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Lars KÜHNE

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To Axel Thue and his legacy

Résumé. Cet article est consacré à une nouvelle démonstration de la transcendance des évaluations du logarithme archimédien en tout nombre algébrique, exception faite de l'unité. Par analogie avec d'autres démonstrations de ce résultat, nous employons une variante de l'approximation de Padé pour le logarithme népérien. Une différence cependant : la construction de ces approximations de Padé est ici obtenue par le lemme de Siegel. La méthode exposée suggère des généralisations qui sont aussi évoquées.

Abstract. This article is devoted to a new proof of transcendence for evaluations of the archimedean logarithm at all algebraic numbers except unity. As in other proofs of the same theorem, a sort of Padé approximation for the natural logarithm is employed. Whereas in previous approaches the used Padé approximants have been obtained rather ad hoc, we construct them here systematically by Siegel's Lemma. The method presented suggests some generalizations, which are also briefly surveyed.

1. Introduction

The following theorem is a classic in transcendence theory and one of its first major achievements.

Theorem 1.1. *Let $z \neq 1$ be an algebraic complex number. Then $\mathfrak{log}(z)$ is transcendent.*

Here, $\mathfrak{log}$ denotes an arbitrary determination of the archimedean logarithm on $\mathbb{C}^\times$ with base e; see Section 2.1 for our conventions on the various logarithms appearing in this article.

The transcendence of $\mathfrak{log}(-1) = i\pi$ was established by Lindemann [44] and the general case of the theorem is a simple subcase of the Lindemann-Weierstrass theorem [47]. All these results rely on an approximation technique used by Hermite [27] in his proof that e is transcendental. The reader is referred to the appendix of Mahler's lecture notes [29] for a survey on

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these 'classical' results. In addition, Theorem 1.1 is also a subcase of much more powerful results obtained by Gelfond [25], Schneider [38], and Baker [10] on linear forms in logarithms (cf. [12, 45]), of the Siegel-Shidlovsky theory on E-functions (cf. [39]), as well as of André's structural results on arithmetic Gevrey series (cf. [5, 6, 7, 13]).

Another approach to Theorem 1.1 started with an important contribution of Mahler [28] and was continued by work of Baker [9], Mignotte [32], Reyssat [37], and others. This line of research yields effective results with impressive numerics in form of completely explicit irrationality or transcendence measures for specific values of the logarithm (e.g. $\log(2), \log(3), \pi, \dots$). For all of this, it is crucial to work with specific Padé approximants of the formal power series $\log(1+X) \in \mathbb{Q}[[X]]$ and its multiples. Though Padé approximation also plays a central role in this article, its aim and scope is rather orthogonal to that of these works.

In fact, we do not strive for new numerical improvements but want to expose the efficacy of 'G-function techniques' in transcendence theory. In this spirit, we use Siegel's lemma for the construction of Padé approximants and hence have little information on them. In particular, we have no explicit formulas at our disposal. Therefore, it is not reasonable to expect the results to keep up numerically with the above results or even with more recent advances (e.g. those in [1, 30, 31]) on certain irrationality and transcendence measures of logarithms. However, the abstract approach here is a proper contribution to the scarcely developed transcendence theory of G-functions. The mathematical interest in this class of functions goes back to Siegel's landmark article [40] on diophantine approximation.[1] In its first part about transcendental numbers he studied in detail the transcendence of values attained by E-functions at algebraic points. In this context, one finds the first definition of G-functions and some results on the diophantine nature of their values at algebraic points, mostly in terms of irrationality measures. However, Siegel did not present a proof of his announcements and it took over fifty years until results of this sort were proven in literature by Bombieri [16], triggering further research in this direction by various authors [2, 4, 20, 21]. But already Siegel (p. 240, *ibid.*) noticed a major deficiency of his announcements on G-functions with respect to transcendence:

> 'Wendet man diese Betrachtungen speziell auf die Funktion $\int_0^x \frac{dt}{1+t} = \log(1+t)$ an, so gelangt man infolge der Einschränkung (74) nicht bis zum Lindemannschen Satz. Es

[1] On this occasion, it should be mentioned that the major technical tool of the whole article is Siegel's Lemma (as we call it today). As is well-known and has been mentioned by Siegel, his 'Hilfssatz' (p. 213, *ibid.*) was motivated and implicitly contained in Thue's work.

> gelingt auf diesem Wege nicht, zu beweisen, daß die Zahlen $\log 2$, $\log 3$, ... transzendent sind [...]'[2]

This problem of deducing Theorem 1.1 within the framework of G-functions, for example with techniques commonly used for E-functions as in the Siegel-Shidlovsky theory, has resisted a satisfactory solution up to the present. Nevertheless, there is mentionable work of Nurmagomedov [35] and Galochkin [24] on values of the logarithm albeit under severe technical limitations. Eventually, the book of André [2] and his later article [3] presented a motivating fusion of 'motivic' ideas around Grothendieck's period conjecture [26] and the transcendence theory of G-functions.[3] Nevertheless, concrete results in this field are still meager. The state of the art seems to be contained in the results of André's article [4]. In particular, his results include the transcendence of π but the G-functions in his proof are not logarithms but hypergeometric functions and the modular curves associated with them intervene in his proofs. Thus, it falls short to be a complete solution to Siegel's problem above, let alone a satisfying one. Finding a direct approach with the logarithm was the main motivation behind the work presented here. In addition, the author wanted to understand – in the setting of [4] – the folklore analogy between commutative algebraic groups and Shimura varieties, e.g. modular curves, which is also present in the conjectures of Zilber and Pink (see [36] for more on this analogy). These conjectures are actually related to G-functions as the André-Oort conjecture has its predecessor in André's book (see [2, Section X.4]).

We summarize our findings in the theorem below, which is a version of Theorem 1.1 excluding the roots of unity and hence the transcendence of π. In its statement, log is just the real-valued logarithm on $\mathbb{R}^{>0}$ with base e. For any algebraic number α, we denote by $h(\alpha)$ its absolute logarithmic Weil height. We refer to Sections 2.1 and 2.2 for precise definitions.

Theorem 1.2. *Let $\mathbb{K}$ be a number field of degree D and $\iota_0 : \mathbb{K} \hookrightarrow \mathbb{C}$ one of its archimedean embeddings. Let further $\alpha \neq 0$ and β be two algebraic numbers in $\mathbb{K}$. Assume additionally that α is neither 1 nor a root of unity. Then, for any $\varepsilon > 0$ there exists an effectively computable constant $c_1 = c_1(D, \varepsilon)$ such that*

$$\begin{aligned}\log |\mathfrak{log}(\iota_0(\alpha)) - \iota_0(\beta)| \\ > -c_1 \max\left\{1, |\mathfrak{log}(\iota_0(\alpha))|^{1+\varepsilon}, h(\alpha)\log(h(\alpha))\right\} \max\left\{1, h(\beta)\right\}.\end{aligned}$$

In particular, $\mathfrak{log}(\iota_0(\alpha))$ is transcendental for all $\alpha \in \bar{\mathbb{Q}} \backslash \mu_\infty(\bar{\mathbb{Q}})$.

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[2] While this article was under consideration, a complete English translation of [40] due to Clemens Fuchs has appeared in [52]. A translation of this excerpt may be found therein on p. 45.

[3] The only transcendence proof in [2] asserts that π is transcendent and is unfortunately incomplete (see the introduction of [4]).

The exclusion of roots of unity is made to simplify our exposition. Because of the above-mentioned works the benefit of proving also the transcendence of π with our method does not seem to outweight a more complicated exposition. Hence, this special case is left to later publication. Much diligence is already needed in order to determine the transcendence measure in the given form and an explicit computation of $c_1(D, \varepsilon)$ is not performed here for reasons of economy. As one may see from the above theorem, the dependence on α is slightly weaker than in some previous results. For example, from [34, Theorem 1] one may deduce that $\log|\mathfrak{log}(\iota_0(\alpha)) - \iota_0(\beta)|$ is bounded from below by

$$-c_2(D) \max\{1, |\mathfrak{log}(\iota_0(\alpha))| \log(|\mathfrak{log}(\iota_0(\alpha))|), h(\alpha)\log(h(\alpha))\} \max\{1, h(\beta)\}$$

with $c_2(D) = 3 \cdot 10^6 D^3 \log(D+2)$. Actually, the best measure one can reasonably hope for is

$$\log|\mathfrak{log}(\iota_0(\alpha)) - \iota_0(\beta)| > -c_3 D^2 \max\left\{\frac{1}{D}, \frac{|\mathfrak{log}(\iota_0(\alpha))|}{D}, h(\alpha), h(\beta)\right\}$$

for some absolute constant c_3; see [46] for a nice exposition of this and related conjectures. The author ignores whether the method presented here is optimal in regard to such improvements. If α is fixed, the dependence on $h(\beta)$ is optimal up to the constant depending on α.

We conclude the introduction by summarizing the contents of this article. In Section 2 we give the necessary background for our proof of Theorem 1.2. After establishing the basic notions in Sections 2.1 and 2.2, we cite at the end of Section 2.2 the fundamental constructive tool, namely a version of Siegel's Lemma due to Bombieri and Vaaler [18]. In Section 2.3 we give the two basic results (Lemmas 2.2 and 2.3) related with the arithmetic size of the logarithm power series and its multiples. Our exposition is rather detailed and self-contained. We do not use any theorem from the general theory of G-functions (see [2, 22]) since the results are rather elementary in our setting and we want more precision on the error terms in order to establish the transcendence measure of Theorem 1.2. In Section 2.4, we use the Picard-Vessiot theory of differential systems on the projective line $\mathbb{P}_1(\mathbb{C})$ in order to derive a zero estimate. Such zero estimates can be also found in [4, 14] but we need slightly better information here. In fact, an easy variation of the argument from [4, 14] rectifies this and yields our needed Lemma 2.6. This lemma is again not stated in utmost generality but in view of our special application.

Our proof of Theorem 1.2 is given in Section 3. In an interpolation step (Section 3.2) we invoke Siegel's Lemma to construct Padé approximants to the logarithm. The approach suggested in [40] and elsewhere is to work with a Padé approximation of the logarithm

$$\mathrm{Log}_1(X) = \sum\nolimits_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}(X-1)^i$$

and its multiples. However, this approach does not yield a proof of Theorem 1.2. To remedy this, we immitate André's use of a Hecke correspondence on the modular curve $X(2)$ [4, Section 8] in relation with Faltings' lemma [23, Lemma 5] on the height difference between two abelian varieties related by an isogeny. In fact, we replace the Hecke correspondence by an étale covering[4] $\mathbb{G}_m \to \mathbb{G}_m$, $X \mapsto X^N$, with N a sufficiently large integer[5]; in other words, one just takes all the N-th roots of the given number α. With increasing N, these roots 'converge' at archimedean places to the N-th roots of unity. Hence, it is at these roots of unity ζ – and not just at 1 – where we have to conduct a simultaneous Padé approximation to the 'logarithms' $\mathrm{Log}_\zeta(X) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i\zeta^i}(X-\zeta)^i$ (and their multiples). Taking N-th roots decreases the height of α by a factor of $1/N$ but in general increases its degree by N. These two effects fairly balance each other and the crucial input comes from the fact that the N-th roots of α 'converge' to the N-th roots of unity with a distance decreasing as $1/N$.

We exploit this in our extrapolation step (Section 3.5), where we aim at establishing a contradiction to the product formula for the lowest non-vanishing derivative of the constructed Padé approximation at a certain N-th root of α. The contribution from the archimedean embedding ι_0 is very small due to the high order of vanishing at the roots of unity and the other places cannot account for this. However, this goes only through if the order of vanishing at the N-th roots of α can be reasonably controlled. Our zero estimate enables us to do so by making an appropriate selection beforehand in Section 3.4. In fact, we just have to eventually replace α with some (small) multiple α^r, $1 \leq r \leq R$, where R is an auxiliary parameter depending only on $[\mathbb{K} : \mathbb{Q}]$ and ε.

Finally, let us mention that the only obstruction that hinders us from extending our method to the multivariate case, Baker's theorem, is the lack of an appropriate zero estimate. Indeed, the rest of our proof is rather rugged as we only need the classical Schwarz lemma in the form of the maximum modulus principle for our Padé approximation. A discussion of this and further aspects of our work is given in the final Section 4. We hope that this motivates future research.

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[4] As Umberto Zannier pointed out to me, a related idea also appears in a work of Bombieri [17]. It actually goes back to Siegel's use of isogenies in order to obtain the finiteness of integral points in his famous theorem [40] (see [17], p. 64). In particular, our Claim 3.1 here could be provided more elementarily as in Lemma 1 of [17], using only the most basic version of Siegel's Lemma.

[5] With the auxiliary parameters R and V introduced in our proof, we have $N = RV$.

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2. Preparations

In this section we recall some well-known facts for the convenience of the reader. In addition, we use it to devise the notations used in this article.

2.1. Notations. By $\mathbb{N}$ (resp. $\mathbb{N}^+$) we denote the set of non-negative (resp. positive) integers. We choose once and for all an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and by a number field we mean nothing else but a finite extension of $\mathbb{Q}$ in $\bar{\mathbb{Q}}$. We denote the open disc of radius r around a point $z \in \mathbb{C}$ by $B_r(z)$. Its closure, the closed disc of radius r around $z \in \mathbb{C}$, is denoted by $\bar{B}_r(z)$. For any complex number $z \in \mathbb{C}^\times$ we write $\arg(z)$ for its argument in $\mathbb{R}/(2\pi\mathbb{Z})$.

