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Simultaneous Padé approximants to the Euler, exponential and logarithmic functions

par Tanguy RIVOAL

On the occasion of Axel Thue's 150th birthday

Résumé. Nous présentons une méthode générale qui permet d'obtenir des approximations simultanées de type Padé pour les fonctions exponentielles et logarithmes.

Abstract. We present a general method to obtain simultaneous explicit Padé type approximations to the exponential and logarithmic functions.

1. Introduction

Thue proved his famous theorem on rational approximations of algebraic numbers by rational numbers [24], by a method which in some sense amounts to the computation of certain inexplicit Padé approximants. This method is ineffective and Thue tried to find effective irrationality measures for large classes of algebraic numbers. In [25] he was in particular able to do this for certain numbers of the form $\sqrt[r]{a/b}$ by means of the diagonal Padé approximants for the binomial functions $\sqrt[r]{1-x}$. See [6] for some historical comments.

Since Hermite's fundamental work on the values of the exponential function, the importance of Padé approximation (in a broad sense) in Diophantine approximation cannot be exaggerated, and we will present some examples below. Our aim is to pursue further in this direction. We present here explicit simultaneous Padé (type) approximants for the three series $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ and $\mathscr{E}(z) = z\int_0^{\infty} \frac{e^{-t}}{1-zt}\mathrm{d}t \sim$

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$\sum_{n=0}^{\infty} n! z^{n+1}$. (The asymptotic expansion holds in a suitable angular sector.) This problem seems to have never been addressed before and its difficulty is due to the fact that these series belong to three different classes of the hypergeometric hierarchy:

$$\begin{aligned}\exp(z) &= {}_1F_1\left[\begin{matrix}1\\1\end{matrix};z\right]\\ -\frac{1}{z}\log(1-z) &= {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};z\right]\\ \frac{1}{z}\mathscr{E}(z) &= {}_2F_0\left[\begin{matrix}1,1\\ \end{matrix};z\right]\end{aligned}$$

where hypergeometric series are defined by

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix};z\right] := \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(1)_k(b_1)_k\cdots(b_q)_k}z^k$$

with $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

For any integer $n\geq 0$, it is known that Legendre polynomial $\mathbf{P}_n^{\boldsymbol{\ell}}(z) := \frac{1}{n!}(z^n(1-z)^n)^{(n)}$ of degree n is simply related to the denominators of the Padé approximants $[n-1/n]$ of $\frac{1}{z}\log(1-z)$ at $z=0$, and that Laguerre polynomial $\mathbf{P}_n^{\mathbf{E}}(z) := \frac{1}{n!}e^z(z^ne^{-z})^{(n)}$ of degree n is also related to the denominators of the Padé approximants $[n-1/n]$ of $\frac{1}{z}\mathscr{E}(z)$ at $z=0$. The "exponential" polynomial $\mathbf{P}_n^{\mathbf{e}}(z) := \frac{1}{n!}e^{-z}z^{2n}(z^{-n}e^z)^{(n)}$ of degree n is the denominator of the Padé approximants $[n-1/n]$ of $\exp(z)$ at $z=0$. (In fact $\mathbf{P}_n^{\mathbf{e}}(z)$ is the generalised Laguerre polynomial $L_n^{(-2n)}(-z)$.) Such expressions built on repeated differentiations are known as Rodrigues formulas and we recall the connection with Padé approximants below.

The sequences $(\mathbf{P}_n^{\boldsymbol{\ell}}(z))_{n\geq 0}$ and $(\mathbf{P}_n^{\mathbf{E}}(z))_{n\geq 0}$ are sequences of orthogonal polynomials for the positive weights $\mathbf{1}_{[0,1]}$ and $e^{-z}\mathbf{1}_{[0,\infty)}$ respectively. This is not the case for the sequence $(\mathbf{P}_n^{\mathbf{e}}(z))_{n\geq 0}$, because $\mathbf{P}_n^{\mathbf{e}}(z)$ does not always have only real roots (for instance for $n=2$), which is a necessary condition for polynomial orthogonality on the real line with respect to integration against a positive measure (see [7]); it is however an orthogonal sequence in a more general sense [14, 21]. For a recent survey on Hermite-Padé approximants and orthogonal polynomials, see [26].

The remainder functions of the above mentioned Padé constructions also have simple expressions in terms of hypergeometric series at the same level of the hierarchy: ([1])

$$\mathbf{P}^{\boldsymbol{\ell}}_n(z)\log\big(1-\frac{1}{z}\big)-\mathbf{Q}^{\boldsymbol{\ell}}_n(z)$$

$$(1.1)\qquad =(-1)^{n-1}\int_0^1\frac{t^n(1-t)^n}{(z-t)^{n+1}}\mathrm{d}t=\frac{n!^2(-z)^{-n-1}}{(2n+1)!}\cdot{}_2F_1\left[\begin{matrix}n+1,n+1\\2n+2\end{matrix};\frac{1}{z}\right],$$

$$\mathbf{P}^{\mathbf{E}}_n(z)\mathscr{E}\big(\frac{1}{z}\big)-\mathbf{Q}^{\mathbf{E}}_n(z)$$

$$(1.2)\qquad =\int_0^\infty\frac{t^n}{(z-t)^{n+1}}e^{-t}\mathrm{d}t=(-1)^n\frac{n!}{z^{n+1}}\cdot{}_2F_0\left[\begin{matrix}n+1,n+1\\ \end{matrix};\frac{1}{z}\right],$$

$$\mathbf{P}^{\mathbf{e}}_n(z)\exp(z)-\mathbf{Q}^{\mathbf{e}}_n(z)$$

$$(1.3)\qquad =\frac{nz^{2n}}{n!^2}\int_0^1 e^{zt}t^n(1-t)^{n-1}\mathrm{d}t=\frac{z^{2n}}{(2n)!}\cdot{}_1F_1\left[\begin{matrix}n+1\\2n+1\end{matrix};z\right],$$

for some (explicitable) polynomials $\mathbf{Q}^{\boldsymbol{\ell}}_n(z)$, $\mathbf{Q}^{\mathbf{E}}_n(z)$ and $\mathbf{Q}^{\mathbf{e}}_n(z)$ of degree $n-1$. We give a proof of these facts in Section 6 for the reader's convenience. All the approximations and formulas in the paper have an analytic meaning (around $z=0$ or $z=\infty$ depending on the case), not a mere formal one. ([2])

There exist many papers devoted to the explicit computations of simultaneous Padé approximants (at various points) of hypergeometric functions in the classes ${}_{p+r}F_{q+r}$ where p,q are fixed for the problem considered and r is an integer ranging in a finite set. See for instance [8, 9, 11, 13, 15, 16, 17, 20]. However, there does not exist so far in the literature any explicit formulas for the simultaneous Padé (type) approximants of type I or II for functions at different levels of the hypergeometric hierarchy.

To do this, we leave the world of hypergeometric polynomials/series in one variable to the more obscure world of multivariate hypergeometric polynomials/series (specialised in one variable): except for one of our theorems, none of the formulas given for the polynomials and remainders are hypergeometric series in one variable. The main idea is the composition, in a suitable sense, of Rodrigues formulas that define Legendre, Laguerre and exponential polynomials; we make this more precise in Section 1.1. The composition of differential operators to define new sequences of polynomials is not a new idea, see [2] for an extensive study of such compositions and [3, 19] for a new number theoretical application, different from the classical ones such as those described in [1, 4, 5]. But it is apparently the first time it is used to construct simultaneous approximations to functions at different levels in the hypergeometric hierarchy.

[1] It is easier to write the formulas at $z=\infty$ for log and $\mathscr{E}$.

[2] In the three cases, when $n=0$, the Padé approximants $[-1/0]$ reduces to $\mathbf{P}_0=1$ and $\mathbf{Q}_0=0$; for consistency, the integral expression in (1.3) must be understood as the integral of $\exp(zt)$ against the Dirac measure at $t=1$, hence equal to e^z. The same remark applies to the integrals in (4.2) and (6.7).

1.1. Principle of the constructions. We explain here the idea behind all the results presented in the paper. For simplicity, we consider only the approximation of two functions but the principle can easily be extended. We consider two series $f(z), g(z) \in \mathbb{C}[[z]]$, whose respective $[k/n]$ and $[\ell/n]$ Padé approximants at some points have their denominators of degree n of the form

$$\mathbf{P}^{\mathbf{f}}_n(z) := \Psi(z)A_n(z)\big(\Psi(z)^{-1}B_n(z)\big)^{(n)}$$
$$\mathbf{P}^{\mathbf{g}}_n(z) := \Phi(z)C_n(z)\big(\Phi(z)^{-1}D_n(z)\big)^{(n)}$$

where $\Psi(z), \Phi(z)$ are suitable functions, $A_n(z), C_n(z) \in \mathbb{C}[z]$, $B_n(z), D_n(z) \in \mathbb{C}(z)$. All these polynomials depend on n and k, or ℓ, but we emphasize only the dependence on n. We require at least that $\mathbf{P}^{\mathbf{f}}_0(z) \equiv 1$, $\mathbf{P}^{\mathbf{g}}_0(z) \equiv 1$, and $\Psi(z)A_n(z)\big(\Psi(z)^{-1}B_n(z)z^k\big)^{(n)}$ and $\Phi(z)C_n(z)\big(\Phi(z)^{-1}D_n(z)z^k\big)^{(n)}$ are polynomials for any integers $k, n \geq 0$, which is the case of the examples studied in this paper. We then observe that the polynomial

$$\mathbf{P}^{\mathbf{f},\mathbf{g}}_{n,m}(z) := \Psi(z)A_n(z)\big(\Psi(z)^{-1}B_n(z)\mathbf{P}^{\mathbf{g}}_m(z)\big)^{(n)} \tag{1.4}$$

coincides with $\mathbf{P}^{\mathbf{f}}_n(z)$ if $m = 0$ and with $\mathbf{P}^{\mathbf{f}}_m(z)$ if $n = 0$. Hence, one expects that (1.4) is the denominator of a simultaneous Padé *type* problem for $f(z)$ and $g(z)$ at $z = 0$. By Padé type, we mean that the order of the approximation is smaller than in a Padé problem, but not trivial neither.

