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Iwasawa theory for symmetric powers of CM modular forms at nonordinary primes, II

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Résumé. Poursuivant l'étude de la théorie d'Iwasawa des puissances symétriques de formes modulaires à multiplication complexe aux nombres premiers supersinguliers entamée par le premier auteur et Antonio Lei, nous démontrons une Conjecture Principale identifiant les fonctions L p-adiques « admissibles » avec les idéaux caractéristiques des modules de Selmer de « pente finie » introduits par le deuxième auteur. Comme ingrédient clé, nous transformons la divisibilité en égalité dans le résultat de Rubin concernant la Conjecture Principale des corps quadratiques imaginaires aux nombres premiers inertes.

Abstract. Continuing the study of the Iwasawa theory of symmetric powers of CM modular forms at supersingular primes begun by the first author and Antonio Lei, we prove a Main Conjecture equating the "admissible" p-adic L-functions to the characteristic ideals of "finite-slope" Selmer modules constructed by the second author. As a key ingredient, we improve Rubin's result on the Main Conjecture of Iwasawa theory for imaginary quadratic fields to an equality at inert primes.

Introduction

The study of the Iwasawa theory of symmetric powers of CM modular forms at supersingular primes was begun by the first author and Antonio Lei in [HL]. They constructed two types of p-adic L-functions: "admissible" ones in the sense of Panchishkin and Dabrowski, and "plus and minus" ones in the sense of Pollack. They also constructed "plus and minus" Selmer modules in the sense of Kobayashi, and, using Kato's Euler system, they compared them to the latter p-adic L-functions via one divisibility in a main conjecture. The present paper performs the analogous comparison between the admissible p-adic L-functions and the "finite-slope" Selmer modules in the sense of the second author. In order to get an identity of characteristic

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ideals, rather than just a divisibility, we improve the work of Rubin on the Main Conjecture of Iwasawa theory for imaginary quadratic fields at inert primes [Ru1, Ru2] to give an equality unconditionally. Rubin's work has since been used by various authors to derive other divisibilities; an examination of these derivations will show that our work upgrades most of these divisibilities to identities.

The first section of this paper is written as a direct continuation of [HL]; all numbered references to equations, theorems, etc. in it are to the two papers commonly, except for bibliographical citations, which are to the references section here. In this section, we recall the relevant setup from [HL], as well as the theory of finite-slope Selmer groups from [P2]. Then we give our results about finite-slope Selmer modules of CM modular forms and their symmetric powers at supersingular primes. The second section is written independently of [HL] and the first section. In it we recall the notations from [Ru1, Ru2] and then treat the Iwasawa theory of imaginary quadratic fields at inert primes.

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7. CM modular forms and their symmetric powers

7.1. Notations and hypotheses of [HL]**.** The prime p is assumed odd. We fix algebraic closures and embeddings $\iota_\infty\colon \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, and use these for the definition of Galois groups and decomposition groups. In particular, we write $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the complex conjugation induced by ι_∞.

We normalize reciprocity maps of class field theory to send uniformizers to arithmetic Frobenius elements. If $E/\mathbb{Q}_p$ is a finite extension, we normalize duals of E-linear Galois representations by $V^* = \mathrm{Hom}_E(V, E(1))$, and Fontaine's functors by $\mathbb{D}_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, \mathbb{B}_{\mathrm{cris}})$ and $\widetilde{\mathbb{D}}_{\mathrm{cris}}(V) = (\mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$.

For $n \leq \infty$ we write $k_n = \mathbb{Q}(\mu_{p^n})$ and $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$. The cyclotomic character χ induces an isomorphism $G_\infty := \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$, and G_∞ factors uniquely as $\Delta \times \Gamma$ in such a way that χ induces isomorphisms $\Delta \cong \mu_{p-1}$ and $\Gamma \cong 1 + p\mathbb{Z}_p$. We fix a topological generator γ_0 of Γ.

For a finite extension E of $\mathbb{Q}_p$ and $G = G_\infty$ or Γ, we write $\Lambda_{\mathcal{O}_E}(G) = \mathcal{O}_E[\![G]\!]$ for the Iwasawa algebra of G with coefficients in $\mathcal{O}_E$ and we write $\Lambda_E(G) = \Lambda_{\mathcal{O}_E}(G) \otimes_{\mathcal{O}_E} E$. Note that $\Lambda_E(G)$ is a product of PIDs if $G = G_\infty$ (resp. a PID if $G = \Gamma$). We let $\mathcal{H}_{r,E}(G)$ be the E-valued r-tempered distributions on G for $r \in \mathbb{R}_{\geq 0}$ and, $\mathcal{H}_{\infty,E}(G) = \bigcup_r \mathcal{H}_{r,E}(G)$. These objects are stable under the involution ι (resp. twisting operator Tw_n for $n \in \mathbb{Z}$)

obtained by continuity and E-linearity by the rule $\sigma \mapsto \sigma^{-1}$ (resp. $\sigma \mapsto \chi(\sigma)^n\sigma$) on group elements $\sigma \in G$. If G acts on M, then M^ι denotes M with G action composed through ι.

We fix an imaginary quadratic field $K \subset \overline{\mathbb{Q}}$, considered as a subfield of $\mathbb{C}$ via ι_∞, with ring of integers $\mathcal{O}$ and quadratic character $\epsilon_K\colon \operatorname{Gal}(K/\mathbb{Q}) \cong \{\pm 1\}$. We assume p is inert in K, i.e. $\epsilon_K(p) = -1$, and write $\mathcal{O}_p$ (resp. K_p) for the completion of $\mathcal{O}$ (resp. K) at $p\mathcal{O}$.

We fix a newform f of weight $k \geq 2$, level $\Gamma_1(N)$ with $p \nmid N$, character ϵ, and CM by K. We write ψ and $\psi^c = \psi \circ c$ for the algebraic Hecke characters of K associated to f, and order them to have types $(k-1, 0)$ and $(0, k-1)$, respectively. We write E for a finite extension of $\mathbb{Q}_p$ containing $\iota_p\iota_\infty^{-1}\psi(\mathbb{A}_{K,f}^\times)$. Note that E contains $\iota_p(K)$ and the images of the coefficients of f under $\iota_p\iota_\infty^{-1}$. We write V_ψ for the one-dimensional E-linear Galois representation attached to ψ, so that when $v \nmid p \operatorname{cond}(\psi)$ the action of Frob_v on V_ψ is by multiplication by $\psi(v)$. We write V_f for the E-linear dual of the two-dimensional Galois representation associated to f by Deligne, with structure map $\rho_f\colon G_\mathbb{Q} \to \mathrm{GL}(V_f)$, satisfying $\det(\rho_f) = \epsilon\chi^{k-1}$. One has $V_f \cong \operatorname{Ind}_K^\mathbb{Q} V_\psi$. Since p is inert in K, the comparison of L-factors between f and ψ gives $x^2 - a_p(f)x + \epsilon(p)p^{k-1} = x^2 - \psi(p)$, and in particular $a_p(f) = 0$ so that f is nonordinary at p. After perhaps enlarging E, we fix a root $\alpha \in E$ of this polynomial, so that the other root is $\overline{\alpha} = -\alpha$, and

$$\psi(p) = \psi^c(p) = -\epsilon(p)p^{k-1} = -\alpha\overline{\alpha} = \alpha^2 = \overline{\alpha}^2.$$

Let $m \geq 1$ be an integer, and write $r = \lfloor m/2 \rfloor$ and $\widetilde{r} = \lceil m/2 \rceil$. We define $V_m = \operatorname{Sym}^m(V_f) \otimes \det(\rho_f)^{-r}$. There exist newforms f_i for $0 \leq i \leq \widetilde{r} - 1$ (Proposition 3.4), of respective weights $k_i = (m-2i)(k-1)+1$, levels $\Gamma_1(N_i)$ with $p \nmid N_i$, characters ϵ_i, and having CM by K (in particular, they are nonordinary at p), such that

$$V_m \cong \bigoplus_{i=0}^{\widetilde{r}-1} \left(V_{f_i} \otimes \chi^{(i-r)(k-1)}\right) \oplus \begin{cases} \epsilon_K^r & m \text{ even}, \\ 0 & m \text{ odd}. \end{cases}$$

As a consequence, the complex L-function (Corollary 3.5), Hodge structure (Lemma 3.6), critical twists (Lemma 3.7), and structure of $\mathbb{D}_{\mathrm{cris}}$ as a filtered φ-module (Lemmas 3.9 and 3.10), for V_m are all computed explicitly. The same computations show that the roots of $x^2 + \epsilon_i(p)p^{k_i-1}$ are $\alpha_i, \overline{\alpha}_i = -\alpha_i$, where

$$\alpha_i = \begin{cases} p^{(r-i)(k-1)} & m \text{ even}, \\ \alpha p^{(r-i)(k-1)} & m \text{ odd}. \end{cases}$$

For η a Dirichlet character of prime-to-p conductor, we denote by L_η its p-adic L-function (Theorem 4.1), considered as an element of $\Lambda_{\mathcal{O}_E}(G_\infty)$ if η is nontrivial and of $[(\gamma_0 - 1)(\gamma_0 - \chi(\gamma_0))]^{-1}\Lambda_{\mathcal{O}_E(G_\infty)}$ if η is the trivial character $\mathbf{1}$. Let $\widetilde{L}_\eta \in \Lambda_{\mathcal{O}_E}(G_\infty)$ then denote the regularized p-adic L-function: if $\eta = \mathbf{1}$ then it is defined in §5.2 by removing the poles of $L_{\mathbf{1}}$, and otherwise it is defined to be L_η. Since the roots $\alpha_i, \overline{\alpha}_i$ of $x^2 + \epsilon_i(p)p^{k_i-1}$ have p-adic valuation $h_i := \frac{k_i-1}{2} < k_i - 1$, there are p-adic L-functions $L_{f_i,\alpha_i}, L_{f_i,\overline{\alpha}_i} \in \mathcal{H}_{h_i,E}(G_\infty)$ (Theorem 4.2). We let $\mathfrak{T}$ denote the collection of tuples $\mathfrak{t} = (\mathfrak{t}_0, \ldots, \mathfrak{t}_{\widetilde{r}-1})$, where each $\mathfrak{t}_i \in \{\alpha_i, \overline{\alpha}_i\}$. For each $\mathfrak{t} \in \mathfrak{T}$, we define the *admissible p-adic L-functions*

$$L_{V_m,\mathfrak{t}} = \iota\left(\prod_{i=0}^{\widetilde{r}-1} \mathrm{Tw}_{(r-i)(k-1)}\, L_{f_i,\mathfrak{t}_i}\right) \cdot \begin{cases} L_{\epsilon_K^r} & m \text{ even}, \\ 1 & m \text{ odd}, \end{cases}$$

as well as their regularized variants $\widetilde{L}_{V_m,\mathfrak{t}}$ where $L_{\epsilon_K^r}$ is replaced by $\widetilde{L}_{\epsilon_K^r}$. (The twist ι and the indexing are our only changes in conventions from [HL]. There, the index set $\mathfrak{S} = \{\pm\}^{\widetilde{r}-1}$ is used, and $\mathfrak{s} \in \mathfrak{S}$ corresponds to $\mathfrak{t} \in \mathfrak{T}$ where $\mathfrak{t}_i = \mathfrak{s}_i p^{(r-i)(k-1)}$ if m is even, and $\mathfrak{t}_i = \mathfrak{s}_i \alpha p^{(r-i)(k-1)}$ if m is odd.) Just as in the case $m = 1$, these functions can be decomposed in terms of appropriate products of twists of “plus and minus” logarithms and “plus and minus” p-adic L-functions (Corollary 6.9); their trivial zeroes and $\mathcal{L}$-invariants are known (Theorem 6.13), using work of Benois [Ben1, Ben2].

