for the Newton map; i.e., its backward orbit $\{\gamma \in \mathbb{P}^1(L) : N_f^n(\gamma) = \alpha \text{ some } n \geq 0\}$ is finite. Equivalently, since roots of f are fixed by N_f , α is dynamically exceptional if and only if it is totally ramified for N_f .

It follows from the Hurwitz formula that if N_f is not purely inseparable, then it has at most two exceptional points.

The following lemma will allow us to move the roots of f and adjust it to be monic without affecting the dynamical properties of the Newton map.

Lemma 2.8 ([7, Prop. 2.4]). *Let $f, g \in L[x]$ be polynomials of degree d related by the formula $g(x) = Af(Bx + C)$, where $A, B, C \in L$ and $AB \neq 0$. Set $\sigma(x) = Bx + C$. Then $N_g = \sigma^{-1} \circ N_f \circ \sigma$.*

Proposition 2.9. *Let $f \in L[x]$ be a squarefree polynomial of degree $d \geq 2$.*

- (i) *If $d = 2$, then every root of f is dynamically exceptional.*
- (ii) *If $d > 2$ and $d \neq 1$ in L , then every root of f is dynamically exceptional if and only if L has positive characteristic and there exist $A, B, C \in L$ and an integer $r \geq 1$ such that*

$$Af(Bx + C) = x^{p^r} - x.$$

- (iii) *If $d > 2$ and $d = 1$ in L , then every root of f is dynamically exceptional if and only if L has positive characteristic and there exist $\lambda, A, B, C \in L$ and an integer $r \geq 1$ such that*

$$Af(Bx + C) = x^{p^r+1} - (\lambda + 1)x^{p^r} + \lambda x.$$

Proof. Suppose first that f is quadratic, and that α, β are the roots of f . Then both α and β are critical fixed points for N_f . A critical fixed point for a quadratic map is totally ramified, so that both α and β are dynamically exceptional for f .

Now assume that $d = \deg(f) > 2$. It was shown in [7, Cor. 1.2] that when L has characteristic zero, a polynomial has at most one dynamically exceptional root. For the remainder of the proof, we may therefore assume that L has characteristic $p > 0$.

Write D for the degree of N_f . Suppose that α, β are distinct roots of f that are totally ramified fixed points for N_f . By applying Lemma 2.8, we may assume without loss that f is monic and that $\alpha = 0$ and $\beta = 1$. By conjugating β to ∞ while fixing 0, we see that N_f is conjugate to $\varphi(x) = x^D$. At least one other fixed point of N_f is totally ramified, so the same must be true of φ . The fixed points of φ are $0, \infty$, and the $(D - 1)^{\text{st}}$ roots of unity. If ζ is such a root of unity, we find that

$$\varphi(x) - \zeta = x^D - \zeta = (x - \zeta)^D$$

if and only if $D = q$ is a power of the characteristic and $\zeta \in \mathbb{F}_q$. For $a \in L \setminus \{0\}$, write $\sigma(x) = ax/(x - 1)$ for a fractional linear transformation that fixes 0 and moves 1 to ∞ . There exists $a \in L$ such that

$$(2.1) \quad N_f(x) = \sigma^{-1} \circ \varphi \circ \sigma(x) = \frac{a^{q-1}x^q}{a^{q-1}x^q - (x - 1)^q} = \frac{a^{q-1}x^q}{a^{q-1}x^q - x^q + 1}.$$

If $d \neq 1$ in L , then N_f fixes infinity, which means $a^{q-1} = 1$. Thus $N_f(x) = x^q$. Proposition 2.3 shows that $f(x) = xg(x^q) + h(x^q)$ for polynomials g, h that are at most linear; computing N_f directly shows that $g = -1$ and $h = x$. (Recall that we have assumed f is monic.)

If instead $d = 1$ in L , then N_f does not fix infinity, so that $a^{q-1} \neq 1$. Replacing a^{q-1} with $1 + 1/\lambda$ in (2.1) shows that

$$N_f(x) = \frac{(\lambda + 1)x^q}{x^q + \lambda}.$$

As in the previous case, $f(x) = xg(x^q) + h(x^q)$ for some linear polynomials g, h ; computing N_f directly shows that $g = x + \lambda$ and $h = -(\lambda + 1)x$.

