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(Recommended by Boris Hasselblatt)

ABSTRACT. In this note we report some advances in the study of thermodynamic formalism for a class of partially hyperbolic systems—center isometries—that includes regular elements in Anosov actions. The techniques are of geometric flavor (in particular, not relying on symbolic dynamics) and even provide new information in the classical case.

For such systems, we give in particular a constructive proof of the existence of the SRB measure and of the entropy maximizing measure. We also establish very fine statistical properties (Bernoulliness), and we give a characterization of equilibrium states in terms of their conditional measures in the stable/unstable lamination, similar to the SRB case. The construction is applied to obtain the uniqueness of quasi-invariant measures associated to Hölder Jacobians for the horocyclic flow.

1. Introduction

Let M be a closed Riemannian manifold and $f : M \to M$ be a diffeomorphism. The study of the statistical properties of f, and in particular its invariant measures, provides a powerful tool to understand the dynamical properties of the map. Let us recall that a Borel probability measure μ on M is f-invariant if $f_*\mu = \mu$, where $f_*\mu(A) = \mu(f^{-1}A)$, and denote by $\mathcal{P}r_f(M)$ the set of all such measures on M.

For maps having rich dynamics, the set $\mathcal{P}r_f(M)$ is usually very complicated, so further restrictions have to be imposed in order to obtain meaningful results. A particularly important choice, both from the theoretical and applied point of view, are the so-called equilibrium states; these are f-invariant measures obtained by a variational principle associated to some real valued map. See part 2.1 for the precise definition.

The study of such types of measures and their properties (thermodynamic formalism) is notably well developed for completely hyperbolic systems (Anosov, or more generally, Axiom A). The reader can consult [5] for an introduction to these topics. Notwithstanding this, under very mild relaxations of the hyperbolicity hypothesis, the panorama becomes much less understood, even in the partially hyperbolic case, which is one of the most extensively researched types of systems besides hyperbolic ones.

In the present note we announce an advance in this theory and report on new methods to study some natural classes of partially hyperbolic systems, as are the ones determined by (regular elements of) Anosov Lie group actions. We point out the existence of other geometrical approaches to study equilibrium states in the partially hyperbolic

setting, as for example the work of Spatzier and Vischer [36], studying isometric extensions of hyperbolic systems, and the recent article of Climenhaga, Pesin, and Zelerowics [11], where the authors use geometric measure theory in a framework similar to ours. Some of our results are comparable to the second cited work, but the methods that we present are completely different, and they allow us to have more control over the equilibrium measures, particularly in terms of their conditional measures along the invariant foliations. This is evidenced by the very fine statistical properties what we are able to prove, not only for the system but also for the referred invariant foliation. In the recent preprint [3], Bonthonneau, Guillarmou, and Weich study SRB measures for abelian actions, instead of an individual regular element as in our case. The techniques in this last cited work are of functional analytic type, using anisotropic Sobolev spaces and Fredholm theory. It seems that our approaches may be complementary to theirs. In any case, the synergy between the other available methods and ours stands as an interesting problem to investigate.

2. Measures along leaves of invariant foliations

2.1. Basic thermodynamic formalism. Consider a compact metric space (M, d), and let $f : M \to M$ be continuous map. Given $x \in M, \epsilon > 0, n \in \mathbb{N}$ we denote by $D(x, \epsilon)$ the open disc of center x and radius ϵ, and by $D(x, \epsilon, n)$ the open (ϵ, n)-Bowen ball centered at x,

$$D(x, \epsilon, n) = \left\{ y \in M : d(f^j x, f^j y) < \epsilon, \; j = 0, \ldots, n-1 \right\};$$

denote

$$s(\epsilon, n) = \inf \left\{ \#E : M = \bigcup_{x \in E} D(x, \epsilon, n) \right\}.$$

Definition 2.1. The topological entropy of f is the quantity

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log s(\epsilon, n)}{n}.$$

Topological entropy is perhaps the most important topological invariant for continuous maps, measuring (loosely speaking) the exponential rate of expansion between orbits. We refer the reader to [37] for a introduction to this theory, and the proof of the facts below.