Throughout this article, an arbitrary branch of the complex logarithm is denoted by $\mathfrak{log}$ to distinguish it from the ordinary real logarithm $\log : \mathbb{R}^\times \to \mathbb{R}$ with base e. This slightly uncommon notation is justified by the fact that $\mathfrak{log}$ is the main arithmetic object of our study and log is just a mere abetter providing more convenient sums instead of products in estimations. Particularly in places where both log and $\mathfrak{log}$ are present, our distinction between them clarifies the exposition. In various situations the choice of branch for $\mathfrak{log}$ is subject to further conditions. We extend the real logarithm log to $[0, \infty)$ by $\log(0) = -\infty$ and use the obvious interpretation of $-\infty$ in inequalities. For a non-negative real number r we define

$$\log^+(r) = \log \max\{1, r\}$$

and

$$\log\log^+(r) = \log^+(\log^+(r)).$$

Furthermore, we write

$$\mathrm{Log}_\alpha(X) = \sum\nolimits_{i=1}^{\infty} \frac{(-1)^{i-1}}{i\alpha^i}(X-\alpha)^i \in \mathbb{Q}(\alpha)[[X-\alpha]]$$

for any $\alpha \in \bar{\mathbb{Q}}$, $\alpha \in \mathbb{C}$ or $\alpha \in \mathbb{C}_p$. Let $\mathbf{F}$ be an arbitrary field. For a m-tuple $\underline{P} = (P_1, P_2, \ldots, P_m) \in \mathbf{F}[X]^m$ of polynomials we set

$$\deg(\underline{P}) = \max_{1 \le i \le m} \{\deg(P_i)\}.$$

The coefficients of a polynomial $P \in \mathbf{F}[X]$ are denoted by $(P)_i$, $i \in \mathbb{N}$, $0 \le i \le \deg(P)$, such that

$$P(X) = \sum_{0 \le i \le \deg(P)} (P)_i X^i.$$

In the same way, we denote the coefficients of a power series $P \in \mathbf{F}[[X]]$ by $(P)_i$, $i \in \mathbb{N}$. If $\mathbf{F}$ is equipped with a norm $|\cdot|$, e.g. $(\mathbf{F}, |\cdot|) = (\mathbb{C}, |\cdot|_{\mathbb{C}})$ or $(\mathbf{F}, |\cdot|) = (\mathbb{C}_p, |\cdot|_{\mathbb{C}_p})$, we write

$$|\underline{\alpha}| = \max_{1\leq i\leq n} \{|\alpha_i|\}$$

for any tuple $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{F}^n$ and

$$|P| = \max_{0\leq i\leq \deg(P)} \{|(P)_i|\}$$

for any polynomial $P \in \mathbf{F}[X]$. Finally, for a m-tuple $\underline{P} = (P_1, P_2, \ldots, P_m) \in \mathbf{F}[X]^m$ of polynomials we set

$$|\underline{P}| = \max_{1\leq j\leq m} \{|P_j|\} = \max_{\substack{1\leq j\leq m\\ 0\leq i\leq \deg(\underline{P})}} \{|(P_j)_i|\}.$$

2.2. Values, norms, and heights. Let $\mathbb{K}$ be some fixed number field. Any archimedean (resp. non-archimedean) embedding $\iota : \mathbb{K} \hookrightarrow \mathbb{C}$ (resp. $\iota : \mathbb{K} \hookrightarrow \mathbb{C}_p$) induces a norm $|\cdot|_\iota$ on $\mathbb{K}$ if we set

$$|\xi|_\iota = |\iota(\xi)|_{\mathbb{C}} \text{ (resp. } |\xi|_\iota = |\iota(\xi)|_{\mathbb{C}_p}),$$

where $|\cdot|_{\mathbb{C}}$ (resp. $|\cdot|_{\mathbb{C}_p}$) is the standard absolute value on $\mathbb{C}$ (resp. $\mathbb{C}_p$). For a n-tuple $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$ we define

$$|\underline{\xi}|_\iota = \max_{1\leq i\leq n} \{|\xi_1|_\iota, \ldots, |\xi_n|_\iota\}.$$

By

$$|P|_\iota = |\iota(P)|_{\mathbb{C}} \text{ (resp. } |P|_\iota = |\iota(P)|_{\mathbb{C}_p})$$

we obtain a compatible norm on $\mathbb{K}[X]$ and by

$$|\underline{P}|_\iota = |\iota(\underline{P})|_{\mathbb{C}} \text{ (resp. } |\underline{P}|_\iota = |\iota(\underline{P})|_{\mathbb{C}_p})$$

a norm on tuples of such polynomials. The set of archimedean (resp. non-archimedean) embeddings of $\mathbb{K}$ is denoted by $\mathcal{E}_{\mathbb{K},\infty}$ (resp. $\mathcal{E}_{\mathbb{K},f}$). For the union $\mathcal{E}_{\mathbb{K},\infty} \cup \mathcal{E}_{\mathbb{K},f}$ of all embeddings of $\mathbb{K}$ into $\mathbb{C}$ or $\mathbb{C}_p$ we simply write $\mathcal{E}_{\mathbb{K}}$. Any non-zero algebraic number $\xi \in \mathbb{K}$ satisfies $|\xi|_\iota = 1$ for all but finitely many $\iota \in \mathcal{E}_{\mathbb{K}}$ and the product formula

$$1 = \prod\nolimits_{\iota\in\mathcal{E}_{\mathbb{K}}} |\xi|_\iota$$

holds. For an extension of number fields $\mathbb{K} \subseteq \mathbb{L}$ and two embeddings $\iota \in \mathcal{E}_{\mathbb{K}}$ and $\kappa \in \mathcal{E}_{\mathbb{L}}$ we write $\kappa|\iota$ and say 'κ divides ι' if the restriction of κ to $\mathbb{K}$ equals ι. In this situation, by separability there exist exactly $[\mathbb{L} : \mathbb{K}]$ embeddings $\kappa \in \mathcal{E}_{\mathbb{L}}$ dividing a given embedding $\iota \in \mathcal{E}_{\mathbb{K}}$ and trivially

$$|\xi|_\iota^{[\mathbb{L}:\mathbb{K}]} = \prod\nolimits_{\substack{\kappa\in\mathcal{E}_{\mathbb{L}}\\ \kappa|\iota}} |\xi|_\kappa$$

for any $\xi \in \mathbb{K}$. Following the usual conventions, the unique archimedean embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$ is denoted by ∞ and the unique non-archimedean embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_p$ plainly by p.

The (absolute logarithmic) Weil height $h(\xi)$ of an algebraic number $\xi \in \mathbb{K}$ is defined by

$$h(\xi) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \log^+ |\xi|_\iota.$$

Note that $h(\xi)$ is well-defined and does not dependent on the choice of a particular number field $\mathbb{K}$ containing some $\xi \in \bar{\mathbb{Q}}$. Hence with every algebraic number $\xi \in \bar{\mathbb{Q}}$ we have associated a unique Weil height $h(\xi)$. For a n-tuple $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$ we define its (affine) Weil height by

$$h(\underline{\xi}) = h(\xi_1, \ldots, \xi_n) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \log^+ \max\{|\xi_1|_\iota, \ldots, |\xi_n|_\iota\}$$

Furthermore, we define the Weil height $h(P)$ (resp. $h(\underline{P})$) of a non-zero polynomial $P \in \mathbb{K}[X]$ (resp. a m-tuple $\underline{P} = (P_1, \ldots, P_m) \in \mathbb{K}[X]^m$ of polynomials) by

$$h(P) = \sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \log^+ |P|_\iota (\text{resp. } h(\underline{P}) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \log^+ \max\{|P_1|_\iota, \ldots, |P_m|_\iota\}).$$

Again, none of these heights depends on the choice of $\mathbb{K}$ so that we are free to use them all without any reference to a specific subfield $\mathbb{K}$ of $\bar{\mathbb{Q}}$. For any of the above heights h, we write h_f for its non-archimedean part. This means that h_f is defined analogously to h by replacing sums over all $\iota \in \mathcal{E}_{\mathbb{K}}$ with sums over all non-archimedean embeddings $\iota \in \mathcal{E}_{\mathbb{K},f}$, i.e. by 'omitting' the archimedean ones. Similarly, we define $h_{\mathcal{S}}$ for any subset $\mathcal{S} \subseteq \mathcal{E}_{\mathbb{K},\infty}$ by summing only over the archimedean embeddings $\iota \in \mathcal{S}$.

We now recall Siegel's Lemma as follows:[6]

Lemma 2.1. *(*[18, Theorem 14]*) Let $\mathbb{L}^{(l)}$, $l = 1, \ldots, L$, be a collection of number fields over some fixed number field $\mathbb{K}$. Consider the linear subspace*

[6] The heights we use here are slightly different from those of *loc. cit.* and a word about the differences is necessary. In Section II.1 of *loc. cit.,* the height of a $(M \times N)$-matrix A, $M < N$, is defined by using (at non-archimedean places) its Grassmann coordinates, i.e. the determinants of the $(M \times M)$-minors of A. This procedure does not work if the matrix A has rank strictly less than M. This is why Theorem 14 of *loc. cit.* must actually demand that the system of linear equations

$$\sum_{n=1}^{N} a_{mn}^{(l)} x_n = 0,\ m = 1, \ldots, M_l,\ l = 1, \ldots, L,$$

here is of full rank $\sum_{l=1}^{L} [\mathbb{L}^{(l)} : \mathbb{K}] M_l$. However, this is not necessary for us: Indeed, the proof of our Lemma 2.1 can be immediately reduced to the case of full rank because we are only using affine (or inhomogeneous in the terminology of *loc. cit.*) heights for measuring the system of linear equations. The reader may note also that (2.6) on p. 15 of *loc. cit.* allows a straightforward majorization of the height bound in Theorem 14 of *loc. cit.* by that of Lemma 2.1 here.

V of $\bar{\mathbb{Q}}^N$ defined by the linear equations

$$\sum_{n=1}^{N} a_{mn}^{(l)} x_n = 0,\ m = 1, \ldots, M_l,\ l = 1, \ldots, L,$$

having coefficients $a_{mn}^{(l)} \in \mathbb{L}^{(l)}$. Then, if $N > \sum_{l=1}^{L} [\mathbb{L}^{(l)} : \mathbb{K}] M_l$ there exists a non-zero element $\underline{\xi} \in V(\mathbb{K})$ such that its logarithmic Weil height $h(\underline{\xi})$ is less than

$$\frac{\sum_{l=1}^{L} [\mathbb{L}^{(l)} : \mathbb{K}] M_l}{N - \sum_{l=1}^{L} [\mathbb{L}^{(l)} : \mathbb{K}] M_l} \left(\max_{\substack{1 \le m \le M_l \\ 1 \le l \le L}} h(a_{m1}^{(l)}, \ldots, a_{mN}^{(l)}) + \frac{1}{2} \log(N) \right) + c_4(\mathbb{K}),$$

where $c_4(\mathbb{K})$ depends only on the number field $\mathbb{K}$; in fact, one may use

$$c_4(\mathbb{K}) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \left(\frac{1}{2} \mathrm{disc}(\mathbb{K}) + s(\mathbb{K}) \log\left(\frac{2}{\pi}\right) \right)$$

with $\mathrm{disc}(\mathbb{K})$ *the discriminant and $s(\mathbb{K})$ the number of complex places of* $\mathbb{K}$. *In particular, $c_4(\mathbb{Q}) \le 1/2$.*

2.3. Some estimates. Given algebraic numbers $\alpha \in \mathbb{K}^\times$ and $\beta \in \mathbb{K}$ in some fixed number field $\mathbb{K}$ we investigate now the arithmetic sizes of the formal power series

$$Y^u(X) = (\beta + \mathrm{Log}_\alpha(X))^u \in \mathbb{K}[[X - \alpha]],\ u \in \mathbb{N}.$$

The next two lemmas correspond to Lemma 2g in [2, Section I.1.4] and the Proposition on p. 79 in [2, Section IV.5.4]. Nevertheless, they are proven directly and with more precision here.

Lemma 2.2. *For any non-negative integers T and U the quantity*

$$\frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\iota \in \mathcal{E}_{\mathbb{K},f}} \log^+ \left(\max_{\substack{0 \le u \le U \\ 0 \le m \le T}} \left| \frac{1}{m!} \frac{\partial^m Y^u}{\partial X^m}(\alpha) \right|_\iota \right)$$

is majorized by

$$T h_f(\alpha^{-1}) + U h_f(\beta) + T \left(1 + \frac{c_5}{\max\{1, \log^+(T)\}} \right) (1 + \log^+(U)),$$

where $c_5 > 0$ is some absolute constant.