Similarly, we can consider

$$\mathbf{P}^{\mathbf{g},\mathbf{f}}_{m,n}(z) := \Phi(z)C_m(z)\big(\Phi(z)^{-1}D_m(z)\mathbf{P}^{\mathbf{g}}_n(z)\big)^{(m)}$$

for the same purpose. Usually, $\mathbf{P}^{\mathbf{f},\mathbf{g}}_{n,m}(z)$ and $\mathbf{P}^{\mathbf{g},\mathbf{f}}_{m,n}(z)$ are distinct, leading to another simultaneous approximation, but this is not always the case. Moreover, the polynomials

$$\begin{aligned} &\Psi(z)A_n(z)\big(\Psi(z)^{-1}B_n(z)z^m\mathbf{P}^{\mathbf{g}}_m(\pm 1/z)\big)^{(n)} \\ &\Psi(z)A_n(z)\big(\Psi(z)^{-1}B_n(z)\mathbf{P}^{\mathbf{g}}_m(\pm z - a)\big)^{(n)} \end{aligned} \tag{1.5}$$

(and the two similar polynomials involving $\mathbf{P}^{\mathbf{f}}_n$ instead of $\mathbf{P}^{\mathbf{g}}_m$) can be used to obtain solutions of simultaneous Padé type problem for $f(z)$ and $g(\pm 1/z)$ at $z = 0$ and $z = \infty$, respectively $f(z)$ and $g(\pm z - a)$ at $z = 0$ and $z = a$. Depending on the structure of the polynomials, other "composite" polynomials are possible: see Theorem 5 (which even yields Padé approximants) and Theorem 7.

Here, we use this procedure with (generalisations of) $\mathbf{P}^{\boldsymbol{\ell}}_n(z)$, $\mathbf{P}^{\boldsymbol{E}}_n(z)$ and $\mathbf{P}^{\boldsymbol{e}}_n(z)$, which correspond to approximations at $z = 0$ for $\exp(z)$ and $z = \infty$ for $\log(1 - 1/z)$ and $\mathscr{E}(1/z)$. For instance, when we set the parameters in Theorem 1 to $a = b = c = n$, $d = 3n$, $f = 2n$ for any integer $n \geq 0$, we obtain three explicit non trivial polynomials $\mathbf{P}_{4n}(z), \mathbf{Q}_{4n}(z), \widetilde{\mathbf{Q}}_{4n}(z)$, of degree at most $4n$, such that

$$\mathbf{P}_{4n}(z)\log\big(1-\frac{1}{z}\big)-\mathbf{Q}_{4n}(z)=\mathcal{O}\big(\frac{1}{z^{n+1}}\big)$$
$$\mathbf{P}_{4n}(z)\exp(z)-\widetilde{\mathbf{Q}}_{4n}(z)=\mathcal{O}(z^{5n+1}).$$

We don't exhaust all the possibilities suggested above, mainly to avoid notational complications and repetitions of arguments. We focus mainly on approximations generated by polynomials of type (1.4) and we give only two examples of approximations generated by (1.5), for $\exp(z)$ and $\log(1-z)$ at $z=0$.

1.2. Organisation of the paper. The main theorems (Theorems 1 to 7) are given in Sections 2 to 5, which correspond to various couplings of the functions $\exp(z)$, $\log(1-z)$ and $\mathscr{E}(z)$. In Section 6, as a warm up, we remind the reader of the constructions of the Padé approximants $[n-1/n]$ of these three functions. The proofs of Theorems 1, 2, 5 and 6 are given in Sections 7, 8 and 9 respectively. In the final section, we use the same principle of composition of Rodrigues type differential operators to present explicit simultaneous Padé approximants to $\log(1-z)$ and $(1-z)^\alpha$ of type II at $z=0$ (and also polylogarithms). It is likely that simultaneous Padé (type) approximants to $\exp(z)$, $\log(1-1/z)$, $\mathscr{E}(1/z)$ and $(1-1/z)^\alpha$ could be obtained by the methods of this paper. When our approximations are only Padé type approximants, it would be interesting to find other Padé conditions (on some functions) to embed the problem into a Padé problem of type II, or even a mixed one with type I, similar to the problems considered in [11] for instance.

To avoid complicated notations, almost all the polynomials and remainders of the approximations will be denoted by the same bold letters (possibly with a hat or a tilde) without mention of the obvious parameters.

2. Simultaneous Padé type approximants for exp and log

Motivations from diophantine approximations are at the origin of this paper. Indeed, it is a classical fact that diagonal Padé approximants of $\exp(z)$ and $\log(1-z)$ yield the irrationality of $\exp(p/q)$ and $\log(1+1/q)$ for any integers $p, q \geq 1$, and even good irrationality measures for these numbers (see [1, 5]). This is an indication that explicit simultaneous Padé type approximants to $\exp(z)$ and $\log(1-z)$ might lead for instance to the linear independence of 1, e and $\log(2)$ over $\mathbb{Q}$.

We present here four quite general results concerning simultaneous approximations of $\exp(z)$ and $\log(1-1/z)$ at $z=0$ and $z=\infty$ for the first two, and of $\exp(z)$ and $\log(1-z)$ at $z=0$ for the last two. In particular, the first and third approximations involve multi-parameters integrals in the spirit of [18, 27]. Unfortunately, none of these constructions seems to be strong enough to obtain the desired diophantine results but they might be a step in this direction.

We set $\alpha^+ = \max(\alpha, 0)$ and, for any integers $a, b, c, d, f \geq 0$, we define $\mathbf{P}^{\boldsymbol{\ell}}_{a,b,c}(z) = \frac{1}{c!}(z^a(1-z)^b)^{(c)}$ and $\mathbf{P}^{\mathbf{e}}_{d,f}(z) = \frac{1}{d!}z^{d+f+1}e^{-z}(z^{-f-1}e^z)^{(d)}$, polynomials of respective degree at most $(a+b-c)^+$ and d, which generalise the Legendre and exponential polynomials.

2.1. First approximations. For any integers $a, b, c, d, f \geq 0$, we define

$$\mathbf{L}(z) := \frac{(-1)^{c-1}}{d!f!} \int_0^1 \int_0^\infty \frac{x^a(1-x)^b y^f (x-y)^d}{(z-x)^{c+1}} e^{-y} \mathrm{d}x\mathrm{d}y, \tag{2.1}$$

$$\mathbf{E}(z) := \frac{1}{c!d!} \sum_{k=f+d+1}^{\infty} \frac{(k-f-d)_d}{k!} e^z (z^{a+k}(1-z)^b e^{-z})^{(c)} \tag{2.2}$$

and the polynomial of type (1.4)

$$\begin{aligned} \mathbf{P}(z) :&= \frac{1}{c!}(z^a(1-z)^b \mathbf{P}^{\mathbf{e}}_{d,f}(z))^{(c)} \\ &= \sum_{j=0}^{d} \sum_{k=0}^{b} (-1)^{d-j+k} \binom{b}{k} \binom{a+j+k}{c} \binom{f+d-j}{f} \frac{z^{j+k+a-c}}{j!}. \end{aligned} \tag{2.3}$$

It is clear that

$$\deg(\mathbf{P}) \leq a+b+d-c, \ \mathrm{ord}_{z=0}(\mathbf{P}) \geq (a-c)^+, \ \mathrm{ord}_{z=1}(\mathbf{P}) \geq (b-c)^+.$$

In particular, $\mathbf{P}(z) \equiv 0$ if $a+b+d < c$.

Theorem 1. *For any integers $a, b, c, d, f \geq 0$ such that $a+b+d \geq c$, there exist three polynomials $\mathbf{Q}_0(z)$, $\mathbf{Q}_1(z)$ and $\mathbf{Q}_2(z)$ such that*

$$\deg(\mathbf{Q}_0) \leq b+d, \quad \deg(\mathbf{Q}_1) \leq a+d,$$

$$\deg(\mathbf{Q}_2) \leq a+b+f, \quad \mathrm{ord}_{z=0}(\mathbf{Q}_2) \geq (a-c)^+, \quad \mathrm{ord}_{z=1}(\mathbf{Q}_2) \geq (b-c)^+$$

and

$$\mathbf{L}(z) = \mathbf{P}(z)\log\big(1-\frac{1}{z}\big) - z^{a-c}\mathbf{Q}_0(z) - (1-z)^{b-c}\mathbf{Q}_1(z) = \mathcal{O}\big(\frac{1}{z^{c+1}}\big), \tag{2.4}$$

$$\mathbf{E}(z) = \mathbf{P}(z)\exp(z) - \mathbf{Q}_2(z) = \mathcal{O}(z^{(f+d+a-c+1)^+}). \tag{2.5}$$

Explicit expressions for the $\mathbf{Q}_j$'s can be obtained from the proof. In general, it is not true that

$$\mathbf{L}(z) = -\int_0^1 \frac{\mathbf{P}(t)}{z-t}\mathrm{d}t$$

but this is true at least when $a \geq c$ and $b \geq c$.