Finally, for $\theta = \mathbf{1}$ or ϵ_K, recall the Selmer groups $\mathrm{Sel}_{k_\infty}(A_\theta^*)$ of equation (8), whose Pontryagin duals $\mathrm{Sel}_{k_\infty}(A_\theta^*)^\vee$ are finitely generated, torsion $\Lambda_{\mathcal{O}_E}(G_\infty)$-modules.

7.2. Finite-slope Selmer complexes.

7.2.1. ***Rigid analytic Iwasawa algebras.*** For $G = G_\infty$ or Γ, we write $\mathcal{H}_E(G)$ for the E-valued locally analytic distributions on G; explicitly, $\mathcal{H}_E(\Gamma)$ is given by

$$\left\{ \sum_{n\geq 0} c_n \cdot (\gamma_0 - 1)^n \in E[\![\gamma_0 - 1]\!] \,\middle|\, \lim_{n\to\infty} |c_n| s^n = 0 \ \text{ for all } 0 \leq s < 1 \right\},$$

and $\mathcal{H}_E(G_\infty) = \mathcal{H}_E(\Gamma) \otimes_E E[\Delta]$. This ring contains $\mathcal{H}_{\infty,E}(G)$, and the subalgebra $\Lambda_E(G)$ (hence also $\mathcal{H}_{\infty,E}(G)$) is dense for a Fréchet topology. Although the ring is not Noetherian, it is a product of Bézout domains if $G = G_\infty$ (resp. is a Bézout domain if $G = \Gamma$).

The topological ring $\mathcal{H}_E(G)$ is moreover a Fréchet–Stein algebra, so that the coadmissible $\mathcal{H}_E(G)$-modules (in the sense of [ST]) form an abelian full subcategory of all $\mathcal{H}_E(G)$-modules. Coadmissible $\mathcal{H}_E(G)$-modules include the finitely generated ones, and have similar properties to finitely generated modules over a product of PIDs if $G = G_\infty$ (resp. over a PID if $G = \Gamma$), including a structure theory and a notion of characteristic ideal.

The rule $P \mapsto P\mathcal{H}_E(G)$ provides a bijection between prime (resp. maximal) ideals in $\Lambda_E(G)$ and topologically closed prime (resp. maximal) ideals in $\mathcal{H}_E(G)$, and when P is maximal one has $\Lambda_E(G)/P^n \xrightarrow{\sim} \mathcal{H}_E(G)/P^n$ for all $n \geq 0$. In particular, the algebra map $\Lambda_E(G) \to \mathcal{H}_E(G)$ is faithfully flat so that the operation $M \mapsto M \otimes_{\Lambda_E(G)} \mathcal{H}_E(G)$ on $\Lambda_E(G)$-modules is exact and fully faithful. If M is a finitely generated, torsion $\Lambda_E(G)$-module, then it follows from these facts that the natural map $M \xrightarrow{\otimes 1} M \otimes_{\Lambda_E(G)} \mathcal{H}_E(G)$ is a bijection, and M admits a canonical structure of $\mathcal{H}_E(G)$-module. It then follows that $\mathrm{char}_{\mathcal{H}_E(G)} M = (\mathrm{char}_{\Lambda_E(G)} M)\mathcal{H}_E(G)$, and, since $\mathcal{H}_E(G)^\times = \Lambda_E(G)^\times$, all generators of this ideal actually belong to $\mathrm{char}_{\Lambda_E(G)} M$.

7.2.2. ***Galois cohomology.*** Write S for the set of primes dividing Np, write $\mathbb{Q}_S$ for the maximal extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ unramified outside $S \cup \{\infty\}$, and let $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ denote the corresponding quotient of $G_{\mathbb{Q}}$. Recall that $k_\infty \subset \mathbb{Q}_S$, and that the natural map from G_∞ to the quotient $\mathrm{Gal}(k_\infty/\mathbb{Q})$ of $G_{\mathbb{Q},S}$ is an isomorphism; we henceforth identify G_∞ with this quotient of $G_{\mathbb{Q},S}$. The embedding ι_p determines a decomposition group $G_p \subset G_{\mathbb{Q},S}$, and choosing additional algebraic closures and emeddings $\iota_\ell \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ similarly determines decomposition groups $G_\ell \subset G_{\mathbb{Q},S}$ for each $\ell \mid N$. If X is a continuous representation of $G_{\mathbb{Q},S}$ and G is one of $G_{\mathbb{Q},S}$ or G_v with $v \in S$, we write $\mathbf{R}\Gamma(G, X)$ for the class in the derived category of the complex of continuous cochains of G with coefficients in X, and we write $H^*(G, X)$ for its cohomology.

We write $\Lambda_E(G_\infty)^\iota$ (resp. $\mathcal{H}_E(G_\infty)^\iota$) for $\Lambda_E(G_\infty)$ (resp. $\mathcal{H}_E(G_\infty)$) considered with G_∞-action, and hence also $G_{\mathbb{Q},S}$-action, with $g \in G_\infty$ acting by multiplication by $g^{-1} \in G_\infty \subset \Lambda_E(G_\infty)^\times = \mathcal{H}_E(G_\infty)^\times$. If V is a continuous E-linear $G_{\mathbb{Q},S}$-representation, then its classical Iwasawa cohomology over $G = G_{\mathbb{Q},S}$ or G_v ($v \in S$) is defined by choosing a $G_{\mathbb{Q},S}$-stable $\mathcal{O}_E$-lattice $T \subset V$ and forming $[\varprojlim_n H^*(G \cap \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}(\mu_{p^n})), T)] \otimes_{\mathcal{O}_E} E$; a variant of Shapiro's lemma [Nek, (8.4.4)] identifies it with $H^*(G, V \otimes_E \Lambda_E(G_\infty)^\iota)$, and in particular it is canonically independent of the choice of lattice T. The natural map

$$H^*(G, V \otimes_E \Lambda_E(G_\infty)^\iota) \otimes_{\Lambda_E(G_\infty)} \mathcal{H}_E(G_\infty) \to H^*(G, V \otimes_E \mathcal{H}_E(G_\infty)^\iota)$$

is an isomorphism. We define $\mathbf{R}\Gamma_{\mathrm{Iw}}(G, V) = \mathbf{R}\Gamma(G, V \otimes_E \mathcal{H}_E(G_\infty)^\iota)$ and $H^*_{\mathrm{Iw}}(G, V) = H^*(G, V \otimes_E \mathcal{H}_E(G_\infty)^\iota)$. We refer to $H^*_{\mathrm{Iw}}(G, V)$ as the *rigid analytic Iwasawa cohomology*, or, because we have no use for classical Iwasawa cohomology in this paper, simply the *Iwasawa cohomology.* The complex of $\mathcal{H}_E(G_\infty)$-modules $\mathbf{R}\Gamma_{\mathrm{Iw}}(G, V)$ is perfect for the degrees $[0, 2]$, and in particular the Iwasawa cohomology groups are coadmissible $\mathcal{H}_E(G_\infty)$-modules.

For details on the claimed properties of Iwasawa cohomology, we refer to [P2, §1].

7.2.3. *(φ, G_∞)-modules.* There is an equivalence $V \mapsto \mathbb{D}_{\mathrm{rig}}(V)$ between the categories continuous E-linear G_p-representations and étale (φ, G_∞)-modules over $\mathcal{R}_E = \mathcal{R} \otimes_{\mathbb{Q}_p} E$, where $\mathcal{R}$ is the Robba ring; see [P2, Theorem 2.3].

Given any (φ, G_∞)-module D over $\mathcal{R}$, we define $\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D)$ to be the class of

$$[D \xrightarrow{1-\psi} D]$$

in the derived category, where ψ is the canonical left inverse to φ and the complex is concentrated in degrees $1, 2$, and we define $H^*_{\mathrm{Iw}}(G_p, D)$ to be its cohomology, referring to the latter as the *Iwasawa cohomology* of D. This complex of $\mathcal{H}_E(G_\infty)$-modules is perfect for the degrees $[0, 2]$, so that the Iwasawa cohomology groups are coadmissible $\mathcal{H}_E(G_\infty)$-modules. Note the comparison

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, V) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, \mathbb{D}_{\mathrm{rig}}(V)). \tag{26}$$

For details on the claimed properties of Iwasawa comohomology of (φ, G_∞)-modules, we refer to [P1] and [KPX, §§4.3–4.4].