Proof. The case $T = 0$ is straightforward and is excluded in the following. From $Y^u(X) = Y^1(X)^u$ we infer that

(2.1)

$$(Y^u)_m = \sum_{m^{(1)} + \ldots + m^{(u)} = m} (Y^1)_{m^{(1)}} \cdots (Y^1)_{m^{(u)}} \text{ for all } m \in \mathbb{N} \text{ and } u \ge 1.$$

Additionally, $(Y^0)_m = 1$ if $m = 0$ and $(Y^0)_m = 0$ if $m \neq 0$. Let $\iota : \mathbb{K} \hookrightarrow \mathbb{C}_p$ be a non-archimedean embedding. By ultrametricity it follows that

$$\max_{\substack{0\leq u\leq U\\ 0\leq m\leq T}} |(Y^u)_m|_\iota \leq \max\{1, |\alpha|^{-1}\}^T \max\{1, |\beta|\}^U \Theta_p(U,T),$$

where we set $\Theta_p(U,T) = 1$ if $U = 0$ and

$$\Theta_p(U,T) = \max_{1\leq u\leq U} \max_{\substack{k_1,k_2,\ldots,k_u\in\mathbb{N}^+\\ k_1+k_2+\ldots+k_u\leq T}} \left|\frac{1}{k_1k_2\cdots k_u}\right|_p$$

elsewise. Thus, summing up over all non-archimedean embeddings of $\mathbb{K}$ dividing some rational prime p we infer that

$$\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+\left(\max_{\substack{0\leq u\leq U\\ 0\leq m\leq T}} \left|\frac{1}{m!}\frac{\partial^m Y^u}{\partial X^m}(\alpha)\right|_\iota\right)$$

is bounded from above by

$$T\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+(|\alpha|_\iota^{-1}) + U\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+(|\beta|_\iota) + [\mathbb{K}:\mathbb{Q}]\log\Theta_p(U,T).$$

Now, to bound the maximum of all

$$\left|\frac{1}{k_1k_2\cdots k_u}\right|_p$$

occurring in $\Theta_p(U,T)$ we may restrict to the case where $k_1 \geq k_2 \geq \ldots \geq k_u$. This assumption implies in particular $k_i \leq T/i$. Hence

$$\log\Theta_p(U,T) \leq \sum_{i=1}^{U} \max_{1\leq k\leq T/i}\{\log|k^{-1}|_p\},$$

and since

$$\max_{1\leq k\leq T/i}\{\log|k^{-1}|_p\} \leq \left\lfloor\frac{\log(T/i)}{\log(p)}\right\rfloor \log(p),$$

this implies already that

$$\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+\left(\max_{\substack{0\leq u\leq U\\ 0\leq m\leq T}} \left|\frac{1}{m!}\frac{\partial^m Y^u}{\partial X^m}(\alpha)\right|_\iota\right)$$

is bounded from above by

$$T\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+(|\alpha|_\iota^{-1}) + U\sum_{\substack{\iota\in\mathcal{E}_{\mathbb{K},f}\\ \iota|p}} \log^+(|\beta|_\iota) + [\mathbb{K}:\mathbb{Q}]\sum_{i=1}^{U}\Xi(T/ip)\log(T/i),$$

where $\Xi(x) = 1$ if $x \in [0,1]$ and $\Xi(x) = 0$ elsewise. Now, a summation over all rational primes p gives the upper bound

$$T \sum_{\iota \in \mathcal{E}_{\mathbb{K},f}} \log^+(|\alpha|_\iota^{-1}) + U \sum_{\iota \in \mathcal{E}_{\mathbb{K},f}} \log^+(|\beta|_\iota) + [\mathbb{K}:\mathbb{Q}] \sum_{i=1}^{U} \pi(T/i) \log(T/i)$$

on

$$\sum_{\iota \in \mathcal{E}_{\mathbb{K},f}} \log^+ \left(\max_{\substack{0 \le u \le U \\ 0 \le m \le T}} \left| \frac{1}{m!} \frac{\partial^m Y^u}{\partial X^m}(\alpha) \right|_\iota \right),$$

where π is the prime counting function defined by

$$\pi(l) = \#\{p \mid p \text{ prime}, p \le l\}.$$

By the prime number theorem (see the discussion below Theorem 6.9 in [33]),

$$\pi(l) \le \frac{l}{\max\{1, \log(l)\}} + \frac{c_5 l}{\max\{1, \log(l)\}^2}$$

for some absolute constant $c_5 > 0$ and any positive integer l. It follows that

$$\begin{aligned} \sum_{i=1}^{U} \pi(T/i) \log(T/i) &\le T \left(1 + \frac{c_5}{\max\{1, \log(T)\}}\right) \left(\sum_{i=1}^{U} \frac{1}{i}\right) \\ &\le T \left(1 + \frac{c_5}{\max\{1, \log(T)\}}\right) (1 + \log^+(U)). \end{aligned}$$

□

We also need an archimedean analogue of Lemma 2.2.

Lemma 2.3. *Let $\mathcal{T}$ be a subset of $\mathcal{E}_{\mathbb{K},\infty}$. For all non-negative integers T and U the real number*

$$\frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{\iota \in \mathcal{T}} \log^+ \left(\max_{\substack{0 \le u \le U \\ 0 \le m \le T}} \left| \frac{1}{m!} \frac{\partial^m Y^u}{\partial X^m}(\alpha) \right|_\iota \right)$$

is bounded from above by

$$T h_{\mathcal{T}}(\alpha^{-1}) + U h_{\mathcal{T}}(\beta) + \log(2)(T+U).$$

Proof. Let $\iota : \mathbb{K} \hookrightarrow \mathbb{C}$ be an archimedean embedding in $\mathcal{T}$. An upper bound for the number of summands in (2.1) is

$$\binom{m+u-1}{u-1}.$$

A standard estimate yields

$$\max_{\substack{0 \le u \le U \\ 0 \le m \le T}} \log^+ \binom{m+u-1}{u-1} \le \log(2)(T+U).$$

Thus, replacing the triangle inequality in the above argument with the ultrametric one we conclude that

$$\sum_{\iota\in\mathcal{T}} \log^+\left(\max_{\substack{0\leq u\leq U\\ 0\leq m\leq T}} \left|\frac{1}{m!}\frac{\partial^m Y^u}{\partial X^m}(\alpha)\right|_\iota\right)$$

is majorized by

$$T\sum_{\iota\in\mathcal{T}} \log^+(|\alpha|_\iota^{-1}) + U\sum_{\iota\in\mathcal{T}} \log^+(|\beta|_\iota) + \log(2)[\mathbb{K}:\mathbb{Q}](T+U).$$

□

Now, given non-zero complex number $w\in\mathbb{C}^\times$ and two complex numbers $v, v'\in\mathbb{C}$ we state upper bounds for the Taylor coefficients of the formal power series

$$g_u(z) = (v+\mathrm{Log}_w(z))^u - \left(v'+\mathrm{Log}_w(z)\right)^u \in \mathbb{C}[[z-w]]$$

at the point w in $\mathbb{C}^\times$.

Lemma 2.4. *Assume that* $|v-v'|\leq 1$*. Then,*

$$\max_{\substack{0\leq u\leq U\\ 0\leq m\leq T}} \left\{\log\left|\frac{1}{m!}\frac{\partial^m g_u}{\partial z^m}(w)\right|\right\} \leq \log(|v-v'|) + T\log^+(|w|^{-1}) + \log(2)(T+U).$$

Proof. We write

$$f_1(z) = v + \mathrm{Log}_w(z)$$

and

$$f_2(z) = v' + \mathrm{Log}_w(z).$$

Then,

$$g_u(z) = f_1(z)^u - f_2(z)^u$$

and $(1/m!)(\partial^m g_u/\partial z^m)(w)$ equals

$$\sum_{m^{(1)}+\ldots+m^{(u)}=m} \left(\prod_{i=1}^{u}\left(\frac{1}{m^{(i)}!}\frac{\partial^{m^{(i)}} f_1}{\partial z^{m^{(i)}}}\right) - \prod_{i=1}^{u}\left(\frac{1}{m^{(i)}!}\frac{\partial^{m^{(i)}} f_2}{\partial z^{m^{(i)}}}\right)\right)(w). \tag{2.2}$$

From the proof of Lemma 2.3 we know that the above sum has at most 2^{T+U} summands. Since

$$\frac{1}{m!}\frac{\partial^m f_1}{\partial z^m}(w) = \frac{1}{m!}\frac{\partial^m f_2}{\partial z^m}(w) \text{ for all } m\neq 0,$$

the summand in (2.2) associated with $(m^{(1)},\ldots,m^{(u)})$ is zero if $m^{(i)}\neq 0$ for all $1\leq i\leq u$. Therefore, each non-zero summand has at least one factor of the form $|v-v'|$ and is hence bounded by

$$|v-v'|\max\{1,|w|^{-1}\}^m.$$

The lemma follows now directly from the above estimates and the triangle inequality. □

Next, we give a trivial estimate on the evaluation of polynomials.

Lemma 2.5. *Let* $\mathbf{C}$ *be either* $\mathbb{C}$ *or* $\mathbb{C}_p$*,* P *a polynomial in* $\mathbf{C}[X]$ *and* w *an element of* $\mathbf{C}$*. If* $\mathbf{C} = \mathbb{C}_p$*, then*

$$\log \left| \frac{1}{m!} \frac{\partial^m P}{\partial z^m}(w) \right| \leq \deg(P) \log^+(|w|) + \log(|P|)$$

for any non-negative integer m*. If* $\mathbf{C} = \mathbb{C}$*, the above inequality holds with an additional summand* $2\deg(P)$ *on the right hand side.*

Proof. We have

$$\frac{1}{m!} \frac{\partial^m P}{\partial z^m}(w) = \sum_{0 \leq i \leq \deg(P)-m} \binom{i+m}{i} (P)_{i+m} w^i.$$

Hence, if $\mathbf{C} = \mathbb{C}_p$ it follows that

$$\begin{aligned} \log \left| \frac{1}{m!} \frac{\partial^m P}{\partial z^m}(w) \right| &\leq \log \left(\max_{0 \leq i \leq \deg(P)-m} \left| \binom{i+m}{i} (P)_{i+m} w^i \right| \right) \\ &\leq \deg(P) \log^+(|w|) + \log(|P|) \end{aligned}$$

since binomial coefficients are natural numbers. If $\mathbf{C} = \mathbb{C}$ one obtains the same bound except for an additional occurrence of

$$\log\left(\deg(P) - m + 1\right) + \log \left(\max_{0 \leq i \leq \deg(P)-m} \binom{i+m}{i} \right), \tag{2.3}$$

and one easily infers the lemma from this by using the inequalities

$$\log \left(\max_{0 \leq i \leq \deg(P)-m} \binom{i+m}{i} \right) \leq \deg(P) \log(2)$$

and

$$\log\left(\deg(P) - m + 1\right) \leq \log(\deg(P) + 1).$$

□

2.4. Some differential algebra. All facts from the Galois theory of linear differential equations used here can be found in the book [43]. For each $\xi \in \mathbb{P}_1(\mathbb{C})$ we denote by t_ξ a local uniformizer and by d/dt_ξ the tangent vector dual to t_ξ. For the sequel, the precise choice of t_ξ is easily seen to be irrelevant. We work with the universal Picard-Vessiot field (L_ξ, ∂_ξ) associated with $(\mathbb{C}((t_\xi)), d/dt_\xi)$, cf. [43, Section 3.2] for the details of this construction. We may choose for each $w \in \mathbb{C} \backslash \mathbb{Z}$ a non-zero solution $t_\xi^w \in L_\xi$ of $\partial_\xi Y = w t_\xi^{-1} Y$ such that $t_\xi^{w_1 + w_2} = t_\xi^{w_1} t_\xi^{w_2}$ for all $w_1, w_2 \in \mathbb{C}$. Additionally, we choose a logarithm $l_\xi \in L_\xi$, which is just a fixed solution Y of $\partial_\xi Y = t_\xi^{-1}$

in L_ξ. Recall from [43, Section 3.1] that a regular singular differential system is determined by an equation

$$\left(\frac{dY_1}{dX}, \dots, \frac{dY_m}{dX}\right)^t = \mathbf{A}(Y_1, \dots, Y_m)^t,\ \mathbf{A} \in \mathbb{C}(X)^{m\times m}, \tag{2.4}$$

with the property that for each $\xi \in \mathbb{P}_1(\mathbb{C})$ there exists a matrix $\mathbf{Y} = (Y_{ij}) \in \mathrm{GL}_m(L_\xi)$ such that $\frac{d\mathbf{Y}}{dX} = \mathbf{AY}$ and each entry Y_{ij} is a finite sum

$$\sum\nolimits_k t_\xi^{\varepsilon_{i,j,k}} P_{i,j,k}(t_\xi) l_\xi^k \ (\varepsilon_{i,j,k} \in \mathbb{C},\ P_{i,j,k}(X) \in \mathbb{C}[[X]],\ P_{i,j,k}(0) \neq 0).$$

In this context, we say that $\mathbf{A}$ is non-singular[7] at ξ if all Y_{ij} are contained in $\mathbb{C}[[t_\xi]]$ and otherwise that $\mathbf{A}$ is singular at ξ. The local exponent $\varepsilon(\mathbf{A}, \xi)$ of (2.4) at a singular point $\xi \in \mathbb{P}_1(\mathbb{C})$ is defined as the minimum $\min_{i,j,k}\{\mathrm{Re}(\varepsilon_{i,j,k})\}$.[8] Since we are solely interested in differential systems related with (rather specific) G-functions, we will always have $\varepsilon_{i,j,k} \in \mathbb{Q}$ in this article. We remark that our proof of the result below is also true if $\mathbb{C}$ is replaced with any algebraically closed field of characteristic zero in case all $\varepsilon_{i,j,k}$ are rational.