2.2. Second approximations. Here, we consider the opposite composition of polynomials. For any integers $a, b, c, d, f \geq 0$, we define

$$\widetilde{\mathbf{L}}(z) := \frac{(-1)^{c-1}}{d!} z^{d+f+1} e^{-z} \int_0^1 t^a (1-t)^b \frac{\partial^d}{\partial z^d}\Big(\frac{z^{-f-1}e^z}{(z-t)^{c+1}}\Big)\mathrm{d}t \tag{2.6}$$

$$\widetilde{\mathbf{E}}(z) := \sum_{\substack{j,k\geq 0 \\ j+k \geq f+d+c-a+1}} (-1)^j \binom{b}{j}\binom{a+j}{c} \frac{(j+k+a-c-f-d)_d}{k!d!} z^{j+k+a-c} \tag{2.7}$$

and the polynomial of type (1.4)

$$\begin{aligned}\widetilde{\mathbf{P}}(z) &:= \frac{1}{d!} z^{d+f+1} e^{-z} (z^{-f-1} e^z \mathbf{P}^{\boldsymbol{\ell}}_{a,b,c}(z))^{(d)} \\ &= \sum_{j=0}^{d}\sum_{k=0}^{b} (-1)^{d-j+k} \binom{d}{j}\binom{b}{k}\binom{a+k}{c} \frac{(f+c-a-k+1)_{d-j}}{d!} z^{a+k-c+j}.\end{aligned} \tag{2.8}$$

We have $\deg(\widetilde{\mathbf{P}}) \leq a+b+d-c$ and, moreover, if $a+b<c$, then $\widetilde{\mathbf{P}}(z) \equiv 0$.

Theorem 2. *For any integers $a, b, c, d, f \geq 0$, there exist three polynomials $\widetilde{\mathbf{Q}}_0(z)$, $\widetilde{\mathbf{Q}}_1(z)$, and $\widetilde{\mathbf{Q}}_2(z)$ such that*

$$\deg(\widetilde{\mathbf{Q}}_0) \leq b+d, \quad \deg(\widetilde{\mathbf{Q}}_1) \leq a+2d, \quad \deg(\widetilde{\mathbf{Q}}_2) \leq f+c-a$$

and

$$\widetilde{\mathbf{L}}(z) = \widetilde{\mathbf{P}}(z)\log\big(1-\frac{1}{z}\big) - z^{a-c}\widetilde{\mathbf{Q}}_0(z) - (1-z)^{b-c-d}\widetilde{\mathbf{Q}}_1(z) = \mathcal{O}\big(\frac{1}{z^{c-d+1}}\big), \tag{2.9}$$

$$\widetilde{\mathbf{E}}(z) = \widetilde{\mathbf{P}}(z)\exp(z) - z^{a-c}\widetilde{\mathbf{Q}}_2(z) = \mathcal{O}(z^{(f+d+c-a+1)^+ + a-c}). \tag{2.10}$$

Explicit expressions for the $\widetilde{\mathbf{Q}}_j$'s can be obtained from the proof. In general, it is not true that

$$\widetilde{\mathbf{L}}(z) = -\int_0^1 \frac{\widetilde{\mathbf{P}}(t)}{z-t}\mathrm{d}t.$$

The choice $a=c$, $b=c+d \leq f$ with $d \leq c$ is such that all the degrees of the involved polynomials are less than the two orders of approximations (viewed at $z=0$ say), which makes this example close to Padé approximants of type II.

We now present two simultaneous approximations results for $\exp(z)$ and $\log(1-z)$ at $z=0$ (or $z=\infty$, which is the same thing up to a change of variable). We state them in less generality than the two previous theorems, because the general approximations (with c replaced by a or b at certain obvious places) are quite complicated to write down, but this could be done in principle.

2.3. Third approximations. For any integers $c, d, f \geq 0$, we define

$$\mathbf{L}(z) := \frac{(-1)^{c-1}}{d!f!}\int_0^1\int_0^\infty \frac{x^c(1-x)^c y^f(1-xy)^d}{(z-x)^{c+1}}e^{-y}\mathrm{d}x\mathrm{d}y, \tag{2.11}$$

$$\mathbf{E}(z) := \frac{1}{c!d!}\sum_{k=d+f+1}^{\infty}\frac{(k-d-f)_d}{k!}e^{1/z}(z^{c+d-k}(1-z)^c e^{-1/z})^{(c)} \tag{2.12}$$

and the polynomial (of type (1.5)) of degree $c+d$

(2.13)

$$\begin{aligned}\mathbf{P}(z) :&= \frac{1}{c!}(z^{c+d}(1-z)^c\mathbf{P}^{\mathbf{e}}_{d,f}(1/z))^{(c)},\\ &= \sum_{j=0}^{d}\sum_{k=0}^{c}(-1)^{d-j+k}\binom{c}{k}\binom{c+d-j+k}{c}\binom{d+f-j}{f}\frac{z^{d-j+k}}{j!}.\end{aligned}$$

Theorem 3. *For any integers $c, d, f \geq 0$ such that $c \geq d$, there exist two polynomials $\mathbf{Q}_1(z)$ and $\mathbf{Q}_2(z)$ of respective degree $c+d$ and $2c+f$ such that*

$$\mathbf{L}(z) = \mathbf{P}(z)\log\big(1-\frac{1}{z}\big) - \mathbf{Q}_1(z) = \mathcal{O}\big(\frac{1}{z^{c+1}}\big), \tag{2.14}$$

$$\mathbf{E}(z) = \mathbf{P}(z)\exp\big(\frac{1}{z}\big) - z^{-c}\mathbf{Q}_2(z) = \mathcal{O}\big(\frac{1}{z^{f+1}}\big). \tag{2.15}$$

Explicit expressions for $\mathbf{Q}_1(z)$ and $\mathbf{Q}_2(z)$ can be obtained from the proof.

2.4. Fourth approximations. For any integers $c, d, f \geq 0$, we define

$$\widetilde{\mathbf{L}}(z) := \frac{(-1)^{c-1}}{d!}z^{d+f+1}e^{-z}\int_0^1 t^c(1-t)^c\frac{\partial^d}{\partial z^d}\Big(\frac{z^{2c-f}e^z}{(1-zt)^{c+1}}\Big)\mathrm{d}t \tag{2.16}$$

$$\widetilde{\mathbf{E}}(z) := \sum_{j=0}^{d}\binom{d}{j}\sum_{k=f+j+1}^{\infty}\frac{(k-f-j)_j}{k!d!}z^{k+f-j}(z^n\mathbf{P}^{\boldsymbol{\ell}}_c(1/z))^{(j)} \tag{2.17}$$

and the polynomial (of type (1.5)) of degree $c+d$

(2.18)

$$\begin{aligned}\widetilde{\mathbf{P}}(z) :&= \frac{1}{d!}e^{-z}z^{d+f+1}(z^{c-f-1}e^z\mathbf{P}^{\boldsymbol{\ell}}_c(1/z))^{(c)}\\ &= \sum_{j=0}^{d}\sum_{k=0}^{c}(-1)^{d-j+k}\binom{c}{k}\binom{c+k}{c}\binom{c+d+f-k-j}{d-j}\frac{z^{n-k+j}}{j!}.\end{aligned}$$

Theorem 4. *For any integers $c, d, f \geq 0$ such that $c \geq 2d$, there exist two polynomials $\widetilde{\mathbf{Q}}_1(z)$ and $\widetilde{\mathbf{Q}}_2(z)$ of respective degree $c + 2d$ and f such that*

$$\widetilde{\mathbf{L}}(z) = \widetilde{\mathbf{P}}(z)\log(1-z) - (1-z)^{-d}\widetilde{\mathbf{Q}}_1(z) = \mathcal{O}(z^{2c+1}), \tag{2.19}$$

$$\widetilde{\mathbf{E}}(z) = \widetilde{\mathbf{P}}(z)\exp(z) - \widetilde{\mathbf{Q}}_2(z) = \mathcal{O}(z^{d+f+1}). \tag{2.20}$$

Explicit expressions for $\widetilde{\mathbf{Q}}_1(z)$ and $\widetilde{\mathbf{Q}}_2(z)$ can be obtained from the proof.

The proofs of Theorems 1 and 2 are given in Section 7. The proofs of Theorems 3 and 4 are similar and are omitted.

3. Simultaneous Padé approximants for log and $\mathscr{E}$

The results presented in this section concern the functions $\log(1 - 1/z)$ and $\mathscr{E}(1/z)$ at $z = \infty$. Since we don't expect that any new diophantine result can be deduced from them, they are not given at the same level of generality as in Section 2. However, this would be possible.

Our first example is not a composition of $P_n^{\boldsymbol{\ell}}(z)$ and $P_n^{\boldsymbol{E}}(z)$ of type (1.4) or (1.5), but an alternative one alluded to at the end of Section 1.1. For any integers $m, n \geq 0$, let us define

$$\begin{aligned}\mathbf{P}(z) &:= \frac{1}{m!n!}(e^z(z^{n+m}(1-z)^{n+m}e^{-z})^{(n)})^{(m)}\\ &= \sum_{j=0}^{n}\sum_{k=0}^{m+n}\binom{m+n}{k}\binom{m+j+k}{m}\binom{m+n+k}{m+j+k}\frac{(-z)^{j+k}}{j!},\end{aligned} \tag{3.1}$$

$$\mathbf{Q}_1(z) := -\int_0^1 \frac{\mathbf{P}(z)-\mathbf{P}(t)}{z-t}\mathrm{d}t, \quad \mathbf{Q}_2(z) := \int_0^\infty \frac{\mathbf{P}(z)-\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t,$$

and

$$\mathbf{Q}_3(z) := \int_0^1 \frac{\mathbf{P}(z)-\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t.$$

The polynomial $\mathbf{P}(z)$ is of degree $m + 2n$ and the polynomials $\mathbf{Q}_j(z)$ are of degree $m + 2n - 1$ for $j = 1, 2, 3$.

