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Centralizers of hyperbolic and kinematic-expansive flows

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Abstract. We show that generic C^∞ hyperbolic flows commute with no C^∞-diffeomorphism other than a time-t map of the flow itself. Kinematic-expansivity, a substantial weakening of expansivity, implies that C^0 flows have quasidiscrete C^0-centralizer, and additional conditions broader than transitivity then give discrete C^0-centralizer. We also prove centralizer-rigidity: a diffeomorphism commuting with a generic hyperbolic flow is determined by its values on any open set.

1. Introduction

It is natural to expect a dynamical system to have no symmetries unless it is quite special. Symmetries correspond to the existence of a diffeomorphism or flow that commutes with the given dynamics, so one expects flows to "typically" have small centralizers, i.e., to commute with few flows or diffeomorphisms. Our topological results imply that expansive continuous flows have essentially discrete centralizer; the natural condition for this is a weakening of expansivity that does not allow reparameterizations and hence requires only a kinematic or dynamical separation of orbits rather than a geometric one (Definition 2.1), and the results imply that these flows commute with no other *flow*.

Smale's list of problems for the next century [32] includes questions regarding how typically centralizers are trivial. Given his interest in classifying dynamical systems up to conjugacy, the centralizer also describes the exact extent of nonuniqueness of a conjugacy between dynamical systems. Our main result is that hyperbolic flows generically commute with no *diffeomorphism*.

Definition 1.1. A C^r flow $\Phi\colon \mathbb{R}\times M \to M$ is a 1-parameter group of C^r diffeomorphisms $t \mapsto \Phi(t,\cdot) = \varphi^t$ for $t \in \mathbb{R}$ on a closed C^r-manifold M with $0 < r \le \infty$. We will use the notation Φ when we are speaking about a flow, and φ^t for the time t-map of the flow, or to emphasize the dependence on t. A C^0 flow is a continuous $\mathbb{R}$-action by homeomorphisms of a compact topological space X (and not assumed to be generated by a vector field).

- Two diffeomorphisms f, g are said to *commute* if $f \circ g = g \circ f$.
- For $r \ge 0$ a C^r diffeomorphism $f\colon M \to M$ *commutes with a flow* Φ if $f \circ \varphi^t = \varphi^t \circ f$ for all $t \in \mathbb{R}$.[1] We denote the set of such diffeomorphisms by $\mathscr{C}^r(\Phi)$ and say that Φ has trivial C^r-centralizer if $\mathscr{C}^r(\Phi) = \{\varphi^t \mid t \in \mathbb{R}\}$.[2]

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[1] If Φ, f are continuous then $\{ t \in \mathbb{R} \mid f\circ\varphi^t = \varphi^t \circ f \}$ is a closed subgroup of $(\mathbb{R},+)$, hence either $\{0\}$, $\mathbb{R}$, or $p\mathbb{Z}$.

[2] Triviality or discreteness of a centralizer of a flow Φ means triviality of the closed centralizer subgroup modulo its closed normal subgroup Φ.

- Two C^r flows Φ, Ψ commute if all ϕ^t, ψ^s do, so $\psi^s \circ \varphi^t = \varphi^t \circ \psi^s$ for all $s, t \in \mathbb{R}$. The set of flows Ψ that commute with Φ is called the C^r-$\mathbb{R}$-*centralizer* of Φ, denoted $Z^r(\Phi)$.
- We say that $Z^r(\Phi)$ is trivial (or that Φ *has trivial* C^r-$\mathbb{R}$-*centralizer*) if $Z^r(\Phi)$ consists of all constant-time reparameterizations of Φ. In other words, $\Psi \in Z^r(\Phi) \Rightarrow \psi^t = \varphi^{ct}$ for some $c \in \mathbb{R}$ and all $t \in \mathbb{R}$.
- A flow Φ has *discrete* C^r-centralizer if there is a $\delta > 0$ such that $f \in C^r$, $f \circ \varphi^t = \varphi^t \circ f$ for all t, and $d_{C^0}(f, \mathrm{Id}) < \delta$ imply that $f = \varphi^\tau$ for some τ.
- A flow Φ has *collinear* C^0-$\mathbb{R}$-centralizer if each orbit of a commuting flow Ψ is contained in an orbit of Φ.
- A flow Φ has *quasitrivial* C^0-$\mathbb{R}$-centralizer if a commuting flow Ψ is of the form $\psi^t(\cdot) = \varphi^{t\cdot\tau(\cdot)}(\cdot)$ for a continuous Φ-invariant τ.
- A flow Φ has *quasidiscrete* centralizer if there is a $\delta > 0$ such that $f \circ \varphi^t = \varphi^t \circ f$ for all t and $d_{C^0}(f, \mathrm{Id}) < \delta$ imply that there is a continuous Φ-invariant $\tau\colon X \to \mathbb{R}$ such that $f(\cdot) = \varphi^{\tau(\cdot)}(\cdot)$.

Remark 1.2. Evidently (by contraposition), having (quasi-)discrete centralizer implies having (quasi-)trivial $\mathbb{R}$-centralizer.

The difference between having quasitrivial and trivial $\mathbb{R}$-centralizer is the existence of continuous invariant functions (Proposition 3.12) or, equivalently, having a nontrivial constant flow as a topological factor.

Triviality of the $\mathbb{R}$-centralizer simply means that a commuting flow consists of the same set of maps:

Proposition 1.3. *If two flows are the same* set *of maps, then they are the same* group *of maps: if* Φ, Ψ *are continuous flows with* $\{\psi^t \mid t \in \mathbb{R}\} = \{\varphi^s \mid s \in \mathbb{R}\}$*, then* $\exists\, c \in \mathbb{R}\ \forall\, t \in \mathbb{R}$ *such that* $\psi^t = \varphi^{ct}$.

Proof. The continuous function $t \mapsto d_{C^0}(\varphi^t, \mathrm{id})$ attains its minimum value 0 on a closed additive subgroup of $\mathbb{R}$, i.e., either on $\mathbb{R}$ (so the claim is trivial), on $\{0\}$, or on $P\mathbb{Z}$ for some $P > 0$ (the flow is periodic). In the nontrivial cases, $t \mapsto \tau(t)$ is well-defined on $\mathbb{R}$ (respectively, $\mathbb{R}/P\mathbb{Z}$) by $\psi^t = \varphi^{\tau(t)}$ (since $t \mapsto \varphi^t$ is injective) and continuous at 0 since $\forall \epsilon > 0\ \exists \delta > 0\colon d_{C^0}(\varphi^t, \mathrm{id}) < \delta \Rightarrow |t| < \epsilon$. At the same time, $\varphi^{\tau(t+s)} = \psi^{t+s} = \psi^t \circ \psi^s = \varphi^{\tau(t)} \circ \varphi^{\tau(s)} = \varphi^{\tau(t)+\tau(s)}$, so injectivity gives $\tau(t+s) = \tau(t) + \tau(s)$ for all s, t. This implies that τ is both linear on $\mathbb{Q}$ (respectively, $\mathbb{Q}/P\mathbb{Z} \subset \mathbb{R}/P\mathbb{Z}$) and continuous (let $s \to 0$), so $\tau(t) \equiv ct$ for some $c \in \mathbb{R}$. □

Corollary 1.4. Φ *has trivial* C^r-$\mathbb{R}$-*centralizer iff* $Z^r(\Phi) = \{\varphi^t \mid t \in \mathbb{R}\}$.

If a flow has nontrivial centralizer, then it is part of an $\mathbb{R}^k$-action, which are often necessarily algebraic and hence far from generic [19, Corollary 5], [17, 18, 29]. Hyperbolicity entails a complicated and tightly interwoven structure on the phase space that is both topologically rigid and smoothly unclassifiable. Therefore, this is a natural context in which to expect $\mathbb{R}$-centralizers to be generically trivial, and there are a number of prior results to that effect.

1.1. Prior results on discrete-time centralizers and commuting diffeomorphisms. Extending results of Anderson [1], Palis [25] proved that among C^∞ Axiom A diffeomorphisms with strong transversality there is an open and dense set with discrete centralizer. Palis and Yoccoz [26] extended this: a large class of Axiom A diffeomorphisms with strong transversality has trivial centralizer. In [11] this was shown to hold for generic (non-Anosov) Axiom A diffeomorphisms with the no-cycles condition (which is weaker than strong transversality). Rocha and Varandas [28] have shown that the centralizer of C^r-generic diffeomorphisms restricted to hyperbolic basic sets is trivial.

In the C^1-topology, Bonatti, Crovisier, and Wilkinson proved that diffeomorphisms generically have trivial centralizer [5–7] but jointly with Vago [4] they found that on any compact manifold there is a nonempty open set of C^1-diffeomorphisms with a C^1-dense subset of C^∞-diffeomorphisms whose C^∞-centralizer is uncountable, hence nontrivial. Our results instead concern flows.

1.2. Continuous-time centralizers: commuting flows. For commuting flows there have been fewer results. Sad [30] proved that there is an open and dense set of C^∞ Axiom A flows with strong transversality that have trivial $\mathbb{R}$-centralizer, and Fisher–Hasselblatt [13] proved that C^r-flows which C^1-stably have trivial $\mathbb{R}$-centralizer are C^1 dense among partially hyperbolic C^r flows, among volume-preserving partially hyperbolic C^r flows, among symplectic partially hyperbolic C^r flows, and among contact partially hyperbolic C^r flows. Recently, Bonomo, Rocha, and Varandas [8] proved that the $\mathbb{R}$-centralizer of any Komuro-expansive[3] flow with nonresonant singularities is trivial. These results were extended by Bonomo and Varandas [9] to show triviality of the $\mathbb{R}$-centralizer of the restriction to a basic set. Recently, Leguil, Obata, and Santiago [21] established genericity of collinearity of the $\mathbb{R}$-centralizer and investigated when an $\mathbb{R}$-centralizer for a flow on a manifold is not necessarily trivial, but is "small" in a certain sense, establishing two criteria that imply this. Our results in the realm of topological dynamics implement a suggestion which Obata related to us that a much weaker notion than expansivity (being *kinematic-expansive*; Definition 2.1) implies quasidiscrete centralizer.

We first determine $Z^r(\Phi)$ for a kinematic-expansive flow Φ. Although there exist open sets of Anosov diffeomorphisms with trivial centralizer [3], there are many examples of Anosov diffeomorphisms with nontrivial centralizer. This is not the case for Anosov flows. It has been said to be "well-known and elementary" that Anosov flows have trivial $\mathbb{R}$-centralizer [12, Corollary 9.1.4].[4] We extend this to kinematic-expansive flows *that have no differentiability*—noting that results about C^0-centralizers seem to be exceedingly rare.[5]

Theorem 1.5. *For separating and kinematic-expansive flows (Definition 2.1) we have the following:*

(1) Separating flows have collinear *C^0-$\mathbb{R}$-centralizer.*
(2) Transitive separating flows (hence also transitive kinematic-expansive flows) have trivial C^0-$\mathbb{R}$-centralizer.
(3) Fixed-point-free kinematic-expansive flows have quasitrivial C^0-$\mathbb{R}$-centralizer.
(4) If Φ is fixed-point-free and kinematic-expansive on a connected space and has at most countably many chain-components (Definition 2.8), all of which are topologically transitive, then Φ has trivial C^0-$\mathbb{R}$-centralizer.

Remark 1.6. A Bowen–Walters-expansive flow is kinematic-expansive, and its fixed points are isolated points of the space [12, Remark 1.7.3]. We note that these latter conditions suffice for the results such as Theorems 1.5 and 1.8 where we state the assumption that the flow is fixed-point-free and kinematic-expansive.

[3] Komuro-expansivity is broader than the usual notion of expansivity (Remark 2.2) and equivalent when there are no fixed points; sometimes it is called expansivity as well (Definition 2.1) [8, Section 2.1.2], [2]. Important classes of geometric Lorenz attractors are Komuro-expansive, as this form of expansivity is more compatible with the coexistence of regular and singular orbits for the flow.

[4] "il est bien connu (et élémentaire) qu'un champ de vecteurs qui commute avec un champ d'Anosov est nécessairement un multiple constant de ce champ" [14, p. 262]

[5] The results in [24] are similar to our results on centralizers of kinematic-expansive flows, but Oka defines a so-called unstable centralizer and studies it for Bowen–Walters-expansive flows.

Since the original notion of Bowen–Walters-expansivity was modeled on hyperbolic dynamics, it is much more restrictive than kinematic-expansivity. There are transitive (indeed, minimal) kinematic-expansive flows with no hyperbolicity at all, such as special flows over an irrational rotation [22]. Indeed, any compact surface admits a kinematic-expansive flow [2, Theorem 5.6], while nontrivial hyperbolicity requires higher dimension by the Poincaré–Bendixson Theorem.