Conversely, if f is of either of the two special forms in the statement, then N_f is purely inseparable (Proposition 2.3), which implies that all of its fixed points are totally ramified. □

3. Success of Newton’s method over global height fields

3.1. Global height fields. We follow the conventions in [3, §1].

Definition 3.1. A field K will be called a **global height field** if it is equipped with a collection M_K of inequivalent nontrivial absolute values $|\cdot|_v$ such that the following two conditions hold for each $\alpha \in K \setminus \{0\}$:

- (Adelic Property) The set $\{v \in M_K : |\alpha|_v \neq 1\}$ is finite, and
- (Product Formula) $\prod_{v \in M_K} |\alpha|_v = 1$.

Examples are given by number fields and function fields $k(X)$, where X is a geometrically irreducible normal projective variety defined over a field k .¹ See also [5] and [8, §2] for further interesting examples. Every finite extension K'/K may be endowed with the structure of a global height field where each absolute value on K' restricts to one in M_K , up to equivalence. We write $M_{K'}$ for the corresponding set of absolute values on K' normalized as in [3, §1.3.6, 1.3.12]. (The normalization is unimportant for the results presented here.)

Let K be a global height field with a fixed algebraic closure K^a . The absolute logarithmic height function $h : \mathbb{P}^1(K^a) \rightarrow \mathbb{R}_{\geq 0}$ is defined as follows. Let $\alpha = (\alpha_0 : \alpha_1) \in \mathbb{P}^1(K^a)$, and let K'/K be a finite extension containing the coordinates of α . Then we define

$$h(\alpha) = \sum_{w \in M_{K'}} \log \max\{|\alpha_0|_w, |\alpha_1|_w\}.$$

¹Fix a nontrivial numerically effective divisor H on X . Each prime divisor Y on X gives rise to an absolute value $|\cdot|_Y$ defined by $|f|_Y = \exp(-n_Y \text{ord}_Y(f))$ for $f \in k(X)$, where $n_Y = Y \cdot H^{\dim(X)-1}$.

By the product formula, the sum is independent of the choice of homogeneous coordinates for α , and the normalizations of the elements of $M_{K'}$ are arranged so that $h(\alpha)$ is independent of the choice of extension K'/K containing α .

3.2. Arithmetically exceptional roots. Throughout this section, K will denote a fixed global height field.

Definition 3.2. Let (x_n) be a Newton approximation sequence associated to some squarefree polynomial $f \in K[x]$ and some $x_0 \in K$. (Recall that we assume (x_n) is not eventually periodic.) A root $\alpha \in K^a$ of f will be called **arithmetically exceptional** for f if the Newton approximation sequence (x_n) converges to α in K_v for only finitely many places $v \in M_K$.

Definitions 2.7 and 3.2 allow us to restate Theorem 1.2 as an equivalence between arithmetic and dynamical exceptionality:

Theorem 3.3. *Let (x_n) be a Newton approximation sequence associated to some squarefree polynomial $f \in K[x]$ and a point $x_0 \in K$. Then a root of f is arithmetically exceptional if and only if it is dynamically exceptional.*

The proof is a consequence of the following result of the second author. Note that although it was originally stated for function fields over a finite constant field, its proof uses only the properties of a global height field.

Theorem 3.4 ([11, Cor. 3.3]). *Let $\varphi \in K(x)$ be a rational function of degree at least 2. Let $\beta, \gamma \in \mathbb{P}^1(K)$. Suppose that γ is periodic and not exceptional for φ . If β is not preperiodic, then there exist infinitely many places v of K such that $|\varphi^n(\beta) - \gamma|_v < 1$ for some $n \geq 0$.*

Proof of Theorems 1.2 and 3.3. By passing to a finite extension if necessary, we may assume that all of the roots of f lie in K . Let $\alpha \in K$ be such a root. Write f as $f(x) = (x - \alpha)g(x)$ for some polynomial $g \in K[x]$ that does not vanish at α . Define S_α to be the finite set of places $v \in M_K$ such that at least one of the following is true:

- The leading coefficient of f' or g' has v -adic absolute value different from 1;
- $|\alpha - \beta|_v \neq 1$ for some root β of f' or g' ;
- $|\beta|_v \leq 1$ for some root β of f' ; and
- $|\alpha|_v \leq 1$.