Theorem 5. *For any $m, n \geq 0$, we have*

$$\begin{aligned}\mathbf{P}(z)\log\left(1-\frac{1}{z}\right) - \mathbf{Q}_1(z) &= \mathcal{O}\left(\frac{1}{z^{m+1}}\right)\\ &= \frac{(-1)^{m-1}}{n!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{m+1}}\cdot {}_2F_0\left[\begin{matrix}-n, m+1\\ -\end{matrix}; -\frac{1}{z-t}\right]\mathrm{d}t.\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathbf{P}(z)\mathscr{E}\left(\frac{1}{z}\right) - \mathbf{Q}_2(z) &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right)\\ &= \frac{(-1)^{n}}{m!}\int_0^\infty \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+1}}\cdot {}_2F_0\left[\begin{matrix}-m, n+1\\ -\end{matrix}; \frac{1}{z-t}\right]e^{-t}\mathrm{d}t.\end{aligned} \tag{3.3}$$

$$\mathbf{P}(z)\int_0^1 \frac{e^{-t}}{z-t}\mathrm{d}t - \mathbf{Q}_3(z) = \mathcal{O}(\frac{1}{z^{n+1}}) \tag{3.4}$$
$$= \frac{(-1)^n}{m!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+1}} \cdot {}_2F_0\left[\begin{matrix}-m, n+1\\ -\end{matrix}; \frac{1}{z-t}\right] e^{-t}\mathrm{d}t.$$

The theorem shows that $\mathbf{P}(z)$ is a denominator of the Padé problem of type II $[m+2n-1, m+2n-1, m+2n-1/m+2n]$ at $z=\infty$ for the three functions $\log(1-1/z)$, $\mathscr{E}(1/z)$ and $\int_0^1 \frac{e^{-t}\mathrm{d}t}{z-t}$. It is a translation of the fact that $(\mathbf{P}(z))_{m,n\geq 0}$ is a sequence of multiple orthogonal polynomials with respect to the weights $\mathbf{1}_{[0,1]}$, $e^{-z}\mathbf{1}_{[0,1]}$, $e^{-z}\mathbf{1}_{[0,\infty)}$. If $n=0$, we get the Padé approximants $[m-1/m]$ for $\log(1-1/z)$ at $z=\infty$, while if $m=0$, we get the Padé approximants of type II $[2n-1, 2n-1/2n]$ for $\mathscr{E}(1/z)$ and $\int_0^1 \frac{e^{-t}\mathrm{d}t}{z-t}$ at $z=\infty$.

Many similar results can be obtained along the same lines. We sketch below two of them.

We first consider a "composition" of $\mathbf{P}_n^{\boldsymbol{\ell}}(z)$ and $\mathbf{P}_n^{\mathbf{E}}(z)$ of type (1.4). For any $m, n \geq 0$, let us define the polynomial, of degree $m+n$,

$$\widehat{\mathbf{P}}(z) := \frac{1}{m!}(z^m(1-z)^m\mathbf{P}_n^{\mathbf{E}}(z))^{(m)}. \tag{3.5}$$

Then there exist two polynomials $\widehat{\mathbf{Q}}_1(z)$ and $\widehat{\mathbf{Q}}_2(z)$, each of degree $m+n-1$, such that for any $m, n \geq 0$, we have

$$\widehat{\mathbf{P}}(z)\log\left(1-\frac{1}{z}\right) - \widehat{\mathbf{Q}}_1(z) = \mathcal{O}(\frac{1}{z^{m+1}}), \tag{3.6}$$

$$\widehat{\mathbf{P}}(z)\mathscr{E}(\frac{1}{z}) - \widehat{\mathbf{Q}}_2(z) = \mathcal{O}(\frac{1}{z^{n-2m+1}}), \tag{3.7}$$

If $m=0$, resp. $n=0$, we get the Padé approximants $[n-1/n]$ for $\mathscr{E}(\frac{1}{z})$ at $z=\infty$, resp. the Padé approximants $[m-1/m]$ for $\log(1-1/z)$ at $z=\infty$.

For any $m, n \geq 0$, let us define the polynomial, of degree $m+n$,

$$\widetilde{\mathbf{P}}(z) := \frac{1}{m!n!}((1-z)^m e^z(z^{n+m}e^{-z})^{(n)})^{(m)}. \tag{3.8}$$

Then there exist two polynomials $\widetilde{\mathbf{Q}}_1(z)$ and $\widetilde{\mathbf{Q}}_2(z)$, each of degree $m+n-1$, such that for any $m, n \geq 0$, we have

$$\widetilde{\mathbf{P}}(z)\log\left(1-\frac{1}{z}\right) - \widetilde{\mathbf{Q}}_1(z) = \mathcal{O}(\frac{1}{z^{m+1}}), \tag{3.9}$$

$$\widetilde{\mathbf{P}}(z)\mathscr{E}(\frac{1}{z}) - \widetilde{\mathbf{Q}}_2(z) = \mathcal{O}(\frac{1}{z^{n-m+1}}), \tag{3.10}$$

If $m=0$, resp. $n=0$, we get the Padé approximants $[n-1/n]$ for $\mathscr{E}(\frac{1}{z})$ at $z=\infty$, resp. the Padé approximants $[m-1/m]$ for $\log(1-1/z)$ at $z=\infty$.

The proof of Theorem 5 is given in Section 8.

4. Simultaneous Padé approximants for exp and $\mathscr{E}$

Our next result concerns the function $\exp(-z)$ and $\mathscr{E}(1/z)$ at $z=0$ and $z=\infty$ respectively. For any integers $m,n\geq 0$, let us define the polynomial of type (1.4)

$$\begin{aligned}\mathbf{P}(z) &:= \frac{1}{m!}e^z(z^m e^{-z}\mathbf{P}_n^{\mathbf{e}}(-z))^{(m)} = \frac{1}{m!n!}e^z(z^{2n+m}(z^{-n}e^{-z})^{(n)})^{(m)}\\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{n+j}\binom{2n-k-1}{n-1}\binom{m+k}{m-j}\frac{z^{k+j}}{k!j!} \qquad (4.1)\end{aligned}$$

and

$$\mathbf{Q}_1(z) := (-1)^n\sum_{k=0}^{n-1}\binom{2n-k-1}{n}\binom{m+k}{m}\frac{(-z)^k}{k!},$$

$$\mathbf{Q}_2(z) := \int_0^\infty \frac{\mathbf{P}(z)-\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t.$$

The polynomial $\mathbf{P}(z)$ is of degree $m+n$, $\mathbf{Q}_1(z)$ of degree $n-1$ and $\mathbf{Q}_2(z)$ of degree $m+n-1$.

Theorem 6. *For any $m,n\geq 0$, we have*

$$\begin{aligned}\mathbf{P}(z)\exp(-z)-\mathbf{Q}_1(z) &= \mathcal{O}(z^{2n})\\ &= \frac{n}{n!^2}\binom{m+2n}{m}z^{2n}\int_0^1 e^{-zt}t^n(1-t)^{n-1}\cdot {}_1F_1\left[\begin{matrix}-m\\2n+1\end{matrix};zt\right]\mathrm{d}t\\ &= \binom{m+2n}{m}\frac{z^{2n}}{(2n)!}\cdot {}_2F_2\left[\begin{matrix}m+2n+1,n+1\\2n+1,2n+1\end{matrix};-z\right], \qquad (4.2)\end{aligned}$$

$$\begin{aligned}&\mathbf{P}(z)\mathscr{E}\Big(\frac{1}{z}\Big)-\mathbf{Q}_2(z) = \mathcal{O}\Big(\frac{1}{z^{m+1}}\Big)\\ &= (-1)^{m+n}\binom{m+2n}{n}\int_0^\infty \frac{t^m}{(z-t)^{m+1}}\cdot {}_2F_1\left[\begin{matrix}-n,m+1\\m+n+1\end{matrix};-\frac{t}{z-t}\right]e^{-t}\mathrm{d}t. \qquad (4.3)\\ &\sim (-1)^{m+n}\binom{m+2n}{n}\frac{m!}{z^{m+1}}\cdot {}_3F_1\left[\begin{matrix}m+2n+1,m+1,m+1\\m+n+1\end{matrix};\frac{1}{z}\right].\end{aligned}$$

If $n=0$, resp. $m=0$, we get the Padé approximants $[m-1/m]$ for $\mathscr{E}(\frac{1}{z})$ at $z=\infty$, resp. the Padé approximants $[n-1/n]$ for $\exp(-z)$ at $z=0$.

The polynomial $\mathbf{P}(z)$ is also equal to $\frac{1}{m!n!}e^z z^{2n}(z^{-n}(z^m e^{-z})^{(m)})^{(n)}$, i.e., the operators defining $\mathbf{P}_m^{\mathbf{E}}(z)$ and $\mathbf{P}_n^{\mathbf{e}}(-z)$ "commute" in some sense.

The proof of Theorem 6 is given in Section 9.

5. Simultaneous Padé type approximants for exp, log and $\mathscr{E}$

We now present a problem that involves the three functions $\exp(-z)$, $\log(1-1/z)$ and $\mathscr{E}(1/z)$ simultaneously. Theorem 7 is just one example of what can be done.

For any integers $k, m, n \geq 0$, let us define

$$\mathbf{P}(z) := \frac{1}{k!m!n!}\big(e^z(z^{k+2m+n}(1-z)^{k+n}(z^{-m}e^{-z})^{(m)})^{(n)}\big)^{(k)} \tag{5.1}$$

which is of degree $k+m+2n$.