We define $\widetilde{\mathbb{D}}_{\mathrm{cris}}(D) = D[1/t]^{G_\infty}$ (where $t \in \mathcal{R}$ is Fontaine's $2\pi i$) and $\mathbb{D}_{\mathrm{cris}}(D) = \widetilde{\mathbb{D}}_{\mathrm{cris}}(\mathrm{Hom}_{\mathcal{R}_E}(D, \mathcal{R}_E))$, and we say that D is crystalline if $\dim_E \mathbb{D}_{\mathrm{cris}}(D) = \mathrm{rank}_{\mathcal{R}_E} D$. Note the comparisons

$$\mathbb{D}_{\mathrm{cris}}(V) \cong \mathbb{D}_{\mathrm{cris}}(\mathbb{D}_{\mathrm{rig}}(V)), \qquad \widetilde{\mathbb{D}}_{\mathrm{cris}}(V) \cong \widetilde{\mathbb{D}}_{\mathrm{cris}}(\mathbb{D}_{\mathrm{rig}}(V)).$$

The functor $\widetilde{\mathbb{D}}_{\mathrm{cris}}$ provides an exact, rank-preserving equivalence of exact $\otimes$-categories with Harder–Narasimhan filtrations, from crystalline (φ, G_∞)-modules over $\mathcal{R}_E$ to filtered φ-modules over E, under which those (φ, G_∞)-modules of the form $\mathbb{D}_{\mathrm{rig}}(V)$ correspond to the weakly admissible filtered φ-modules. In particular, if we tacitly equip any $E[\varphi]$-submodule of a filtered φ-module with the induced filtration, then for D crystalline $\widetilde{\mathbb{D}}_{\mathrm{cris}}$ induces a functorial, order-preserving bijection

$$\{t\text{-saturated } (\varphi, G_\infty)\text{-submodules of } D\} \leftrightarrow \{E[\varphi]\text{-stable subspaces of } \widetilde{\mathbb{D}}_{\mathrm{cris}}(D)\}.$$

For details on the relationship between (φ, G_∞)-modules and the crystalline theory, we refer the reader to [Ber].

7.2.4. *Selmer complexes.* In the remainder of this subsection, we assume given a continuous E-representation V of $G_{\mathbb{Q},S}$ that is crystalline at p, as well as a fixed $E[\varphi]$-stable $F \subseteq \mathbb{D}_{\mathrm{cris}}(V|_{G_p})$, and we associate to these data an Iwasawa-theoretic Selmer complex.

We begin by defining a local condition for each $v \in S$, by which we mean an object U_v in the derived category together with a morphism $i_v \colon U_v \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v, V)$. If $v \neq p$, we denote by $I_v \subset G_v$ the inertia subgroup, and we let $U_v = \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v/I_v, V^{I_v})$ and i_v be the inflation map. If $v = p$, we write $F^\perp \subseteq \widetilde{\mathbb{D}}_{\mathrm{cris}}(V)$ for the orthogonal complement of F, and then $D_F^+ := \widetilde{\mathbb{D}}_{\mathrm{cris}}^{-1}(F^\perp) \subseteq \mathbb{D}_{\mathrm{rig}}(V)$ and $D_F^- = \mathbb{D}_{\mathrm{rig}}(V)/D_F^+$. Then we let $U_v = \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^+)$, and we let i_v be the functorial map to $\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, \mathbb{D}_{\mathrm{rig}}(V)) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, V)$.

We now define the *Selmer complex* $\mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V)$ to be the mapping fiber of the morphism

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G_{\mathbb{Q},S}, V) \oplus \bigoplus_{v\in S} U_v \xrightarrow{\bigoplus_{v\in S}\mathrm{res}_v - \bigoplus_{v\in S} i_v} \bigoplus_{v\in S} \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v, V),$$

where $\mathrm{res}_v\colon \mathbf{R}\Gamma_{\mathrm{Iw}}(G_{\mathbb{Q},S}, X) \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v, X)$ denotes restriction of cochains to the decomposition group. We write $\widetilde{H}^*_{F,\mathrm{Iw}}(\mathbb{Q}, V)$ for its cohomology, referring to it as the *extended Selmer groups*. By our knowledge of the constituents of the mapping fiber, $\mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V)$ is a perfect complex of $\mathcal{H}_E(G_\infty)$-modules for the range $[0, 3]$.

We will have need for a version without imposing local conditions at p. Namely, we write $\mathbf{R}\widetilde{\Gamma}_{(p),\mathrm{Iw}}(\mathbb{Q}, V)$ for the mapping fiber of

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G_{\mathbb{Q},S}, V) \oplus \bigoplus_{v\in S^{(p)}} U_{v,\mathrm{Iw}} \xrightarrow{\bigoplus_{v\in S^{(p)}}\mathrm{res}_v - \bigoplus_{v\in S^{(p)}} i_v} \bigoplus_{v\in S^{(p)}} \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v, V),$$

where $S^{(p)} = S\backslash\{p\}$, and we write $\widetilde{H}^*_{(p),\mathrm{Iw}}(\mathbb{Q}, V)$ for its cohomology. Because the formation of Iwasawa complexes is exact, we have an exact triangle

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^+) \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, V) \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^-) \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^+)[1],$$

where we have used the comparision (26). This triangle identifies the term $\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^-)$ to $X[1]$, where X is the mapping fiber of $\mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^+) \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, V)$. On the other hand, comparing the definitions of the complexes $\mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V)$ and $\mathbf{R}\widetilde{\Gamma}_{(p),\mathrm{Iw}}(\mathbb{Q}, V)$ as mapping fibers, we obtain an exact triangle

$$X \to \mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V) \to \mathbf{R}\widetilde{\Gamma}_{(p),\mathrm{Iw}}(\mathbb{Q}, V) \to X[1],$$

from which we deduce the exact triangle

$$\begin{aligned}(27)\quad \mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V) &\to \mathbf{R}\widetilde{\Gamma}_{(p),\mathrm{Iw}}(\mathbb{Q}, V) \\ &\to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_p, D_F^-) \to \mathbf{R}\widetilde{\Gamma}_{F,\mathrm{Iw}}(\mathbb{Q}, V)[1].\end{aligned}$$

7.3. The Main Conjecture for *f* and its symmetric powers. We remind the reader of the fixed newform f of weight k, level $\Gamma_1(N)$ with $p \nmid N$ and character ϵ, with CM by K, and the roots $\alpha, \overline{\alpha}$ of $x^2 + \epsilon(p)p^{k-1}$.

Since the elements $\alpha, \overline{\alpha}$ are distinct, the φ-eigenspace with eigenvalue α determines an $E[\varphi]$-stable subspace $F_\alpha \subseteq \mathbb{D}_{\mathrm{cris}}(V_f)$. We apply the constructions of Iwasawa-theoretic extended Selmer groups, with their associated ranks and characteristic ideals, to the data of V_f equipped with F_α.

The following is the "finite-slope" form of the Main Conjecture of Iwasawa theory for f.

Theorem 7.1. *Assume that p does not divide the order of the nebentypus ϵ. The coadmissible $\mathcal{H}_E(G_\infty)$-module $\widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f)$ is torsion, and*

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)} \widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f) = (\mathrm{Tw}_{-1} L_{f,\alpha}).$$

Proof. We reproduce the argument of [P1, §5], adapted to the normalizations of this paper.

In the notation of §7.2.4, the object $D^-_{F_\alpha}$ is crystalline, and $\widetilde{\mathbb{D}}_{\mathrm{cris}}(D^-_{F_\alpha})$ has φ-eigenvalue α^{-1} and Hodge–Tate weight 0. This implies $H^2_{\mathrm{Iw}}(G_p, D^-_{F_\alpha}) = 0$. (If k is odd, $\epsilon(p) = -1$, and $\alpha = +p^{(k-1)/2}$ then $H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})_{\mathrm{tors}} \cong E(\chi^{(k-1)/2})$ is nonzero, but this "exceptional zero" does not affect the present proof.)

Write $f^c = f \otimes \epsilon^{-1}$ for the eigenform with Fourier coefficients complex conjugate to those of f, and recall the duality $\mathrm{Hom}_E(V_{f^c}, E) \cong V_f(1-k)$. Let $z'_{f^c} \in \widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, \mathrm{Hom}_E(V_{f^c}, E))$ denote Kato's zeta element derived from elliptic units (denoted $z^{(p)}_\gamma(f^*)$ for suitable $\gamma \in \mathrm{Hom}_E(V_{f^c}, E)$ in [Ka, §15]), and let

$$z_f = \mathrm{Tw}_{k-1}\, z'_{f^c} \in \widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, \mathrm{Hom}_E(V_{f^c}, E)(k-1)) \cong \widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f).$$

For a crystalline (φ, G_∞)-module D satisfying $\mathrm{Fil}^1 \mathbb{D}_{\mathrm{dR}}(D) = 0$, recall the dual of the big exponential map treated in [Nak, §3]:

$$\mathrm{Exp}^*_{D^*} \colon H^1_{\mathrm{Iw}}(G_p, D) \to \widetilde{\mathbb{D}}_{\mathrm{cris}}(D) \otimes_E \mathcal{H}_E(G_\infty).$$

One has morphisms

$$\widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f) \xrightarrow{\mathrm{loc}_{V_f}} H^1_{\mathrm{Iw}}(G_p, V_f) \cong H^1_{\mathrm{Iw}}(G_p, \mathbb{D}_{\mathrm{rig}}(V_f))$$

and, by naturality in D, a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(G_p, \mathbb{D}_{\mathrm{rig}}(V_f)) & \xrightarrow{\mathrm{Col}_\alpha} & H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha}) \\ \mathrm{Exp}^*_{V_f^*} \downarrow & & \downarrow \mathrm{Exp}^*_{D^{-,*}_{F_\alpha}} \\ \widetilde{\mathbb{D}}_{\mathrm{cris}}(V_f) \otimes_E \mathcal{H}_E(G_\infty) & \to & \widetilde{\mathbb{D}}_{\mathrm{cris}}(D^-_{F_\alpha}) \otimes_E \mathcal{H}_E(G_\infty). \end{array}$$

Write $\mathrm{loc}_\alpha = \mathrm{Col}_\alpha \circ \mathrm{loc}_{V_f}$, where the maps loc_{V_f} and Col_α are as in the preceding two displays. Identifying $\widetilde{\mathbb{D}}_{\mathrm{cris}}(D_{F_\alpha}^-) = \mathrm{Hom}_E(Ee_\alpha, E)$, [Ka, Theorem 16.6(2)] shows that

$$(28)\quad (\mathrm{Tw}_1 \, \mathrm{Exp}^*_{D_{F_\alpha}^{-,*}} \mathrm{loc}_\alpha \, z_f)(e_\alpha) = (\mathrm{Exp}^*_{\mathrm{Hom}_E(V_f,E)} \mathrm{loc}_{V_f(1)} \mathrm{Tw}_1 \, z_f)(e_\alpha) = L_{f,\alpha},$$

after perhaps rescaling e_α. In particular, loc_α is a nontorsion morphism.