Let $v \in M_K \setminus S_\alpha$, and suppose that $n \geq 0$ is such that $|x_n - \alpha|_v < 1$. By definition,

$$\begin{aligned} x_{n+1} - \alpha &= N_f(x_n) - \alpha = x_n - \alpha - \frac{(x_n - \alpha)g(x_n)}{(x_n - \alpha)g'(x_n) + g(x_n)} \\ &= (x_n - \alpha)^2 \cdot \frac{g'(x_n)}{f'(x_n)}. \end{aligned}$$

As $v \notin S_\alpha$, the denominator of this last expression has v -adic absolute value 1. Hence, $|x_{n+1} - \alpha|_v \leq |x_n - \alpha|_v^2$. By induction, $|x_{n+\ell} - \alpha|_v \leq |x_n - \alpha|_v^{2^\ell}$ for every $\ell \geq 0$. It follows that the Newton approximation sequence (x_n) converges to α in the v -adic topology for $v \in M_K \setminus S_\alpha$ if and only if x_n lies in the strict open disk of radius 1 about α for some $n \geq 0$.

Now suppose that α is not dynamically exceptional for f . Theorem 3.4 may be applied in our setting with $\beta = x_0$ and $\gamma = \alpha$ to show there are infinitely many places v for which $|x_n - \alpha|_v < 1$ for some n . So α is not arithmetically exceptional by the preceding paragraph.

Finally, suppose that α is dynamically exceptional. Then

$$N_f(x) - \alpha = c \frac{(x - \alpha)^{\deg(N_f)}}{f'(x)}$$

for some nonzero $c \in K$. Setting $x = x_n$, we see that

$$(3.1) \quad x_{n+1} - \alpha = c \frac{(x_n - \alpha)^{\deg(N_f)}}{f'(x_n)}.$$

Let S'_α be S_α along with the finitely many other v for which $|c|_v \neq 1$. Take $v \notin S'_\alpha$, and suppose that $|x_{n+1} - \alpha|_v < 1$ for some $n \geq 0$. Then either $|f'(x_n)|_v > 1$ or $|x_n - \alpha|_v < 1$. If the former is true, then our hypothesis on v shows that $|x_n - \beta|_v > 1$ for some root β of f' , in which case $|x_n|_v > 1$. But (3.1) shows that $|x_{n+1} - \alpha|_v = |x_n|_v^{\deg(N_f) - \deg(f')} \geq 1$, a contradiction. So we are forced to conclude that $|x_n - \alpha|_v < 1$ whenever $|x_{n+1} - \alpha|_v < 1$. By induction, $|x_n - \alpha|_v < 1$ if and only if $|x_0 - \alpha|_v < 1$. As (x_n) is not eventually periodic, there are only finitely many places v for which this last inequality can hold. Hence, α is arithmetically exceptional. \square

4. Failure of Newton’s method

4.1. Global fields. Throughout this subsection, we let K be a global field — that is, a finite extension of the field of rational numbers \mathbb{Q} or of the function field $\mathbb{F}_p(T)$ for some rational prime p . Let K^a be a fixed algebraic closure of K . We write \mathfrak{o}_K for the integral closure of \mathbb{Z} or $\mathbb{F}_p[T]$ inside K . In the function field setting, this is equivalent to saying that K is the function field of a smooth connected projective curve C/\mathbb{F}_p endowed with a finite morphism $C \rightarrow \mathbb{P}^1_{\mathbb{F}_p}$, and \mathfrak{o}_K is the ring of rational functions on C that are regular away from the points lying above $\infty \in \mathbb{P}^1_{\mathbb{F}_p}$.

Write M_K^0 for the set of finite places of K ; each place $v \in M_K^0$ may be canonically identified with a unique prime ideal $\mathfrak{p}_v \in \mathfrak{o}_K$, and conversely. We will interchange v and \mathfrak{p}_v as needed without further comment. The norm of $v \in M_K^0$, denoted Nv , is the order of the residue field associated to \mathfrak{p}_v : $Nv = \#\mathfrak{o}_K/\mathfrak{p}_v$. The (natural) lower density of a set of places $S \subset M_K^0$

is defined to be

$$\underline{\delta}(S) = \liminf_{X \rightarrow \infty} \frac{\#\{v \in S : Nv \leq X\}}{\#\{v \in M_K^0 : Nv \leq X\}}.$$

Remark 4.1. Only finitely many places of K are ignored by focusing on M_K^0 , so this omission has no effect on any of the results in this section.

The proof of Theorem 1.5 will follow the general strategy of [7], but in place of the primitive prime divisor result from [6], we use a result of Benedetto et al. for number fields [2] and the corresponding result of the second author for function fields [11]. For the statement, we recall that if K is a function field of characteristic p and $\varphi \in K(x)$ is an inseparable rational function, then it may be written as $\varphi(x) = \psi(x^{p^r})$ for some $r \geq 1$ and some separable rational function ψ . The degree of ψ is called the **separable degree** of φ . Equivalently, the separable degree of φ is the number of K^a -rational solutions to the equation $\varphi(x) = \beta$ for a generic choice of β . To say that the separable degree of φ is greater than 1 is to say that φ does not induce a bijection on $\mathbb{P}^1(K^a)$.