Theorem 7. *For any integers $k, m, n \geq 0$, there exist four polynomials $\mathbf{Q}_j(z)$ ($j = 1, 2, 3, 4$), each of degree $k+m+2n-1$ such that*

$$\mathbf{P}(z)\exp(-z) - \mathbf{Q}_1(z) = \mathcal{O}(z^{2m}) \tag{5.2}$$

$$\mathbf{P}(z)\log\big(1-\frac{1}{z}\big) - \mathbf{Q}_2(z) = \mathcal{O}\big(\frac{1}{z^{k+1}}\big), \tag{5.3}$$

$$\mathbf{P}(z)\mathscr{E}\big(\frac{1}{z}\big) - \mathbf{Q}_3(z) = \mathcal{O}\big(\frac{1}{z^{n+1}}\big), \tag{5.4}$$

$$\mathbf{P}(z)\int_0^1 \frac{e^{-t}}{z-t}\mathrm{d}t - \mathbf{Q}_4(z) = \mathcal{O}\big(\frac{1}{z^{n+1}}\big). \tag{5.5}$$

It is possible to give explicit expression for the polynomials and remainder terms but they are not illuminating. If $m = n = 0$, $\mathbf{P}(z) = \mathbf{P}_k^{\boldsymbol{\ell}}(z)$ and we get the Padé approximants $[n-1/n]$ to $\log(1-1/z)$ at $z=\infty$. If $k = m = 0$, we get the Padé approximants of type II $[2n-1, 2n-1/2n]$ for $\mathscr{E}(1/z)$ and $\int_0^1 \frac{e^{-t}\mathrm{d}t}{z-t}$ at $z = \infty$. If $k = n = 0$, $\mathbf{P}(z) = \mathbf{P}_m^{\mathbf{e}}(-z)$ and we get the Padé approximants $[n-1/n]$ to $\exp(-z)$ at $z = 0$.

We skip the proof of Theorem 7 because it is completely similar to those presented in the previous sections.

6. Padé approximants for $\exp(z), \log(1-1/z)$ and $\mathscr{E}(1/z)$

We recall here how to prove the assertions made at the beginning of the introduction concerning the (nearly) diagonal Padé approximants to $\log(1-z)$, $\mathscr{E}(z)$ and $\exp(z)$.

For $\log(1-z)$, we first define the polynomial

$$\mathbf{Q}_n^{\boldsymbol{\ell}}(z) := -\int_0^1 \frac{\mathbf{P}_n^{\boldsymbol{\ell}}(z) - \mathbf{P}_n^{\boldsymbol{\ell}}(t)}{z-t}\mathrm{d}t,$$

which is of degree $n-1$. It is obvious that

$$\mathbf{P}_n^{\boldsymbol{\ell}}(z)\log\big(1-\frac{1}{z}\big) - \mathbf{Q}_n^{\ell}(z) = -\int_0^1 \frac{\mathbf{P}_n^{\boldsymbol{\ell}}(t)}{z-t}\mathrm{d}t$$

and after n integrations by parts, we get the integral expression given for the remainder $\mathbf{R}_n^{\boldsymbol{\ell}}(z)$ in (1.1)

For $\mathscr{E}(z)$, we first define the polynomial

$$\mathbf{Q}_n^{\mathbf{E}}(z) := \int_0^\infty \frac{\mathbf{P}_n^{\mathbf{E}}(z) - \mathbf{P}_n^{\mathbf{E}}(t)}{z-t} e^{-t}\mathrm{d}t,$$

which is of degree $n-1$. It is also obvious that

$$\mathbf{P}_n^{\mathbf{E}}(z)\mathscr{E}(\frac{1}{z}) - \mathbf{Q}_n^{\mathbf{E}}(z) = \int_0^\infty \frac{\mathbf{P}_n^{\mathbf{E}}(t)}{z-t} e^{-t}\mathrm{d}t$$

and after n integrations by parts, we get the integral expression given for the remainder $\mathbf{R}_n^{\mathbf{E}}(z)$ in (1.2).

For $\exp(z)$, we proceed differently. We have

$$\begin{aligned}\mathbf{P}_n^{\mathbf{e}}(z)e^z &= \frac{1}{n!} z^{2n}(z^{-n}e^z)^{(n)} = \sum_{k=0}^\infty \frac{1}{k!n!} z^{2n}(z^{k-n})^{(n)} \\ &= \sum_{k=0}^\infty \frac{(k-n)(k-n-1)\cdots(k-2n+1)}{k!} z^k \\ &= \left(\sum_{k=0}^{n-1} + \sum_{k=2n}^\infty\right) \frac{(k-n)(k-n-1)\cdots(k-2n+1)}{k!n!} z^k\end{aligned}$$

because the terms for $k = n, n+1, \ldots, 2n-1$ all vanish. We define the polynomial, of degree $n-1$,

$$\begin{aligned}\mathbf{Q}_n^{\mathbf{e}}(z) : &= \sum_{k=0}^{n-1} \frac{(k-n)(k-n-1)\cdots(k-2n+1)}{k!n!} z^k \\ &= (-1)^n \sum_{k=0}^{n-1} \binom{2n-k-1}{n} \frac{z^k}{k!}\end{aligned}$$

and it is a simple task to transform the remainder

$$\mathbf{R}_n^{\mathbf{e}}(z) := \sum_{k=2n}^\infty \frac{(k-n)(k-n-1)\cdots(k-2n+1)}{k!n!} z^k$$

into the integral given in (1.3).

We now prove the hypergeometric expressions for the polynomials and remainders of the three Padé constructions. The series expansions given in (6.2) and (6.3) are proved by expanding $1/(1-t/z)^{n+1}$ in power series of t. The transformation of such series into hypergeometric form is then straightforward.

$$\mathbf{R}_n^{\boldsymbol{\ell}}(z) = (-1)^{n-1}\sum_{k=0}^{\infty}\frac{k(k-1)\cdots(k-n+1)}{(k+1)\cdots(k+n+1)}\cdot\frac{1}{z^{k+1}}$$

$$= \frac{n!^2}{(2n+1)!(-z)^{n+1}}\cdot {}_2F_1\left[\begin{matrix}n+1, n+1\\ 2n+2\end{matrix}; \frac{1}{z}\right], \tag{6.1}$$

$$\mathbf{R}_n^{\mathbf{E}}(z) \sim \frac{(-1)^n}{n!}\sum_{k=0}^{\infty}k(k-1)\ldots(k-n+1)\cdot\frac{k!}{z^{k+1}}$$

$$= (-1)^n\frac{n!}{z^{n+1}}\cdot {}_2F_0\left[\begin{matrix}n+1, n+1\\ \end{matrix}; \frac{1}{z}\right], \tag{6.2}$$

$$\mathbf{R}_n^{\mathbf{e}}(z) = \sum_{k=0}^{\infty}\frac{k(k-1)\ldots(k-n+1)}{(k+n)!}\cdot z^{k+n}$$

$$= \frac{z^{2n}}{(2n)!}\cdot {}_1F_1\left[\begin{matrix}n+1\\ 2n+1\end{matrix}; z\right]. \tag{6.3}$$

and

$$\mathbf{P}_n^{\boldsymbol{\ell}}(z) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n}z^k = {}_2F_1\left[\begin{matrix}-n, n+1\\ 1\end{matrix}; z\right], \tag{6.4}$$

$$\mathbf{P}_n^{\mathbf{E}}(z) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{z^k}{k!} = \frac{(-z)^n}{n!}\cdot {}_2F_0\left[\begin{matrix}-n, -n\\ \end{matrix}; -\frac{1}{z}\right] = {}_1F_1\left[\begin{matrix}-n\\ 1\end{matrix}; z\right], \tag{6.5}$$

$$\mathbf{P}_n^{\mathbf{e}}(z) = \sum_{k=0}^{n}(-1)^{n-k}\binom{2n-k-1}{n-1}\frac{z^k}{k!} = \frac{n}{n!^2}\int_0^{\infty}e^{-t}t^{n-1}(z-t)^n\mathrm{d}t \tag{6.6}$$

$$= (-1)^n\binom{2n-1}{n-1}\cdot {}_1F_1\left[\begin{matrix}-n\\ -2n+1\end{matrix}; -z\right] = \frac{z^n}{n!}\cdot {}_2F_0\left[\begin{matrix}-n, n\\ \end{matrix}; \frac{1}{z}\right]. \tag{6.7}$$

We observe that $\mathbf{P}_n^{\mathbf{E}}(z)$ and $\mathbf{P}_n^{\mathbf{e}}(z)$ both have expressions, trivially equivalent, that belong to two different classes of the hypergeometric hierarchy. This is an instance of the classical theory of asymptotic expansions at $z=\infty$ of ${}_1F_1[z]$ functions in terms of ${}_2F_0[1/z]$; See [22, §4.6].

7. Proofs of Theorems 1 and 2

Both proofs make use of the following lemma at some point.

Lemma 1. *For any integers $a, b, c \geq 0$, there exist two polynomials $Q_0(z)$ and $Q_1(z)$ such that* $\deg(Q_0) \leq b$, $\deg(Q_1) \leq a$ *and*

$$(-1)^{c-1}\int_0^1 \frac{t^a(1-t)^b}{(z-t)^{c+1}}\mathrm{d}t = \frac{1}{c!}(z^a(1-z)^b)^{(c)}\cdot\log\big(1-\frac{1}{z}\big)-\frac{Q_0(z)}{z^{c-a}}-\frac{Q_1(z)}{(1-z)^{c-b}}.$$

Proof. This follows by integrating c times by parts the left hand side. □

Since

$$\frac{1}{c!}(z^a(1-z)^b)^{(c)}\cdot\log\big(1-\frac{1}{z}\big) = -\frac{1}{c!}\int_0^1 \frac{(t^a(1-t)^b)^{(c)}}{z-t}\mathrm{d}t + \text{polynomial}(z),$$

Lemma 1 quantifies the difference between

$$\frac{1}{c!}\int_0^1 \frac{(t^a(1-t)^b)^{(c)}}{z-t}\mathrm{d}t \quad \text{and} \quad (-1)^c\int_0^1 \frac{t^a(1-t)^b}{(z-t)^{c+1}}\mathrm{d}t.$$

If $a \geq c$ and $b \geq c$, both integrals are equal as we can see by integrating c times by parts, but this is not true in general.