The exact triangle (27) gives rise to an exact sequence

$$0 \to \widetilde{H}^1_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f) \to \widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f) \xrightarrow{\mathrm{loc}_\alpha} H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha}) \to \widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f) \to \widetilde{H}^2_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f) \to 0.$$

It follows from [Ka, Theorem 12.4] that the finitely generated $\mathcal{H}_E(G_\infty)$-module $\widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f)$ (resp. $\widetilde{H}^2_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f)$) is free of rank 1 (resp. is torsion). Employing the local Euler–Poincaré formula and the fact that loc_α is nontorsion, we see from the preceding exact sequence that $\widetilde{H}^1_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f) = 0$, $\widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f)$ is torsion, and

$$\left(\mathrm{char}_{\mathcal{H}_E(G_\infty)} \frac{\widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f)}{\mathcal{H}_E(G_\infty) z_f}\right) \left(\mathrm{char}_{\mathcal{H}_E(G_\infty)} \widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f)\right) = \left(\mathrm{char}_{\mathcal{H}_E(G_\infty)} \frac{H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})}{\mathcal{H}_E(G_\infty)\, \mathrm{loc}_\alpha \, z_f}\right) \left(\mathrm{char}_{\mathcal{H}_E(G_\infty)} \widetilde{H}^2_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f)\right).$$

Applying Tw_{k-1} to the claim of [Ka, Theorem 12.5(3)] with f^* in place of f, we deduce that

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)} \frac{\widetilde{H}^1_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f)}{\mathcal{H}_E(G_\infty) z_f} = \mathrm{char}_{\mathcal{H}_E(G_\infty)} \widetilde{H}^2_{(p),\mathrm{Iw}}(\mathbb{Q}, V_f),$$

and therefore

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)} \widetilde{H}^2_{F_\alpha,\mathrm{Iw}}(\mathbb{Q}, V_f) = \mathrm{char}_{\mathcal{H}_E(G_\infty)} \frac{H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})}{\mathcal{H}_E(G_\infty)\, \mathrm{loc}_\alpha \, z_f}.$$

Although only a divisibility of characteristic ideals is claimed by Kato, one easily checks that his proof, especially [Ka, Proposition 15.17], gives an equality whenever Rubin's method gives an equality. Under the hypothesis that ϵ has order prime to p, the required extension of Rubin's work is precisely Theorem 8.1 below. It remains to compute the right hand side of the last identity. In fact, one has the exact sequence

$$0 \to H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})_{\mathrm{tors}} \to \frac{H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})}{\mathcal{H}_E(G_\infty)\,\mathrm{loc}_\alpha\, z_f} \xrightarrow{\mathrm{Exp}^*_{D^{-,*}_{F_\alpha}}} \frac{\widetilde{\mathbb{D}}_{\mathrm{cris}}(D^-_{F_\alpha}) \otimes_E \mathcal{H}_E(G_\infty)}{\mathcal{H}_E(G_\infty)\,\mathrm{Exp}^*_{D^{-,*}_{F_\alpha}}\,\mathrm{loc}_\alpha\, z_f} \to \mathrm{coker}\,\mathrm{Exp}^*_{D^{-,*}_{F_\alpha}} \to 0,$$

and because $D^-_{F_\alpha}$ has Hodge–Tate weight zero and $H^2_{\mathrm{Iw}}(G_p, D^-_{F_\alpha}) = 0$, [Nak, Theorem 3.21] shows that

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)}\, H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})_{\mathrm{tors}} = \mathrm{char}_{\mathcal{H}_E(G_\infty)}\,\mathrm{coker}\,\mathrm{Exp}^*_{D^{-,*}_{F_\alpha}},$$

and hence

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)}\, \frac{H^1_{\mathrm{Iw}}(G_p, D^-_{F_\alpha})}{\mathcal{H}_E(G_\infty)\,\mathrm{loc}_\alpha\, z_f} = \mathrm{char}_{\mathcal{H}_E(G_\infty)}\, \frac{\widetilde{\mathbb{D}}_{\mathrm{cris}}(D^-_{F_\alpha}) \otimes_E \mathcal{H}_E(G_\infty)}{\mathcal{H}_E(G_\infty)\,\mathrm{Exp}^*_{D^{-,*}_{F_\alpha}}\,\mathrm{loc}_\alpha\, z_f}.$$

Finally, by (28) the right hand side above is generated by $\mathrm{Tw}_{-1}\, L_{f,\alpha}$. ☐

We now turn to the Main Conjecture of Iwasawa theory for V_m in its "finite-slope" form, beginning with the finite-slope analogue of Definition 5.3. Fix $\mathfrak{t} = (\mathfrak{t}_0, \dots, \mathfrak{t}_{\widetilde{r}-1}) \in \mathfrak{T}$. For each $i = 0, \dots, \widetilde{r}-1$, the elements $\alpha_i, \overline{\alpha}_i$ are distinct, so the φ-eigenspace with eigenvalue $\mathfrak{t}_i p^{(r-i)(k-1)}$ determines an $E[\varphi]$-stable subspace $F_i \subseteq \mathbb{D}_{\mathrm{cris}}(V_{f_i}((i-r)(k-1)))$. We may apply the constructions of Iwasawa-theoretic extended Selmer groups, with their associated ranks and characteristic ideals, to the data of $V_{f_i}((i-r)(k-1))$ equipped with F_i.

Definition 7.2. For $\mathfrak{t} \in \mathfrak{T}$, we define the coadmissible $\mathcal{H}_E(G_\infty)$-module

$$\mathrm{Sel}^{\mathfrak{t}}_{k_\infty}(V_m^*)^\vee := \left(\bigoplus_{i=0}^{\widetilde{r}-1} \widetilde{H}^2_{F_i,\mathrm{Iw}}\,(\mathbb{Q}, V_{f_i}((i-r)(k-1)))^\iota\right) \oplus \begin{cases} \mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee[1/p] & m \text{ even},\\ 0 & m \text{ odd}.\end{cases}$$

Remarks 7.3.

(1) We remind the reader that since $\mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee$ is a finitely generated, torsion $\Lambda_{\mathcal{O}_E}(G_\infty)$-module, it follows from §7.2.1 that

$$\mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee[1/p] = \mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee \otimes_{\Lambda_{\mathcal{O}_E}(G_\infty)} \Lambda_E(G_\infty) \xrightarrow{\sim} \mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee \otimes_{\Lambda_{\mathcal{O}_E}(G_\infty)} \mathcal{H}_E(G_\infty),$$

and therefore $\mathrm{Sel}_{k_\infty}(A^*_{\epsilon^r_K})^\vee[1/p]$ is naturally a finitely generated (hence coadmissible), torsion $\mathcal{H}_E(G_\infty)$-module.

(2) For $\mathfrak{s} \in \mathfrak{S}$ and $i = 0, \dots, \widetilde{r} - 1$ one can build the "plus and minus" Iwasawa-theoretic Selmer complex $\mathbf{R}\widetilde{\Gamma}_{\mathfrak{s}_i,\mathrm{Iw}}(\mathbb{Q}, V_{f_i}((i-r)(k-1)))$ with local condition at p appropriately built from the choice $\mathfrak{s}_i$, with degree 2 cohomology $\widetilde{H}^2_{\mathfrak{s}_i,\mathrm{Iw}}(\mathbb{Q}, V_{f_i}((i-r)(k-1)))$, a finitely generated $\Lambda_E(G_\infty)$-module. Then arithmetic duality for Selmer complexes gives

$$H_f^{1,\mathfrak{s}_i}(k_\infty, A^*_{f_i}((r-i)(k-1)))^\vee[1/p] \cong \widetilde{H}^2_{\mathfrak{s}_i,\mathrm{Iw}}(\mathbb{Q}, V_{f_i}((i-r)(k-1)))^\iota.$$

Since Theorem 5.6 shows the right hand side to be torsion, by §7.2.1 it is naturally a coadmissible $\mathcal{H}_E(G_\infty)$-module. We refer the reader to [P1, end of §4] for details.

(3) Although the notation $\mathrm{Sel}^{\mathfrak{t}}_{k_\infty}(V^*_m)^\vee$ in the finite-slope case was chosen for symmetry with $\mathrm{Sel}^{\mathfrak{s}}_{k_\infty}(A^*_m)^\vee[1/p]$ in the "plus and minus" case, this notation is highly misleading: it is an essential feature of the finite-slope theory that $\mathrm{Sel}^{\mathfrak{t}}_{k_\infty}(V^*_m)^\vee$ is coadmissible but typically *not* finitely generated over $\mathcal{H}_E(G_\infty)$, and therefore does not arise as the Pontryagin dual (with p inverted) of direct limits of finite-layer objects, as $\mathrm{Sel}^{\mathfrak{s}}_{k_\infty}(A^*_m)^\vee[1/p]$ does in (2) above. This fact forces us to work on the other side of arithmetic duality, as in the first summand of Definition 7.2.

Theorem 7.4. *If* $\mathfrak{t} \in \mathfrak{T}$*, the coadmissible* $\mathcal{H}_E(G_\infty)$*-module* $\mathrm{Sel}^{\mathfrak{t}}_{k_\infty}(V^*_m)^\vee$ *is torsion, and*

$$\mathrm{char}_{\mathcal{H}_E(G_\infty)} \mathrm{Sel}^{\mathfrak{t}}_{k_\infty}(V^*_m)^\vee = (\mathrm{Tw}_1\, \widetilde{L}_{V_m,\mathfrak{t}}).$$

Proof. Just as in the proof of Theorem 5.9, this theorem follows from Theorem 5.5 and from Theorem 7.1 applied to each f_i. ☐

8. The Main Conjecture for imaginary quadratic fields at inert primes

8.1. Recall and statement. In the fundamental works [Ru1, Ru2], Rubin perfected the Euler system method for elliptic units. From this he deduced a divisibility of characteristic ideals as in the Main Conjecture of Iwasawa theory. In most cases, he used the analytic class number formula to promote the divisibilities to identities. In this section we extend the use of the analytic class number formula to the remaining cases. The obstruction in these problematic cases is that the control maps on global/elliptic units and class groups are far from being isomorphisms. Our approach is to use base change of Selmer complexes to get a precise description of the failure of control, and then to apply a numerical characterization of μ- and λ-invariants that is valid even in the presence of zeroes of the characteristic ideal at finite-order points. This section is written independently of the preceding notations and hypotheses of this paper and [HL]; we employ notations as in [Ru1], recalled as follows.