The proof of Lemma 2.6 follows closely that given in the appendix of [4, p. 123], which is modeled on the one in [14]. The reader may note that our lemma below is not as general as the 'Lemme de zéros' in [4, p. 123] since we assume that (2.4) is non-singular at $\xi_1, \dots, \xi_n \in \mathbb{P}_1(\mathbb{C})\backslash\{0, \infty\}$. Doing so, we can avoid an m^2n-term that would follow from a direct application of the lemma in *loc. cit.* Although this term is considered as an error term for most applications, such as those in *loc. cit.*, its removal is a crucial factor for our final transcendence measure.

Lemma 2.6. *Assume that the regular singular differential system (2.4) is non-singular except for (possibly)* 0 *and* ∞. *Consider further* n *(distinct) points* $\xi_1, \dots, \xi_n \in \mathbb{P}_1(\mathbb{C})\backslash\{0, \infty\}$ *and for each of these points a (formal) solution*

$$(Y_1^{(i)}, \dots, Y_m^{(i)})^t \in \mathbb{C}[[X - \xi_i]]^m$$

of (2.4). Then,

$$\sum_{i=1}^n \mathrm{ord}_{X=\xi_i}(P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)}) \\ \leq m\left(\deg(\underline{P}) + n + \frac{(m-1)}{2} - \varepsilon(\mathbf{A}, 0) - \varepsilon(\mathbf{A}, \infty)\right)$$

for any m*-tuple* $\underline{P} = (P_1, \dots, P_m) \in \mathbb{C}[X]^m$ *of polynomials such that*

$$P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)} \neq 0$$

[7] Thus, those points that are called by some authors (for historic reasons) apparent singularities are considered as non-singular points here.

[8] It is easy to see that $\varepsilon(\mathbf{A}, \xi)$ only depends on $\mathbf{A}$ and ξ but not on $\mathbf{Y}$.

for all $1 \leq i \leq n$.

Proof. Let (L, ∂) be the Picard-Vessiot field [43, Definition 1.21] associated with (2.4) over $(\mathbb{C}(X), d/dX)$. By [43, Proposition 1.22] the ring of constants in L is the same as the ring of constants of $\mathbb{C}(X)$, which is $\mathbb{C}$. Given $\underline{P} \in \mathbb{C}[X]^m$ we define V as the $\mathbb{C}$-subspace of L that is generated by the elements

$$P_1Y_1 + \ldots + P_mY_m \in L,$$

where $(Y_1, \ldots, Y_m)$ runs through all solutions of (2.4) in L^m. Trivially, V is invariant under the differential Galois group of $L/\mathbb{C}(X)$. In addition, the $\mathbb{C}$-dimension of V is $m_0 \leq m$ by [43, Lemma 1.8]. By [43, Lemma 2.17][9] there exists a scalar differential operator

$$Q = \partial^{m_0} + q_{m_0-1}\partial^{m_0-1} + \ldots + q_0,\ q_i \in \mathbb{C}(X),$$

such that the $\mathbb{C}$-vector space of its solutions

$$\{Z \in L \mid QZ = 0\}$$

is exactly V. For each $\xi \in \mathbb{P}_1(\mathbb{C})$, the differential ring $(\mathbb{C}(X), d/dX)$ embeds canonically in $(\mathbb{C}((t_\xi)), d/dt_\xi)$ and we may consider the universal Picard-Vessiot field (L_ξ, ∂_ξ) introduced above. From this construction, it is apparent that the valuation $\mathrm{ord}_{X=\xi} : \mathbb{C}((t_\xi)) \to \mathbb{Z}$ extends to a valuation $\mathrm{ord}_{X=\xi} : L_\xi \to \mathbb{Q}$ and that every constant in L_ξ is already an element of $\mathbb{C}$. Every linear differential system of size m over $\mathbb{C}((t_\xi))$ has a fundamental matrix in $\mathrm{GL}_m(L_\xi)$, in particular this holds for the linear differential system (2.4). By the uniqueness of the Picard-Vessiot extension ([43, Proposition 1.22]), the differential subfield of L_ξ generated by $\mathbb{C}(X)$ and the entries of a fundamental matrix of (2.4) in $\mathrm{GL}_m(L_\xi)$ is isomorphic (as a differential field) to the Picard-Vessiot field L. Consequently, for each $\xi \in \mathbb{P}_1(\mathbb{C})$ we may consider (L, ∂) as a differential subfield of (L_ξ, ∂_ξ), containing the m_0-dimensional (cf. [43, Lemma 1.10]) solution space V of $QZ = 0$. By abuse of notation we write also ∂ for ∂_ξ from now on; the preceeding remarks show that this does not harm our exposition. Since the Wronskian $w_\xi \in L_\xi$ of the homogeneous differential equation $QZ = 0$ in L_ξ satisfies the linear differential equation

$$\partial w_\xi = q_{m_0-1} w_\xi$$

by [43, Exercise 1.14.5.(b)], we deduce that

$$\mathrm{ord}_{X=\xi}(w_\xi) = \mathrm{res}_{X=\xi}(q_{m_0-1})$$

[9] This lemma is evidently also true for the Picard-Vessiot field extension $L/\mathbb{C}(X)$ instead of the 'universal differential field extension $\mathcal{F}/\mathbb{C}(X)$', see the remark directly after the proof of [43, Lemma 2.17]. For an explicit construction, see [22, Remark III.8.7].

by a formal computation. Proceeding as above for any point $\xi \in \mathbb{P}_1(\mathbb{C})$, we obtain a Wronskian $w_\xi \in L_\xi$ for each $\xi \in \mathbb{P}_1(\mathbb{C})$ and

$$\sum_{\xi \in \mathbb{P}_1(\mathbb{C})} \mathrm{ord}_{X=\xi}(w_\xi) = \sum_{\xi \in \mathbb{P}_1(\mathbb{C})} \mathrm{res}_{X=\xi}(q_{m_0-1}) = 0$$

because q_{m_0-1} is a rational function on $\mathbb{P}_1(\mathbb{C})$. The asserted bound now follows by estimating the orders $\mathrm{ord}_{X=\xi}(w_\xi)$, $\xi \in \mathbb{P}_1(\mathbb{C})$, from below and plugging these into the above equality. For this, we use that the Wronskian w_ξ of $QZ = 0$ equals a non-zero (cf. [43, Lemma 1.12]) multiple of the determinant

$$\det \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_{m_0} \\ \partial Z_1 & \partial Z_2 & \cdots & \partial Z_{m_0} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{m_0-1} Z_1 & \partial^{m_0-1} Z_2 & \cdots & \partial^{m_0-1} Z_{m_0} \end{pmatrix}$$

of an arbitrary $\mathbb{C}$-basis $Z_1, Z_2, \ldots, Z_{m_0}$ of V (considered as a subspace of L_ξ). From this presentation, one infers immediately that

$$\mathrm{ord}_{X=\xi_i}(w_{\xi_i}) \geq \mathrm{ord}_{X=\xi_i}(P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)}) - m_0$$

for any $i \in \{1, \ldots, n\}$. In fact, since $\mathbf{A}$ is non-singular at ξ_i we may choose a fundamental set $Z_1 = P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)}, Z_2, \ldots, Z_{m_0}$ of solutions with $Z_2, \ldots, Z_{m_0} \in \mathbb{C}[[X - \xi_i]]$. Thus,

$$\mathrm{ord}_{X=\xi_i}(\partial^j Z_1) \geq \mathrm{ord}_{X=\xi_i}(P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)}) - j$$

and $\mathrm{ord}_{X=\xi_i}(\partial^j Z_k) \geq 0$ for any $j, k \in \{1, \ldots, m_0\}$. Using Laplace's formula for the first column of the above determinant, we infer that

$$\mathrm{ord}_{X=\xi_i}(w_{\xi_i}) \geq \mathrm{ord}_{X=\xi_i}(P_1 Y_1^{(i)} + \ldots + P_m Y_m^{(i)}) - m_0$$

In the same way, we obtain for any $\xi \in \mathbb{P}_1(\mathbb{C}) \backslash \{0, \xi_1, \ldots, \xi_n, \infty\}$ the inequality $\mathrm{ord}_{X=\xi}(w_\xi) \geq 0$. Finally, using the fact that $\partial_\xi t_\xi^w$, $w \in \mathbb{C}^\times$, is a non-zero multiple of t_ξ^{w-1} and that

$$\partial_\xi l_\xi^k = k t_\xi^{-1} l_\xi^{k-1},\ k \in \mathbb{N}^+,$$

we obtain

$$\mathrm{ord}_{X=0}(w_0) \geq m_0 \left(-\frac{(m_0-1)}{2} + \varepsilon(\mathbf{A}, 0) \right)$$

and

$$\mathrm{ord}_{X=\infty}(w_\infty) \geq m_0 \left(-\deg(\underline{P}) + \varepsilon(\mathbf{A}, \infty) \right).$$

The lemma follows easily by combining the various bounds. $\square$

3. Proof of Theorem 1.2

3.1. Set up (First Part). Throughout the proof of Theorem 1.2 the complex number $\mathfrak{log}(\iota_0(\alpha))$ is always meant with the same determination as in its statement.

To start with, we introduce some auxiliary parameters R, S, T, U, and V subject to later choice. *All these parameters are assumed to be positive integers.* In the proof, we assume various restrictions (R_i) on these parameters. Furthermore, our proof is split up in various claims and we adopt the convention that *restrictions on auxiliary parameters imposed in the statement of any claim are also tacitly valid for the sequel, in particular within the statements and proofs of the following claims.* Eventually, we collect all these restrictions and give an admissible set (R, S, T, U, V) of auxiliary parameters in Claim 3.8.

Next come the main protagonists of our proof: As in the statement of Theorem 1.2 let α be an algebraic number that is not a root of unity. For any $s \in \mathbb{Z}$ we define the *set* $\alpha^{s/RV}$ of (s/RV)-th powers of α to be the $\bar{\mathbb{Q}}$-solution set of

$$(X^{RV} - \alpha^s) \in \bar{\mathbb{Q}}[X].$$

Each set $\alpha^{s/RV}$, $s \in \mathbb{Z}$, has cardinality RV and the various $\alpha^{s/RV}$, $s \in \mathbb{Z}$, are disjoint; indeed, assume $\gamma \in \alpha^{s/RV} \cap \alpha^{s'/RV}$. This would imply $\gamma^{RV} = \alpha^s = \alpha^{s'}$ and hence $\alpha^{s-s'} = 1$ and $s = s'$. We denote by $\alpha^{\cdot/RV}$ the (disjoint) union of all $\alpha^{s/RV}$, $s \in \mathbb{Z}$. Furthermore, $h(\gamma) = \frac{s}{RV} h(\alpha)$ for any element $\gamma \in \alpha^{s/RV}$ because of the multiplicativity of the height. For every $\gamma \in \alpha^{\cdot/RV}$ and every $0 \le u \le U$ we define the formal power series

$$Y_\gamma^u = \left(\frac{s}{RV}\beta + \mathrm{Log}_\gamma(X)\right)^u \in \mathbb{Q}(\gamma)[[X - \gamma]],$$

where $s \in \mathbb{Z}$ denotes the unique integer such that $\gamma \in \alpha^{s/RV}$. Finally, we set

$$\Delta = \iota_0(\beta) - \mathfrak{log}(\iota_0(\alpha)).$$

We may and do assume that $|\Delta| \le 1$.

3.2. Interpolation. An application of Siegel's Lemma gives the following Padé approximation, which comprises the interpolation step of our proof.

Claim 3.1. *For any tuple (R, S, T, U, V) of positive integers such that*

$$(\mathrm{R}_1) \qquad T \ge \max\{2, U, \log(R), \log(V), \log(S+1)\},$$

there exists a tuple

$$\underline{P} = (P_0, \ldots, P_U) \in \mathbb{Q}[X]^{U+1},\ (P_0, \ldots, P_U) \ne (0, \ldots, 0),$$

satisfying

$$\frac{\deg(\underline{P})}{T} \leq \frac{R(S+1)V}{U+1}, \tag{3.1}$$

$$\frac{h(\underline{P})}{T} \leq \frac{1}{S}\left(\frac{R(S+1)V}{U+1}\log(2)+\log(U)+c_7\right)+\frac{1}{2T}, \tag{3.2}$$

where $c_7 > 0$ *is some absolute constant, and*

$$\mathrm{ord}_{X=\gamma}(P_0Y_\gamma^0+\ldots+P_UY_\gamma^U) \geq T \textit{ for all } \gamma \in \alpha^{0/RV}. \tag{3.3}$$

Proof. For the proof, we fix representatives $\zeta^{(1)},\ldots,\zeta^{(L)}$ of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-set $\alpha^{0/RV}$, the set of RV-th roots of unity. For readability we write Y_l^u instead of $Y_{\zeta^{(l)}}^u$. It suffices to prove the existence of $\underline{P} \in \mathbb{Q}[X]^{U+1}$ satisfying (3.1), (3.2), and

$$\mathrm{ord}_{X=\zeta^{(l)}}(P_0Y_l^0+\ldots+P_UY_l^U) \geq T \text{ for all } 1 \leq l \leq L. \tag{3.4}$$