An application of the above result is to Axiom A flows that are quasitransverse, i.e., $T_xW^u(x)\cap T_xW^s(x)=\{0\}$ for all $x\in M$. These flows are expansive (hence kinematic-expansive) and have finitely many chain components, each transitive. Note that Anosov flows (transitive or not) are quasitransverse Axiom A flows without fixed points.

Corollary 1.7. *A fixed-point-free quasitransverse Axiom A flow has trivial C^0-$\mathbb{R}$-centralizer.*

The heuristic reason for this is that commuting flows act "isometrically" on each other's orbits, and this is incompatible with expansivity (or hyperbolicity) unless the orbits coincide; a more explicit argument invokes uniqueness in structural stability (Theorem 2.9). In either approach it remains to make the orbitwise "isometries" coherent (see page 28). We do this in the C^0-category where neither approach is viable. We emphasize again that there is not exactly an abundance of results about C^0-centralizers because the constraints from differentiability make these issues much more manageable (but see, e.g., [27]).

1.3. Diffeomorphisms and homeomorphisms commuting with flows. Returning to the point of view that we are studying the symmetry group of a flow leads us to the core question of which diffeomorphisms or homeomorphisms (rather than flows) commute with a flow, and thereby to our main results—which then imply results about $\mathbb{R}$-centralizers such as Theorem 1.5. We first note two underlying facts for *continuous* flows. Refining an argument by Walters shows that the symmetry group of sufficiently "intricate" flows is essentially discrete, but unlike his result, ours uses less restrictive versions of expansivity called kinematic-expansivity (Definition 2.1) and being separating, respectively; this is done in parts (2), (3), and (5) of Proposition 3.1 below and gives the following:

Theorem 1.8. *Topologically transitive separating flows have discrete centralizer (and hence trivial C^0-$\mathbb{R}$-centralizer). Kinematic-expansive flows without fixed points have quasidiscrete centralizer (and hence quasitrivial C^0-$\mathbb{R}$-centralizer).*

Remark 1.9. In this generality flows may not have trivial C^0-$\mathbb{R}$-centralizer. The "periodic band" $(x,y)\mapsto(x+ty,y)$ on $S^1\times[1,2]$ is kinematic-expansive and commutes with $f(x,y)=\varphi^{h(y)}(x,y)$ for any h (see also Proposition 3.12).

Theorem 1.8 is meant to be an indication of what we prove; note that it is about merely *continuous* flows. We produce discreteness of the C^0-centralizer in greater generality (Propositions 3.1 and 3.9 and Remark 3.14). This has the interesting application, in Theorem 3.15, that any topological conjugacy to a transitive separating flow (say) is locally unique (i.e., unique when chosen near the identity).

In this generality, one should not expect trivial centralizer: the geodesic flow on the usual genus-2 surface has a finite symmetry group generated by reflection and rotation isometries of the double torus, and the suspension of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ has the symmetry coming from $x\mapsto -x$. These and their C^1-perturbations are transitive separating flows with discrete yet nontrivial C^0 centralizer. (We note that Obata [23] recently established triviality of $\mathbb{Z}$-centralizers for generic vector fields in some classes.)

By contrast, our main results say that generic flows of the following kinds commute with no *diffeomorphisms* other than time-t maps of the flow. That is to say we establish much more than triviality of the $\mathbb{R}$-centralizer by showing that the (diffeomorphism-) centralizer is trivial.

Definition 1.10. Following [26], let $\mathscr{A}^r(M)$ be the set of C^r hyperbolic flows (Definition 2.6) on a manifold M that are not transitive Anosov, and $\mathscr{A}_1^r(M)$ be the set of $\Phi \in \mathscr{A}^r(M)$ with a fixed or periodic sink or source (i.e., periodic attractor or repeller).

Specifically, for the latter class, indeed an open dense set of such flows has trivial centralizer:

Theorem 1.11. *For a C^1-open and C^∞-dense set $\mathscr{O}$ of $\Phi \in \mathscr{A}_1^\infty(M)$ we have $\mathscr{C}^\infty(\Phi) = \{\varphi^t \mid t \in \mathbb{R}\}$, that is, if $f \in \mathscr{C}^\infty(\Phi)$, then $f = \varphi^t$ for some $t \in \mathbb{R}$.*

The perturbations performed are done on the wandering points, and this is why our construction does not work for transitive Anosov flows. More specifically, while it is easy to force a commuting diffeomorphism to send each periodic orbit to itself, there is no control over the way each orbit is shifted, and in the presence of recurrence this creates problems (see also Remark 4.5). We also note that the need for considering C^∞ flows arises from Sternberg linearization (Theorem 4.12), which involves a loss of differentiability that is hard to control.

Without assuming the presence of a fixed or periodic sink or source this conclusion remains true in low-dimensional situations; in full generality we obtain trivial centralizer for generic such flows.

Theorem 1.12. *$\mathscr{C}^\infty(\Phi) = \{\varphi^t \mid t \in \mathbb{R}\}$ for a residual set $\mathscr{R}$ of $\Phi \in \mathscr{A}^\infty(M)$ (Definition 1.10). If $\dim(M) = 3$, then $\mathscr{R}$ can be taken C^1-open and C^∞-dense.*

One of our auxiliary results was conjectured in [26, p. 83] for discrete time and is of independent interest: there is an open dense set of Φ in $\mathscr{A}^\infty(M)$ with centralizer-rigidity.

Theorem 1.13 (Rigidity)**.** *There is a C^1-open C^∞-dense of hyperbolic flows Φ (Definition 2.6) such that if $f_1, f_2 \in \mathscr{C}^\infty(\Phi)$, and $f_1 = f_2$ on a nonempty open set, then $f_1 = f_2$ on M.*

Remark 1.14. For (all) transitive C^r flows this is obvious because the set where $f_1 = f_2$ is closed and Φ-invariant; if $f_1(x) = f_2(x)$, then

$$f_1(\varphi^t(x)) = \varphi^t(f_1(x)) = \varphi^t(f_2(x)) = f_2(\varphi^t(x)).$$

Below, we prove this for $\Phi \in \mathscr{V}$, the C^1-open C^∞-dense subset of $\mathscr{A}^\infty(M)$ from Proposition 4.10.

2. Background

We review some basic notions pertinent to hyperbolic sets and introduce kinematic-expansivity.

Definition 2.1 (Nonwandering, transitivity, expansivity [10, 12])**.**

- A point x is *nonwandering* for a flow Φ on X if $x \in U$ open, $T > 0 \Rightarrow \exists\, t > T$ with $\varphi^t(U) \cap U \neq \varnothing$; otherwise it is said to be *wandering*. The (closed) set of nonwandering points is denoted by $NW(\Phi)$.
- The *ω-limit set* of $x \in X$ is $\omega(x) \coloneqq \bigcap_{t \geq 0} \overline{\varphi^{[t,\infty)}(x)} \subset NW(\Phi)$ and the *α-limit set* is $\alpha(x) \coloneqq \bigcap_{t \leq 0} \overline{\varphi^{(-\infty,t]}(x)} \subset NW(\Phi)$. The *limit set* of Φ is $L(\Phi) \coloneqq \overline{\bigcup_{x \in X} \alpha(x) \cup \omega(x)}$. Φ is said to be (topologically) *transitive* if there is a dense forward semiorbit $\varphi^{[0,\infty)}(x)$.

- Φ is *kinematic-expansive* if for all $\epsilon > 0$ there is a $\delta > 0$, called a *separation constant* (for ϵ), such that if $x, y \in X$ and $d(\varphi^t(x), \varphi^t(y)) < \delta\ \forall t \in \mathbb{R}$, then $y = \varphi^t(x)$ for some $|t| < \epsilon$.
- Φ is *separating* [21, Definition 2.3] if there is a $\delta > 0$ such that if $x, y \in X$ and $d(\varphi^t(x), \varphi^t(y)) < \delta\ \forall t \in \mathbb{R}$, then $y \in \varphi^{\mathbb{R}}(x)$.

Remark 2.2. Kinematic-expansivity is not invariant under orbit-equivalence or time-changes [2, Tables 1, 2]—and neither is the centralizer: time-changing $(x, y) \mapsto (x + ty, y)$ on $S^1 \times [1,2]$ (which is kinematic-expansive) to $(x, y) \mapsto (x + y, y)$ (which is not) adds $(x, y) \mapsto (x, f(y))$ to the centralizer. The property of a flow that all time-changes are kinematic-expansive is *strong kinematic-expansivity* (likewise with "separating").

Kinematic-expansivity suffices for the Bowen–Walters theory of existence [12, Exercise 4.25] and uniqueness [12, Remark 7.3.21] of equilibrium states. This is plausible because kinematic-expansivity is closely related to expansivity of the time-1 map, which also serves as a natural motivation for the notion of kinematic-expansivity.

Kinematic-expansivity does not allow reparameterizations and hence requires only a "kinematic" separation of orbits rather than a "geometric" one like Bowen–Walters-expansivity, which was defined in order to deal with hyperbolic (and symbolic) flows in an axiomatic way: Φ is *expansive* (or Bowen–Walters-expansive for emphasis) if for all $\epsilon > 0$ there is a $\delta > 0$, called an *expansivity constant* (for ϵ), such that if $x, y \in X$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous, $s(0) = 0$, and $d(\varphi^t(x), \varphi^{s(t)}(y)) < \delta\ \forall t \in \mathbb{R}$, then $y = \varphi^t(x)$ for some $|t| < \epsilon$. If "$s\colon \mathbb{R} \to \mathbb{R}$ continuous" is replaced by "$s\colon \mathbb{R} \to \mathbb{R}$ an increasing homeomorphism", then this is called Komuro-expansivity (or *geometric* expansivity [2])—though this notion is also due to Bowen and Walters [10].

Bowen–Walters-expansivity, Komuro-expansivity, kinematic-expansivity, and being separating are progressively less restrictive (see also [2, 15, 16]), and for our arguments the differences are manifest in connection with fixed points. Bowen–Walters-expansivity implies that (without loss of generality) there are no fixed points—they are isolated points of X [12, Remark 1.7.3]. Being separating (and hence kinematic-expansivity) implies that the set of fixed points is discrete, but we usually further need them to be isolated points of the space. Being separating suffices by itself for some of our centralizer arguments, and for others it does so if one also assumes topological transitivity.

Proposition 2.3. *A separating flow does not have arbitrarily small positive periods, and the set of fixed points is discrete, hence finite.*

Proof. If there are arbitrarily small positive periods or the fixed points accumulate, then there are points x_n with periods $0 \le p_n := \inf\{\, t > 0 \,|\, \varphi^t(x_n) = x_n \,\} \to 0$ which, by passing to a subsequence, converge to a (necessarily fixed) point x with a δ-neighborhood that contains either a fixed point $y \ne x$ or the orbit of a periodic nonfixed point y. Then $d(\varphi^t(x), \varphi^t(y)) < \delta$ for all $t \in \mathbb{R}$, so Φ is not separating. □

Definition 2.4 (Hyperbolic set, Axiom A)**.** Let M be a smooth manifold and Φ a smooth flow on M. A compact Φ-invariant set Λ is a *hyperbolic set* for Φ if there are a finite number of hyperbolic fixed points $\{p_1, ..., p_k\}$, a closed set Λ' such that $\Lambda = \Lambda' \cup \{p_1, ..., p_k\}$, a Φ-invariant splitting $T_{\Lambda'}M = E^s \oplus E^c \oplus E^u$, and constants $C \ge 1$, $\lambda \in (0,1)$ such that

- $E^c(x) := \mathbb{R}X(x) \ne \{0\}$ for all $x \in \Lambda'$, where $X(x) := \frac{d}{dt}\varphi^t(x)\big|_{t=0}$,
- $\|D\varphi^t\restriction_{E^s_x}\| \le C\lambda^t$ for all $t > 0$ and all $x \in \Lambda'$, and
- $\|D\varphi^{-t}\restriction_{E^u_x}\| \le C\lambda^t$ for all $t > 0$ and all $x \in \Lambda'$.

A flow Φ on a connected manifold M is said to be an *Anosov flow* if M is a hyperbolic set for Φ.

A flow Φ satisfies *Axiom A* if $NW(\Phi)$ is hyperbolic and is the closure of the periodic orbits.[6]

A hyperbolic set Λ for Φ is said to be *locally maximal* if there is a neighborhood V of Λ (an *isolating neighborhood*) such that $\Lambda = \Lambda_\Phi^V := \bigcap_{t\in\mathbb{R}} \varphi^t(V)$. A locally maximal hyperbolic set Λ for a flow Φ is a *basic set* if $\Phi\restriction_\Lambda$ is topologically transitive.