We take K to be an imaginary quadratic field, and p an odd prime inert in K. Let K_0 be a finite abelian extension with $\Delta = \mathrm{Gal}(K_0/K)$ and $\delta = [K_0 : K]$, and assume that $p \nmid \delta$. Let K_∞ be an abelian extension of K containing K_0, such that $\Gamma = \mathrm{Gal}(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$. One has $\mathscr{G} = \mathrm{Gal}(K_\infty/K) = \Delta \times \Gamma$. Accordingly, $K_\infty = K_0 \cdot K_\infty^\Delta$, where $\mathrm{Gal}(K_\infty^\Delta/K)$ is identified with Γ.

We write $\Lambda = \Lambda(\mathscr{G}) = \mathbb{Z}_p[\![\mathscr{G}]\!]$. The letter η will always range over the irreducible representations of Δ over $\mathbb{Z}_p$. One has $\mathbb{Z}_p[\Delta] = \bigoplus_\eta \mathbb{Z}_p[\Delta]^\eta$, where $\mathbb{Z}_p[\Delta]^\eta$ is isomorphic to the ring of integers in the unramified extension of $\mathbb{Q}_p$ of degree $\dim(\eta)$, and, accordingly, $\Lambda = \bigoplus_\eta \mathbb{Z}_p[\Delta]^\eta[\![\Gamma]\!]$. The sum map $\mathrm{sum}\colon \mathbb{Z}_p[\Delta] \to \mathbb{Z}_p$, $\sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma$, is identified with the projection onto the component $\mathbb{Z}_p[\Delta]^{\mathbf{1}}$ indexed by the trivial character $\mathbf{1}$; write $\mathbb{Z}_p[\Delta]^!$ for the kernel of the sum map, which is equal to $\bigoplus_{\eta \neq \mathbf{1}} \mathbb{Z}_p[\Delta]^\eta$, and satisfies $\mathbb{Z}_p[\Delta] = \mathbb{Z}_p[\Delta]^{\mathbf{1}} \oplus \mathbb{Z}_p[\Delta]^!$.

Let F be a subextension of K_∞/K_0. If F/K_0 is finite, we associate to it the following objects:

- $A(F) = \mathrm{Pic}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the p-part of its ideal class group,
- $X(F) = \mathrm{Pic}(\mathcal{O}_F, p^\infty) = \varprojlim_n (\mathrm{Pic}(\mathcal{O}_F, p^n) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is the inverse limit of the p-parts of its ray class groups of conductor p^n,
- $U(F) = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times_{\text{pro-}p}$ is the pro-p part of its group of semilocal units,
- $\mathscr{E}(F) = \mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is its group of global units $\otimes \mathbb{Z}_p$, and
- $\mathscr{C}(F)$ is its group of elliptic units $\otimes \mathbb{Z}_p$, as defined in [Ru1, §1].

If F/K_0 is infinite, and $? \in \{A, X, U, \mathscr{E}, \mathscr{C}\}$, we let $?(F) = \varprojlim_{F_0} ?(F_0)$, where F_0 ranges over the finite subextensions of F, obtaining a finitely generated $\mathbb{Z}_p[\![\mathrm{Gal}(F/K)]\!]$-module. Note that Leopoldt's conjecture is known in this case, so by the definition of ray class groups one has a short exact sequence

$$0 \to \mathscr{E}(F) \to U(F) \to X(F) \to A(F) \to 0.$$

Class field theory identifies $A(F)$ (resp. $X(F)$) with the Galois group of the maximal p-abelian extension of F which is everywhere unramified (resp. unramified at primes not dividing p).

The following improvement of Rubin's work is the main result of this section, and the remainder of this section consists of its proof.

Theorem 8.1. *One has the equality of characteristic ideals,*

$$\mathrm{char}_\Lambda A(K_\infty) = \mathrm{char}_\Lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)).$$

In [Ru1, Theorem 4.1(ii)] and [Ru2, Theorem 2(ii)] it is proved that both the ideals $\mathrm{char}_\Lambda A(K_\infty)$ and $\mathrm{char}_\Lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))$ are nonzero at each η-factor, that the first of these ideals divides the second, and that their η-factors are equal when η is nontrivial on the decomposition group of p in Δ. To get equality for the remaining η, we may thus reduce to the case where p is totally split in K_0/K.

In §8.2 we prove the claim (32) below, which is specific to $\mathbb{Z}_p$-extensions, and then in §8.3 we deduce the theorem for both $\mathbb{Z}_p$- and $\mathbb{Z}_p^2$-extensions from this claim. Therefore, it is convenient that from now on we specialize our notation to where K_∞^Δ is any $\mathbb{Z}_p$-extension (and return to considering $\mathbb{Z}_p^2$-extensions later). We index finite subextensions F of K_∞/K_0 as $F = K_n = K_\infty^{\Gamma^{p^n}}$ for $n \geq 0$. Fix a topological generator $\gamma \in \Gamma$, and for brevity write $\Lambda_n = \mathbb{Z}_p[\mathscr{G}/\Gamma^{p^n}] = \Lambda/(\gamma^{p^n} - 1)$.

Since $\operatorname{char}_\Lambda A(K_\infty)$ divides $\operatorname{char}_\Lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))$, the Iwasawa μ- and λ-invariants of $A(K_\infty)$ and $\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)$, considered as a $\mathbb{Z}_p[\![\Gamma]\!]$-modules, satisfy

$$\begin{aligned} \mu(A(K_\infty)) &\leq \mu(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)), \\ \lambda(A(K_\infty)) &\leq \lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)). \end{aligned} \tag{29}$$

We shall improve these inequalities to the claim that for some $\epsilon \in \{0, 1\}$ one has

$$\begin{aligned} \mu(A(K_\infty)) &= \mu(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)), \\ \epsilon + \lambda(A(K_\infty)) &= \lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)), \end{aligned} \tag{30}$$

and additionally

$$\operatorname{rank}_{\mathbb{Z}_p} A(K_\infty)_{\mathscr{G}} = 0, \qquad \operatorname{rank}_{\mathbb{Z}_p} (\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))_{\mathscr{G}} = \epsilon. \tag{31}$$

These computations are equivalent to the claim that

$$(\operatorname{char}_\Lambda \mathbb{Z}_p)^\epsilon \cdot \operatorname{char}_\Lambda A(K_\infty) = \operatorname{char}_\Lambda \mathscr{E}(K_\infty)/\mathscr{C}(K_\infty). \tag{32}$$

8.2. Proof of (32)**.** Recall that we have assumed p is totally split in K_0/K, and that Γ is isomorphic to $\mathbb{Z}_p$.

8.2.1. *Characterization of Iwasawa invariants.* We will need the following numerical characterization of Iwasawa μ- and λ-invariants. For $\{a_n\}$ a sequence of positive real numbers, if there exist real numbers μ, λ such that $\log_p a_n = \mu p^n + \lambda n + O(1)$ as $n \to +\infty$, then these numbers μ, λ are uniquely determined by $\{a_n\}$, and we write $\mu = \mu(\{a_n\})$ and $\lambda = \lambda(\{a_n\})$.

Lemma 8.2. *Let M be a finitely generated, torsion $\mathbb{Z}_p[\![\Gamma]\!]$-module. Then for $n \gg 0$ the quantity $\operatorname{rank}_{\mathbb{Z}_p} M_{\Gamma^{p^n}}$ stabilizes to some integer $r \geq 0$, so that $M_{\Gamma^{p^n}} \approx \mathbb{Z}_p^{\oplus r} \oplus M_{\Gamma^{p^n}}[p^\infty]$, and Iwasawa's μ- and λ-invariants of M satisfy $\mu(M) = \mu(\{\#M_{\Gamma^{p^n}}[p^\infty]\})$ and $\lambda(M) = r + \lambda(\{\#M_{\Gamma^{p^n}}[p^\infty]\})$.*

Proof. One easily sees that if $M \to M'$ is a pseudo-isomorphism, then both sides of the desired identities are invariant under replacing M by M'. Using the structure theorem and additivity over direct sums, it therefore suffices to check the case where $M = \mathbb{Z}_p[\![\Gamma]\!]/(f)$ for prime $f \in \mathbb{Z}_p[\![\Gamma]\!]$. The case where f is relatively prime to all the augmentation ideals $I(\Gamma^{p^k}) = (f_k)$ of Γ^{p^k} for $k \geq 0$, or equivalently where $r = 0$, is well-known. The remaining case is where $f = f_k/f_{k-1}$ for $k \geq 0$ (we set $f_{-1} = 1$), whence one has

$$(\mathbb{Z}_p[\![\Gamma]\!]/(f))_{\Gamma^{p^n}} = \mathbb{Z}_p[\![\Gamma]\!]/(f, f_n) = \mathbb{Z}_p[\![\Gamma]\!]/(f) \approx \mathbb{Z}_p^{\oplus(p-1)p^{k-1}}$$

for $n \geq k$, agreeing with the Iwasawa invariants. □

8.2.2. ***The operation*** $\overset{\mathbf{L}}{\otimes}_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda_n}$**.** Since we use base change in the derived category, we give some generalities on the operation $\overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$. We first compute that $\Lambda_n[0] \cong [\Lambda \xrightarrow{\gamma^{p^n}-1} \Lambda]$ as objects in the derived category of Λ-modules, the latter concentrated in degrees $-1, 0$, so that for any Λ-module (resp. complex of Λ-modules) X one may compute $X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$ as $[X \xrightarrow{\gamma^{p^n}-1} X]$ (resp. as the mapping cone of $\gamma^{p^n} - 1$ on X). The induced map $X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_{n+1} \to X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$ corresponds to the morphism $[X \xrightarrow{\gamma^{p^{n+1}}-1} X] \to [X \xrightarrow{\gamma^{p^n}-1} X]$ given by multiplication by $1 + \gamma^{p^n} + \cdots + \gamma^{(p-1)p^n}$ in shift degree -1, and by the identity in shift degree 0. Alternatively, the Tor spectral sequence degenerates to short exact sequences

$$0 \to H^i(X)_{\Gamma^{p^n}} \to H^i(X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n) \to H^{i+1}(X)^{\Gamma^{p^n}} \to 0, \tag{33}$$

and the natural morphism from the above sequence for $n+1$ to the sequence for n is given by the natural projection on the first term, and by multiplication by $1 + \gamma^{p^n} + \cdots + \gamma^{(p-1)p^n}$ on the last term. The Bockstein homomorphism $\beta = \beta_X$, defined as the connecting homomorphism in the exact triangle

$$\begin{aligned} X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} &\left(\Lambda_n \xrightarrow{\gamma^{p^n}-1} \Lambda/(\gamma^{p^n}-1)^2 \to \Lambda_n \to \Lambda_n[1]\right) \\ &\cong \left(X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n \xrightarrow{\gamma^{p^n}-1} X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda/(\gamma^{p^n}-1)^2 \to X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n \xrightarrow{\beta} X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n[1]\right), \end{aligned}$$

is computed on cohomology as the composite

$$\begin{aligned} H^i(\beta) \colon H^i(X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n) &\twoheadrightarrow H^i(X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n)/H^i(X)_{\Gamma^{p^n}} \\ &\cong H^{i+1}(X)^{\Gamma^{p^n}} \hookrightarrow H^{i+1}(X) \twoheadrightarrow H^{i+1}(X)_{\Gamma^{p^n}} \hookrightarrow H^{i+1}(X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n). \end{aligned}$$

Note that if Z is a finitely generated, torsion Λ-module, then $\operatorname{rank}_{\mathbb{Z}_p} Z^{\Gamma^{p^n}} = \operatorname{rank}_{\mathbb{Z}_p} Z_{\Gamma^{p^n}}$.