7.1. Proof of Theorem 1. We decompose the proof into two parts.

Properties of $\mathbf{L}(z)$**.** It is a trivial observation that $\mathbf{L}(z) = \mathcal{O}(\frac{1}{z^{c+1}})$. We now find its decomposition (2.4). We have

$$\begin{aligned}\frac{1}{d!f!}\int_0^\infty e^{-t}t^f(x-t)^d\mathrm{d}t &= \sum_{j=0}^{d}(-1)^{d-j}\binom{d+f-j}{f}\frac{1}{j!}x^j\\ &= \frac{1}{d!}x^{d+f+1}e^{-x}(x^{-f-1}e^x)^{(d)}\end{aligned}$$

provided that $d, f \geq 0$, which is the case. Hence,

$$\begin{aligned}\mathbf{L}(z) &= (-1)^{c-1}\sum_{j=0}^{d}(-1)^{d-j}\binom{d+f-j}{f}\frac{1}{j!}\int_0^1\frac{x^{a+j}(1-x)^b}{(z-x)^{c+1}}\mathrm{d}x\\ &= \sum_{j=0}^{d}(-1)^{d-j}\binom{d+f-j}{f}\frac{1}{j!}\Big(\frac{1}{c!}(z^{a+j}(1-z)^b)^{(c)}\log\big(1-\frac{1}{z}\big)\\ &\qquad - z^{a+j-c}Q_{0,j}(z)-(1-z)^{b-c}Q_{1,j}(z)\Big)\end{aligned}$$

where $Q_{0,j}(z)$ and $Q_{1,j}(z)$ are as described in Lemma 1, used to get the second equality.

Going backwards, we see that

$$\begin{aligned}&\sum_{j=0}^{d}(-1)^{d-j}\binom{d+f-j}{f}\frac{1}{j!}\frac{1}{c!}(z^{a+j}(1-z)^b)^{(c)}\\ &\qquad = \frac{1}{c!d!}(z^{a+d+f-1}(1-z)^b e^{-z}(z^{-f-1}e^z)^{(d)})^{(c)} = \mathbf{P}(z).\end{aligned}$$

It is also a routine task to see that

$$\sum_{j=0}^{d}(-1)^{d-j}\binom{d+f-j}{f}\frac{1}{j!}\Big(z^{a+j-c}Q_{0,j}(z)+(1-z)^{b-c}Q_{1,j}(z)\Big) \\ = z^{a-c}\mathbf{Q}_0(z)+(1-z)^{b-c}\mathbf{Q}_1(z)$$

where $\mathbf{Q}_0(z)$ and $\mathbf{Q}_1(z)$ are as described in the theorem. This completes the proof of the assertions for $\mathbf{L}(z)$.

Properties of $\mathbf{E}(z)$***.*** We have

$$z^{d+f+1}(z^{-f-1}e^z)^{(d)}=\sum_{k=0}^{\infty}\frac{1}{k!}z^{d+f+1}(z^{k-f-1})^{(d)}=\sum_{k=0}^{\infty}\frac{(k-f-d)_d}{k!}z^k,$$

so that

$$\mathbf{P}(z)\exp(z)=\frac{1}{c!d!}\sum_{k=0}^{\infty}\frac{(k-f-d)_d}{k!}e^z(z^{a+k}(1-z)^b e^{-z})^{(c)}$$

Since $(k-f-d)_d=0$ for $k\in\{f+1,\dots,f+d\}$, it is useful to define the polynomial

$$\mathbf{Q}_2(z):=\frac{1}{c!d!}\sum_{k=0}^{f}\frac{(k-f-d)_d}{k!}e^z(z^{a+k}(1-z)^b e^{-z})^{(c)}.$$

It is a polynomial of degree at most $a+b+f$ and order at $z=0$, resp. $z=1$, equal to $(a-c)^+$, resp. $(b-c)^+$, and

$$\mathbf{P}(z)\exp(z)-\mathbf{Q}_2(z)=\frac{1}{c!d!}\sum_{k=f+d+1}^{\infty}\frac{(k-f-d)_d}{k!}e^z(z^{a+k}(1-z)^b e^{-z})^{(c)}$$

is equal to $\mathbf{E}(z)$. The order at $z=0$ of $\mathbf{E}(z)$ is clearly at least $(a+d+f-c+1)^+$ and its order at $z=1$ is at least $(b-c)^+$.

7.2. Proof of Theorem 2.

We decompose the proof into two parts.

Properties of $\widetilde{\mathbf{L}}(z)$***.*** It is clear that $\widetilde{\mathbf{L}}(z)=\mathcal{O}(\frac{1}{z^{c-d+1}})$. We now set

$$\widehat{\mathbf{L}}(z):=-\int_0^1\frac{\widetilde{\mathbf{P}}(t)}{z-t}\mathrm{d}t=\widetilde{\mathbf{P}}(z)\log\big(1-\frac{1}{z}\big)-q_1(z) \tag{7.1}$$

where

$$q_1(z)=-\int_0^1\frac{\widetilde{\mathbf{P}}(z)-\widetilde{\mathbf{P}}(t)}{z-t}\mathrm{d}t.$$

Expanding $\widetilde{\mathbf{P}}(z)$ in the integral definition of $\widehat{\mathbf{L}}(z)$, we have

$$\widehat{\mathbf{L}}(z) = -\frac{1}{c!d!}\sum_{j=0}^{d}\binom{d}{j}\int_0^1 \frac{t^{d+f+1}e^{-t}(t^{-f-1}e^t)^{(d-j)}(t^a(1-t)^b)^{(c+j)}}{z-t}\mathrm{d}t$$

$$= -\frac{1}{c!d!}\sum_{j=0}^{d}\binom{d}{j}\sum_{\ell=j}^{d}\binom{d-j}{d-\ell}(-1)^{d-\ell}(f+1)_{d-\ell}\int_0^1 \frac{t^\ell(t^a(1-t)^b)^{(c+j)}}{z-t}\mathrm{d}t$$

$$= \frac{1}{d!}\sum_{j=0}^{d}\binom{d}{j}\sum_{\ell=j}^{d}\binom{d-j}{d-\ell}(-1)^{d-\ell}(f+1)_{d-\ell}\Bigg(\frac{1}{c!}\int_0^1 \frac{z^\ell - t^\ell}{z-t}(t^a(1-t)^b)^{(c+j)}\mathrm{d}t \tag{7.2}$$

$$- \frac{z^\ell}{c!}\int_0^1 \frac{(t^a(1-t)^b)^{(c+j)}}{z-t}\mathrm{d}t\Bigg). \tag{7.3}$$

We denote by $q_2(z)$ the polynomial on the line (7.2); it is of degree $d + (a+b-c)^+ - 1$.

Let us now study the integral on the line (7.2). We have

$$-\frac{1}{c!}\int_0^1 \frac{(t^a(1-t)^b)^{(c+j)}}{z-t}\mathrm{d}t = \frac{1}{c!}(z^a(1-z)^b)^{(c+j)}\log\big(1-\frac{1}{z}\big) \tag{7.4}$$
$$+\frac{1}{c!}\int_0^1 \frac{(z^a(1-z)^b)^{(c+j)} - (t^a(1-t)^b)^{(c+j)}}{z-t}\mathrm{d}t.$$

We denote by $q_{2,j}(z)$ the polynomial on the second line of (7.4). By unicity of the decomposition of $\widehat{\mathbf{L}}(z)$, we have

$$q_1(z) = -q_2(z) - \frac{1}{d!}\sum_{j=0}^{d}\binom{d}{j}\sum_{\ell=j}^{d}\binom{d-j}{d-\ell}(-1)^{d-\ell}(f+1)_{d-\ell}z^\ell q_{2,j}(z).$$

By Lemma 1, we know that

$$\frac{1}{c!}(z^a(1-z)^b)^{(c+j)}\cdot\log\big(1-\frac{1}{z}\big) = \tag{7.5}$$
$$\frac{(-1)^{c+j-1}(c+j)!}{c!}\int_0^1 \frac{t^a(1-t)^b}{(z-t)^{c+j+1}}\mathrm{d}t + z^{a-c-j}q_{0,j}(z) + (1-z)^{b-c-j}q_{1,j}(z)$$

with $\deg(q_{0,j}) \le b$ and $\deg(q_{0,j}) \le a$. We then put the right-hand side of (7.5) into (7.4), and the right-hand side of the resulting equation into (7.3). We obtain

$$\widehat{\mathbf{L}}(z) = -q_1(z) + \frac{1}{d!}\sum_{j=0}^{d}\binom{d}{j}\sum_{\ell=j}^{d}\binom{d-j}{d-\ell}(-1)^{d-\ell}(f+1)_{d-\ell}\Bigg(z^{a+\ell-c-j}q_{0,j}(z) \tag{7.6}$$
$$+ z^\ell(1-z)^{b-c-j}q_{1,j}(z) + \frac{(-1)^{c+j-1}(c+j)!}{c!}z^\ell\int_0^1 \frac{t^a(1-t)^b}{(z-t)^{c+j+1}}\mathrm{d}t\Bigg)$$
$$= -q_1(z) + z^{a-c}\widetilde{\mathbf{Q}}_0(z) + (1-z)^{b-c-d}\widetilde{\mathbf{Q}}_1(z)$$
$$+ \frac{(-1)^{c-1}}{d!}z^{d+f+1}e^{-z}\int_0^1 t^a(1-t)^b\frac{\partial^d}{\partial z^d}\Big(\frac{z^{-f-1}e^z}{(z-t)^{c+1}}\Big)\mathrm{d}t, \tag{7.7}$$

where $\widetilde{\mathbf{Q}}_0(z)$ and $\widetilde{\mathbf{Q}}_1(z)$ are as defined in the statement of the theorem. To conclude, we simply compare the two expressions (7.1) and (7.7) of $\widehat{\mathbf{L}}(z)$.