A set Λ such that $\Phi\restriction_\Lambda$ is topologically transitive is an *attractor* of Φ if there is an open set U such that $\bigcap_{t\ge0}\varphi^t(\overline{U}) = \Lambda$. (Then Λ is closed and Φ-invariant.) A *repeller* is an attractor for φ^{-t}.

If a flow Φ satisfies *Axiom A*, then $NW\big(\Phi\restriction_{NW(\Phi)}\big) = NW(\Phi)$. If Λ is a basic set, then $NW(\Phi\restriction_\Lambda) = \Lambda$. The nonwandering set of an Axiom A flow is a finite union of disjoint basic sets by Smale's Spectral Decomposition Theorem 2.10.

The *local strong stable manifold* and *local strong unstable manifold* of a point x are characterized as follows:

$$W_\epsilon^s(x) = \big\{ y \,\big|\, d(\varphi^t(y),\varphi^t(x)) < \epsilon \text{ for } t>0,\, d(\varphi^t(x),\varphi^t(y)) \xrightarrow{t\to+\infty} 0 \big\},$$

$$W_\epsilon^u(x) = \big\{ y \,\big|\, d(\varphi^{-t}(y),\varphi^{-t}(x)) < \epsilon \text{ for } t>0,\, d(\varphi^{-t}(x),\varphi^{-t}(y)) \xrightarrow{t\to+\infty} 0 \big\}.$$

The global *strong stable* and *strong unstable* manifolds

$$\begin{aligned} W^s(x) &:= \bigcup_{t>0} \varphi^{-t}(W_\epsilon^s(\varphi^t(x))) = \big\{ y\in M \,\big|\, d(\varphi^t(x),\varphi^t(y)) \xrightarrow{t\to+\infty} 0 \big\},\\ W^u(x) &:= \bigcup_{t>0} \varphi^{t}(W_\epsilon^u(\varphi^{-t}(x))) = \big\{ y\in M \,\big|\, d(\varphi^{-t}(x),\varphi^{-t}(y)) \xrightarrow{t\to+\infty} 0 \big\} \end{aligned} \tag{2.1}$$

are smoothly injectively immersed manifolds, as are the manifolds

$$W^{cs}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^s(x)) \text{ and } W^{cu}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^u(x)), \tag{2.2}$$

called the *weak stable* and *weak unstable* manifolds (or *center-stable* and *center-unstable* manifolds) of x. Note that $T_xW^{cs}(x) = E_x^s \oplus E_x^c$, $T_xW^{cu}(x) = E_x^c \oplus E_x^u$.

Theorem 2.5 (In-Phase Theorem [12, Theorem 5.3.25])**.** *If Λ is a compact locally maximal hyperbolic set for Φ on M, then $W^s(\Lambda) := \big\{x\in M \,\big|\, \varnothing \ne \omega(x) \subset \Lambda\big\} = \bigcup_{x\in\Lambda} W^s(x)$, and $W^u(\Lambda) := \big\{x\in M \,\big|\, \varnothing \ne \alpha(x) \subset \Lambda\big\} = \bigcup_{x\in\Lambda} W^u(x)$, and for any $\epsilon>0$ there is a neighborhood U of Λ with $\bigcap_{t\ge0}\varphi^{-t}(U) \subset W_\epsilon^s(\Lambda) := \bigcup_{x\in\Lambda} W_\epsilon^s(x)$ (and likewise for W^u).*

If Φ is an Axiom A flow, then $M = \bigcup_{i=1}^m W^s(\Lambda_i) = \bigcup_{i=1}^m W^u(\Lambda_i)$ with each union disjoint, where $\{\Lambda_i\}_{i=1}^m$ is the spectral decomposition. Furthermore, there is an open and dense set of points that are contained in the basin $W^s(\Lambda_i)$ of an attractor Λ_i and the basin $W^u(\Lambda_j)$ of a repeller Λ_j.

Definition 2.6 (Hyperbolic flow)**.** If Φ is an Axiom A flow, define a partial ordering $\gg$ on the basic sets $\Lambda_1,\dots,\Lambda_n$ from the spectral decomposition by

$$\Lambda_i \gg \Lambda_j \text{ if } (W^u(\Lambda_i)\smallsetminus\Lambda_i) \cap (W^s(\Lambda_j)\smallsetminus\Lambda_j) \ne \varnothing.$$

A *k-cycle* consists of a sequence of basic sets $\Lambda_{i_1} \gg \Lambda_{i_2} \gg \cdots \gg \Lambda_{i_k} \gg \Lambda_{i_1}$. We say that Φ has *no cycles* if there are no cycles among the basic sets; in that case Φ is said to be a *hyperbolic flow*.

Remark 2.7. Hyperbolicity of a flow is equivalent to hyperbolicity of the chain recurrent set (Definition 2.8) [12, Theorem 5.3.47] and implied by *strong transversality* of an Axiom A flow assumed in [26]: $W^s(x)$ and $W^u(x)$ are transverse for all $x \in M$.

[6]Following Bowen, our Axiom A allows for hyperbolic fixed points, whereas Smale's original Axiom A excluded singularities (he used "Axiom A' " for Bowen's Axiom A).

Definition 2.8 (Chain-recurrence)**.** An *ϵ-chain* for a flow Φ on a space X is a map $g\colon I \to X$ on a nontrivial interval $I \subset \mathbb{R}$ such that

$$d(g(t+\tau), \varphi^{\tau}(g(t))) < \epsilon, \quad \text{for } t, t+\tau \in I \text{ and } |\tau| < 1.$$

A point $x \in X$ is *chain recurrent* if it lies on periodic ϵ-chains for every $\epsilon > 0$ (the set $\mathscr{R}(\Phi) \supset NW(\Phi)$ of such points is the chain-recurrent set), and chain-recurrent points x, y are *chain-equivalent* if the pair lies on periodic ϵ-chains for every $\epsilon > 0$. The equivalence classes are called the *chain-components*.

The Anosov Shadowing Theorem [12, Theorem 5.4.1] implies in particular the *shadowing property* (ϵ-chains are close to orbits)[7], hence Axiom A, as well as the next result, that hyperbolic dynamics topologically do not change under perturbation. We will, however, see that since derivatives can change under perturbations, so can the centralizer.

Theorem 2.9 (Strong structural stability of hyperbolic sets)**.** *Suppose Λ is a compact hyperbolic set for a C^1 flow Φ on M. Then there are*

- *a C^1-neighborhood U of Φ,*
- *a C^0-neighborhood V of the inclusion $\iota := \mathrm{Id}\restriction_{\Lambda}$ of Λ in M, and*
- *a continuous map $h\colon U \to C^0(\Lambda, M)$, $\Psi \mapsto h_{\Psi}$*

such that $h_{\Phi} = \iota$ and for each $\Psi \in U$

- *h_{Ψ} is a (Hölder) continuous embedding,*
- *h_{Ψ} is the transversely unique map in V for which $\psi^{\tau(t)} \circ h_{\Psi} = h_{\Psi} \circ \varphi^{t}\restriction_{\Lambda}$, where τ is given by the Shadowing Theorem, and*
- *the* continuation *$\Lambda_{\Psi} := h_{\Psi}(\Lambda)$ is a hyperbolic set for Ψ.*

If $\Psi = \Phi$ is Anosov, then transverse uniqueness comes close to establishing the "well-known and elementary" triviality of centralizers of Anosov flows.

The shadowing property together with expansivity gives

Theorem 2.10 (Spectral Decomposition)**.** *The chain-recurrent set of an expansive flow with the shadowing property has finitely many chain-components, and each is topologically transitive.*

3. Centralizers for continuous flows

In this section we prove Theorem 1.5 about centralizers for kinematic-expansive flows. Walters observed that expansive homeomorphisms have discrete centralizers [33, Theorem 2], and likewise, kinematic-expansivity of a flow (Definition 2.1) ensures that centralizers are discrete—provided we add hypotheses to control "longitudinal" phenomena. The Walters argument shows that a commuting homeomorphism preserves orbits, but unlike in discrete time we need to further establish that the shift along them is constant. We begin by showing that this is the case on orbit closures and by giving conditions under which this shift can be taken continuous.

Proposition 3.1. *Consider a continuous flow Φ on X and $f \in \mathscr{C}^0(\Phi)$.*

(1) If Φ is separating, δ as in the definition, $d_{C^0}(f, \mathrm{Id}) < \delta$, then $f(x) \in \mathscr{O}(x)$, for all $x \in X$.

(2) If Φ is kinematic-expansive, $\epsilon > 0$, $\delta > 0$ a separation constant for ϵ, $d_{C^0}(f, \mathrm{Id}) < \delta$, then $\forall x \in X$ $f(x) \in \varphi^{(-\epsilon,\epsilon)}(x)$.

(3) $f(x) \in \mathscr{O}(x) := \varphi^{\mathbb{R}}(x) \Rightarrow \exists\, \tau = \tau(\mathscr{O}(x)) \in \mathbb{R}\colon f\restriction_{\overline{\mathscr{O}(x)}} = \varphi^{\tau}\restriction_{\overline{\mathscr{O}(x)}}$.

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[7] This is also known as the pseudo-orbit tracing property.

(4) *If Φ is fixed-point-free, $f(x) \in \mathcal{O}(x)$ for all $x \in X$, and if $x \mapsto \tau(\mathcal{O}(x))$ from* (3) *can be chosen to be less than half the smallest positive period in absolute value, then τ can be chosen continuously on X.*

(5) *If Φ is fixed-point-free and kinematic-expansive, $\epsilon > 0$ is less than half the smallest positive period of Φ in absolute value, $\delta > 0$ is a separation constant for ϵ, and $d_{C^0}(f, \mathrm{Id}) < \delta$, then one can continuously define $x \mapsto \tau(\mathcal{O}(x))$ on X such that $f\restriction_{\overline{\mathcal{O}(x)}} = \varphi^{\tau(\mathcal{O}(x))}\restriction_{\overline{\mathcal{O}(x)}}$.*

Proof. (1) & (2): If $d_{C^0}(f, \mathrm{Id}) < \delta$, then $d(\varphi^t(x), \varphi^t(f(x))) = d(\varphi^t(x), f(\varphi^t(x))) < \delta$ for all x and all t, so the definition of separating and kinematic-expansivity implies the respective result.

(3): If $f(x) = \varphi^\tau(x)$, then $f(\varphi^t(x)) = \varphi^t(f(x)) = \varphi^{t+\tau}(x) = \varphi^\tau(\varphi^t(x))$, so $f\restriction_{\mathcal{O}(x)} = \varphi^\tau\restriction_{\mathcal{O}(x)}$. The claim follows by continuity of f and φ^τ.

(4): Under the constraints in the assumption, τ is uniquely determined for each $x \in X$, and if $x_n \to x$, then $n \mapsto \tau(x_n)$ has an accumulation point $\tau_0 = \lim_{k\to\infty} \tau(x_{n_k})$, and continuity of Φ and f gives $\varphi^{\tau_0}(x) = \lim_{k\to\infty} \varphi^{\tau(x_{n_k})}(x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = f(x) = \varphi^{\tau(x)}(x)$, so $\tau(x) = \tau_0 = \lim_{n\to\infty} \tau(x_n)$.

(5): This follows from the preceding items. □

Remark 3.2. Note that (4) also holds if the fixed points of Φ are isolated points of X, for instance, if Φ is Bowen–Walters-expansive.

Taking $\mathcal{O}(x)$ dense in Proposition 3.1(3) gives (the first sentence of Theorem 1.8 and hence) Theorem 1.5(2).[8] Specifically, (1) and (3) imply

Proposition 3.3 (Discrete centralizer)**.** *If Φ is a topologically transitive separating continuous flow on X, δ a corresponding constant, $f \in \mathscr{C}^0(\Phi)$, and $d_{C^0}(f, \mathrm{Id}) < \delta$, then $f = \varphi^\tau$ for some τ—so Φ has discrete centralizer and hence trivial C^0-$\mathbb{R}$-centralizer.*

Without transitivity we need to combine continuity of the time-shift with topological dynamics to conclude that it must be constant.

Definition 3.4. A continuous function is said to be a *constant of motion* for a flow Φ if it is Φ-invariant, i.e., constant on orbits (and hence on orbit closures) of Φ. We say that a flow Φ *has no constant of motion* if every continuous Φ-invariant function is constant. (Apparently, Thom conjectured that this is C^1-generic [23, §1].)

Remark 3.5. This notion works equally well with C^r functions invariant under a C^r flow, but we only use it in the C^0 category.