In fact, a general element in $\alpha^{0/RV}$ is of the form $\sigma(\zeta^{(l)})$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and

$$\sigma(P_0Y_l^0+\ldots+P_UY_l^U) = P_0Y_{\sigma(\zeta^{(l)})}^0+\ldots+P_UY_{\sigma(\zeta^{(l)})}^U$$

would imply (3.3) for $\sigma(\zeta^{(l)})$. Since the orbit of $\zeta^{(l)}$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ has size $[\mathbb{Q}(\zeta^{(l)}):\mathbb{Q}]$ and the set $\alpha^{0/RV}$ has RV elements in total we conclude that

$$\textstyle\sum_{l=1}^{L}[\mathbb{Q}(\zeta^{(l)}):\mathbb{Q}] = RV. \tag{3.5}$$

The Taylor expansion of P_u, $0 \leq u \leq U$, at $\zeta^{(l)}$, $1 \leq l \leq L$, is

$$\sum_{0\leq i\leq \deg(\underline{P})} \frac{1}{i!}\frac{\partial^i P_u}{\partial X^i}(\zeta^{(l)})(X-\zeta^{(l)})^i,$$

and

$$\frac{1}{i!}\frac{\partial^i P_u}{\partial X^i}(\zeta^{(l)}) = \sum_{0\leq j\leq \deg(\underline{P})-i}\binom{j+i}{j}(P_u)_{j+i}(\zeta^{(l)})^j.$$

Thus,

$$P_u = \sum_{0\leq i\leq \deg(\underline{P})}\left(\sum_{0\leq j\leq \deg(\underline{P})-i}\binom{j+i}{j}(P_u)_{j+i}(\zeta^{(l)})^j\right)(X-\zeta^{(l)})^i,$$

and

$$P_0Y_l^0+\ldots+P_UY_l^U \in \mathbb{Q}(\zeta^{(l)})[[X-\zeta^{(l)}]]$$

equals

$$\sum_{i=0}^{\infty}\left(\sum_{\substack{0\le u\le U\\ i^{(1)}+i^{(2)}=i\\ 0\le j\le \deg(\underline{P})-i^{(1)}}} \binom{j+i^{(1)}}{j}\,(P_u)_{j+i^{(1)}}\,(\zeta^{(l)})^j\,(Y_l^u)_{i^{(2)}}\right)(X-\zeta^{(l)})^i.$$

Therefore, condition (3.4) is equivalent to the following set of homogeneous linear equations (on substituting $k = j + i^{(1)}$):

(3.6)

$$\sum_{\substack{0\le u\le U\\ 0\le k\le \deg(\underline{P})}}\left(\sum_{k-i\le j\le k}\binom{k}{j}(\zeta^{(l)})^j\,(Y_l^u)_{i-k+j}\right)(P_u)_k = 0,\quad \begin{array}{c} 0\le i\le (T-1),\\ 1\le l\le L.\end{array}$$

For each $1 \le l \le L$ these are T equations in $(U+1)(\deg(\underline{P})+1)$ unknowns $(P_u)_k$ and with coefficients in $\mathbb{Q}(\zeta^{(l)})$. Now, the inequality (3.1) is immediately satisfied if we set

$$\deg(\underline{P}) = \left\lfloor T\frac{R(S+1)V}{U+1}\right\rfloor.$$

Indeed, this implies directly (3.1) and furthermore

$$R(S+1)TV \le (U+1)(\deg(\underline{P})+1).$$

Combining this with (3.5) yields that the positive integer

$$T\sum_{l=1}^{L}[\mathbb{Q}(\zeta^{(l)}):\mathbb{Q}] = RTV$$

is bounded from above by

$$(U+1)(\deg(\underline{P})+1).$$

Thus, by elementary linear algebra the above system (3.6) of homogeneous linear equations admits non-trivial $\mathbb{Q}$-rational solutions. It remains to obtain such a solution with bounded height by an application of Lemma 2.1: For this, we have to bound the heights of the coefficients

$$\eta_{l,i,u,k} = \sum_{k-i\le j\le k}\binom{k}{j}(\zeta^{(l)})^j\,(Y_l^u)_{i-k+j}\,,$$

where $1 \le l \le L$, $0 \le i \le (T-1)$, $0 \le u \le U$, and $0 \le k \le \deg(\underline{P})$. In the case that ι is a non-archimedean embedding we have

$$\log^+\left(\max_{\substack{0\le u\le U\\ 0\le k\le \deg(\underline{P})}}|\eta_{l,i,u,k}|_\iota\right) \le \log^+\left(\max_{\substack{0\le u\le U\\ 0\le m\le (T-1)}}|(Y_l^u)_m|_\iota\right)$$

for all $1 \leq l \leq L$ and $0 \leq i \leq (T-1)$. If ι is archimedean a similar estimate is obtained by noting that

$$\log^+(|\eta_{l,i,u,k}|_\iota) \leq \log^+\left(\max_{\substack{0\leq u\leq U\\0\leq m\leq (T-1)}} |(Y_l^u)_m|_\iota\right) + \log\left(\max_{0\leq j\leq i}\binom{k}{j}\right) + \log(T);$$

since we have

$$\max_{\substack{0\leq j\leq (T-1)\\0\leq k\leq \deg(\underline{P})}}\binom{k}{j} \leq 2^{\deg(\underline{P})},$$

it follows that

$$\log^+\left(\max_{\substack{0\leq u\leq U\\0\leq k\leq \deg(\underline{P})}} |\eta_{l,i,u,k}|_\iota\right) \leq \log^+\left(\max_{\substack{0\leq u\leq U\\0\leq m\leq (T-1)}} |(Y_l^u)_m|_\iota\right) + \log(2)\deg(\underline{P}) + \log(T)$$

for each archimedean embedding ι. Invoking Lemmas 2.2 and 2.3, a summation over all embeddings of $\mathbb{Q}(\zeta^{(l)})$ and division by $[\mathbb{Q}(\zeta^{(l)}) : \mathbb{Q}]$ gives (taking into account the assumption (R_1))

$$\begin{aligned} h(\eta_{l,i}) &\leq \log(2)\deg(\underline{P}) + T\left(1 + \frac{c_5}{\max\{1, \log^+(T)\}}\right)(1+\log(U)) \\ &\quad + \log(T) + \log(2)\,(T+U) \\ &< T\left(\frac{R(S+1)V}{U+1}\log(2) + \log(U) + c_6\right), \end{aligned}$$

where $\eta_{l,i}$ denotes the $(U+1)(\deg(\underline{P})+1)$-tuple having entries $\eta_{l,i,u,k}$, $0 \leq u \leq U$, $0 \leq k \leq \deg(\underline{P})$, in some arbitrary (e.g., lexicographic) order and c_6 is some positive absolute constant. Now, Lemma 2.1 implies that there exists a non-zero solution $\underline{P} \in \mathbb{Q}[X]^{U+1}$ of the equation system (3.6) such that its height $h(\underline{P})$ is less than

$$\frac{RTV\left(\max_{\substack{0\leq i\leq (T-1)\\1\leq l\leq L}} h(\eta_{l,i}) + \frac{1}{2}\log\left((U+1)(\deg(\underline{P})+1)\right)\right)}{(U+1)(\deg(\underline{P})+1) - RTV} + c_4(\mathbb{Q}).$$

Using $(U+1)(\deg(\underline{P})+1) - RTV \geq RSTV$, this can be furthermore bounded from above by

$$\frac{1}{S}\left(\max_{\substack{0\leq i\leq (T-1)\\1\leq l\leq L}} h(\eta_{l,i}) + \frac{1}{2}\left(\log(RTV(S+1)) + 1\right)\right) + c_4(\mathbb{Q}).$$

With (3.1), (R_1), and $c_4(\mathbb{Q}) = 1/2$ this implies immediately

$$h(\underline{P})/T \leq \frac{1}{S}\left(\frac{R(S+1)V}{U+1}\log(2) + \log(U) + c_7\right) + \frac{1}{2T},$$

where one may take $c_7 = c_6 + 5/2$. $\square$

3.3. Set up (Second Part). Before we can conduct the extrapolation step we need to introduce some further objects and notations. For readability we leave out the reference to R and V in some definitions given here. This comes at the moderate price that they tacitly depend on these two auxiliary parameters. The extrapolation step takes place at a point in one of the sets $\alpha^{r/RV} \subseteq \bar{\mathbb{Q}}$, $1 \leq r \leq R$, chosen with the help of a zero estimate. For any such $\gamma \in \alpha^{r/RV}$ we write

$$T_\gamma = \mathrm{ord}_{X=\gamma}(P_0 Y_\gamma^0 + \ldots + P_U Y_\gamma^U).$$

Since $\underline{P}$ is non-zero and Y_γ^1 is transcendental over $\bar{\mathbb{Q}}[X]$, the order T_γ is always finite. For each $0 \leq r \leq R$ we define $\mathcal{Z}_r$ as the set of all pairs (γ, ι) consisting of an element $\gamma \in \alpha^{r/RV}$ and an archimedean embedding $\iota \in \mathcal{E}_{\mathbb{K}(\gamma)}$ dividing ι_0. Recall that the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{K})$ acts naturally[10] on $\mathcal{Z}_r$ and denote the corresponding equivalence relation by $\sim$. By $\zeta(\gamma, \iota)$ we denote the (unique) complex root of unity ζ that satisfies

$$\arg(\iota(\gamma)) = \frac{r}{RV}\arg(\iota_0(\alpha)) + \arg(\zeta).$$

Evidently, for each $r \in \{0, \ldots, R\}$ the induced map

$$\mathcal{Z}_r/\!\sim\ \longrightarrow \mu_{RV}(\mathbb{C}),\ \ [(\gamma,\iota)]_\sim \longmapsto \zeta(\gamma,\iota),$$

is a bijection. For any $(\gamma, \iota) \in \mathcal{Z}_r$ the formal series Y_γ^u, $0 \leq u \leq U$, yield complex power series

$$Y^u_{(\gamma,\iota)} = \iota(Y_\gamma^u) = \left(\frac{r}{RV}\iota_0(\beta) + \mathrm{Log}_{\iota(\gamma)}(z)\right)^u \in \mathbb{C}[[z - \iota(\gamma)]],\ 0 \leq u \leq U.$$

These series define holomorphic functions on the open disc with center $\iota(\gamma) \in \mathbb{C}^\times$ and radius $|\iota_0(\alpha)|^{r/RV}$, which we also denote by $Y^u_{(\gamma,\iota)}$.

Claim 3.2. *Assume that*

(R_2) $$V > |\mathfrak{log}(\iota_0(\alpha))|\,.$$

Then, for any $(\gamma,\iota) \in \mathcal{Z}_r$, $1 \leq r \leq R$, *we have*

$$\log|\iota(\gamma) - \zeta(\gamma,\iota)| < -\log(V) + \log|\mathfrak{log}(\iota_0(\alpha))| + 1.$$

---

[10] In fact, with each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{K})$ is associated the bijection

$$(\gamma,\iota) \longmapsto (\sigma(\gamma), \iota \circ \sigma^{-1}).$$

Proof. For any $(\gamma,\iota) \in \mathscr{Z}_r$ the real part of the complex logarithm $\mathfrak{log}$ at γ does not depend on a choice of branch and equals

$$\left(\frac{r}{RV}\right) \operatorname{Re} \mathfrak{log}(\iota_0(\alpha)).$$

From our assumption on V we infer that this is bounded from above by

$$\frac{1}{V} \left|\operatorname{Re} \mathfrak{log}(\iota_0(\alpha))\right| < 1.$$

This implies immediately that $\iota(\gamma)$ is contained in the annulus $e^{-1} < |z| < e$. The assertion can be inferred from this by an elementary consideration. □

Claim 3.3. *Assume that*

$$V > e \left|\mathfrak{log}(\iota_0(\alpha))\right|. \tag{R_3}$$

Then $\iota(\gamma) \in B_1(\zeta(\gamma,\iota))$ *for any* $(\gamma,\iota) \in \mathscr{Z}_r$. *In addition, the holomorphic function* $Y^u_{(\gamma,\iota)}$ *extends (uniquely) to* $B_1(\zeta(\gamma,\iota))$ *and its Taylor expansion at* $\zeta(\gamma,\iota)$ *is given by the power series*

$$\left(\operatorname{Log}_{\zeta(\gamma,\iota)}(z) + \frac{r}{RV}\Delta\right)^u \in \mathbb{C}[[z-\zeta(\gamma,\iota)]],\ 0 \leq u \leq U.$$

Proof. By Claim 3.2, our assumption on V implies

$$\log|\iota(\gamma) - \zeta(\gamma,\iota)| < -\log(V) + \log\left|\mathfrak{log}(\iota_0(\alpha))\right| + 1 < 0,$$

which means $|\iota(\gamma) - \zeta(\gamma,\iota)| < 1$. It follows immediately that

$$\left|\operatorname{Log}_{\zeta(\gamma,\iota)}(\iota(\gamma))\right| < 1 + \pi/3$$

For the following, fix some representative $\eta \in 2\pi\mathbb{Q}$ of $\arg(\zeta(\gamma,\iota))$. There exists a single branch of the complex logarithm on $B_1(\zeta(\gamma,\iota))$ with Taylor expansion

$$\frac{r}{RV}\mathfrak{log}(\iota_0(\alpha)) + i\eta + \operatorname{Log}_{\iota(\gamma)}(z)$$

at $\iota(\gamma)$ and Taylor expansion

$$i\eta + \operatorname{Log}_{\zeta(\gamma,\iota)}(z)$$

at $\zeta(\gamma,\iota)$; for both Taylor expansions give rise to a branch of the complex logarithm and thus they are either equal or their difference has value bounded from below by 2π. By evaluation at $z = \iota(\gamma)$ we infer that this difference is actually of value less than $2 + \pi/3 < 2\pi$. Hence, it must be zero. Now, the series $Y^1_{(\gamma,\iota)}$ has Taylor expansion

$$\frac{r}{RV}\iota_0(\beta) + \operatorname{Log}_{\iota(\gamma)}(z)$$

at $\iota(\gamma) \in B_1(\zeta(\gamma,\iota))$. From the above consideration, it follows that its analytic continuation to $B_1(\zeta(\gamma,\iota))$ has Taylor expansion

$$\frac{r}{RV}\iota_0(\beta) - \frac{r}{RV}\mathfrak{log}(\iota_0(\alpha)) + \mathrm{Log}_{\zeta(\gamma,\iota)}(z) = \frac{r}{RV}\Delta + \mathrm{Log}_{\zeta(\gamma,\iota)}(z)$$

at $\zeta(\gamma,\iota)$. □

3.4. Zero estimate. We now dwell on finding an algebraic number $\gamma_0 \in \alpha^{r/RV}$, $1 \leq r \leq R$, that is favorable for our extrapolation step. For this, we use our zero estimate and the various notations introduced in Section 2.4.