Properties of $\widetilde{\mathbf{E}}(z)$**.** We have

$$\begin{aligned}\widetilde{\mathbf{P}}(z)\exp(z) &= \frac{1}{c!d!}\sum_{k=0}^{\infty}\frac{z^{d+f+1}}{k!}\big(z^{k-f-1}(z^a(1-z)^b)^{(c)}\big)^{(d)}\\ &= \frac{1}{d!}\sum_{j=0}^{b}(-1)^j\binom{b}{j}\binom{a+j}{c}\sum_{k=0}^{\infty}\frac{z^{d+f+1}}{k!}\big(z^{j+k+a-c-f-1}\big)^{(d)}\\ &= \frac{1}{d!}\sum_{j=0}^{d}(-1)^j\binom{b}{j}\binom{a+j}{c}\sum_{k=0}^{\infty}\frac{(j+k+a-c-f-d)_d}{k!}z^{j+k+a-c}\end{aligned}$$

Since $(j+k+a-c-f-d)_d=0$ for $j+k\in\{f+c-a+1,\dots,f+d+c-a\}$, we can define the polynomial

(7.8)
$$\widetilde{\mathbf{Q}}_2(z) := \frac{1}{d!}\sum_{\substack{j,k\geq 0\\ j+k\leq f+c-a}}(-1)^j\binom{b}{j}\binom{a+j}{c}\frac{(j+k+a-c-f-d)_d}{k!}z^{j+k}$$

so that

$$\begin{aligned}&\widetilde{\mathbf{P}}(z)\exp(z)-z^{a-c}\widetilde{\mathbf{Q}}_2(z)\\ &= \sum_{\substack{j,k\geq 0\\ j+k\geq f+d+c-a+1}}(-1)^j\binom{b}{j}\binom{a+j}{c}\frac{(j+k+a-c-f-d)_d}{k!d!}z^{j+k+a-c},\end{aligned}$$

which is equal to $\widetilde{\mathbf{E}}(z)$ by the definition (2.7). It is a simple observation that the order of $\widetilde{\mathbf{E}}(z)$ at $z=0$ is $\geq (f+d+c-a+1)^{+}+a-c$. It is also easy to determine the degree of $\widetilde{\mathbf{Q}}_2(z)$ from (7.8).

8. Proof of Theorem 5

Before proving this theorem, we state a lemma, whose proof is straightforward by Leibniz formula.

Lemma 2. *For any $c,z\in\mathbb{C}$, any integers $\ell,s\geq 0$, we have*

$$e^{-ct}\left(\frac{e^{ct}}{(z-t)^{\ell+1}}\right)^{(s)} = \sum_{k=0}^{s}c^{s-k}\binom{s}{k}\frac{(\ell+k)!}{\ell!}\frac{1}{(z-t)^{\ell+k+1}} \tag{8.1}$$

where the differentiation is with respect to t.

Proof of Theorem 5. The first estimate to be proved can be restated as follows:

$$\mathbf{R}_1(z) := -\int_0^1\frac{\mathbf{P}(t)}{z-t}\mathrm{d}t = \mathcal{O}\Big(\frac{1}{z^{m+1}}\Big).$$

After m successive integrations by parts, followed by n integrations by parts, we have

$$
\begin{aligned}
\mathbf{R}_1(z) &= \frac{(-1)^{m-1}}{m!n!}\int_0^1 \frac{e^t(t^{m+n}(1-t)^{m+n}e^{-t})^{(n)}}{(z-t)^{m+1}}\mathrm{d}t \\
&= \frac{(-1)^{m-1}}{m!n!}\int_0^1 \left(\frac{e^t}{(z-t)^{m+1}}\right)^{(n)} t^{m+n}(1-t)^{m+n}e^{-t}\mathrm{d}t \\
&= (-1)^{m-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(m+k)!}{m!n!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{m+k+1}}\mathrm{d}t \qquad (8.2)\\
&= \frac{(-1)^{m-1}}{n!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{m+1}}\cdot {}_2F_0\left[\begin{matrix}-n, m+1\\ -\end{matrix}; -\frac{1}{z-t}\right]\mathrm{d}t,
\end{aligned}
$$

where we used Lemma 2 with $c=1$, $\ell=m$ and $s=n$. It is clear that the integrals on the right-hand side of (8.2) can be expanded as power series of $1/z$ with order at least $m+1$, which proves the claim.

Similarly, the second estimate to be proved can be restated as follows:

$$
\mathbf{R}_2(z) := \int_0^\infty \frac{\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t = \mathcal{O}(\frac{1}{z^{n+1}}).
$$

After m successive integrations by parts, followed by n integrations by parts, we have

$$
\begin{aligned}
\mathbf{R}_2(z) &= \frac{(-1)^m}{m!n!}\int_0^\infty \left(\frac{e^{-t}}{z-t}\right)^{(m)} e^t(t^{m+n}(1-t)^{m+n}e^{-t})^{(n)}\mathrm{d}t \\
&= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{k!}{m!n!}\int_0^\infty \frac{(t^{m+n}(1-t)^{m+n}e^{-t})^{(n)}}{(z-t)^{k+1}}\mathrm{d}t \qquad (8.3)\\
&= \sum_{k=0}^{m}(-1)^{n+k}\binom{m}{k}\frac{(n+k)!}{m!n!}\int_0^\infty \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+k+1}}e^{-t}\mathrm{d}t, \qquad (8.4)\\
&= \frac{(-1)^n}{m!}\int_0^\infty \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+1}}{}_2F_0\left[\begin{matrix}-m, n+1\\ -\end{matrix}; \frac{1}{z-t}\right]e^{-t}\mathrm{d}t,
\end{aligned}
$$

where we used Lemma 2 with $c=-1$, $\ell=0$ and $s=m$ to get the second equality. Again, it is clear that the integral on the right-hand side of (8.4) can be expanded as a power series of $1/z$ with order at least $n+1$, which proves the claim.

Finally, the third estimate to be proved can be restated as follows:

$$
\mathbf{R}_3(z) := \int_0^1 \frac{\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t = \mathcal{O}(\frac{1}{z^{n+1}}).
$$

After m successive integrations by parts, followed by n integrations by parts, we have

$$
\begin{aligned}
\mathbf{R}_3(t) &= \frac{(-1)^m}{m!n!}\int_0^1 \left(\frac{e^{-t}}{z-t}\right)^{(m)} e^t(t^{m+n}(1-t)^{m+n}e^{-t})^{(n)}\mathrm{d}t \\
&= \sum_{k=0}^m (-1)^k\binom{m}{k}\frac{k!}{m!n!}\int_0^1 \frac{(t^{m+n}(1-t)^{m+n}e^{-t})^{(n)}}{(z-t)^{k+1}}\mathrm{d}t \qquad (8.5)\\
&= \sum_{k=0}^m (-1)^{k+n}\binom{m}{k}\frac{(n+k)!}{m!n!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+k+1}}e^{-t}\mathrm{d}t, \qquad (8.6)\\
&= \frac{(-1)^n}{n!}\int_0^1 \frac{t^{m+n}(1-t)^{m+n}}{(z-t)^{n+1}}\, {}_2F_0\left[\begin{matrix}-m, n+1\\ -\end{matrix}; \frac{1}{z-t}\right] e^{-t}\mathrm{d}t,
\end{aligned}
$$

where we used Lemma 2 with $c=-1$, $\ell=0$ and $s=m$ to get the second equality. Again, it is clear that the integral on the right hand side of (8.6) can be expanded as a power series of $1/z$ with order at least $n+1$, which proves the claim. □

We close this section by mentioning that Theorem 5 is simply the translation of the multiple orthogonalities satisfied by $\mathbf{P}(t)$. The proof appears in disguise in the above lines.

Proposition 1. *For any $m,n\geq 0$, we have*

$$
\begin{aligned}
\int_0^1 t^k\mathbf{P}(t)\,\mathrm{d}t = 0, &\quad k\in\{0,\ldots,m-1\},\\
\int_1^\infty t^k\mathbf{P}(t)e^{-t}\,\mathrm{d}t = 0, &\quad k\in\{0,\ldots,n-1\},\\
\int_0^1 t^k\mathbf{P}(t)e^{-t}\,\mathrm{d}t = 0, &\quad k\in\{0,\ldots,n-1\}.
\end{aligned}
$$

9. Proof of Theorem 6

Let us prove the first part. We have

$$\mathbf{P}(z)e^{-z} = \frac{1}{m!}(z^m e^{-z}\mathbf{P}_n^{\mathbf{e}}(-z))^{(m)}.$$

Furthermore, since $\mathbf{P}_n^{\mathbf{e}}(-z) = \frac{1}{n!}e^z z^{2n}(z^{-n}e^{-z})^{(n)}$, we have

$$e^{-z}\mathbf{P}_n^{\mathbf{e}}(-z) = \frac{1}{n!}z^{2n}(z^{-n}e^{-z})^{(n)} = \sum_{k=0}^\infty (-1)^k\frac{(k-2n+1)_n}{k!n!}z^k,$$

so that

$$\mathbf{P}(z)\exp(-z) = \sum_{k=0}^{\infty}(-1)^k \frac{(k-2n+1)_n}{k!}\frac{1}{m!}(z^{k+m})^{(m)}$$
$$= \left(\sum_{k=0}^{n-1} + \sum_{k=2n}^{\infty}\right)(-1)^k \frac{(k-2n+1)_n(k+1)_m}{k!m!} z^k$$

and the result follows after setting

$$\mathbf{Q}_1(z) := \sum_{k=0}^{n-1}(-1)^k \frac{(k-2n+1)_n(k+1)_m}{k!m!} z^k$$

and

$$\mathbf{R}_1(z) := \sum_{k=2n}^{\infty}(-1)^k \frac{(k-2n+1)_n(k+1)_m}{k!m!} z^k = \mathcal{O}(z^{2n})$$

and after simplifications of the expressions.