Remark 3.6. The existence of a nontrivial constant of motion $h\colon X \to \mathbb{R}$ is an obstruction to discreteness (versus quasidiscreteness) of the centralizer of a flow Φ because $x \mapsto \varphi^{\epsilon \arctan(h(x))}(x)$ is in the centralizer for any ϵ without being a time-t map. Thus, having no constant of motion is necessary for having discrete centralizer, and this part of our work can be seen as seeking additional assumptions to produce a sufficient condition for discreteness of the centralizer. We note that the absence of a constant of motion is independent of kinematic-expansivity; the construction in [2, Theorem 5.6] can be modified by the insertion of a "periodic band" (Remark 1.9) around a sink or source to produce a kinematic-expansive flow with a constant of motion on any compact surface.

Since the time shift in Proposition 3.1 is a constant of motion, we now seek assumptions (more general than topological transitivity) under which a continuous flow has no

[8] Indeed, Proposition 3.1(3) implies that if Φ has a dense orbit that is invariant under $f \in \mathscr{C}^0(\Phi)$ then $f = \varphi^\tau$ for some $\tau \in \mathbb{R}$.

constant of motion. For instance, Theorem 1.5 does not properly generalize the corresponding statement for Anosov flows because Anosov flows need not be transitive. The following sidesteps that hypothesis:

Proposition 3.7. *A continuous flow on a connected space has no constant of motion if it has countably many chain-components (Definition 2.8) and each of them is topologically transitive.*

Proof. If a continuous function $h\colon X \to \mathbb{R}$ is constant on orbit closures of the flow Φ, then

- h is constant on each chain-component, and
- if $x \in X$, then $h(\{x\}) = h\big(\overline{\varphi^{\mathbb{R}}(x)}\big) = h(\omega(x))$ (so $h(X) = h(L(\Phi))$), and $\omega(x)$ is contained in a chain-component [12, Propositions 1.5.7(4), 1.5.37].

Thus, $h(X) = h(\mathscr{R}(\Phi)) \subset \mathbb{R}$ is connected and at most countable, hence a point. □

Remark 3.8. The transitivity assumption on chain-components is C^1-generically not needed, i.e., for a C^1-residual set of flows, every continuous invariant function is constant on each chain-component [21, Lemma 6.17].

Theorem 1.5 follows from Proposition 3.7 and Proposition 3.1(5)—as does the next result.

Proposition 3.9 (Discrete centralizer)**.** *A fixed-point-free kinematic-expansive flow on a connected space with countably many chain-components, each transitive, has discrete C^0-centralizer and hence trivial C^0-$\mathbb{R}$-centralizer.*

By the Spectral Decomposition (Theorem 2.10) this gives in particular

Theorem 3.10. *Expansive flows with the shadowing property on a connected space have discrete centralizer and hence trivial C^0-$\mathbb{R}$-centralizer.*

Remark 3.11 (Hayashi). Variants of Theorem 1.5 arise by showing that h in the proof of Proposition 3.7 is constant on chain-components C under hypotheses other than transitivity:

- The closing property—$\forall \epsilon > 0\, \exists \delta > 0$: δ-chains are ϵ-shadowed by a closed orbit; if $x \sim y$ take $x_i \to x$, $y_i = \varphi^{t_i}(x_i) \to y$, hence $h(x_i) = h(y_i)$—but this argument also establishes transitivity.
- The shadowing property—either by an analogous argument or because together with expansivity it implies the closing property—and hence transitivity in this context.
- $\exists\, x \in C\ \forall\, y \in C\ \exists\, x_i \to x,\ t_i \in \mathbb{R}$ with $\varphi^{t_i}(x_i) \to y$ (see also Remark 3.14).

We expand on the preceding results and on the last of these suggestions by spelling out more carefully what these basic arguments establish, starting with an evident consequence of the definitions.

Proposition 3.12. *A flow has trivial (discrete) $\mathbb{R}$-centralizer iff it has quasitrivial (quasidiscrete) $\mathbb{R}$-centralizer and no constant of motion.*

Combined with Theorem 1.8 itself, this gives another criterion for having trivial C^0-$\mathbb{R}$-centralizer:

Proposition 3.13. *If Φ is a fixed-point-free kinematic-expansive continuous flow on X that has no constant of motion, then Φ has discrete C^0-centralizer and hence trivial C^0-$\mathbb{R}$-centralizer.*

Remark 3.14. Here are a few sufficient conditions for having no constant of motion; keep in mind that they are of interest in the presence of kinematic-expansivity (or of having quasidiscrete centralizer) because it is in that context that these then imply discrete centralizer.

- Φ does not have the identity as a (nontrivial) topological factor. (A nonconstant invariant continuous function defines such a factor map—and vice versa; this is actually a characterization.)
- The limit set $L(\Phi)$ (Definition 2.1) is contained in an at most countable union of *elongational limit sets* $\overline{\mathscr{E}}(x) := \overline{\bigcup_{n\in\mathbb{N}}\mathscr{E}^n(x)}$, where $\mathscr{E}^n(x) := \mathscr{E}(\mathscr{E}^{n-1}(x))$ and

  $$\mathscr{E}(A) := \Big\{ \lim_{i\to\infty} \varphi^{t_i}(x_i) \,\Big|\, \lim_{i\to\infty} x_i \in A,\ t_i \in \mathbb{R} \Big\} = \bigcap \big\{ \overline{\varphi^{\mathbb{R}}(O)} \,\big|\, A \subset O \text{ open} \big\},$$

  the *elongation* of A. (A constant of motion is constant on $\overline{\mathscr{E}}(x)$ and takes all its values on $L(\Phi)$.)
- More strongly, one can replace the elongational limit sets in the previous item by the *elongational hulls* of points x, the smallest set containing x that is closed under application of $\mathscr{E}$.

One can contemplate what the nature of a kinematic-expansive flow with a constant of motion might be. The restriction Φ_s of such a flow to a level set is itself kinematic-expansive. (We note that if the restriction to any level set is expansive *and has the shadowing property* then the flow is not expansive. Thus, any expansive such examples decompose into expansive flows none of which have the shadowing property. This illustrates how kinematic-expansivity is a substantial generalization.)

We previously remarked on uniqueness of conjugacies, and this is an interesting issue in this topological context because conjugacies are not often smooth. Thus (since $k^{-1}h$ is in the centralizer of Φ below), we note the following:

Theorem 3.15 (Local uniqueness of conjugacies)**.** *Suppose Φ is continuous flow with discrete C^0-centralizer. If Ψ is topologically conjugate to Φ via a homeomorphism h, then h is locally unique, i.e., there is a $\delta > 0$ such that if k is a conjugacy between Φ and Ψ with $d_{C^0}(h,k) < \delta$, then $h = k\circ\varphi^t$ for some small t.*

Remark 3.16. Our explorations of when a flow has no constant of motion are also pertinent to quasitriviality of the diffeomorphism-centralizer: the (diffeomorphism-)centralizer of a flow Φ is said to be quasitrivial if it consists of maps of the form $\varphi^{T(\cdot)}(\cdot)$; if the flow has no constant of motion, then $T(\cdot)$ is necessarily constant and the centralizer is trivial. And this in turn then yields uniqueness (rather than local uniqueness) of conjugacies.

Proposition 3.1 also implies, in particular,

Proposition 3.17. *Let Φ be a C^r Axiom A flow on a closed manifold M and let $\epsilon > 0$ be an expansive constant for $\Phi\restriction_{NW(\Phi)}$. If $f \in \mathscr{C}^0(M)$ and $d_0(f,\mathrm{Id}) < \epsilon$, then $f(x) \in \mathscr{O}(x)$ for all $x \in NW(\Phi)$.*

This result points to two issues in identifying the centralizer. The first is having to deal with wandering points. Hyperbolicity helps describe the centralizer of an Axiom A flow on the nonwandering set, but we will need perturbation methods to "control" centralizers on the wandering set. The second is that the discreteness of the centralizer in Proposition 3.17 helps show that any commuting *flow* is a constant-time reparameterization, but for a diffeomorphism far from the identity, much more is needed to show that it is a time-t map of the flow. That is the substance of the next sections.

4. Centralizers for Axiom A flows

We now prove Theorems 1.11 and 1.12. We first outline the arguments. The first step (Subsection 4.1) ensures that the commuting diffeomorphism fixes the various attractors and repellers as well as their basins. This can often be established for commuting *homeomorphisms* by considering periods of closed orbits, but since we work in the smooth category and want to utilize fixed points as well, we use that a closed orbit and its image under a commuting diffeomorphism must have conjugate derivatives (Lemma 4.2), which allows us by perturbation to force the commuting diffeomorphism to fix a periodic orbit or fixed point in each attractor/repeller, and hence that whole set itself. It then clearly fixes the entire basins as well (Lemma 4.1). Thus, there is a C^1-open C^∞-dense set $\mathscr{U}$ of flows such that any diffeomorphism g commuting with a flow in $\mathscr{U}$ fixes the basins of each attractor or repeller, and each attractor or repeller contains at least one fixed or periodic point whose orbit is fixed by the commuting diffeomorphism g.

In Subsection 4.2 we show that once a commuting diffeomorphism has been identified on an open subset of one of these basins, then it is globally identified. This is done in 2 parts. For the open and dense set $\mathscr{U}$, Theorem 4.9 ensures that if two commuting diffeomorphisms agree in an open set of a basin, then they agree for the entire basin, and Theorem 1.13 then links the basins of the attractors and repellers to let us conclude that there is an open and dense set $\mathscr{V}$ of flows such that if two commuting maps agree on an open set, then they agree on the entire manifold. This reduces the proof of Theorems 1.11 and 1.12 to a local problem analogous to the results in [26] for maps: it remains to show that on a basin of an attractor or repeller any commuting diffeomorphism is a time-t map of the flow.

We previously mentioned that the heart of the problem is in controlling nonwandering points, and accordingly, this remaining portion of the proof is the most difficult. We carry it out in 2 parts. Subsections 4.3 and 4.4 explain the reduction to an algebraic problem. More specifically, normal-forms theory allows us to translate the local problem to an algebraic one, and Lemma 4.20 uses Theorems 4.9 and 1.13 to establish that the solution of the algebraic problem does indeed imply the solution of the dynamical problem and hence Theorems 1.11 and 1.12. Finally, the perturbations to solve the algebraic problem are carried out in Section 4.5.

4.1. Fixing the basins. The first step towards limiting what diffeomorphisms commute with a hyperbolic flow is to see that typically a commuting *homeomorphism* fixes the "large scale" or "combinatorial" structure of the flow, namely the pieces of the chain decomposition (including the various attractors and repellers; see Definition 2.8) and their respective basins. The latter is an easy consequence of the former, which suggests that this is a C^0-open circumstance.

Lemma 4.1. *Let Φ be a C^r Axiom A flow on a closed manifold M, $f \in \mathscr{C}^0(\Phi)$, and $x \in M$. Then*

$$f(W^s(x,\Phi)) = W^s(f(x),\Phi) \text{ and } f(W^u(x,\Phi)) = W^u(f(x),\Phi).$$

The set of fixed points of a flow Φ is invariant under any $f \in \mathscr{C}^0(\Phi)$, as is the set of T-periodic *orbits* for any $T > 0$ and, crucially, the period of each. Any $f \in \mathscr{C}^1(\Phi)$ furthermore conjugates the derivatives as follows.

Lemma 4.2. *If $f \in \mathscr{C}^0(\Phi)$, then the chain-recurrent set $\mathscr{R}(\Phi)$ is f-invariant, and if $p \in M$ is a fixed point or T-periodic point of Φ, then so is $f(p)$ (i.e., with the same period). If, furthermore, $f \in \mathscr{C}^1(\Phi)$, then the derivatives of φ^T at p and $f(p)$ are (linearly) conjugate.*

Proof. If $\varphi^t(p) = p$, then $f(p) = f(\varphi^t(p)) = \varphi^t(f(p))$. If $p \in M$ is a fixed point, then this holds for all $t \in \mathbb{R}$ and so $f(p)$ is a fixed point for Φ. If $p \in M$ is T-periodic, then this holds for $t = T$, so $f(p)$ is T-periodic. Differentiation of $f(\varphi^t(p)) = \varphi^t(f(p))$ for $t = T$ then gives

$$D\varphi^T(f(p))Df(p) = Df(\varphi^T(p))D\varphi^T(p) = Df(p)D\varphi^T(p). \qquad \square$$

In particular, the spectrum of $D\varphi^T(p)$ and $D\varphi^T(f(p))$ is the same; later this will be important for establishing triviality of the centralizer. The spectral decomposition of an Axiom A flow Φ is invariant under any $f \in \mathscr{C}^0(\Phi)$ because f preserves chain-recurrence and chain-equivalence and hence permutes the chain-components of Φ.

Lemma 4.3. *There is a C^0-open and C^∞-dense set of Axiom A flows Φ such that every $f \in \mathscr{C}^0(\Phi)$ fixes each chain-component of Φ that does not consist of a fixed point.*

Proof. For each chain-component with a periodic orbit consider the least period in that chain component. These being pairwise distinct is a C^0-open condition and implies that these chain-components are each f-invariant (Lemma 4.2), as are then their basins (Lemma 4.1).