It so happens that it is also important to consider some weighted versions of the previous quantity. For $\varphi : M \to \mathbb{R}$ a continuous map (called the *potential* in this theory), denote

$$S(\epsilon, n) = \inf \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : M = \bigcup_{x \in E} D(x, \epsilon, n) \right\}, \quad S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ f^i.$$

Definition 2.2. The topological pressure associated to the system (f, φ) is

$$P_{top}^f(\varphi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log S(\epsilon, n)}{n}.$$

Observe that $h_{\text{top}}(f) = P_{\text{top}}(f, 0)$. There is also a metric version of the previous concepts; to state it let us recall that $\mu \in \mathscr{P}r_f(M)$ is ergodic ($\mu \in \mathscr{E}rg_f(M)$) if any $\psi \in L^2(M, \mu)$ satisfying $f \circ \psi = \psi$-a.e. is constant almost everywhere.

Definition 2.3. Let μ be an ergodic measure. Then the metric entropy of f with respect to μ is

$$h_\mu(f) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{-\log D(x, \epsilon, n)}{n},$$

and if φ is a potential, the metric pressure of (f, φ) with respect to μ is

$$P_\mu(f, \varphi) = h_\mu(f) + \int \varphi \, d\mu.$$

The limit above exists μ-a.e., see [7]. We then have the following:

Theorem (Variational Principle - Walters)**.** *It holds that*

$$P_{\text{top}}(f, \varphi) = \sup_{\mu \in \mathcal{E}rg_f(M)} P_\mu(\varphi).$$

Definition 2.4. $\mu \in \mathcal{E}rg_f(M)$ is an (ergodic) equilibrium state if $P_{\text{top}}(f, \varphi) = P_\mu(f, \varphi)$.

2.2. Center isometries. We now specify the type of map to which our results apply.

Definition 2.5. Let M be a closed manifold. A diffeomorphism $f : M \to M$ is a center isometry if there exist a continuous splitting of the tangent bundle of the form $TM = E^u \oplus E^c \oplus E^s$ and a (at least continuous) Riemannian metric $\|\cdot\|$ such that

   (1) $\dim E^u, \dim E^s \geq 1$;
   (2) E^u, E^s, E^c are Df-invariant in the sense that for every $x \in M$, $Df(E_x^*) = E_{fx}^*$;
   (3) for every $x \in M$, for every unit vector $v^* \in E_x^*$, $\|D_x f(v^c)\| = 1$, $\|D_x f^n(v^s)\| < 1$, and $\|D_x f^{-1}(v^u)\| < 1$.

Typical examples of such maps are group extensions of Anosov systems, and regular elements of Anosov actions [20]. We refer the reader to the survey article [19] for basic information on these systems, and in particular for a discussion of the following:

   (∗) all bundles $E^s, E^c, E^c, E^{cs} = E^c \oplus E^s, E^{cu} = E^c \oplus E^u$ are integrable to foliations $\mathcal{W}^s, \mathcal{W}^c, \mathcal{W}^u, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ that are invariant under f, that is, f permutes their leaves. Moreover, $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ are sub-foliated by leaves of $\mathcal{W}^s, \mathcal{W}^c$ and $\mathcal{W}^u, \mathcal{W}^c$, respectively. Finally, $\mathcal{W}^s$ is exponentially contracting in the sense that f exponentially contracts the corresponding induced distances on leaves, whereas $\mathcal{W}^u$ is exponentially expanding.

Our first central result is the next theorem.

Theorem 2.1. *Let $f : M \to M$ be a center isometry of class $\mathcal{C}^2$ such that every leaf of $\mathcal{W}^s, \mathcal{W}^u$ is dense. Let $\varphi : M \to \mathbb{R}$ be a Hölder potential that is either*

   (1) *constant on leaves of $\mathcal{W}^c$, or*
   (2) $\varphi = -\log \det Df\big|E^u$ *(SRB case).*

Then there exist $P \in \mathbb{R}$ and families of measures $\mu^u = \{\mu_x^u\}_{x \in M}$, $\mu^s = \{\mu_x^s\}_{x \in M}$, $\mu^{cu} = \{\mu_x^{cu}\}_{x \in M}$, $\mu^{cs} = \{\mu_x^{cs}\}_{x \in M}$ such that for every $x \in M$

   (1) *the measure μ^*, $* \in \{u, s, cu, cs\}$, is a Radon measure on $W^*(x)$ of full (relative) support, and $y \in W^*(x)$ implies $\mu_x^* = \mu_y^*$;*
   (2) *the following quasi-invariance properties are satisfied:*
       (a) $\mu_{fx}^\sigma = e^{P - \varphi} f_* \mu_x^\sigma$, $\quad \sigma \in \{u, cu\}$,
       (b) $\mu_{fx}^\sigma = e^{\varphi - P} f_* \mu_x^\sigma$, $\quad \sigma \in \{s, cs\}$.