The series representation of $\mathbf{R}_1(z)$ is hypergeometric, for we have

$$\mathbf{R}_1(z) = \binom{m+2n}{m}\frac{z^{2n}}{(2n)!}\, {}_2F_2\left[\begin{matrix} m+2n+1, n+1 \\ 2n+1, 2n+1 \end{matrix}; -z\right].$$

We now observe that, by Euler's integral identity,

$${}_2F_2\left[\begin{matrix} m+2n+1, n+1 \\ 2n+1, 2n+1 \end{matrix}; -z\right]$$
$$= n\binom{2n}{n}\int_0^1 t^n(1-t)^{n-1}\cdot {}_1F_1\left[\begin{matrix} m+2n+1 \\ 2n+1 \end{matrix}; -zt\right]\mathrm{d}t$$
$$= n\binom{2n}{n}\int_0^1 t^n(1-t)^{n-1}e^{-tz}\cdot {}_1F_1\left[\begin{matrix} -m \\ 2n+1 \end{matrix}; zt\right]\mathrm{d}t$$

where in the last step we used Kummer's transformation. This proves the first part of the theorem.

For the second part, we have to prove that

$$\mathbf{R}_2(z) := \int_0^{\infty}\frac{\mathbf{P}(t)}{z-t}e^{-t}\mathrm{d}t = \mathcal{O}(\frac{1}{z^{m+1}}).$$

This follows after m successive integrations by parts:

$$\mathbf{R}_2(z) = (-1)^m\int_0^{\infty}\frac{t^m\mathbf{P}_n^{\mathbf{e}}(-t)}{(z-t)^{m+1}}e^{-t}\mathrm{d}t,$$

which is obviously $\mathcal{O}(1/z^{m+1})$. To get the integral expression of $\mathbf{R}_2(z)$, we integrate by parts n consecutive times:

$$\begin{aligned}\mathbf{R}_2(z) &= (-1)^{m+n}\int_0^\infty \frac{1}{n!t^n}\left(\frac{t^{m+2n}}{(z-t)^{m+1}}\right)^{(n)} e^{-t}\mathrm{d}t\\ &= (-1)^{m+n}\binom{m+2n}{n}\int_0^\infty \frac{t^m}{(z-t)^{m+1}}\cdot {}_2F_1\left[\begin{matrix}-n, m+1\\ m+n+1\end{matrix}; -\frac{t}{z-t}\right] e^{-t}\mathrm{d}t.\end{aligned}$$

where we used Leibniz's formula.

The asymptotic expansion of $\mathbf{R}_2(z)$ is

$$\mathbf{R}_2(z) \sim \sum_{k=0}^\infty \frac{1}{z^{k+1}}\int_0^\infty t^k\mathbf{P}(t)e^{-t}\mathrm{d}t.$$

These integrals are easily computed by successive integrations by parts:

$$\begin{aligned}\int_0^\infty t^k\mathbf{P}(t)e^{-t}\mathrm{d}t &= (-1)^m\frac{(k-m+1)_m}{m!n!}\int_0^\infty t^{k+2n}(t^{-n}e^{-t})^{(n)}\mathrm{d}t\\ &= (-1)^{m+n}\frac{(k-m+1)_m(k+n+1)_n}{m!n!}\int_0^\infty t^k e^{-t}\mathrm{d}t\\ &= (-1)^{m+n}\frac{(k-m+1)_m(k+n+1)_n}{m!n!}k!.\end{aligned}$$

This finishes the proof of the theorem.

The polynomial $\mathbf{P}(z)$ is not a hypergeometric polynomial in one variable, but is a specialisation in one variable of a hypergeometric polynomial in two variables. It is thus remarkable that the remainders $\mathbf{R}_1(z)$, $\mathbf{R}_2(z)$ are hypergeometric series in one variable, as well as $\mathbf{Q}_1(z)$:

$$\begin{aligned}\mathbf{R}_1(z) &= \binom{m+2n}{m}\frac{z^{2n}}{(2n)!}\cdot {}_2F_2\left[\begin{matrix}m+2n+1, n+1\\ 2n+1, 2n+1\end{matrix}; \frac{1}{z}\right]\\ \mathbf{R}_2(z) &\sim (-1)^{m+n}\binom{m+2n}{n}\frac{m!}{z^{m+1}}\cdot {}_3F_1\left[\begin{matrix}m+2n+1, m+1, m+1\\ n+m+1\end{matrix}; \frac{1}{z}\right]\\ \mathbf{Q}_1(z) &= (-1)^n\binom{2n-1}{n}\cdot {}_2F_2\left[\begin{matrix}1-n, m+1\\ 1, 1-2n\end{matrix}; -z\right].\end{aligned}$$

10. Simultaneous Padé approximants for $\log(1-z)$ and $(1-z)^\alpha$

For any integer $n \geq 0$ and any real numbers $\alpha, \beta \in (-1, 1)$, let us consider the differential operator $D^n_{\alpha,\beta}$ defined by

$$D^n_{\alpha,\beta}(\Phi(z)) := \frac{1}{n!\,z^\alpha(1-z)^\beta}\left(z^{n+\alpha}(1-z)^{n+\beta}\Phi(z)\right)^{(n)}.$$

Such a differential operator maps polynomials on polynomials.

The following lemma follows by considering $\Phi(z) = z^k$ for any $k \geq 0$.

Lemma 3. *For any polynomial $\Phi(z) \in \mathbb{C}[z]$, we have*

$$D^n_{\alpha_1,\beta_1}\big(D^n_{\alpha_2,\beta_2}(\Phi(z))\big) = D^n_{\alpha_2,\beta_2}\big(D^n_{\alpha_1,\beta_1}(\Phi(z))\big),$$

provided that $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$.

The commutativity of the differential operators in Lemma 3 is in contrast with the various polynomials $\mathbf{P}(z)$ considered in our previous theorems, for which the underlying differential operators do not commute in general (except in the case of $\exp(-z)$ and $\mathscr{E}(1/z)$).

We now consider a multiset $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_J \ldots \alpha_J\}$ of reals numbers in $(-1, 1)$ such that $\alpha_m - \alpha_n \notin \mathbb{Z}$ for any $n \neq m$ and each α_m is repeated ℓ_m times. We set $\boldsymbol{\alpha} = \{a_1, a_2, \ldots, a_s\}$ where $s = \sum_{j=1}^{J} \ell_j$ and we define a polynomial of degree sn by

$$\mathbf{P}(z) := D^n_{a_1,-a_1}\big(D^n_{a_2,-a_2}(\ldots D^n_{a_s,-a_s}(1)\ldots)\big). \tag{10.1}$$

The order chosen for the a's is not important by Lemma 3.

By multiple integrations by parts and by Lemma 3, we see that, under the above conditions,

$$\int_0^1 t^k \mathbf{P}(t)\, t^{\alpha_j}(1-t)^{-\alpha_j} \log(t)^\rho \mathrm{d}t = 0$$

for all $k \in \{0, 1, \ldots, n-1\}$, $j \in \{1, \ldots, J\}$ and $\rho \in \{0, \ldots, \ell_j - 1\}$. In other words, the polynomials $\mathbf{P}(z)$, $n \geq 0$ form a sequence of multiple orthogonal polynomials on $[0, 1]$ for the s weights $t^{\alpha_j}(1-t)^{-\alpha_j}\log(t)^\rho$, $0 \leq \rho \leq \ell_j - 1$, $1 \leq j \leq J$. This translates into simultaneous Padé approximants of type II for the family of functions

$$\Psi_{\alpha_j,\rho}(z) := \int_0^1 \frac{t^{\alpha_j}(1-t)^{-\alpha_j}}{z-t} \log(1/t)^\rho \mathrm{d}t.$$

We set

$$\mathbf{Q}_{j,\rho}(z) := \int_0^1 \frac{\mathbf{P}(z) - \mathbf{P}(t)}{z-t}\, t^{\alpha_j}(1-t)^{-\alpha_j} \log(1/t)^\rho \mathrm{d}t$$

which is of degree $sn - 1$.

Theorem 8. *In the above conditions, we have*

$$\mathbf{P}(z)\Psi_{\alpha_j,\rho}(z) - \mathbf{Q}_{j,\rho}(z) = \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big). \tag{10.2}$$

When $\alpha_j \in \mathbb{Q}$, the coefficients of the polynomials $\mathbf{P}(z)$ and $\frac{\sin(\pi\alpha)}{\pi\alpha}\mathbf{Q}_{j,\rho}(z)$ are rational numbers.

The functions $\Psi_{\alpha,\rho}(z)$ can sometimes be expressed in term of elementary functions. For instance, if $\alpha \in (-1, 1), \alpha \neq 0$, we have

$$\Psi_{\alpha,0}(1/z) = \frac{\pi}{\sin(\pi\alpha)}\Big((1-z)^{-\alpha} - 1\Big)$$

while if $\alpha = 0$, $\frac{1}{\rho!}\Psi_{0,\rho}(1/z) = \mathrm{Li}_{\rho+1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^{\rho+1}}$, and in particular $\Psi_{0,0}(z) = -\log(1-1/z)$.

As an application of Theorem 8, consider any finite set S of functions in the infinite set

$$\{\log(1-z), \mathrm{Li}_2(z), \mathrm{Li}_3(z), \ldots, (1-z)^{\alpha_1}, (1-z)^{\alpha_2}, \ldots, (1-z)^{\alpha_s} \ldots, \}$$

where the only assumption of the α's is that $\alpha_m \in \mathbb{Q}$ and $\alpha_n - \alpha_m \notin \mathbb{Z}$ for $n \neq m$. By standard arguments in number theory we omit, Theorem 8 enables us to prove the linear independence over $\mathbb{Q}$ of the values of the functions in S evaluated at any rational point x sufficiently close to 0 (depending on $\boldsymbol{\alpha} \subset \mathbb{Q}$).

The construction seems to be new in this generality but see [23] for related considerations. In the literature, one can find the case of Theorem 8 where $\boldsymbol{\alpha} = \{0, 0, \ldots, 0\}$ (Hata [12]) as well as simultaneous Hermite-Padé (type I) for the functions $(1-z)^{\alpha_1}, \ldots, (1-z)^{\alpha_s}$ (Chudnovski [9]). In both cases, arithmetical applications of the type mentioned above are given in precise form.

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