That these least periods are pairwise distinct is C^∞-dense as follows: in each of these chain-components pick a periodic point p_i with that least period and a function ρ_i which is C^∞-close to 1 and with $\rho_i \equiv 1$ outside a small neighborhood of p_i chosen such that the time-change Φ' generated by the vector field $\prod_i \rho_i X$, where X generates Φ, has distinct least periods. $\square$

Remark 4.4. Lemma 4.3 illustrates the presence of "longitudinal" effects specific to flows. This is related to the fact that *conjugacies* between flows are rarer than orbit-equivalence, which is insensitive to time-changes, the very construction that gives rise to Φ' in this proof. However, fixed points of Φ have no meaningful longitudinal aspects, but C^1-techniques make sure they are fixed by f: there is a C^1-open and C^∞-dense set $\mathscr{U}_0$ of Axiom A flows such that each attractor or repeller has a fixed or periodic point where the derivative of the period or time-1 map is not conjugate to the corresponding derivative at any other such periodic orbit with the same period or fixed point. Lemma 4.1 then further implies that the basin of each attractor or repeller is fixed for any commuting diffeomorphism.

Remark 4.5. Here is what (little) we can say about transitive Anosov flows Φ. C^r-generically no two closed orbits have the same period, so each is invariant under any $f \in \mathscr{C}^0(\Phi)$. This removes some possible obvious elements from the centralizer, such as might arise for the geodesic flow of a negatively curved surface from symmetries of the surface. Proposition 3.1 provides a little more information. If $x = \lim_{n\to\infty} x_n$ with x_n Φ-periodic, then $\lim_{n\to\infty} \tau(x_n) = \infty$ with (any choice of) $\tau(x_n)$ as in Proposition 3.1(3), unless $\varphi^{\mathbb{R}}(x)$ is f-invariant. If $\varphi^{\mathbb{R}}(x)$ is dense, then $f(x) \neq \varphi^t(x)$ for any $t \in \mathbb{R}$ by Proposition 3.1(3), unless $f \equiv \varphi^\tau$. But it is not apparent how this can be used to show that the centralizer is trivial.

In closing we note that we have not so far used the no-cycles assumption.

4.2. Rigidity: local coincidence to global coincidence. The goal of this section is a global rigidity result, Theorem 1.13, which may be of independent interest.

The first step, Theorem 4.9, is at the heart of reducing the proof of Theorems 1.11 and 1.12 to a local problem by fixing the commuting diffeomorphism on stable and unstable sets *once it has been fixed on an open subset*; it is obtained by a minor modification of the discrete-time arguments in [1]. (For fixed or periodic attractors it is immediate from Theorem 4.15.)

Definition 4.6. Denote the spectrum of an $n\times n$ matrix A by $\Sigma(A)=\{\lambda_1,\dots,\lambda_n\}$, the eigenvalues of A repeated with multiplicity. A is said to be *stable hyperbolic* if $\mathrm{Re}\lambda<0$ for all $\lambda\in\Sigma(A)$. Define the function

$$(\lambda,m)\mapsto\gamma(\lambda,m):=\lambda-(m_1\lambda_1+\cdots+\lambda_n m_n).$$

Definition 4.7. A linear map $A\colon\mathbb{R}^n\to\mathbb{R}^n$ is *nonresonant* if $\mathrm{Re}\lambda_i\neq\mathrm{Re}\lambda_1^{m_1}\cdots\mathrm{Re}\lambda_n^{m_n}$ whenever $0\le m_j\in\mathbb{Z}$ with $\sum m_j\ge 2$. A point $p=\varphi^t(p)$ is nonresonant if $D\varphi^t(p)$ is. A stable hyperbolic matrix is *nonresonant* if $\mathrm{Re}\gamma(\lambda,m)\neq 0$ for any m where $|m|\ge 2$ and any $\lambda\in\Sigma(A)$.

For a flow there is a similar, but slightly different notion of nonresonance that we use for the fixed points of the flow.

Remark 4.8. There is a C^1-open C^∞-dense set $\mathscr{U}$ of $\Phi\in\mathscr{U}_0$ (from Remark 4.4) for which each attractor contains a fixed or periodic point that is nonresonant and where the derivative is not conjugate to the corresponding derivative at any other such fixed point or periodic orbit with the same period. Each attractor or repeller of such a flow then satisfies the hypotheses of the next theorem.

Theorem 4.9. *Let Φ be a C^∞ flow on a manifold M and $\Lambda\subset M$ be a transitive hyperbolic attractor containing a fixed or periodic point p that is nonresonant. If $f_1,f_2\in\mathscr{C}^\infty(\Phi)$, and there exists an open set $V\subset W^s(\Lambda)$ such that $f_1\restriction_V=f_2\restriction_V$, then $f_1\restriction_{W^s(\Lambda)}=f_2\restriction_{W^s(\Lambda)}$.*

We delay the proof of the result until the next section. The desired global rigidity result, Theorem 1.13, is now obtained by linking the basins of attractors and repellers. In [26] the maps are assumed to have strong transversality in order to link the basins, but after perturbations this can be done for Axiom A maps with the no cycles property [11]:

Proposition 4.10. *There is a C^1-open C^∞-dense $\mathscr{V}\subset\mathscr{U}\subset\mathscr{A}^r(M)$ ($\mathscr{U}$ from Remark 4.8) for $1\le r\le\infty$, such that if Λ and Λ' are attractors for $\Phi\in\mathscr{V}$ with*

$$\overline{W^s(\Lambda)}\cap\overline{W^s(\Lambda')}\neq\varnothing,$$

then there exists a hyperbolic repeller $\hat\Lambda$, such that

$$W^s(\Lambda)\cap W^u(\hat\Lambda)\neq\varnothing \text{ and } W^s(\Lambda')\cap W^u(\hat\Lambda)\neq\varnothing.$$

The proof is almost identical to its discrete-time counterpart [11, Proposition 3.2] as the proof uses properties of no cycles and perturbations on the wandering points, and these hold for flows as well.

This in turn implies Theorem 1.13:

Proof of Theorem 1.13. Let $\Lambda_1,\dots,\Lambda_k$ denote the hyperbolic attractors of Φ. If $f\in\mathscr{C}^\infty(\Phi)$ is the identity on a nonempty open $U\subset M$, then there is an i such that $\mathrm{int}(W^s(\Lambda_i)\cap U)\neq\varnothing$, hence $f\restriction_{W^s(\Lambda_i)}=\mathrm{Id}\restriction_{W^s(\Lambda_i)}$ by Theorem 4.9. Now for j such that

$$\overline{W^s(\Lambda_i)}\cap\overline{W^s(\Lambda_j)}\neq\varnothing$$

there is a repeller $\hat\Lambda$ with

$$W^s(\Lambda_i)\cap W^u(\hat\Lambda)\neq\varnothing\neq W^s(\Lambda_j)\cap W^u(\hat\Lambda).$$

Therefore, f is the identity on $W^u(\hat\Lambda)\cup W^s(\Lambda_j)$ since the intersection of the basins is an open set. Hence, f is the identity on the open and dense set of points contained in the basin of an attractor or repeller. Continuity of f implies that f is the identity on all of M. □

4.3. Linearization theorems for flows and maps. We now take the first step in the reduction to an algebraic problem. Under the nonresonance condition we have generically established, standard normal-form theory becomes the theory of smooth linearization [31] on the local stable manifold of a hyperbolic fixed or periodic sink or source or a periodic point for a hyperbolic attractor, and it implies that *any element of the centralizer is simultaneously linearized.*

Definition 4.11. An $n\times n$ stable hyperbolic matrix A satisfies the *Sternberg condition of order* $N\ge 2$ if $\mathrm{Re}\gamma(\lambda,m)\neq 0$ for all $\lambda\in\Sigma(A)$ and $m=(m_1,\dots,m_n)\in\mathbb{N}^n$ with $|m|:=m_1+\cdots+m_n\le N$.

Theorem 4.12 (Sternberg's Theorem)**.** *Let $Q\ge 2$ and R be C^{2Q} on an open set $U\subset\mathbb{R}^n$ containing the origin. If $D^kR(0)=0$ for $k=0,1$ and A is a stable hyperbolic matrix (i.e., all its eigenvalues are inside the unit circle) such that A satisfies the Sternberg condition of order Q, then the flow Φ on $\mathbb{R}^n$ generated by $x'=Ax+R(x)$ admits a $C^{\lfloor Q/\rho\rfloor}$-linearization near 0, with ρ defined by*

$$\rho:=\rho(A):=\frac{\max\{|\mathrm{Re}\lambda|\mid\lambda\in\Sigma(A)\}}{\min\{|\mathrm{Re}\lambda|\mid\lambda\in\Sigma(A)\}}.$$

We remark that a similar result holds for an unstable hyperbolic matrix simply by taking the inverse of the flow.

We say that a stable hyperbolic matrix is *nonresonant* if $\mathrm{Re}\gamma(\lambda,m)\neq 0$ for any m and any $\lambda\in\Sigma(A)$. The following immediate corollary is the main application of Sternberg's Theorem.

Corollary 4.13. *If $f\in C^\infty$ and $x'=f(x)=Ax+R(x)$, where A is a nonresonant stable hyperbolic matrix, then there exists a C^∞-smooth linearization.*

This is more natural in the form of a local restatement.

Theorem 4.14. *If $f\colon\mathbb{R}^n\to\mathbb{R}^n$ is a C^∞ diffeomorphism and the origin is a hyperbolic sink for f with $Df(0)$ nonresonant, then there exists a C^∞-smooth linearization of f in a neighborhood of the origin.*

Next, the nonresonance assumption implies that the centralizer of a nonresonant linear system consists of linear maps [12, Theorem 6.8.22], and as a consequence, the smooth linearization from Theorem 4.14 for the stable manifold of a sink simultaneously smoothly linearizes any smooth map in the centralizer.

Theorem 4.15. *Let $A\colon\mathbb{R}^n\to\mathbb{R}^n$ be a nonresonant stable hyperbolic matrix and Φ_A the linear flow generated by A. If g is a C^∞ homeomorphism such that $g\varphi_A^t=\varphi_A^t g$ for all $t\in\mathbb{R}$, then g is linear.*

These results imply that if we consider either a hyperbolic sink or source, or a periodic point for a hyperbolic attractor of a C^∞ flow, then not only is the stable manifold of such a point linearizable, but any element of the centralizer is simultaneously linearized.

Proof of Theorem 4.9. We first suppose that p is a nonresonant stable hyperbolic fixed point for Φ. Then there exists a neighborhood U of p and $\Phi\restriction_U$ is linear with the appropriate smooth coordinate system by Corollary 4.13. Let $f=f_1\circ f_2^{-1}$. Then f is a C^∞ homeomorphism of M and $f\varphi^t=\varphi^t f$ for all $t\in\mathbb{R}$. Hence, $f\restriction_U$ is linear in the same smooth coordinate system by Theorem 4.15.

There exists some $t>0$ such that $\varphi^t(V)\cap U$ is an open set. By hypothesis, $f\restriction_{\varphi^t(V)}$ is the identity. So in the local coordinate system f is a linear diffeomorphism on U that is the identity on a nonempty open subset of U. Hence, f is the identity on U. Now for any $y \in W^s(p)$ there exists some t such that $\varphi^t(y) \in U$. Then $f(y) = (\varphi^{-t} f \varphi^t)(y) = y$ and $f\restriction_{W^s(p)}$ is the identity. Hence, $f_1 = f_2$ on $W^s(p)$.

More generally, we let p be a nonresonant hyperbolic periodic point contained in Λ. Since $W^s(\mathscr{O}(p)) = W^{cs}(p)$ is dense in $W^s(\Lambda)$ (see for instance [12, Proposition 6.2.8]) by the In-Phase Theorem 2.5 and by the Spectral Decomposition Theorem, there exists some $T_0 > 0$ such that $W^s(\varphi^{T_0}(p)) \cap V$ contains an open set in $W^s(\varphi^{T_0}(p))$. Let $\pi(p)$ be the period of p. As above we define $f = f_1 \circ f_2^{-1}$. Then f is a C^∞ homeomorphism of M and $f\varphi^t = \varphi^t f$ for all $t \in \mathbb{R}$.

By Theorem 4.14 there exists a neighborhood U of $\varphi^{T_0}(p)$ in $W^s(\varphi^{T_0}(p))$ and a smooth coordinate system such that $\varphi^{\pi(p)}$ is linear on U. Then there is some $n \in \mathbb{N}$ such that $\varphi^{n\pi(p)}(V)$ contains an open set in U. Then, as above, f is the identity in U. Hence, f is the identity on $W^s(\varphi^{T_0}(p))$.