Theorem 4.2 ([2, Thm. 1.1], [11, Thm. 4]). *Let K be a global field, and let $\varphi \in K(x)$ be a rational function of degree at least 2. If K is a function field, suppose further that φ has separable degree at least 2. Let $\gamma \in \mathbb{P}^1(K)$ be a point with infinite φ -orbit, and let $\mathcal{B} \subset \mathbb{P}^1(K)$ be a finite set of preperiodic points for φ . Then the set of places $\mathcal{P} \subset M_K^0$ such that $\varphi^n(\gamma) \not\equiv \beta \pmod{\mathfrak{p}}$ for any $n \geq 0$, any $\beta \in \mathcal{B}$, and any $\mathfrak{p} \in \mathcal{P}$, has positive lower density.*

Proof of Theorem 1.5. We wish to show that the set of places v of the global field K at which (x_n) fails to converge v -adically to a root of f has positive lower density. Let us suppose first that N_f is not purely inseparable. Let \mathcal{B} denote the set of fixed points of N_f . (See Proposition 2.4.) A continuity argument shows that if (x_n) converges in $\mathbb{P}^1(K_{\mathfrak{p}})$ for a prime $\mathfrak{p} \in M_K^0$, then it converges to an element of \mathcal{B} . Applying Theorem 4.2 with $\gamma = x_0$ shows that the sequence (x_n) fails to v -adically approach \mathcal{B} for a set of places v of K with positive lower density.

Now we suppose that $N_f(x)$ is purely inseparable; hence, K necessarily has positive characteristic p . After conjugating if necessary, we may replace N_f with x^q for q some power of p . It suffices to show that (x_n) converges v -adically to an element of $\mathbb{P}^1(\mathbb{F}_q)$ for at most finitely many places v . Let $\alpha \in \mathbb{F}_q$. Write $y_0 = x_0 - \alpha$. A suitable induction hypothesis shows that $x_n = \alpha + y_0^{q^n}$. Then for any place v , we see that $|x_n - \alpha|_v = |y_0|_v^{q^n}$, so that (x_n) converges v -adically to α only for those places v for which $|y_0|_v < 1$. The set of such places v is finite. A similar argument applies in the case $\alpha = \infty$ if we set $y_0 = 1/x_0$. □

4.2. Function fields over global height fields. Before beginning the proof of Theorem 1.8, we recall that if k is a global height field with height function h_k , and if $\varphi \in k(x)$ is a rational function of degree at least 2, there is a canonical height function $\hat{h}_{k,\varphi} : \mathbb{P}^1(k^a) \rightarrow \mathbb{R}_{\geq 0}$ characterized by the following two properties:

- $\hat{h}_{k,\varphi}(\varphi(\alpha)) = \deg(\varphi) \cdot \hat{h}_{k,\varphi}(\alpha)$ for any $\alpha \in \mathbb{P}^1(k^a)$.
- The difference $|\hat{h}_{k,\varphi}(\alpha) - h_k(\alpha)|$ for $\alpha \in \mathbb{P}^1(k^a)$ may be bounded by a constant that does not depend on α .

Recall also that if K/k is a function field as in the statement of Theorem 1.8, we say that a rational function $\varphi \in K(x)$ is isotrivial if there is $\sigma \in \text{PGL}_2(K^a)$ such that $\sigma \circ \varphi \circ \sigma^{-1} \in k^a(x)$. That is, up to a coordinate change and a finite extension of K , the function φ is defined over the constant field. We prove the following fact about canonical heights:

Lemma 4.3. *Let K/k be a function field over a global height field k , let $\varphi \in K(x)$ be an isotrivial rational function, and let $x_0 \in K$. Suppose that the isotriviality of φ is witnessed by $\sigma \in \text{PGL}_2(K^a)$; that is, $\sigma \circ \varphi \circ \sigma^{-1} \in k(x)$. The canonical height $\hat{h}_{K,\varphi}(x_0) \rightarrow \infty$ as $n \rightarrow \infty$ unless x_0 is preperiodic or $\sigma(x_0) \in \mathbb{P}^1(k^a)$.*

Proof. If x_0 is preperiodic, then $\hat{h}_{K,\varphi}(x_0) = 0$, so we assume that x_0 is not preperiodic. Let $\psi := \sigma \circ \varphi \circ \sigma^{-1} \in k(x)$ and define $\xi_n = \sigma(x_n)$. Observe that $\xi_{n+1} = \psi(\xi_n)$ for every $n \geq 0$. If $\xi_0 \in k^a$, then $h_K(\xi_n) = 0$ for all n . This implies $\hat{h}_{K,\psi}(\xi_n) = 0$ for all n . Since canonical heights are conjugation invariant, we conclude that $\hat{h}_{K,\varphi}(x_n) = 0$ for all n .