Claim 3.4. *There exists some* $\gamma_0 \in \alpha^{r_0/RV}$, $r_0 \in \{1,\ldots,R\}$, *such that*

$$\frac{T_{\gamma_0}}{T} \leq \frac{S}{R} + \frac{(R+1)(U+1)}{RT} + \frac{U(U+1)}{2R^2VT}.$$

Proof. For any $(\gamma,\iota) \in \mathcal{Z}_r$ the vector

$$\left(Y^0_{(\gamma,\iota)},\ldots,Y^U_{(\gamma,\iota)}\right)^t \in \mathbb{C}[[z-\iota(\gamma)]]^{U+1}$$

is easily seen to be a formal solution of the differential system (2.4) with $m = U+1$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1/X & 0 & \ldots & 0 & 0 \\ 0 & 2/X & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & U/X & 0 \end{pmatrix}$$

at the point $\iota(\gamma) \in \mathbb{P}_1(\mathbb{C})$; for this means nothing else but

$$\frac{d}{dz}Y^0_{(\gamma,\iota)}(z) = 0$$

and

$$\begin{aligned} \frac{d}{dz}Y^u_{(\gamma,\iota)}(z) &= \frac{d}{dz}\left(\frac{r}{RV}\iota_0(\beta) + \mathrm{Log}_{\iota(\gamma)}(z)\right)^u \\ &= \frac{u}{z}\left(\frac{r}{RV}\iota_0(\beta) + \mathrm{Log}_{\iota(\gamma)}(z)\right)^{u-1} \\ &= \frac{u}{z}Y^{u-1}_{(\gamma,\iota)}(z) \end{aligned}$$

for each $u \in \{1,\ldots,U\}$. It is easy to see (e.g. by writing down explicit fundamental matrices) that 0 and ∞ are the only singular points of $\mathbf{A}$. In addition, since the solutions of (2.4) in L_0 with $\mathbf{A}$ and m as above are of the form

$$(Y_0(z),\ldots,Y_U(z)),$$

where

$$Y_j(z) = \sum_{i=0}^{j} c_i \binom{j}{i} l_0^{j-i},\ c_i \in \mathbb{C},\ 0 \leq j \leq U,$$

we have $\varepsilon(\mathbf{A},0)=0$. By symmetry, we also have $\varepsilon(\mathbf{A},\infty)=0$. We apply Lemma 2.6 for the $n=R(R+1)V$ distinct points

$$\bigcup_{r=0}^{R}\{\iota(\gamma)\mid(\gamma,\iota)\in\mathscr{Z}_r\}\subseteq\mathbb{P}_1(\mathbb{C})\backslash\{0,\infty\}$$

and the (formal) solutions

$$(Y^0_{(\gamma,\iota)},\dots,Y^U_{(\gamma,\iota)})^t\in\mathbb{C}[[X-\gamma]]^{U+1},\ (\gamma,\iota)\in\mathscr{Z}_r.$$

We infer that

$$\sum_{r=0}^{R}\sum_{\gamma\in\alpha^{r/RV}}\mathrm{ord}_{X=\gamma}(P_0Y^0_\gamma+\ldots+P_UY^U_\gamma)=RVT+\sum_{r=1}^{R}\sum_{\gamma\in\alpha^{r/RV}}T_\gamma$$

is bounded from above by

$$(U+1)\deg(\underline{P})+(U+1)R(R+1)V+\frac{(U+1)U}{2}.$$

By using the bound (3.1) and the pigeonhole principle on the R^2V points $\bigcup_{r=1}^{R}\alpha^{r/RV}$, we deduce that there must exist some $\gamma_0\in\alpha^{r_0/RV}$, $r_0\in\{1,\dots,R\}$, as claimed. $\square$

To simplify notation, we write $T_0=T_{\gamma_0}$, $\mathbb{K}_0=\mathbb{K}(\gamma_0)$, and $\zeta_0^{(\iota)}=\zeta(\gamma_0,\iota)$ in the following. We also define

$$\beta_0=\frac{1}{T_0!}\frac{\partial^{T_0}}{\partial X^{T_0}}(P_0Y^0_{\gamma_0}+\ldots+P_UY^U_{\gamma_0})(\gamma_0)=\sum_{\substack{0\le u\le U\\ i+j=T_0}}\frac{1}{i!}\frac{\partial^iP_u}{\partial X^i}(\gamma_0)\frac{1}{j!}\frac{\partial^jY^u_\gamma}{\partial X^j}(\gamma_0).$$

By its construction, β_0 is a non-zero algebraic number.

3.5. Extrapolation. For use in the proof of Claim 3.7 below, we deduce now in Claims 3.5 and 3.6 upper bounds on the archimedean and non-archimedean values of β_0. In fact, Claim 3.7 follows directly from inserting these bounds in the product formula for the non-zero algebraic number β_0. We denote by $\mathcal{S}$ the set of archimedean embeddings of $\mathbb{K}_0$ extending ι_0 and by $\overline{\mathcal{S}}$ its complement in $\mathcal{E}_{\mathbb{K}_0,\infty}$. The terms $\Theta_1,\dots,\Theta_4$ should be considered negligible; this is justified later.

Claim 3.5. *The quantity $T^{-1}h_f(\beta_0)$ is bounded from above by*

$$\frac{U}{T}\log(V)+\left(1+\frac{c_5}{\max\{1,\log^+(T_0)\}}\right)\frac{T_0}{T}\log(U)\\+\frac{h_f(\underline{P})}{T}+\frac{R(S+1)}{U+1}h_f(\alpha)+\frac{T_0}{TV}h_f(\alpha^{-1})+\frac{U}{T}h_f(\beta)+\Theta_1,$$

where

$$\Theta_1 = \log(R)\frac{U}{T} + \left(1 + \frac{c_5}{\max\{1, \log^+(T_0)\}}\right)\frac{T_0}{T}.$$

Similarly, the real number $T^{-1}h_{\overline{\mathcal{S}}}(\beta_0)$ is bounded from above by

$$\frac{h_{\overline{\mathcal{S}}}(\underline{P})}{T} + \frac{R(S+1)}{U+1}h_{\overline{\mathcal{S}}}(\alpha) + \frac{T_0}{TV}h_{\overline{\mathcal{S}}}(\alpha^{-1}) + \frac{U}{T}h_{\overline{\mathcal{S}}}(\beta) + \Theta_2,$$

where

$$\Theta_2 = 2\frac{R(S+1)V}{U+1} + \left(\frac{T_0}{T} + \frac{U}{T}\right)(1 + \log(2)).$$

Proof. For any non-archimedean embedding $\iota : \mathbb{K}_0 \hookrightarrow \mathbb{C}_p$ we infer from ultrametricity that

$$\log^+(|\beta_0|_\iota) \le \log^+\left(\max_{\substack{0\le u\le U\\0\le i\le T_0}}\left|\frac{1}{i!}\frac{\partial^i P_u}{\partial X^i}(\gamma)\right|_\iota\right) + \log^+\left(\max_{\substack{0\le u\le U\\0\le j\le T_0}}\left|\frac{1}{j!}\frac{\partial^j Y_\gamma^u}{\partial X^j}(\gamma)\right|_\iota\right).$$

Using the triangle inequality instead of ultrametricity, we deduce for any archimedean embedding $\iota : \mathbb{K}_0 \hookrightarrow \mathbb{C}$ dividing some embedding in $\mathcal{E}_{\mathbb{K}_0,\infty}$ the above bound with an additional summand

$$\log(T_0+1) + \log(U+1)$$

on the right hand side. Lemma 2.5 implies directly that the quantity

$$T^{-1}\log^+\left(\max_{\substack{0\le u\le U\\0\le i\le T_0}}\left|\frac{1}{i!}\frac{\partial^i P_u}{\partial X^i}(\gamma_0)\right|_\iota\right)$$

is bounded from above by

$$\frac{\deg(\underline{P})}{T}\log^+(|\gamma|_\iota) + \frac{\log(|\underline{P}|_\iota)}{T} \le \frac{R(S+1)}{U+1}\log^+(|\alpha|_\iota) + \frac{\log(|\underline{P}|_\iota)}{T}$$

if ι is non-archimedean and by

$$\frac{R(S+1)}{U+1}\log^+(|\alpha|_\iota) + \frac{\log^+(|\underline{P}|_\iota)}{T} + 2\frac{R(S+1)V}{U+1}$$

if ι is archimedean. In addition, Lemma 2.2 yields immediately that

$$\frac{1}{[\mathbb{K}_0:\mathbb{Q}]}\sum_{\iota\in\mathcal{E}_{\mathbb{K}_0,f}}\log^+\left(\max_{\substack{0\le u\le U\\0\le j\le T_0}}\left|\frac{1}{j!}\frac{\partial^j Y_{\gamma_0}^u}{\partial X^j}(\gamma)\right|_\iota\right) \tag{3.7}$$

is bounded from above by

$$\frac{T_0}{V}h_f(\alpha^{-1}) + U(h_f(\beta) + \log(RV)) + T_0\left(1 + \frac{c_5}{\max\{1, \log^+(T_0)\}}\right)(1 + \log(U)).$$

The non-archimedean part of the claim now follows. Finally, if we replace $\sum_{\iota\in\mathcal{E}_{\mathbb{K}_0,f}}$ with $\sum_{\iota\in\overline{\mathcal{S}}}$ in (3.7) Lemma 2.3 gives us the upper bound

$$\frac{T_0}{V}h_{\overline{\mathcal{S}}}(\alpha^{-1})+Uh_{\overline{\mathcal{S}}}(\beta)+(T_0+U)\log(2).$$

on the outcome. With these preparations, the asserted bounds can be deduced by elementary manipulations. $\square$

The most important contribution to the product formula for β_0 comes from the archimedean embeddings that divide ι_0. For each such $\iota\in\mathcal{E}_{\mathbb{K}_0,\infty}$ the complex number $\iota(\beta_0)$ is the T_0-th Taylor coefficient of the holomorphic function $F_0^{(\iota)}$ given around $\iota(\gamma_0)$ by the series

$$(P_0Y^0_{(\gamma_0,\iota)}+\ldots+P_UY^U_{(\gamma_0,\iota)})(z)\in\mathbb{C}[[z-\iota(\gamma_0)]].$$

Recall from Claim 3.3 that $F_0^{(\iota)}(z)$ coincides with

$$\sum\nolimits_{u=0}^{U}\iota(P_u)(z)\left(\mathrm{Log}_{\zeta_0^{(\iota)}}(z)+\frac{r_0}{RV}\Delta\right)^u$$

on the non-empty overlap $\mathcal{D}$ of $B_1(\zeta_0^{(\iota)})$ and $B_{|\iota(\gamma_0)|}(\iota(\gamma_0))$. We define further

$$G_0^{(\iota)}(z)=\sum\nolimits_{u=0}^{U}\iota(P_u)(z)\mathrm{Log}_{\zeta_0^{(\iota)}}(z)^u,\ z\in B_1(\zeta_0^{(\iota)}).$$

The construction in Claim 3.1 ensures $\mathrm{ord}_{z=\zeta_0^{(\iota)}}(G_0^{(\iota)})\geq T$; for there exists some $\zeta\in\alpha^{0/RV}$ together with an archimedean embedding $\kappa:\mathbb{K}_0(\zeta)\to\mathbb{C}$ such that $\kappa(\zeta)=\zeta_0^{(\iota)}$ and we conclude that

$$G_0^{(\iota)}(z)=\sum\nolimits_{u=0}^{U}\iota(P_u)(z)\kappa(Y_\zeta^u)(z)=\kappa(P_0Y_\zeta^0+\ldots+P_UY_\zeta^U)(z),\ z\in\mathcal{D},$$

because the polynomials P_u, $0\leq u\leq U$, have their coefficients in $\mathbb{Q}$.