Now for $y \in W^s(\mathscr{O}(p))$ there exists some t such that $\varphi^t(y) \in W^s(\varphi^{T_0}(p))$. Then $f(y) = (\varphi^{-t} f \varphi^t)(y) = y$. Since $W^s(\mathscr{O}(p))$ is dense in $W^s(\Lambda)$ and f is the identity on $W^s(\mathscr{O}(p))$ we see that f is the identity on $W^s(\Lambda)$ so $f_1 = f_2$ on $W^s(\Lambda)$. $\square$

4.4. Lie group of commuting matrices. By simultaneous linearization we have reduced the problem to an algebraic one. We now define the Lie group of matrices that commute with a linear contraction. In the case where p is a periodic point the linearization is a linear contracting map and when p is fixed there will be a linear contracting flow.

The parametric version of Sternberg's linearization [1] implies that the linearization depends continuously on the flow. (While stated for discrete time, the result holds for flows either by adapting the proof or by using a standard argument to show that the linearizing diffeomorphism for the time-1 map linearizes the flow [12, Proof of Theorem 5.5.1].)

Remark 4.16. For Φ in the open and dense set $\mathscr{V} \subset \mathscr{A}^\infty(M)$ defined in Proposition 4.10 and the nonresonant fixed or periodic point $p = p(\Phi)$ whose orbit is fixed by any commuting diffeomorphism (Remark 4.4), we have an embedding $\mathscr{E}(p,\Phi)\colon \mathbb{R}^n \to M$ (with n necessarily the stable dimension) such that:

(1) $\mathscr{E}(p,\Phi)(\mathbb{R}^n) = W^s(p,\Phi) = W^s(p)$;
(2) $\Phi \mapsto \mathscr{E}(p,\Phi)$ is continuous (where we use the continuation of p);[9] and
(3) for each connected component $\mathscr{W}$ of $\mathscr{V}$ there exist $r, s \in \mathbb{N}$ with $r + 2s = n$, coordinates $x_1, \dots, x_{r+s} \in \mathbb{R}^r \times \mathbb{C}^s$, and a continuous map $\lambda = (\lambda_1, \dots, \lambda_{r+s})\colon \mathscr{W} \to (\mathbb{R}^*)^r \times (\mathbb{C}^* \setminus \mathbb{R}^*)^s$ (where "$*$" is short for "$\setminus\{0\}$") such that
   - if p is a hyperbolic fixed point, then $\Psi(p,\Phi) := \mathscr{E}(p,\Phi)^{-1} \circ \varphi^t\restriction_{W^s(p)} \circ \mathscr{E}(p,\Phi)$ is a linear contracting flow on $\mathbb{R}^n$ that is diagonalized in the x_i-coordinates with eigenvalues λ_i and depends continuously on $\Phi \in \mathscr{W}$;
   - if p is a hyperbolic periodic point with period $\pi(p)$, then $A(p,\Phi) := \mathscr{E}(p,\Phi)^{-1} \circ \varphi^{\pi(p)}\restriction_{W^s(p)} \circ \mathscr{E}(p,\Phi)$ is a linear contracting map on $\mathbb{R}^n$ that is diagonalized in the x_i-coordinates with eigenvalues λ_i and depends continuously on $\Phi \in \mathscr{W}$.

We now consider elements of the centralizer for these linearized maps. If $h \in \mathrm{Diff}^\infty(\mathbb{R}^n)$ commutes with a linear nonresonant map, then h is linear and furthermore diagonal with respect to the coordinates described above [20]. We fix a neighborhood $\mathscr{V}$ of the matrix (either the matrix for the linearized map $A(p,\Phi)$ or the matrix for the linearized flow $\Psi(p,\Phi)$) such that for any $B \in \mathscr{V}$ the sign of the real and imaginary parts of the eigenvalues for B agree with those of this matrix.

[9] Φ' near Φ has a unique hyperbolic p' near p which varies continuously with Φ'.

Remark 4.17 (Linear centralizer group). The set of invertible diagonal matrices that commute with an invertible diagonal matrix A is an abelian group isomorphic to the disconnected Lie group $Z \coloneqq Z_{r,s} \coloneqq \mathbb{R}^{r+s} \times (\mathbb{Z}/2\mathbb{Z})^r \times (S^1)^s$ [26]. The cyclic subgroup $\langle \epsilon \rangle$ generated by any

$$\epsilon \in \left\{(\theta_1,\ldots,\theta_{r+s},\epsilon_1,\ldots,\epsilon_{r+s}) \in Z_{r,s} \,\middle|\, \theta_i = 1\ \forall i,\ \epsilon_j = 1\ \forall j > r\right\} \sim (\mathbb{Z}/2\mathbb{Z})^r$$

is discrete in Z,

$$Z_0 \coloneqq Z_{r,s,\epsilon} \coloneqq Z_{r,s}/\langle \epsilon \rangle$$

is a disconnected abelian Lie group, and

$$Z_1 \coloneqq Z'_{r,s,\epsilon} \coloneqq \ker \chi / \langle \epsilon \rangle$$

is the maximal compact subgroup of Z_0, where the surjective homomorphism χ from $Z_{r,s}$ to the hyperplane Σ in $\mathbb{R}^{r+s}$ determined by $\sum_{i=1}^{r+s} \theta'_i = 0$, is defined by

$$\chi(\theta_1,\ldots,\theta_{r+s},\epsilon_1,\ldots,\epsilon_{r+s}) = (\theta_1 - \theta_{\text{ave}},\ldots,\theta_{r+s} - \theta_{\text{ave}}) = (\theta'_1,\ldots,\theta'_{r+s}).$$

Here, θ_{ave} is the average value of $\theta_1,\ldots,\theta_{r+s}$.

We will break the proof into two cases. The first case examines elements of Z_1 (whose orbits remain in a compact region, Subsection 4.5.1), and the second case considers the orbits under elements in $Z_0 \smallsetminus Z_1$, which tend to infinity (Subsection 4.5.2). The compact case will be the harder of the two.

For any $B \in \mathscr{V}$ (with $\mathscr{V}$ as defined after Remark 4.16 with r, s, ϵ constant), the abstract group Z is naturally isomorphic to the centralizer $Z(B)$ of B, Z_0 is isomorphic to $Z(B)/\langle B \rangle$ where $\langle B \rangle$ is the cyclic group generated by B, Z_1 represents rescalings[10] of Φ (as opposed to rescalings of time) on $W^s(\mathscr{O}(p))$ modulo period-maps, and $Z_0/Z_1 \simeq \mathbb{R}^{r+s-1}$.

Remark 4.18 (Fundamental domain, orbit space). As in [26, Section 3.5, p. 87], we define *fundamental domains* F, the *spaces* S_A *and* S_B *of orbits* of A and B in $\mathscr{V}$, and a canonical diffeomorphism of S_B onto S_A. The fundamental domains S_A and S_B consist of one point per orbit. In continuous time the fundamental domains are homeomorphic to S^{n-1}. In discrete time the fundamental domains are annuli, and identifying the boundaries of each annulus by A or B, respectively, makes them compact connected manifolds.

Each attractor (or repeller) of $\Phi \in \mathscr{U}_0$ (Remark 4.4) contains a fixed or periodic point p such that if $g \in \mathscr{C}^\infty(\Phi)$, then $g(\mathscr{O}(p)) = \mathscr{O}(p)$. Furthermore, each fixed or periodic point p of $\Phi \in \mathcal{V} \subset \mathscr{U}$ as in Proposition 4.10 is nonresonant.

Remark 4.19 (Linearization). If p is $\pi(p)$-periodic, then there is a unique $\tau \in [0, \pi(p))$ such that $g' \coloneqq g \circ \varphi^\tau$ is the identity on $\mathscr{O}(p)$, and

$$g'(W^s(p)) = W^s(p), \quad g'(W^u(p)) = W^u(p).$$

Since g' restricted to $W^s(p)$ is smooth, there is a linearization of g' restricted to $W^s(p)$, and we denote the corresponding element in Z_0 by $\bar{g}$.

Furthermore, if $g'\restriction_{W^s(p)} = \text{Id}$, then $g'\restriction_{W^s(\mathscr{O}(p))} = \text{Id}$ and so g' is the identity on an open set, as $W^s(\mathscr{O}(p))$ is open, and hence on M (Theorem 1.13), i.e., $g = \varphi^{-\tau}$. So to complete our proof of Theorems 1.11 and 1.12 we will perturb Φ in a way that forces $g'\restriction_{W^s(p)} = \text{Id}$.

[10] plus possibly some rotations (complex part) and flips (real part)

4.5. Perturbations near attractors. We now adapt the perturbation techniques from [26] to continuous time. We use the structural stability of hyperbolic attractors to make perturbations such that there is an open and dense set (or residual set depending on the situation) of flows such that no nontrivial element in the group Z_0 is in the centralizer for the perturbed system. Lemma 4.20 below shows how this implies these flows have a trivial centralizer.

In the proof of Theorem 1.11, we use the attractor or repeller that is a fixed point or single periodic orbit and perturb the flow to first obtain triviality of the centralizer on the basin of attraction. As described above this then extends to triviality of the centralizer on the entire manifold.

For elements in Z_1 the orbit of a point consists of points in a discrete group of invariant tori and this is the reason that Z_1 is referred to as the compact part; whereas, in $Z_0 \smallsetminus Z_1$ we have orbits that tend toward infinity and approach certain eigendirections. We then will work with each of these separately as the orbits have very different behavior.

One place where the different behavior of the orbits is seen is in the orbit space defined in Remark 4.18. The proof of Theorem 1.11 uses the orbit space and carries out different perturbations for elements in Z_1 and those in $Z_0 \smallsetminus Z_1$. The first step is to show that after a perturbation there is no nontrivial element of the compact part, Z_1, that commutes with the linearized system. We then perturb further so that no element from the noncompact part, $Z_0 \smallsetminus Z_1$, is in the centralizer for the linearized system. The next definitions and comments further demonstrate why we divide the compact and noncompact components into different sections.

For a diagonal matrix $B \in \mathscr{V}$ whose diagonal entries are the eigenvalues $(\lambda_1, \dots, \lambda_{r+s})$ and a diagonal matrix D whose diagonal elements are $(\mu_1, \dots, \mu_{r+s})$ we let, as in Remark 4.17,

$$\theta_i = \frac{\ln|\mu_i|}{\ln|\lambda_i|}, \theta_i' = \theta_i - \frac{1}{r+s}\sum_{j=1}^{r+s}\theta_j, \text{ and } \epsilon_i = \frac{\mu_i}{\exp\theta_i\rho_i(B)},$$

where $\exp\rho_i(B) = |\lambda_i|$ for $1 \le i \le r$, and $\exp\rho_i(B) = \lambda_i$ for $r < i \le r+s$. We then have an isomorphism from diagonal matrices onto $Z_{r,s}$ given by $\Theta_B(D) = (\theta_1, \dots, \theta_{r+s}, \epsilon_1, \dots, \epsilon_{r+s})$.

For $i \in \{1, \dots, r+s\}$ let

$$W_i \coloneqq \bigoplus_{j\in\{1,\dots,r+s\}} \begin{cases} \{0\} & \text{if } j \ne i, \\ \mathbb{R} & \text{if } i = j. \end{cases}$$

For $\Gamma \subset \{1, \dots, r+s\}$, let $W_\Gamma \coloneqq \bigoplus_{i\in\Gamma} W_i$, $\Gamma^c \coloneqq \{1, \dots, r+s\} \smallsetminus \Gamma$, and π_Γ the projection with kernel W_{Γ^c} and image W_Γ. The set W_Γ is B-invariant and so projecting onto S_B we obtain a submanifold $\widetilde{W_\Gamma}$ of S_B. We then have an open and dense set

$$W = \mathbb{R}^n \smallsetminus \bigcup_{\Gamma, \Gamma\ne\{1,\dots,r+s\}} W_\Gamma = \left\{ x \in \mathbb{R}^r \times \mathbb{C}^s \,\middle|\, x_i \ne 0 \,\forall\, 1 \le i \le r+s \right\}.$$

This set projects to an open and dense set $\widetilde{W} = S_B \smallsetminus \bigcup_{\Gamma,\Gamma\ne\{1,\dots,r+s\}} \widetilde{W_\Gamma}$ in S_B.

For the map B, we let $\tilde{\pi}^B_\Gamma$ be the induced smooth map from $S_B \smallsetminus \widetilde{W_{\Gamma^c}}$ to $\widetilde{W_\Gamma}$. Then the map $\tilde{\pi}^B_\Gamma$ commutes with the action of Z_0 on S_B [26, Section 3.6], and $\widetilde{W}$.