Finally, we assume that ξ_0 is not defined over (an extension of) the constant field. There necessarily exists a place z of K such that $|\xi_0|_z > 1$. Write $\psi = g/h$ for some coprime polynomials $g, h \in k[x]$. The coefficients of g and h all have z -adic absolute value 1; hence, $\max\{|g(\xi_0)|_z, |h(\xi_0)|_z\} = |\xi_0|_z^d$, where $d = \deg(\psi) = \max\{\deg(g), \deg(h)\}$. Since $\xi_n = g(\xi_{n-1})/h(\xi_{n-1})$ for all $n \geq 1$, we can proceed by induction to show that

$$\max\{|g(\xi_{n-1})|_z, |h(\xi_{n-1})|_z\} = |\xi_0|_z^{dn}.$$

It follows that

$$h_K(\xi_n) = \sum_{y \in M_K} \log \max\{|g(\xi_{n-1})|_y, |h(\xi_{n-1})|_y\} \geq \log |\xi_0|_z^{dn}.$$

Hence, $h_K(\xi_n) \rightarrow \infty$ as $n \rightarrow \infty$. Since the canonical height $\hat{h}_{K,\psi}$ differs from the usual height h_K by a bounded amount, $\hat{h}_{K,\psi}(\xi_n) \rightarrow \infty$, and conjugation invariance of canonical heights shows that $\hat{h}_{K,\varphi}(x_n) \rightarrow \infty$ as well. \square

Using this lemma, a theorem of Baker, and the specialization result of Call and Silverman, we now prove Theorem 1.8.

Proof of Theorem 1.8. Suppose first that the Newton map N_f is not isotrivial. Since x_0 is not preperiodic for N_f , we apply Baker's theorem [1, Cor. 1.8] to conclude that the canonical height $\hat{h}_{K,N_f}(x_0)$ is positive. Write $x_{n,y}$ for the specialization of x_n at $y \in Y(k^a)$, and similarly write $N_{f,y}$ for the induced rational function with coefficients in k^a . Write $\deg(Y)$ for the degree of the ample divisor associated to h_Y . Then Call/Silverman specialization [4, Thm. 4.1] shows that

$$\hat{h}_{k,N_{f,y}}(x_{0,y}) \sim \hat{h}_{K,N_f}(x_0) \cdot \frac{h_Y(y)}{\deg(Y)} \text{ as } h_Y(y) \rightarrow \infty.$$

If $h_Y(y)$ is sufficiently large, then $\hat{h}_{k,N_{f,y}}(x_{0,y}) > 0$. It follows that $x_{0,y}$ is not preperiodic for the specialized map $N_{f,y}$, and, in particular, this means that it cannot ever encounter a fixed point of the map $N_{f,y}$. This is sufficient to prove that (x_n) does not converge y -adically to a root of f .

Now suppose that N_f is isotrivial, and choose $\sigma \in \text{PGL}_2(K^a)$ such that $\psi := \sigma \circ \varphi \circ \sigma^{-1} \in k^a(x)$. Set $\xi_n = \sigma(x_n)$. Note that $\xi_n = \psi(\xi_{n-1})$ for each $n \geq 1$. If $\xi_0 \notin k^a$ then we apply Lemma 4.3 to conclude that $\hat{h}_{K,\psi}(\xi_n) > 0$ for all $n \geq n_0$. If $n_0 > 0$, then replace x_0 with x_{n_0} . The strategy in the previous paragraph applies again to show that (x_n) does not converge y -adically to a root of f as long as $h_Y(y)$ is sufficiently large.

Finally, suppose that N_f is isotrivial as in the last paragraph, and that $\xi_0 = \sigma(x_0) \in k^a$. Without loss of generality, we will assume it lies in $\mathbb{P}^1(k)$. The entire sequence (ξ_n) must also be defined over k . But (ξ_n) is therefore discrete in $\mathbb{P}^1(K_y)$ for every place y , and so it cannot converge unless it is eventually constant. This contradicts the hypothesis that (x_n) is not eventually periodic. We conclude that Newton approximation fails to converge for any places y in this case. \square

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