Claim 3.6. *For each $\iota\in\mathcal{S}$, the quantity $T^{-1}\log(|\beta_0|_\iota)$ is bounded from above by*

$$\frac{1}{T}\log(|\Delta|)+\frac{R(S+1)}{U+1}\log^+(|\alpha|_\iota)+\frac{\log^+(|\underline{P}|_\iota)}{T}+\frac{T_0}{TV}\log^+(|\alpha|_\iota^{-1})+\Theta_3, \tag{3.8}$$

where

$$\Theta_3=\left(\frac{T_0}{T}+\frac{U}{T}+\frac{1}{T}\right)\log(2)+2\frac{R(S+1)V}{U+1},$$

or by

$$\min\left\{0,-1+\frac{T_0}{T}\right\}(\log(V)-\log|\mathfrak{log}(\iota_0(\alpha))|-1)+\frac{\log^+(|\underline{P}|_\iota)}{T}+\frac{T_0}{TV}\log^+(|\alpha|_\iota^{-1})+\Theta_4, \tag{3.9}$$

where

$$\Theta_4 = \frac{R(S+1)V}{U+1}\left(\log\left(\frac{3}{2}\right)+2\right)+3\left(\frac{T_0}{T}+\frac{U}{T}\right)+\frac{\log(2)}{T}+\log(2).$$

Proof. Clearly, we have

$$\iota(\beta_0) = \sum_{\substack{0\le u\le U\\ i+j=T_0}} \iota\left(\frac{1}{i!}\frac{\partial^i P_u}{\partial z^i}(\gamma_0)\right)\left(\frac{1}{j!}\frac{\partial^j h_u}{\partial z^j}(\iota(\gamma_0))\right)+\frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}(\iota(\gamma_0))$$

with

$$h_u(z) = \left(\mathrm{Log}_{\zeta_0^{(\iota)}}(z)+\frac{r_0}{RV}\Delta\right)^u - \mathrm{Log}_{\zeta_0^{(\iota)}}(z)^u,\ z\in\mathcal{D}.$$

The assertion can be shown by estimating $|\iota(\beta_0)|$ from above by

$$\max\left\{\sum_{\substack{0\le u\le U\\ i+j=T_0}} \iota\left(\frac{1}{i!}\frac{\partial^i P_u}{\partial z^i}(\gamma_0)\right)\left(\frac{1}{j!}\frac{\partial^j h_u}{\partial z^j}(\iota(\gamma_0))\right), \frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}(\iota(\gamma_0))\right\}$$

and giving separate estimations for both entries of the maximum. As in the proof of Claim 3.5 we can show that

$$T^{-1}\log^+\left(\max_{\substack{0\le u\le U\\ 0\le i\le T_0}}\left|\frac{1}{i!}\frac{\partial^i P_u}{\partial X^i}(\gamma_0)\right|_\iota\right)$$

is bounded by

$$\frac{R(S+1)}{U+1}\log^+(|\alpha|_\iota)+\frac{\log^+(|\underline{P}|_\iota)}{T}+2\frac{R(S+1)V}{U+1}.$$

In addition, Lemma 2.4 implies

$$\max_{\substack{0\le u\le U\\ 0\le j<T_0}}\left\{\log\left|\frac{1}{j!}\frac{\partial^j h_u}{\partial z^j}(\iota(\gamma_0))\right|\right\}$$
$$\le \log(|\Delta|)-\log(V)+\frac{T_0}{V}\log^+(|\alpha|_\iota^{-1})+(T_0+U)\log(2).$$

This gives the first alternative of the claim. It remains to bound the absolute value of $\frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}(\iota(\gamma_0))$, making use of the fact that $G_0^{(\iota)}$ has a zero of high order at $\zeta_0^{(\iota)}$. For this, we invoke the maximum modulus principle for the holomorphic function $\frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}$ on the disc $B_{1/2}(\iota(\gamma_0))$. Since $\frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}$ has a $\max\{0,T-T_0\}$-fold zero at $\zeta_0^{(\iota)}$, this yields the majorization

$$\max\{0,T-T_0\}\log\left(2|\iota(\gamma_0)-\zeta_0^{(\iota)}|\right)\ +\ \log\left(\max_{|z-\iota(\gamma_0)|=1/2}\left|\frac{1}{T_0!}\frac{\partial^{T_0}G_0^{(\iota)}}{\partial z^{T_0}}(z)\right|\right)$$

of

$$\log \left| \frac{1}{T_0!} \frac{\partial^{T_0} G_0^{(\iota)}}{\partial z^{T_0}} (\iota(\gamma_0)) \right|.$$

(This is true regardless of whether $\zeta_0^{(\iota)} \in B_{1/2}(\iota(\gamma_0))$.) Furthermore, Claim 3.2 tells us that

$$\log \left(|\iota(\gamma_0) - \zeta_0^{(\iota)}| \right) < -\log(V) + \log |\mathfrak{log}(\iota_0(\alpha))| + 1.$$

To bound the other summand, we write

$$\frac{1}{T_0!} \frac{\partial^{T_0} G_0^{(\iota)}}{\partial z^{T_0}} (z) = \sum_{\substack{0 \leq u \leq U \\ i+j=T_0}} \left(\frac{1}{i!} \frac{\partial^i \iota(P_u)}{\partial z^i} (z) \right) \left(\frac{1}{j!} \frac{\partial^j \mathrm{Log}^u_{\zeta_0^{(\iota)}}}{\partial z^j} (z) \right).$$

Lemma 2.5 implies that

$$\frac{1}{T} \log \left(\max_{|z - \iota(\gamma_0)| = 1/2} \max_{0 \leq i \leq T_0} \left| \frac{1}{i!} \frac{\partial^i \iota(P_u)}{\partial z^i} (z) \right| \right)$$

is bounded from above by

$$\frac{R(S+1)V}{U+1} \left(\log \left(\frac{3}{2} \right) + 2 \right) + \frac{\log(|\underline{P}|_\iota)}{T}.$$

Additionally, by Lemma 2.3 the quantity

$$\frac{1}{T} \log \left(\max_{0 \leq j \leq T_0} \left| \frac{1}{j!} \frac{\partial^j \mathrm{Log}^u_{\zeta_0^{(\iota)}}}{\partial z^j} (\iota(\gamma_0)) \right| \right)$$

is less than

$$\frac{T_0}{TV} \log^+(|\alpha|_\iota^{-1}) + \frac{U}{T} \log^+(|\mathrm{Log}_{\zeta_0^{(\iota)}}(\iota(\gamma_0))|) + \left(\frac{T_0}{T} + \frac{U}{T} \right) \log(2).$$

Since $\left| \mathrm{Log}_{\zeta_0^{(\iota)}}(\iota(\gamma_0)) \right| < 1 + \pi/3$ (compare with the proof of Claim 3.2), we may conclude by an elementary manipulation, obtaining the bound in the second alternative of the claim. □

From now on, we keep fixed some

$$\varepsilon_0 \in \left(0, \frac{1}{(8 + 6c_5)[\mathbb{K} : \mathbb{Q}] + 6} \right) \subset \left(0, \frac{1}{14} \right).$$

As one might expect, the relation with the ε in the statement of Theorem 1.2 is such that we have to take $\varepsilon_0 \to 0$ if $\varepsilon \to 0$.

Claim 3.7. *Assume that*

$$(\mathrm{R}_4) \qquad S = \left\lceil \varepsilon_0^{-1} \right\rceil, \ R = \left\lceil \varepsilon_0^{-1} S \right\rceil, \ \textit{and } U = V.$$

Furthermore, assume

$$T \geq U \max\left\{\varepsilon_0^{-1}, h(\beta)\right\}. \tag{R$_5$}$$

Then, there exist constants $c_8(\varepsilon_0)$ *and* $c_9(\varepsilon_0)$ *– specified in the proof below – such that*

$$V \geq \max\{h(\alpha), c_8(\varepsilon_0)\, |\mathfrak{log}(\iota_0(\alpha))|^{c_9(\varepsilon_0)}\}, \tag{R$_6$}$$

implies

$$\begin{aligned} &\frac{1}{T}\log(|\Delta|) + \frac{R(S+1)}{U+1}\log^+(|\alpha|_{\iota_0}) + \Theta_3 \\ &\qquad > \min\left\{0, -1 + \frac{T_0}{T}\right\}(\log(V) - \log|\mathfrak{log}(\iota_0(\alpha))| - 1) + \Theta_4. \end{aligned} \tag{3.10}$$

In addition, we have $c_9(\varepsilon_0) \to 1$ *if* $\varepsilon_0 \to 0$.

Proof. We argue by contradiction and assume that (3.10) is false. This allows us to use Claim 3.6 for all $\iota \in \mathcal{S}$ in its second alternative (3.9). Our plan is to derive a contradiction from the product formula

$$0 = T^{-1}h_f(\beta_0) + T^{-1}h_{\mathcal{S}}(\beta_0) + T^{-1}h_{\overline{\mathcal{S}}}(\beta_0)$$

by estimating both $T^{-1}h_f(\beta_0)$ and $T^{-1}h_{\overline{\mathcal{S}}}(\beta_0)$ (resp. $T^{-1}h_{\mathcal{S}}(\beta_0)$) from above with the help of Claim 3.5 (resp. Claim 3.6). After elementary manipulations, we obtain from the product formula that

$$\begin{aligned} 0 \leq\ & \frac{1}{[\mathbb{K}:\mathbb{Q}]}\min\left\{0, -1 + \frac{T_0}{T}\right\}(\log(V) - \log|\mathfrak{log}(\iota_0(\alpha))| - 1) \\ &+ \frac{U}{T}\log(V) + \left(1 + \frac{c_5}{\max\{1, \log^+(T_0)\}}\right)\frac{T_0}{T}\log(U) \\ &+ \frac{h(\underline{P})}{T} + \left(\frac{R(S+1)}{U+1} + \frac{T_0}{TV}\right)h(\alpha) + \frac{U}{T}h(\beta) \\ &+ \Theta_1 + \Theta_2 + \Theta_4. \end{aligned} \tag{3.11}$$

Now, we give upper bounds for most of the terms in (3.11); the gist is that the first one dominates all other. To be precise, the conditions (R_4), (R_5), and (R_6) force them to be less then some small multiple of $\log(V)$ plus some absolute constant. First of all, using $S \geq \varepsilon_0^{-1}$, $(S+1)/S \leq 2$, $R \leq 4\varepsilon_0^{-2}$, $T \geq \varepsilon_0^{-1}$, and $U = V$ we obtain from (3.2) that

$$\frac{h(\underline{P})}{T} < \varepsilon_0\log(V) + \left(8\varepsilon_0^{-2}\log(2) + \varepsilon_0 c_7 + \varepsilon_0/2\right).$$

Employing the inequalities $R \geq \varepsilon_0^{-1}S$, $(R+1)/R \leq 2$, $T \geq \varepsilon_0^{-1}U \geq \varepsilon_0^{-1}$, and $R \geq \varepsilon_0^{-2}$ we infer from Claim 3.4 that

$$\frac{T_0}{T} \leq \frac{S}{R} + \frac{(R+1)(U+1)}{RT} + \frac{U(U+1)}{2R^2VT} \leq \varepsilon_0 + 4\varepsilon_0 + \varepsilon_0^5 < 6\varepsilon_0.$$

(Note that $\varepsilon_0 < 1/14$ and hence $T_0/T < 1$.) Thus, we have

$$\left(1 + \frac{c_5}{\max\{1, \log^+(T_0)\}}\right) \frac{T_0}{T} \log(U) < 6\varepsilon_0(1 + c_5) \log(V),$$

and

$$\begin{aligned}\Theta_1 = \log(R)\frac{U}{T} + \left(1 + \frac{c_5}{\max\{1, \log^+(T_0)\}}\right) \frac{T_0}{T} \\ \leq -2\varepsilon_0 \log(\varepsilon_0/2) + 6\varepsilon_0(1 + c_5) \leq 4e^{-1} + 6\varepsilon_0(1 + c_5)\end{aligned}$$

because $R \leq 4\varepsilon_0^{-2}$ and the function $-x\log(x)$ obtains e^{-1} as its maximum on $(0,1)$. Since $S + 1 \leq 3\varepsilon_0^{-1}$, $R \leq 4\varepsilon_0^{-2}$, $T \geq \varepsilon_0^{-1}U$, and $U = V$ we also have

$$\Theta_2 = 2\frac{R(S+1)V}{U+1} + \left(\frac{T_0}{T} + \frac{U}{T}\right)(1 + \log(2)) < 24\varepsilon_0^{-3} + 12\varepsilon_0$$

Similiarly, one deduces

$$\begin{aligned}\Theta_4 &< \frac{R(S+1)V}{U+1}\left(\log\left(\frac{3}{2}\right) + 2\right) + 3\left(\frac{T_0}{T} + \frac{U}{T}\right) + \frac{\log(2)}{T} + \log(2) \\ &< 29\varepsilon_0^{-3} + 22\varepsilon_0 + \log(2)\end{aligned}$$

so that in summary we achieve the rough estimate

$$\Theta_1 + \Theta_2 + \Theta_4 < 53\varepsilon_0^{-3} + 37 + 6(1 + c_5).$$

Since $V \geq h(\alpha)$ and $U = V$, we have

$$\frac{R(S+1)}{U+1} h(\alpha) < 12\varepsilon_0^{-3}$$

and

$$\frac{T_0}{TV} h(\alpha) \leq 6\varepsilon_0.$$

Note that by (R_3) we have

$$\log(V) - \log|\mathfrak{log}(\iota_0(\alpha))| - 1 > 0,$$

and thus $T_0/T < 6\varepsilon_0 < 1$ yields

$$\begin{aligned}\min\left\{0, -1 + \frac{T_0}{T}\right\}(\log(V) - \log|\mathfrak{log}(\iota_0(\alpha))| - 1) \\ < (-1 + 6\varepsilon_0)(\log(V) - \log|\mathfrak{log}(\iota_0(\alpha))| - 1).\end{aligned}$$