In [26, Section 3.3] it is shown that $\Gamma(D) \coloneqq \left\{ i \in \{1, \dots, r+s\} \,\middle|\, \theta_i' = \min\theta_j' \right\}$ satisfies $\Gamma(D) = \{1, \dots, r+s\}$ if and only if $D \in Z_1$. Thus, if $D \in Z_0 \smallsetminus Z_1$, then $\Gamma(D)$ is a proper nonempty subset of $\{1, \dots, r+s\}$, so the projection $\pi_{\Gamma(D)}$ has nontrivial kernel, but also does not map all points to the origin. By contrast, in the compact case the projection $\pi_{\Gamma(D)}$ is the identity map, and this is not useful in the arguments that follow. This is one reason we deal with the compact case separately.

The proof of Theorem 1.12 where $\dim(M) = 3$ is also split into two components. The compact part is handled by modifying some of the arguments for the proof of Theorem 1.11. The noncompact part is handled very easily by the low-dimensionality. These combine to give us an open and dense set of flows with trivial centralizer.

The proof of Theorem 1.12, in arbitrary dimension, uses an attractor or repeller that is not a fixed point or single periodic orbit. In fact, the construction can be done for any flow that has such an attractor or repeller, but in higher dimension we only obtain a residual set of flows with trivial centralizer. The argument uses the homoclinic points of a periodic point contained in the attractor or repeller. (Note that for an attractor or repeller that consists of a fixed point or single periodic orbit there are no homoclinic points.) Using the homoclinic points we do not need to separate the argument into the compact and noncompact part, since we do not need to use the orbit space to complete the proof. The proof of Theorem 1.12 is therefore appended to the subsection with the considerations of the noncompact part. The next lemma shows that the arguments described indeed prove Theorems 1.11 and 1.12.

Lemma 4.20. *For $g \in \mathscr{C}^\infty(\Phi)$ we have $g = \varphi^t$ for some $t \in \mathbb{R}$ if and only if $\bar{g} = 1_{Z_0}$ where $\bar{g}$ is the corresponding element in Z_0 given by the linearization as defined in Remark 4.19.*

Proof. If $g = \varphi^t$ for some $t \in \mathbb{R}$ then $\bar{g} = 1_{Z_0}$ by definition of Z_0. Suppose that $\bar{g} = 1_{Z_0}$. Then by definition of Z_0 we have $g' = g\varphi^\tau = \varphi^s$ for some $s \in \mathbb{R}$ on $W^s(\mathscr{O}(p))$, so $g = \varphi^{s-\tau}$ on $W^s(\mathscr{O}(p))$. By Theorems 4.9 and 1.13 this implies that $g = \varphi^{s-\tau}$ on M. □

We now proceed to make the perturbations so each $g \in \mathscr{C}^\infty(\Phi)$ satisfies $\bar{g} = 1_{Z_0}$.

4.5.1. ***Compact part of the centralizer.*** In the proof of Theorem 1.11 the perturbation for the compact part of the centralizer, nontrivial elements in Z_1, is different than for the noncompact part as we described above. In this section we perturb the flow so that no nontrivial element in Z_1 commutes with it, and we show that the set of flows we obtain is open and dense in $\mathscr{A}_1^\infty(M)$. We will be working in the orbit space and examine the effect of the perturbations on the elements in the orbit space.

We now show how to perturb the flows in the case of Theorem 1.11 to obtain an open and dense set of flows whose centralizer has no compact part. Take $p \in M$ either a fixed hyperbolic sink or a periodic hyperbolic sink. We treat the following 3 cases in parallel. In each case we will examine certain "exceptional sets" or "exceptional properties" and perturb the vector field generating the flow so that no element g in the centralizer can have $\bar{g} \in Z_1 \smallsetminus \{1_{Z_1}\}$ by examining the "exceptional sets" or "exceptional properties."

Case 1: The basin of p is not contained in the basin of a single repeller.
Case 2: The basin of p is contained in the basin of a fixed or periodic source.
Case 3: The basin of p is contained in the basin of a single repeller, which is not fixed or periodic.

Case 1. Let $J(p,\Phi)$ be the complement of the center unstable manifolds of the repellers in $W^s(p)$. The set $J(p,\Phi)$ is a nonempty, nowhere dense, closed, flow invariant set in $W^s(p)$. For the linearization A of $W^s(p)$ we denote the orbit space of $W^s(p)$ as described in Remark 4.18 as $S_A = S(p)$. Then $J(p,\Phi)$ projects to a nonempty, nowhere dense, closed Φ-invariant set $\tilde{J}(p,\Phi) \subset S(p)$. Since any $g \in \mathscr{C}^\infty(\Phi)$ fixes the repellers and the unstable set of a repeller, it leaves $J(p,\Phi)$ invariant. Likewise, the action induced by g on the space of orbits leaves $\tilde{J}(p,\Phi)$ invariant. Below we use the set $\tilde{J}(p,\Phi)$ to find a perturbation of the flow in such a way that $\tilde{J}(p,\Phi)$ is not invariant for any element of $Z_1 \smallsetminus \{1_{Z_1}\}$.

Case 2. Let q be the fixed or periodic source with $W^s(p) \smallsetminus \{p\} \subset W^u(\mathscr{O}(q)) \smallsetminus \{\mathscr{O}(q)\}$, $S(p)$ the orbit space for the fundamental domain of $W^s(p)$ (Remark 4.18), and $S(q)$ the orbit space for the fundamental domain for $W^u(q)$. Then each point in $S(p)$ corresponds to a unique point in $S(q)$, and an element of the Lie group that commutes with the linearization on $W^u(q)$ induces an action on $S(q)$ and hence an action on $S(p)$. Below we will perturb the flow in a neighborhood of a fundamental domain of $S(p)$ so the perturbation does not change $S(q)$ and so that that no nontrivial element in the compact part can be in the centralizer.

Case 3. There is a foliation of $W^s(p) \smallsetminus \{p\}$ by center-unstable manifolds of the repeller that is invariant and preserved by any element in $\mathscr{C}^\infty(\Phi)$. There is then an invariant foliation of $S(p)$ given by the image of the foliation under the projection to the space of orbits. We let $\mathscr{F}(p,\Phi)$ be the leaves of the center-unstable foliation and $\tilde{\mathscr{F}}(p,\Phi)$ be the foliation on the space of orbits. Below we perturb the flow so that no nontrivial element in the compact part leaves the foliation invariant.

Definition 4.21. We define the elements of Z_1 that correspond to elements in $\mathscr{C}^\infty(\Phi)$ as follows, according to the three different cases on page 39. With notations as in Remark 4.16(3), set

$$B \coloneqq \begin{cases} A & \text{if } p \text{ or } q \text{ is periodic,} \\ \Psi & \text{if } p \text{ or } q \text{ is fixed.} \end{cases}$$

Case 1: Let $Z_1(p,\Phi) = \left\{ \bar{g} \in Z_1(B(p,\Phi)) \,\middle|\, g(\tilde{J}(p,\Phi)) \subset \tilde{J}(p,\Phi) \right\}$.

Case 2: Let

$$Z_1(p,\Phi) \coloneqq \left\{ (\bar{g},\bar{g}') \in Z_1(B(p,\Phi)) \times Z_1'(B(q,\Phi)) \,\middle|\, \bar{g},\bar{g}' \text{ induce identical actions on } S(p) \right\}.$$

Case 3: Let $Z_1(p,\Phi) = \left\{ \bar{g} \in Z_1(B(p,\Phi)) \,\middle|\, \forall x \in S,\, T_x\bar{g}(T_x\tilde{\mathscr{F}}(p,\Phi)) = T_{\bar{g}(x)}\tilde{\mathscr{F}}(p,\Phi) \right\}$.

In each case, $Z_1(p,\Phi)$ is a closed subgroup of Z_1. Furthermore, for any closed subgroup Z_2 of Z_1 the set

$$\mathscr{V}_{Z_2} \coloneqq \left\{ \Phi \in \mathscr{V} \,\middle|\, Z_1(p,\Phi) \subset Z_2 \right\}$$

is open in $\mathscr{V}$ [26, Lemma, p. 94].

We perturb the flow Φ to a new flow Φ' such that the two flows only differ in the interior of a fundamental domain and such that $\Phi' = \Phi$ in a neighborhood of p. Since the two flows agree in a neighborhood of the boundary of the fundamental domain, they both induce an action on $S(p)$ that is a diffeomorphism if p is periodic and a smooth flow if p is fixed.

To obtain the needed perturbations, note that the open dense set $\widetilde{W} \subset S(p)$ is invariant under each element of Z_0. Since the axes correspond to the eigendirections for the matrices, W consists of vectors that are not eigenvectors, so $\widetilde{W}$ is the set of orbits in the orbit space that are not orbits for eigenvectors. The reason to work in $\widetilde{W}$ is that we can ensure after a perturbation that any element g of the centralizer such that $\bar{g} \in Z_1(p,\Phi)$ is the identity in Z_1; otherwise, we may have the identity in only some of the coordinates and would possibly need to make a series of perturbations.

We now show how a symmetry can be robustly broken:

Lemma 4.22. *If $g \in Z_1 \smallsetminus \{1_{Z_1}\}$, then $\left\{ \Phi \in \mathscr{U} \,\middle|\, g \in Z_1(p,\Phi) \right\}$ is nowhere dense in $\mathscr{U}$.*

Proof. **Case 1**. Since $\widetilde{W}$ is open and dense in $S(p)$, we may assume (by possibly passing to a small perturbation) that $\tilde{J}(p,\Phi) \cap \widetilde{W} \neq \varnothing$. Let $x \in \tilde{J}(p,\Phi) \cap \widetilde{W}$ such that gx is not in the image of the boundary of the fundamental domain in S. Since $\tilde{J}(p,\Phi)$ is nowhere dense we can make a perturbation Φ_0 which is the identity in the neighborhood of x, but supported in a neighborhood of gx. To do this we modify $\tilde{J}(p,\Phi)$ so that it no longer contains gx; so $gx \notin \tilde{J}(p,\Phi) = \tilde{\Phi}_0(\tilde{J}(p,\Phi))$. So $g \notin Z_1(A(p,\Phi))$ for p periodic or $g \notin Z_1(\Psi(p,\Phi))$ for p fixed.

Case 2. We can ensure that the fundamental domains of $W^s(p)$ and $W^u(q)$ are disjoint and perturb the flow Φ so the support intersects the fundamental domain for $W^s(p)$, but not that of $W^u(q)$. Then for $g_1 \in Z_1(B(p,\Phi))$ and $g_2 \in Z_1(B(q,\Phi))$ where $(g_1,g_2) \neq (1_{Z_1(B(p,\Phi))}, 1_{Z_1(B(q,\Phi))})$, we have $(g_1,g_2) \in Z_1(p,\Phi)$ if the induced actions on $S(p)$ are the same. For a point $x \in S(p)$ we can use a small perturbation Φ_0 arbitrarily close to the identity such that $g_2(x) \neq \tilde{\Phi}_0 g_1 \tilde{\Phi}_0^{-1}(x)$. Then $(g_1,g_2) \notin Z_1(p,\Phi')$, and this is an open condition.

Case 3. The proof is similar to the proof for Case 1. We let $x \in \widetilde{W}$ such that gx is not in the boundary of the image of the fundamental domain in $S(p)$. We now perturb the flow Φ to a new flow Φ' such that the two flows agree in a neighborhood of the boundary of the fundamental domain and

$$T_{gx}\tilde{\mathscr{F}}(p,\Phi') \neq T_{gx}\tilde{\mathscr{F}}(p,\Phi) = T_x g(T_x\tilde{\mathscr{F}}(p,\Phi)).$$

So $g \notin Z_1(A(p,\Phi))$ for p periodic or $g \notin Z_1(\Psi(p,\Phi))$ for p fixed. □

We now show how Lemma 4.22 implies that there is an open and dense set of flows such that there is no compact part of the centralizer. For a closed subgroup Z_2 of Z_1 we let

$$\mathscr{U}_{Z_2} = \{\Phi \,|\, Z_1(\Phi) \subset Z_2\}.$$

The set $\mathscr{U}_{\{1_{Z_1}\}}$ is open (by the discussion before Lemma 4.22) and dense in $\mathscr{U}$: Let O be open in $\mathscr{U}$. Then there exists some $\Phi \in O$ such that $Z_1(p,\Phi)$ is minimal among the $Z_1(p,\Phi')$ for $\Phi' \in O$. Then for Φ' near Φ in O we have $Z_1(p,\Phi') \subset Z_1(p,\Phi)$ [26, Lemma 5.2]. By minimality this implies that $Z_1(p,\Phi') = Z_1(p,\Phi)$. Lemma 4.22 now implies that $Z_1(p,\Phi) = \{1_{Z_1}\}$. So there is an open and dense $\mathscr{U}_0 \subset \mathscr{A}_1^\infty(M)$ such that for $\Phi \in \mathscr{U}_0$ and $g \in \mathscr{C}^\infty(\Phi)$ where $\bar{g} \in Z_1(p,\Phi)$ we have $\bar{g} = 1_{Z_1}$.