If X satisfies $X = X^\Gamma$, then the above computations reduce to $X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n \cong X[1] \oplus X$, in such a way that the natural map $X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_{n+1} \to X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$ is identified with multiplication by p in shift degree -1, and with the identity map in shift degree 0. The Bockstein homomorphism

$$\beta\colon X[1] \oplus X = X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n \to X \underset{\Lambda}{\overset{\mathbf{L}}{\otimes}} \Lambda_n[1] = X[2] \oplus X[1]$$

is the identity map on $X[1]$ and zero on the other factors. In this scenario, we write $\beta^{-1} = \beta_X^{-1}\colon X[2] \oplus X[1] \to X[1] \oplus X$ for the map that is inverse to this identity map on $X[1]$ and zero on the other factors. Any morphism $f\colon Y \to X$ gives rise to a morphism $f \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n\colon Y \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n \to X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n = X[1] \oplus X$. Writing $f \otimes_\Lambda \Lambda_n$ for the projection of $f \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$ onto the second component, X, the commutative diagram

$$\begin{array}{ccccc} Y \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n & \xrightarrow{f \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n} & X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n & = & X[1] \oplus X \\ \beta_Y \downarrow & & \beta_X \downarrow & & \searrow\sim \\ Y \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n[1] & \xrightarrow{f \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n[1]} & X \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n[1] & = & X[2] \oplus X[1] \end{array}$$

shows that the projection of $f \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$ onto the first component, $X[1]$, is computed by $\beta_X^{-1} \circ (f \otimes_\Lambda \Lambda_n)[1] \circ \beta_Y$.

8.2.3. ***Invariants of number fields.*** We now return to the setting of the theorem, recalling Nekovář's constructions of the fundamental invariants of number fields in terms of Selmer complexes. Throughout, $n \geq 0$ ranges over nonnegative integers.

There is unique $\mathbb{Z}_p^2$-extension of K, and it contains all $\mathbb{Z}_p$-extensions of K. This extension is unramified at all primes not dividing p, and Lubin–Tate theory shows it is totally ramified at p. The same ramification behavior is therefore true of any $\mathbb{Z}_p$-extension, as well as of K_∞/K_0 because p is totally split in K_0/K. In particular, if S_n denotes the set of places of K_n lying over p, then the restriction maps $S_{n+1} \to S_n$ are bijections, and S_n is a principal homogeneous Δ-set. Fixing once and for all $v_0 \in S_0$, with unique lift $v_n \in S_n$, declaring v_n to be a basepoint of S_n gives an identification $\mathbb{Z}_p[S_n] \cong \mathbb{Z}_p[\Delta]$ of $\mathbb{Z}_p[\Delta]$-modules. We write inv for the composite of the semilocal restriction map, the invariant maps of local class field theory, and this identification:

$$\mathrm{inv}\colon H^2(G_{K,\{p\}}, \mathbb{Z}_p(1)) \to \bigoplus_{v \in S_n} H^2(G_{K_{n,v}}, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p[S_n] \cong \mathbb{Z}_p[\Delta].$$

For brevity we write

$$\mathbf{R}\Gamma_n = \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K_n,\{p\}}, \mathbb{Z}_p(1)),$$
$$\mathbf{R}\Gamma_{\mathrm{Iw}} = \mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty/K_0, \mathbb{Z}_p(1)) = \mathbf{R}\varprojlim_n \mathbf{R}\Gamma_n,$$

and $H^i_? = H^i(\mathbf{R}\Gamma_?)$ for $? \in \{n, \mathrm{Iw}\}$. Then one has by [Nek, (9.2.1.2)] the computations

$$H^i_n = 0,\ i \neq 1,2, \qquad H^1_n = \mathcal{O}^\times_{K_n,\{p\}} \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$
$$0 \to \mathrm{Pic}(\mathcal{O}_{K_n,\{p\}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_n \xrightarrow{\mathrm{inv}} \mathbb{Z}_p[\Delta] \xrightarrow{\mathrm{sum}} \mathbb{Z}_p \to 0,$$

and, passing to inverse limits (Mittag-Leffler holds by compactness),

$$H^i_{\mathrm{Iw}} = 0,\ i \neq 1,2, \qquad H^1_{\mathrm{Iw}} = \varprojlim_n (\mathcal{O}^\times_{K_n,\{p\}} \otimes_{\mathbb{Z}} \mathbb{Z}_p),$$
$$0 \to \varprojlim_n (\mathrm{Pic}(\mathcal{O}_{K_n,\{p\}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p) \to H^2_{\mathrm{Iw}} \xrightarrow{\mathrm{inv}} \mathbb{Z}_p[\Delta] \xrightarrow{\mathrm{sum}} \mathbb{Z}_p \to 0.$$

Let $U^- = \mathbb{Z}_p[\Delta][-1] \oplus \mathbb{Z}_p[\Delta][-2]$, considered as a perfect complex of Λ-modules, or as a complex of Λ_n-modules. One constructs a map $i^-_n \colon \mathbf{R}\Gamma_n \to U^-$ via the local valuation maps in degree one and the local invariant maps in degree two, and obtains a map $i^-_{\mathrm{Iw}} \colon \mathbf{R}\Gamma_{\mathrm{Iw}} \to U^-$ from the i^-_n by taking the inverse limit on n. By taking mapping fibers of i^-_n and i^-_{Iw}, one obtains complexes $\mathbf{R}\widetilde{\Gamma}_{f,n}$ of Λ_n-modules and a perfect complex $\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}}$ of Λ-modules sitting in exact triangles

$$\mathbf{R}\widetilde{\Gamma}_{f,n} \to \mathbf{R}\Gamma_n \xrightarrow{i^-_n} U^- \to \mathbf{R}\widetilde{\Gamma}_{f,n}[1]$$

and

$$\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \to \mathbf{R}\Gamma_{\mathrm{Iw}} \xrightarrow{i^-_{\mathrm{Iw}}} U^- \to \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}}[1].$$

Writing $\widetilde{H}^i_{f,?} = H^i(\mathbf{R}\widetilde{\Gamma}_{f,?})$ for $? \in \{n, \mathrm{Iw}\}$, one has by [Nek, (9.5.2.2)] the computations

$$\widetilde{H}^i_{f,n} = \begin{cases} 0 & i \neq 1,2,3 \\ \mathscr{E}(K_n) & i = 1 \\ A(K_n) & i = 2 \\ \mathbb{Z}_p & i = 3, \end{cases} \qquad \text{and} \qquad \widetilde{H}^i_{f,\mathrm{Iw}} = \begin{cases} 0 & i \neq 1,2,3 \\ \mathscr{E}(K_\infty) & i = 1 \\ A(K_\infty) & i = 2 \\ \mathbb{Z}_p & i = 3. \end{cases}$$

We note especially the connecting map $U^- \to \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}}[1]$ in degree two is equal to the sum map $\mathbb{Z}_p[\Delta] \to \mathbb{Z}_p$, by global class field theory's computation of the Brauer group.

8.2.4. ***Control of invariants.*** The following result summarizes how the invariants of number fields behave under base change in our situation.

Proposition 8.3.

(1) For each n one has two diagrams, called crosses, *each of whose row and column is a short exact sequence, given by*

$$\begin{array}{ccccc} & & \mathscr{E}(K_\infty)_{\Gamma^{p^n}} & & \\ & & \downarrow & & \\ \mathbb{Z}_p[\Delta] & \to & H^1(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) & \to & \mathscr{E}(K_n), \\ & & \downarrow & & \\ & & A(K_\infty)^{\Gamma^{p^n}} & & \end{array}$$

$$\begin{array}{ccccc} & & A(K_\infty)_{\Gamma^{p^n}} & & \\ & & \downarrow & & \\ \mathbb{Z}_p[\Delta] & \to & H^2(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) & \to & A(K_n). \\ & & \downarrow & & \\ & & \mathbb{Z}_p & & \end{array}$$

One also has the computation $H^3(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) \cong \mathbb{Z}_p$.

(2) The composite arrow from the top point to the right point of the first (resp. second) cross gives the control map $\mathscr{E}(K_\infty)_{\Gamma^{p^n}} \to \mathscr{E}(K_n)$ (resp. $A(K_\infty)_{\Gamma^{p^n}} \to A(K_n)$).

(3) There is a morphism of diagrams from the crosses associated to $n+1$ to the crosses associated to n. At the upper (resp. lower, right, left) point of a cross, the morphism is explicitly computed by the natural projection (resp. multiplication by $1 + \gamma^{p^n} + \cdots + \gamma^{(p-1)p^n}$, the norm maps, multiplication by p).