Inserting the above estimates in (3.11) we infer

$$\begin{aligned}\left(\frac{1 - 6\varepsilon_0}{[\mathbb{K} : \mathbb{Q}]} - (8 + 6c_5)\varepsilon_0\right) \log(V) \\ < \frac{1 - 6\varepsilon_0}{[\mathbb{K} : \mathbb{Q}]} \log|\mathfrak{log}(\iota_0(\alpha))| + \frac{1 - 6\varepsilon_0}{[\mathbb{K} : \mathbb{Q}]} + 71\varepsilon_0^{-3} + 51 + 6c_5 + c_7.\end{aligned}$$

By our assumption on ε_0, the coefficient of $\log(V)$ is positive and it follows that

$$V < c_8(\varepsilon_0)\,|\mathfrak{log}(\iota_0(\alpha))|^{c_9(\varepsilon_0)}$$

with

$$c_8(\varepsilon_0) = \exp\left(\frac{1-6\varepsilon_0+\left(71\varepsilon_0^{-3}+51+6c_5+c_7\right)[\mathbb{K}:\mathbb{Q}]}{1-6\varepsilon_0-(8+6c_5)\varepsilon_0[\mathbb{K}:\mathbb{Q}]}\right)$$

and

$$c_9(\varepsilon_0) = 1 + \frac{(8+6c_5)\varepsilon_0[\mathbb{K}:\mathbb{Q}]}{1-6\varepsilon_0-(8+6c_5)\varepsilon_0[\mathbb{K}:\mathbb{Q}]}.$$

This contradicts our assumption on the size of V. The assertion on the asymptotic behavior of $c_9(\varepsilon_0)$ as $\varepsilon_0 \to 0$ is obvious. □

To conclude the proof of Theorem 1.2, we use the product formula

$$0 = T^{-1}h_f(\beta_0) + T^{-1}h_{\mathcal{S}}(\beta_0) + T^{-1}h_{\overline{\mathcal{S}}}(\beta_0)$$

a second time. However, Claim 3.7 tells us that we may assume that Claim 3.6 yields the inequality (3.8) for every $\iota \in \mathcal{S}$. Hence, we obtain

$$\begin{aligned}\frac{1}{T[\mathbb{K}:\mathbb{Q}]}\log(|\Delta|) \geq &-\frac{U}{T}\log(V) - \left(1+\frac{c_5}{\max\{1,\log^+(T_0)\}}\right)\frac{T_0}{T}\log(U)\\ &-\frac{h(\underline{P})}{T} - \frac{R(S+1)}{U+1}h(\alpha) - \frac{T_0}{TV}h(\alpha) - \frac{U}{T}h(\beta)\\ &-\Theta_1-\Theta_2-\Theta_3.\end{aligned}$$

Furthermore, as in the proof of Claim 3.7 we estimate

$$\Theta_3 = \left(\frac{T_0}{T}+\frac{U}{T}+\frac{1}{T}\right)\log(2) + 2\frac{R(S+1)V}{U+1} < 24\varepsilon_0^{-3} + 6\varepsilon_0$$

and hence

$$\Theta_1+\Theta_2+\Theta_3 < 48\varepsilon_0^{-3} + 26 + 6c_5.$$

We infer easily that

$$\log(|\Delta|) > -T[\mathbb{K}:\mathbb{Q}]\left(\varepsilon_0\,(8+6c_5)\log(V) + 66\varepsilon_0^{-3} + 34 + 6c_5 + c_7\right),$$

proceeding again as in the proof of Claim 3.7. It remains to state precisely a set of auxiliary parameters.

Claim 3.8. *Inserting successively*

$$S = \left\lceil \varepsilon_0^{-1} \right\rceil,$$
$$R = \left\lceil \varepsilon_0^{-1} S \right\rceil,$$
$$V = \left\lceil \max\{2e, h(\alpha), c_8(\varepsilon_0)\, |\mathfrak{log}(\iota_0(\alpha))|^{c_9(\varepsilon_0)}\} \right\rceil,$$
$$U = V, \text{ and}$$
$$T = \left\lceil U \max\left\{\varepsilon_0^{-1}, h(\beta)\right\} \right\rceil,$$

gives an admissible set of auxiliary parameters, i.e. the restrictions R_i *(*$i \in \{1, \dots, 6\}$*) are fulfilled.*

Proof. At large, this is straightforward. For (R_2) and (R_3), one has to read off $c_8(\varepsilon_0) > e$ and $c_9(\varepsilon_0) > 1$ from their definitions. The only part that involves some computation is to make sure that

$$T \geq \max\{\log(R), \log(S+1)\}$$

as needed for restriction (R_1). Since $R \leq 4\varepsilon_0^{-2}$ and $S + 1 \leq 3\varepsilon_0^{-1}$ this boils down to

$$\varepsilon_0^{-1} \geq -2\log(\varepsilon_0) + \log(4)$$

for all $0 < \varepsilon_0 \leq 1/14$. The difference of both sides of this inequality has negative derivative for $\varepsilon_0 \in (0, 1)$ and the inequality is true for $\varepsilon_0 = 1/14$. $\square$

With our choice of auxiliary parameters we obtain that $\log(|\Delta|)$ is less than

$$-2V \max\left\{\varepsilon_0^{-1}, h(\beta)\right\} [\mathbb{K} : \mathbb{Q}] \left(\varepsilon_0 \,(8 + 6c_5) \log(V) + 66\varepsilon_0^{-3} + 34 + 6c_5 + c_7\right),$$

where $V = \left\lceil \max\{2e, h(\alpha), c_8(\varepsilon_0)\, |\mathfrak{log}(\iota_0(\alpha))|^{c_9(\varepsilon_0)}\} \right\rceil$. Our main theorem is an easy reformulation of this since $c_9(\varepsilon_0) \to 1$ as $\varepsilon_0 \to 0$.

4. Comments

We conclude with some comments concerning the most natural generalizations of the strategy exposed in this article.

4.1. Elliptic curves. One may extend our approach to elliptic curves, replacing $\mathbb{G}_m$ with an elliptic curve

$$E : y^2 = 4x^3 - g_2 x - g_3,\ g_2, g_3 \in \bar{\mathbb{Q}}.$$

In fact, the logarithm $\int_1^z \frac{dx}{x}$ on $\mathbb{G}_m$ has an elliptic analog $\int_O^P \frac{dx}{y}$, where $O = (0 : 1 : 0)$ is the identity of E. The counterpart of our Theorem 1.2 would state that $\int_O^P \frac{dx}{y}$ is transcendental if P is a non-torsion algebraic point of E. As for tori, one should consider the preimages of P under the isogeny $[N] : E \to E$ for some sufficiently large N. The N^2 preimages of P are 'converging' to the torsion points of order N with a distance decreasing as N^{-1}. In addition, their degrees increase as N^2 and their Néron-Tate heights decrease as N^{-2} so that these effects seem to balance each other. Furthermore, one can transfer our zero estimate from $\mathbb{P}^1$ to E. It is mandatory to work out details but this is left to a future publication.

4.2. Polylogarithms. It should be mentioned explicitly that our method does not extend to higher polylogarithms. In fact, we make essential use of the functional equation

$$\mathfrak{log}(z^n) = n\mathfrak{log}(z). \tag{4.1}$$

Higher polylogarithms satisfy many functional equations but none of them is as useful as the above one. In fact, if $\mathfrak{log}$ obtains an algebraic value at an algebraic point $\alpha \neq 1$, then using (4.1) we infer the existence of countably many such points. Doing anything similar with polylogarithms meets with immense difficulties despite their many functional equations. For illustration, let us consider the dilogarithm $\mathrm{Li}_2(z)$, whose precise definition and basic properties can be found in [51]. It satisfies a functional equation very close to (4.1), namely the 'distribution relation'

$$\mathrm{Li}_2(z) = n \sum_{w^n=z} \mathrm{Li}_2(w).$$

Actually, this is just an instance out of a whole cornucopia of relations, see p. 9 *loc. cit.* Like the distribution relation, they all contain strictly more than two evaluations of Li_2 and hence provide not much additional knowledge in case Li_2 would obtain an algebraic value at an algebraic point.

4.3. Multivariate generalizations. In another direction, one may try to replace $\mathbb{G}_m$ with $\mathbb{G}_m^n$. The straightforward generalization of the functions (cf. Section 3.1)

$$\frac{s}{N}\beta + \mathrm{Log}_\gamma(X),\ \gamma \in \alpha^{s/N},$$

on $\mathbb{G}_m$ are the functions

$$\frac{s}{N}\beta_0 + \beta_1 \mathrm{Log}_{\gamma_1}(X_1) + \beta_2 \mathrm{Log}_{\gamma_2}(X_2) + \ldots + \beta_n \mathrm{Log}_{\gamma_n}(X_n),\ \gamma_i \in \alpha_i^{s/N},$$

on $\mathbb{G}_m^n$. These are indeed multivariate G-functions in the sense of [8] and the reader may note that most of our proof can be adapted neatly to this situation.

However, a serious problem remains unsettled: For illustration, let

$$Y(X_1, X_2) \in \bar{\mathbb{Q}}[[X_1, X_2]]$$

be a general (non-algebraic) multivariate G-function. We want to construct – similar to Section 3.2 – a tuple of polynomials $\underline{P} \in \bar{\mathbb{Q}}[X_1, X_2]^{U+1}$ such that

$$P_0(X_1,X_2)+P_1(X_1,X_2)Y(X_1,X_2)+\ldots+P_U(X_1,X_2)Y(X_1,X_2)^U$$

vanishes at $(0,0)$ with order T. Ignoring all but the dependences on T and U, it is easy to see that we need $\deg(\underline{P}) \approx T/U^{1/2}$ for this construction. However, in case there is a univariate G-function Y_0 such that $Y(X_1,X_2)=Y_0(X_1)$ this can be improved to $\deg(\underline{P}) \approx T/U$. This 'non-generic' case causes problems in finding an appropriate generalization of the zero estimates in Section 2.4. In fact, Bolibrukh [15] provided a multivariate generalization but his result incorporates also the above non-generic case. This is not sufficient for our application. It seems intriguing to reach a better understanding of multivariate zero estimates – even at a single point – and 'non-generic' cases. Concerning the algebraic independence of logarithms in algebraic numbers, there is another impediment to our method. To prove, for example, the transcendence of $\log(2)\log(3)$ we have to work with G-functions of the form

$$Y(X_1,X_2)=\mathrm{Log}_1(X_1)\mathrm{Log}_1(X_2).$$

Unfortunately, the higher derivatives of Y at the N-th roots of $(2,3)\in\mathbb{G}_m^2$ are no longer in $\bar{\mathbb{Q}}$ but in $\bar{\mathbb{Q}}\log(2)+\bar{\mathbb{Q}}\log(3)$. For now, this presents an unsurmountable obstruction.

4.4. Non-archidemean estimates. With a slight modification, we can also obtain a result similar to Theorem 1.2 for p-adic logarithms. We want to sketch this briefly. For simplicity, we restrict to the p-adic completion $\mathbb{Q}_p$ of the rationals. Recall that the logarithm $\mathfrak{log}_p$ on the p-adic numbers $\mathbb{Q}_p$ is defined by its Taylor series at $1\in\mathbb{Q}_p$, which converges on the principal units U_p^1. Imposing multiplicativity, there is a canonical extension of $\mathfrak{log}_p$ to all units U_p in $\mathbb{Q}_p$ because of the decomposition $U_p=U_p^1\times\mu_{p-1}$. In addition, each choice of $\mathfrak{log}_p(p)$ gives an extension of the logarithm to all of $\mathbb{Q}_p$, e.g. demanding $\mathfrak{log}_p(p)=0$ gives Iwasawa's logarithm. Evidently, such extensions to all of $\mathbb{Q}_p$ are irrelevant in transcendence theory and we restrict to U_p here.

At archimedean places, the N-th roots of an algebraic number $\alpha\in U_p$ 'converge' to the N-th roots of unity as N goes to infinity. This is not the case at non-archimedean places. For them, taking N-th roots does not decrease the distance to the roots of unity but even increases the distance whenever $p|N$. Nevertheless, α^{p^M} converges to a $(p-1)$-th root of unity as $M\to\infty$. This provides a way to adapt our procedure to the non-archimedean case. For given algebraic $\alpha\in U_p$ we consider the N-th roots of α^{p^M}, where both M and N are sufficiently large integers such that $(N,p)=1$. The proof uses a Padé approximation to the logarithm at the $(p-1)$-th roots of unity. We skip the details in favor of another comment.

It is well-known (cf. [11, 41, 42]) that for the abc-conjecture the dependence of the transcendence measure for p-adic logarithms on p is most important. In fact, the simple case $2^n + 1 = c$ of the abc-conjecture boils down to lower bounds on $\mathfrak{log}_p(2)$ that depend on p very weakly. Regrettably, this dependence is a major weakness of all currently available results on linear forms in non-archimedean logarithms (cf. [48, 49, 50]). Our method is no exception; the reason why p must intervene crucially in our bounds is the fact that we have to interpolate simultaneously at all roots in μ_{p-1} and the cardinality of this set increases with p. In a way, this is dual to the reason why p appears in other approaches, where one replaces 2 with 2^{p-1} to make sure that it is a principal unit.[11]

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Lars Kühne
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa
Italy
E-mail: `lars.kuhne@sns.it`