4.5.2. ***Noncompact part of the centralizer.*** We now show how to perturb the flows to eliminate the noncompact part of the centralizer. For each of the three different cases we define $Z_0(p,\Phi)$ as we did for $Z_1(p,\Phi)$ (Definition 4.21).

Theorem 1.11 follows from the next proposition.

Proposition 4.23. *$Z_0(p,\Phi)$ is trivial for each Φ in an open dense set $\mathscr{V}_1 \subset \mathscr{V} \subset \mathscr{A}_1^\infty$.*

Proof. As in the proof of Lemma 4.22 we divide this proof into the three different cases on page 39. In each case we use an invariant closed exceptional set or exceptional properties. Then we can choose a point $x \in S_A$ in the exceptional set and make a perturbation such that $\tilde{\pi}_\Gamma^A(x)$ is not in the exceptional set for any proper nonempty subset Γ of $\{1,..,r+s\}$. The next lemma then shows that if there is a nontrivial element in the centralizer there is some Γ such that $\tilde{\pi}_\Gamma^A(x)$ is in the exceptional set, a contradiction.

Lemma 4.24 ([26, Lemma 1])**.** *Let $A \in \mathscr{V}$ and $h \in Z_0 \smallsetminus Z_1$. Then*

$$\lim_{n\to\infty} d(h^n x, h^n(\tilde{\pi}^A_{\Gamma(h)}(x))) = 0$$

for any x in $S_A \smallsetminus \widetilde{W}_{\Gamma(h)^c}$ and there is a subsequence with $\lim_{k\to\infty} h^{n_k} x = \tilde{\pi}^A_{\Gamma(h)}(x)$.

In Case 1, say that $\Phi \in \mathscr{V}$ belongs to $\mathscr{V}_1$ if and only if there exists $x \in \tilde{J}(p,\Phi) \cap \widetilde{W}$ such that for any nontrivial proper subset Γ of $\{1,\dots,r+s\}$ the point $\tilde{\pi}_\Gamma^A \in \widetilde{W}_\Gamma$ does not belong to $\tilde{J}(p,\Phi)$. The set $\mathscr{V}_1$ is open because $\tilde{\pi}_E^A$ and $\tilde{J}(p,\Phi)$ depend continuously on $\Phi \in \mathscr{V}$. For any $\Phi \in \mathscr{V}$, we choose a special small perturbation Φ' such that for some $x \in \tilde{J}(p,\Phi')$ we have $\tilde{\pi}_\Gamma^A(x) \notin \tilde{J}(p,\Phi')$ for any nonempty proper subset Γ of $\{1,\dots,r+s\}$. To perturb the flow we make a small change to the vector field generating the flow. We fix Γ and a small neighborhood of a point corresponding to $\tilde{\pi}_\Gamma^A(x)$ so that $x \in \tilde{J}(p,\Phi')$, but $\tilde{\pi}_\Gamma^A(x) \notin \tilde{J}(p,\Phi')$. We do this for each Γ. Then Φ' belongs to $\mathscr{V}_1$, so $\mathscr{V}_1$ is dense.

For $\Phi \in \mathcal{V}_1$, $\bar{g} \in Z_0(p,\Phi) \smallsetminus Z_1(p,\Phi)$, and $x \in \tilde{J}(p,\Phi) \cap \widetilde{W}$ as in the definition of $\mathcal{V}_1$, Lemma 4.24 implies $\tilde{\pi}^A_{\bar{g}}(x) \in \tilde{J}(p,\Phi)$, a contradiction. So $Z_0(p,\Phi)$ is trivial.

In Case 2, we define $\widetilde{W}$ for p as described above as an open and dense set of S_A and define $\widetilde{W}'$ similarly for the point q. We say $\Phi \in \mathcal{V}$ belongs to $\mathcal{V}_1$ if there belongs some point $x \in \widetilde{W} \cap \widetilde{W}'$ such that for any proper nonempty set Γ we have $\tilde{\pi}^A_\Gamma(x) \in \widetilde{W}'$. Since the sets $\widetilde{W}$ and $\widetilde{W}'$ and the function $\tilde{\pi}^A$ depend continuously on Φ we see that $\mathcal{V}_1$ is open. To see that it is dense again perturb the flow near a point in $\widetilde{W} \cap \widetilde{W}'$. To do this let $x \in \widetilde{W} \cap \widetilde{W}'$ and perturb the flow to a flow Φ' such that the vector fields for the flows agree on a small neighborhood of the fundamental domain for S_A and in a neighborhood of the orbit of x and such that $\tilde{\pi}^A_\Gamma(x) \in \widetilde{W}'$ for any proper nonempty subset of Γ.

For $\Phi \in \mathcal{V}_1$ and $(g_1, g_2) \in (Z_0 \times Z_0') \smallsetminus (Z_1 \times Z_1')$ such that (g_1, g_2) have the same action on S_A we have $g_1 \in Z_0 \smallsetminus Z_1$ and $g_2 \in Z_0' \smallsetminus Z_1'$. By Lemma 4.24, $\tilde{\pi}^A_{g_1}(x)$ is a limit point of $\{g_1^n(x)\}_{n\geq 0}$ and so the limit set of $\{g_2^n(x)\}_{n\geq 0}$ is contained in $S_A \smallsetminus \widetilde{W}'$. This contradicts the choice of x for the flow.

In Case 3, we let $\mathcal{V}_1$ be the set of $\Phi \in \mathcal{V}$ such that for some point $x \in \widetilde{W}$ and any proper nonempty set Γ we have $T_x\tilde{F}(p,\Phi)$ transverse to W_{Γ^c} and $T_{\tilde{\pi}^A_\Gamma(x)}\tilde{F}(p,\Phi)$ is transverse to W_Γ. The set $\mathcal{V}_1$ is open since the tangent spaces depend continuously and transversality is then an open condition. To see that it is dense we perturb the flow to obtain the transversality conditions.

To see that $\Phi \in \mathcal{V}_1$ and $g \in Z_0 \smallsetminus Z_1$ does not leave $\tilde{F}(p,\Phi)$ invariant we use the next lemma to obtain the contradiction.

Lemma 4.25 ([26, Lemma 2])**.** *Let $\Phi \in \mathcal{V}$, $h \in Z_0 \smallsetminus Z_1$, $x \in S_A \smallsetminus \widetilde{W}_{\Gamma(h)^c}$, and $\{n_k\}$ a sequence of integers satisfying the conclusion of Lemma 4.24. If V is a subspace of T_xS transverse to $\widetilde{W}_{\Gamma(h)^c}$ such that*

$$\lim_{k\to\infty} T_x h^{n_k}(V) = V_0 \subset T_{\tilde{\pi}^A_h(x)} S_A,$$

then either $V_0 \subset W_\Gamma(h)$ or $W_\Gamma(h) \subset V_0$.

This proves Proposition 4.23 and hence Theorem 1.11. □

4.5.3. ***Proof of Theorem 1.12.*** Since the conclusion of Theorem 1.11 is stronger than the conclusion of Theorem 1.12 the theorem holds if there is an attractor or repeller that is a fixed point or single periodic orbit. We may then assume that all attractors and repellers for our flow are neither a single periodic orbit nor a fixed point.

Let $\mathcal{V}$ be an open dense set of flows from Theorem 1.13 and such that all attractors or repellers contain a periodic orbit that is nonresonant and such that the orbit is fixed by each element of the centralizer as in Lemma 4.3. Let $\Phi \in \mathcal{V}$ and Λ be an attractor and $p \in \Lambda$ be a periodic point of Λ such that $g(p) \in \mathcal{O}(p)$ for each $g \in \mathcal{C}^\infty(\Phi)$. The set of homoclinic points related to p is $J(\Phi) = W^s(p) \cap W^{cu}(p) \smallsetminus \{p\}$. For each $q \in J(\Phi)$ there is a unique point $q' \in W^u(p)$ such that $q' = \varphi^s(q)$ where $0 \leq s < \pi(p)$ and $\pi(p)$ is the period of p. There exist linearizations of $W^s(p)$ and $W^u(p)$ as described in Remark 4.16. If $\dim(E^s) = n_1$ and $\dim(E^u) = n_2$, then there is a map h from $J(\Phi)$ into $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ given by

$$h(q) = (\mathcal{E}^s(p,\Phi)^{-1}(q), \mathcal{E}^u(p,\Phi)^{-1}(q')),$$

where $\mathcal{E}^s(p,\Phi)$ is the linearization of $W^s(p)$ and $\mathcal{E}^u(p,\Phi)$ is the linearization of $W^u(p)$. The map h is injective and we let $\tilde{J}(p,\Phi) = h(J(p,\Phi))$. This is a discrete closed set in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Furthermore, the set $\tilde{J}(p,\Phi)$ is invariant under the transformation $A(\Phi) = (A_s, A_u^{-1})$ where A_s is the linearization of the flow on $W^s(p)$, and A_u is the linearization of the flow on $W^u(p)$. Furthermore, for any element $g \in C^\infty(\Phi)$ and $\bar{g}_s$ the linearization for g of $W^s(p)$ and $\bar{g}_u$ the linearization for g of $W^u(p)$ the set $\tilde{J}(p,\Phi)$ is invariant for

$(\bar{g}_s, \bar{g}_u^{-1})$. The proof of Theorem 1.12 follows from this next proposition. This proposition and its proof are almost identical to the proof of Proposition 1 of [26, p. 92, p. 95].

Proposition 4.26. *If* $\dim(M) = 3$ *there is an open and dense set* $\mathscr{U}$ *of flows such that if* $\Phi \in \mathscr{U}$ *then no* $g \in Z_0 \setminus \{1_{Z_0}\}$ *leaves* $\tilde{J}(p, \Phi)$ *invariant. If* $\dim(M) \geq 4$*, then there is residual set of flows* $\mathscr{R} \subset \mathscr{V}$ *such that no* $g \in Z_0 \setminus \{1_{Z_0}\}$ *leaves* $\tilde{J}(p, \Phi)$ *invariant for* $\Phi \in \mathscr{R}$.

Proof. Let $x, y \in J(p, \Phi)$ such that y is not in the orbit of x. Let $h(x) = (x_1, x_2)$ and $h(y) = (y_1, y_2)$. Fix x_1' sufficiently close to x_1 and select a small neighborhood V of $\varphi^{\pi(p)}(x)$ so that it does not intersect the closed set consisting of p the orbit of y and the backward orbit of x. We now let Φ' be a perturbation of the flow so the flows agree outside of V and the stable manifold of p in V is the same for the two flows. This can be done in such a way that $h(x) = (x_1', x_2)$ for the perturbed map.

Since the flows Φ' and Φ agree in a neighborhood of p the linearizations are the same. Furthermore, it is not possible for some $g \in Z_0$ to satisfy $g(x) = y$ for both Φ and Φ'. Since the set $\tilde{J}(p, \Phi)$ is discrete we know that if $\bar{g} \in Z_0 \setminus \{1_{Z_0}\}$ that the set of Φ such that $\tilde{J}(p, \Phi)$ is g-invariant is nowhere dense.

Since the homoclinic points $J(p, \Phi)$ are countable there is a residual set $\mathscr{R}$ of $\Phi \in \mathscr{V}$ such that if $g \in Z_0$ and $g(\tilde{J}(p, \Phi)) = \tilde{J}(p, \Phi)$, then $g = 1_{Z_0}$.

We now assume that $\dim(M) = 3$. Modifying the above argument, together with the proof of Case 1 in Lemma 4.22, and the argument just after the proof of Lemma 4.22 we see that there is a C^1-open and C^r-dense set of flows Φ such that no $g \in Z_1 \setminus 1_{Z_1}$ leaves $\tilde{J}(p, \Phi)$ invariant. In this case the stable and unstable manifolds are one-dimensional and the transformation $A(\Phi) = (A_s, A_u^{-1}) = (\lambda_1, \lambda_2)$. Let D be a diagonal matrix with diagonal entries (μ_1, μ_2). For the associated action of D, denoted by h, we see that $h \in Z_0 - Z_1$ if and only if

$$\frac{\log|\mu_1|}{\log|\lambda_1|} \neq \frac{\log|\mu_2|}{\log|\lambda_2|}.$$

Then there exists some $k, l \in \mathbb{Z}$ such that $A(\Phi)^k h^l$ is a contraction—which contradicts $\tilde{J}(p, \Phi)$ being discrete and invariant. So Φ has trivial centralizer. □

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