Proof. By control for Galois cohomology, the natural map $\mathbf{R}\Gamma_{\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n \to \mathbf{R}\Gamma_n$ is an isomorphism, compatible with varying n. Since $U^- = (U^-)^\Gamma$, one has the computation $U^- \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n \cong U^-[1] \oplus U^-$. It follows from the definition of i^-_{Iw} as an inverse limit that $i^-_{\mathrm{Iw}} \otimes_\Lambda \Lambda_n = i^-_n$, so that $i^-_{\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n = (\beta^{-1}_{U^-} \circ i^-_n[1] \circ \beta_{\mathbf{R}\Gamma_n}, i^-_n)$. Thus we have a commutative diagram

$$\begin{array}{ccccccc} \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n & \to & \mathbf{R}\Gamma_n & \xrightarrow{i^-_{\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n} & U^-[1] \oplus U^- & \to & \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n[1] \\ & & =\downarrow & & \mathrm{pr}_2 \downarrow & & \\ \mathbf{R}\widetilde{\Gamma}_{f,n} & \to & \mathbf{R}\Gamma_n & \xrightarrow{i^-_n} & U^- & \to & \mathbf{R}\widetilde{\Gamma}_{f,n}[1], \end{array}$$

which we complete to a morphism of exact triangles via a morphism

$$\mathrm{BC}_n \colon \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda_n \to \mathbf{R}\widetilde{\Gamma}_{f,n}.$$

Taking mapping fibers of the resulting morphism of triangles gives an exact triangle

$$\mathrm{Fib}(\mathrm{BC}_n) \to 0 \to U^-[1] \to \mathrm{Fib}(\mathrm{BC}_n)[1],$$

hence an isomorphism $\mathrm{Fib}(\mathrm{BC}_n) \cong U^-$ and an exact triangle

$$U^- \xrightarrow{j_n} \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda_n \xrightarrow{\mathrm{BC}_n} \mathbf{R}\widetilde{\Gamma}_{f,n} \xrightarrow{k_n} U^-[1]. \tag{34}$$

It is easy to compute that j_n is the composite of the inclusion $U^- \hookrightarrow U^- \oplus U^-[-1]$ and the shifted connecting homomorphism $U^- \oplus U^-[-1] \to \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n$. The construction of the snake lemma shows that k_n is the composite

$$\mathbf{R}\widetilde{\Gamma}_{f,n} \to \mathbf{R}\Gamma_n \xrightarrow{i^-_{\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n} U^-[1] \oplus U^- \xrightarrow{\mathrm{pr}_1} U^-[1],$$

or in other words the composite of $\mathbf{R}\widetilde{\Gamma}_{f,n} \to \mathbf{R}\Gamma_n$ with $\beta^{-1}_{U^-} \circ i^-_n[1] \circ \beta_{\mathbf{R}\Gamma_n}$. Of course, the source or target of $H^i(k_n)\colon \widetilde{H}^i_{f,n} \to H^{i+1}U^-$ is zero if $i \neq 1$, and if $i = 1$ this computation simplifies to

$$\mathscr{E}(K_n) \to H^1_n \xrightarrow{\beta} H^2_n \xrightarrow{\mathrm{inv}} \mathbb{Z}_p[\Delta].$$

The kernel of β contains the universal norms in $\mathscr{E}(K_n)$ for K_∞/K_n, and in particular $\mathscr{C}(K_n)$ (see [dS, Proposition II.2.5] for the norm relations), which itself is of finite index in $\mathscr{E}(K_n)$. Since $\mathbb{Z}_p[\Delta]$ is torsion free, it follows that $H^1(k_n) = 0$, too. Since $H^*(k_n) = 0$, the long exact sequence associated to the triangle (34) breaks up into two short exact sequences, giving the rows of the crosses. Then (33) gives the short exact columns of the crosses. The triangle (34) also gives the computation $H^3(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) \cong \mathbb{Z}_p$. Thus we have shown (1).

The claim (2) is obvious. So is the claim (3), except perhaps concerning the left points; this is because the term $U^-[1]$ in the sequence (34) is identified with the first summand of $U^- \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n \cong U^-[1] \oplus U^-$. □

8.2.5. *Iwasawa invariants of class groups.* The following result relates the Iwasawa invariants of $A(K_\infty)$ to the class numbers of the K_n.

Proposition 8.4. *One has* $\mu(A(K_\infty)) = \mu(\{\#A(K_n)\})$ *and* $\lambda(A(K_\infty)) = \lambda(\#\{A(K_n)\})$.

Proof. We consider the second cross in Proposition 8.3:

$$\begin{array}{c} A(K_\infty)_{\Gamma^{p^n}} \\ \downarrow \\ \mathbb{Z}_p[\Delta] \to H^2(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) \to A(K_n). \\ \downarrow \\ \mathbb{Z}_p \end{array}$$

The computation $\mathrm{rank}_{\mathbb{Z}_p} A(K_\infty)_{\Gamma^{p^n}} = \delta - 1$ is immediate. A diagram chase shows that the composite map $\mathbb{Z}_p[\Delta] \to \mathbb{Z}_p$ of this cross is just the degree three part of the base changed and shifted connecting map $(U^- \to \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}}[1]) \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n[-1] = (U^- \oplus U^-[-1] \to \mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n)$ from the proof of Proposition 8.3; after accounting for the shift, we have already noted the latter to be the sum map by global class field theory (the base change operation here passes to Γ^{p^n}-invariants, which accomplishes nothing). In particular, the composite $\mathbb{Z}_p[\Delta] \to \mathbb{Z}_p$ is surjective. Since there is only one Λ-quotient of $\mathbb{Z}_p[\Delta]$ isomorphic to $\mathbb{Z}_p$, and this quotient is a Λ-direct summand, we may canonically refine the diagram to the short exact sequence

$$0 \to \mathbb{Z}_p[\Delta]^! \to A(K_\infty)_{\Gamma^{p^n}} \to A(K_n) \to 0. \tag{35}$$

In particular there is an injection $A(K_\infty)_{\Gamma^{p^n}}[p^\infty] \hookrightarrow A(K_n)$ of finite abelian groups, and we may form the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z}_p[\Delta]^! & \to & \dfrac{A(K_\infty)_{\Gamma^{p^{n+1}}}}{A(K_\infty)_{\Gamma^{p^{n+1}}}[p^\infty]} & \to & \dfrac{A(K_{n+1})}{A(K_\infty)_{\Gamma^{p^{n+1}}}[p^\infty]} & \to 0 \\ & p \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathbb{Z}_p[\Delta]^! & \to & \dfrac{A(K_\infty)_{\Gamma^{p^n}}}{A(K_\infty)_{\Gamma^{p^n}}[p^\infty]} & \to & \dfrac{A(K_n)}{A(K_\infty)_{\Gamma^{p^n}}[p^\infty]} & \to 0, \end{array}$$

where the first downward arrow is computed by Proposition 8.3(3). Note that $A(K_\infty)_{\Gamma^{p^{n+1}}} \to A(K_\infty)_{\Gamma^{p^n}}$ is a surjection of finitely generated $\mathbb{Z}_p$-modules of the same rank, so that the second downward arrow is an isomorphism. Therefore, applying the snake lemma and examining the final column, we get the exact sequence

$$0 \to \mathbb{Z}_p[\Delta]^!/p \to \frac{A(K_{n+1})}{A(K_\infty)_{\Gamma^{p^{n+1}}}[p^\infty]} \to \frac{A(K_n)}{A(K_\infty)_{\Gamma^{p^n}}[p^\infty]} \to 0.$$

This implies that

$$\frac{\#A(K_n)}{\#A(K_\infty)_{\Gamma^{p^n}}[p^\infty]} = p^{(\delta-1)n} \frac{\#A(K_0)}{\#A(K_\infty)_{\Gamma^{p^0}}[p^\infty]},$$

so that

$$\mu(A(K_\infty)) = \mu(\{\#A(K_\infty)_{\Gamma^{p^n}}[p^\infty]\}) = \mu(\{\#A(K_n)\}),$$
$$\lambda(A(K_\infty)) = \delta - 1 + \lambda(\{\#A(K_\infty)_{\Gamma^{p^n}}[p^\infty]\}) = \lambda(\#\{A(K_n)\}),$$

as was desired. □

8.2.6. *Iwasawa invariants of global/elliptic units.* The following result relates the Iwasawa invariants of $\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)$ to the indices of elliptic units inside global units of the K_n.

Proposition 8.5. *For some* $\epsilon \in \{0,1\}$ *one has* $\mu(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)) = \mu(\{\#\mathscr{E}(K_n)/\mathscr{C}(K_n)\})$ *and* $\lambda(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty)) = \epsilon + \lambda(\{\#\mathscr{E}(K_n)/\mathscr{C}(K_n)\})$.

Proof. We consider the first cross in Proposition 8.3:

$$\begin{array}{c} \mathscr{E}(K_\infty)_{\Gamma^{p^n}} \\ \downarrow \\ \mathbb{Z}_p[\Delta] \to H^1(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\otimes}_\Lambda \Lambda_n) \to \mathscr{E}(K_n). \\ \downarrow \\ A(K_\infty)^{\Gamma^{p^n}} \end{array}$$

It follows from [Ru1, Theorem 7.7] and its proof (in which p being unramified in K_0/K renders $\mu(F)$ trivial and $\mathscr{I}_\mu = \Lambda$) that $\mathscr{C}(K_\infty) \approx \Lambda$ and $\mathscr{C}(K_n)$ embeds in $I(\mathscr{G}) \cdot \Lambda_{\Gamma^{p^n}}$ with finite index, where $I(\mathscr{G}) \subset \Lambda$ is the augmentation ideal. Moreover, the norm relations on elliptic units imply that the control map $\mathscr{C}(K_\infty)_{\Gamma^{p^n}} \to \mathscr{C}(K_n)$ is surjective. But any map $\Lambda_{\Gamma^{p^n}} \to I(\mathscr{G}) \cdot \Lambda_{p^n}$ with finite cokernel has kernel that is equal to $(\Lambda_{\Gamma^{p^n}})^{\mathscr{G}}$, which is a $\mathbb{Z}_p$-direct summand of $\Lambda_{\Gamma^{p^n}}$ that is $\mathscr{G}$-isomorphic to $\mathbb{Z}_p$. This control map followed by the inclusion $\mathscr{C}(K_n) \subseteq \mathscr{E}(K_n)$ is equal to the composite

$$\mathscr{C}(K_\infty)_{\Gamma^{p^n}} \to \mathscr{E}(K_\infty)_{\Gamma^{p^n}} \to \mathscr{E}(K_n),$$

which shows that the subset of $\mathscr{C}(K_\infty)_{\Gamma_{p^n}}$ (viewed inside the upper point of the cross) of elements mapping into the left point $\mathbb{Z}_p[\Delta]$ of the cross is again $(\mathscr{C}(K_\infty)_{\Gamma_{p^n}})^{\mathscr{G}}$. In fact, recalling the decomposition $\mathbb{Z}_p[\Delta] = \mathbb{Z}_p[\Delta]^{\mathbf{1}} \oplus \mathbb{Z}_p[\Delta]^{!}$, the image I_n of $(\mathscr{C}(K_\infty)_{\Gamma_{p^n}})^{\mathscr{G}} \to \mathbb{Z}_p[\Delta]$ is contained in $\mathbb{Z}_p[\Delta]^{\mathbf{1}} \approx \mathbb{Z}_p$. We define integers $e_n \geq 0$ and $\epsilon_n \in \{0,1\}$ by $\mathbb{Z}_p[\Delta]^{\mathbf{1}}/I_n \approx \mathbb{Z}_p^{\epsilon_n} \oplus \mathbb{Z}/p^{e_n}$, so that $I_n = p^{e_n}\mathbb{Z}_p[\Delta]^{\mathbf{1}}$ if $\epsilon_n = 0$, and $I_n = 0$ and $e_n = 0$ if $\epsilon_n = 1$. By explicitly computing the map $(\Lambda_{\Gamma^{p^{n+1}}})^{\mathscr{G}} \to (\Lambda_{\Gamma^{p^n}})^{\mathscr{G}}$ we find that, in the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}_p & \to & \mathscr{C}(K_\infty)_{\Gamma^{p^{n+1}}} & \to & \mathscr{C}(K_{n+1}) & \to 0 \\ & f\downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathbb{Z}_p & \to & \mathscr{C}(K_\infty)_{\Gamma^{p^n}} & \to & \mathscr{C}(K_n) & \to 0, \end{array}$$

the map f is multiplication by p (up to a unit). Let v_n be a generator for I_n (a basis vector if $\epsilon_n = 0$, and zero if $\epsilon_n = 1$). The transition maps from these computations for $n+1$ to these computations for n form the square

$$\begin{array}{ccccc} \mathbb{Z}_p & \xrightarrow{\cdot v_{n+1}} & I_{n+1} \subseteq & \mathbb{Z}_p[\Delta] \\ f\downarrow & & & \downarrow p \\ \mathbb{Z}_p & \xrightarrow{\cdot v_n} & I_n \subseteq & \mathbb{Z}_p[\Delta], \end{array}$$

the second downward map being computed by Proposition 8.3(3), and our computation of f shows that this square commutes (up to a unit). This commutativity implies that $v_n = 0$ if and only if $v_{n+1} = 0$, so that $\epsilon_n = \epsilon_{n+1}$

is independent of n; denote it henceforth by ϵ. The commutativity also implies that $e_n = e_{n+1}$ is independent of n; denote it henceforth by e.

Since $\mathscr{E}(K_n)$ is a free $\mathbb{Z}_p$-module, we may choose a splitting

$$H^1(\mathbf{R}\widetilde{\Gamma}_{f,\mathrm{Iw}} \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda_n) \cong \mathbb{Z}_p[\Delta] \oplus \mathscr{E}(K_n).$$

The definition of I_n allows us to modify the first cross in Proposition 8.3 to a short exact sequence

$$(36)\quad 0 \to (\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))_{\Gamma^{p^n}} \to \mathbb{Z}_p[\Delta]/I_n \oplus \mathscr{E}(K_n)/\mathscr{C}(K_n) \to A(K_\infty)^{\Gamma^{p^n}} \to 0.$$

One has $\mathrm{rank}_{\mathbb{Z}_p} A(K_\infty)^{\Gamma^{p^n}} = \mathrm{rank}_{\mathbb{Z}_p} A(K_\infty)_{\Gamma^{p^n}} = \delta - 1$, and combining this with the above sequence gives $\mathrm{rank}_{\mathbb{Z}_p}(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))_{\Gamma^{p^n}} = \epsilon$. The above sequence also gives an exact sequence of finite abelian groups

$$0 \to (\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))_{\Gamma^{p^n}}[p^\infty] \to \mathbb{Z}/p^e \oplus \mathscr{E}(K_n)/\mathscr{C}(K_n) \to A(K_\infty)^{\Gamma^{p^n}}[p^\infty],$$

where $\#A(K_\infty)^{\Gamma^{p^n}}[p^\infty]$ is bounded independently of n. The claim of the proposition follows. □

8.2.7. ***Application of the analytic class number formula.*** The analytic class number formula [Ru1, Theorem 1.3] gives

$$\#A(K_n) = \#\mathscr{E}(K_n)/\mathscr{C}(K_n).$$

In view of Propositions 8.4 and 8.5, it follows that (30) holds. Moreover, the computations $\mathrm{rank}_{\mathbb{Z}_p} A(K_\infty)_{\mathscr{G}} = 0$ and $\mathrm{rank}_{\mathbb{Z}_p}(\mathscr{E}(K_\infty)/\mathscr{C}(K_\infty))_{\mathscr{G}} = \epsilon$ follow from (35) and (36), respectively, so that (31) holds. This establishes (32).

8.3. Conclusion of the proof. We prove the theorem at once for $\mathbb{Z}_p$- and $\mathbb{Z}_p^2$-extensions, using (32).

We still assume K_∞ is the compositum of K_0 with any $\mathbb{Z}_p$-extension. Let K'_∞ denote the compositum of K_0 with the unique $\mathbb{Z}_p^2$-extension of K, and write $\mathscr{G}' = \mathrm{Gal}(K'_\infty/K) = \Delta \times \Gamma'$ with Γ' isomorphic to $\mathbb{Z}_p^2$, $\Lambda' = \Lambda(\mathscr{G}') = \mathbb{Z}_p[\![\mathscr{G}']\!]$, and $\mathrm{proj}\colon \Lambda' \twoheadrightarrow \Lambda$. By Rubin's theorem, there exist $f' \in \Lambda(\mathscr{G}')$ and $f \in \Lambda(\mathscr{G})$ with

$$f' \cdot \mathrm{char}_{\Lambda'} A(K'_\infty) = \mathrm{char}_{\Lambda'} \mathscr{E}(K'_\infty)/\mathscr{C}(K'_\infty),$$
$$f \cdot \mathrm{char}_{\Lambda} A(K_\infty) = \mathrm{char}_{\Lambda} \mathscr{E}(K_\infty)/\mathscr{C}(K_\infty).$$

By [Ru1, Corollary 7.9(i)] one has $\mathrm{proj}(f') = f$ up to a unit in Λ. Since proj is a homomorphism of semilocal rings that is a bijection on local factors and restricts to a local homomorphism on each local factor, it follows that f' is a unit (resp. restricts to a unit over a given local factor) in Λ' if and only if f is a unit (resp. restricts to a unit over the corresponding local factor) in Λ. On the other hand, (32) implies that f divides $\operatorname{char}_\Lambda \mathbb{Z}_p$ in Λ. Since $(\operatorname{char}_\Lambda \mathbb{Z}_p)^\eta = \Lambda^\eta$, the unit ideal, if $\eta \neq \mathbf{1}$, we deduce the identity of the theorem for both $\mathbb{Z}_p$- and $\mathbb{Z}_p^2$-extensions over each such η-factor. We only have left to consider the case where $\eta = \mathbf{1}$, or rather where Δ is trivial and $K_0 = K$.

Lemma 8.6. *Write $R = \mathbb{Z}_p[\![S,T]\!]$, and for $a, b \in \mathbb{Z}_p$ not both divisible by p, write $R_{a,b} = R/((1+S)^a(1+T)^b - 1)$ with $\pi_{a,b}\colon R \twoheadrightarrow R_{a,b}$. We identify $R_{a,b} \cong \mathbb{Z}_p[\![U]\!]$, where $U = \pi_{a,b}(S)$ if $p \nmid b$ and $U = \pi_{a,b}(T)$ otherwise.*

Suppose $g \in R$ is such that for all a, b above, $\pi_{a,b}(g)$ divides U in $R_{a,b}$. Then g is a unit.

Proof. Write $g = x + yS + zT + O((S,T)^2)$ with $x, y, z \in \mathbb{Z}_p$; we are to show that $p \nmid x$. Since $\pi_{0,1}(g)$ divides U in $R_{0,1}$, and $R_{0,1}$ is a UFD with U a prime element, it follows that $\pi_{0,1}(g)$ is either a unit or U times a unit. As $\pi_{0,1}(g) = x + yU + O(U^2)$, the first case is equivalent to $p \nmid x$, and the second case is equivalent to $x = 0$ and $p \nmid y$. But in the second case the identity

$$g = yS + zT + O((S,T)^2) = (1+S)^y(1+T)^z - 1 + O((S,T)^2)$$

would imply $\pi_{y,z}(g) = 0 + O(U^2)$, that is U^2 divides $\pi_{y,z}(g)$ in $R_{y,z}$, contradicting that $\pi_{y,z}(g)$ divides U. □

We continue with Δ trivial and $K_0 = K$. Choose a $\mathbb{Z}_p$-basis $\gamma_1, \gamma_2 \in \Gamma'$, so that $\ker(\Gamma' \twoheadrightarrow \Gamma) = (\gamma_1^a\gamma_2^b)^{\mathbb{Z}_p}$ for some $a, b \in \mathbb{Z}_p$ not both divisible by p. Set $S = \gamma_1 - 1, T = \gamma_2 - 1 \in \Lambda'$, and note that $\ker(\Lambda' \twoheadrightarrow \Lambda)$ is generated by $(1+S)^a(1+T)^b - 1$, so that the map $\Lambda' \twoheadrightarrow \Lambda$ is identified with the map $\pi_{a,b}\colon R \twoheadrightarrow R_{a,b}$ of the preceding lemma. Under this identification, the augmentation ideal $\operatorname{char}_\Lambda \mathbb{Z}_p$ is generated by $U \in R_{a,b}$, so we have that $\pi_{a,b}(f') = f$ divides U. Since $K_\infty^\Delta = K_\infty$ was allowed to be any $\mathbb{Z}_p$-extension of K, and conversely every such pair of a, b arises from some choice of K_∞, the preceding lemma shows that f' is a unit, and therefore so is f. This completes the proof of the theorem.

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