The method of the proof is a generalization of the arguments used by Margulis for studying the entropy maximizing measure ($\varphi \equiv 0$) in mixing hyperbolic flows [24].

A consequence of the above is that the foliation $\mathcal{W}^u$ is absolutely continuous with respect to the family μ^{cs}. To explain this, consider x_0, y_0 in the same leaf of $\mathcal{W}^u$ and let $\text{hol}^u = \text{hol}_{x_0, y_0}^u : A(x_0) \subset W^{cs}(x_0) \to B(y_0) \subset W^{cs}(y_0)$ is the Poincaré map that sends x_0 to y_0. Then

$$(\text{hol}^u)_*^{-1} \mu_{y_0}^{cs} = \text{Jac}_{x_0, y_0}^u \cdot \mu_{x_0}^{cs}$$

where

$$\text{(2.1)} \qquad \text{Jac}^u_{x_0,y_0}(x) = \prod_{j=1}^{\infty} \frac{e^{\varphi \circ f^{-j}(\text{hol}^u x)}}{e^{\varphi \circ f^{-j}(x)}}.$$

Sketch of the proof. By (2) we get that $f^{-n}\mu^{cs}_{x_0} = e^{S_n \varphi \circ f^{-n+1} - nP} \mu^{cs}_{f^{-n}x_0}$, and similarly for y_0. For $x \in A(x_0), y = \text{hol}^u(x) \in B(y_0)$, the points $f^{-n}x, f^{-n}y$ approximate each other exponentially fast with n, therefore by the Hölder assumption on φ, the difference

$$\left| S_n \varphi(f^{-n+1}(x)) - S_n \varphi(f^{-n+1}(x)) \right|$$

is bounded in n, thus implying that $\text{Jac}^u_{x_0,y_0}(x)$ is well defined. Finally, by invariance of the foliations, $\text{hol}^u x_0, y_0 = f^n \circ \text{hol}^u_{f^{-n}x_0, f^{-n}y_0} \circ f^n$, and representing $\text{hol}^u_{f^{-n}x_0, f^{-n}y_0}$ in exponential charts near $f^{-n}x_0$ one sees that this map approximates uniformly the identity. From the above facts follows the claim. $\qquad \square$

Similar considerations can be made to the $\mu^{cu}, \mathcal{W}^s$. We now define on each leaf $W^u(x) \in \mathcal{W}^u$ a projective class of measures $[v^u_x]$ where

$$\text{(2.2)} \qquad v^u_x = \Delta^u_x \mu^u_x, \qquad \Delta^u_x(y) := \prod_{k=1}^{\infty} \frac{e^{\varphi \circ f^{-k}(y)}}{e^{\varphi \circ f^{-k}(x)}}, y \in W^u(x).$$

The function $\Delta^u_x : W^u(x) \to \mathbb{R}$ is continuous, and if $x' \in W^u(x), v^u_{x'} = \Delta^u_{x'}(x) v^u_x$. Furthermore, we have

$$\text{(2.3)} \qquad f^{-1} v^u_{fx} = \Delta_{fx} \circ f \cdot f^{-1} \mu^u_{fx} = \Delta_{fx} \circ f \cdot e^{P-\varphi} \mu^u_x = e^{P-\varphi(x)} v^u_x.$$

3. Construction of the equilibrium state

We will now use the families of measures given in Theorem 2.1 to construct an equilibrium state for the system (f, φ). We keep the assumptions of that theorem for the rest of the article.

Definition 3.1. If $\mathcal{F} \subset M^n$ is a foliation of codimension q we say that an open set $U \subset M$ is foliation box of $\mathcal{F}$ if it is homeomorphic to $C = (-1,1)^{n-q} \times (-1,1)^q$ by a homeomorphism sending $\mathcal{F}|U$ to the horizontal foliation of C. In this case the embedded discs of U corresponding to the vertical foliation of C will be called the vertical slices of U.

Fix a foliation box U of $\mathcal{W}^u$ together with a vertical slice W, that without loss of generality can be assumed to be a disc in some $W^{cs}(x_0)$. For $A \subset U$ open, define the function $\alpha_{U,W,A} : W \to \mathbb{R}$ by $\alpha_{U,W,A}(w) = v^u_w(A \cap W^u(w, U))$, where $W^u(u, U)$ is the connected component of $W^u(w) \cap U$ that contains w. This function is Borel measurable (in fact, semi-continuous).

Now consider the Borel regular measure $\text{m}_{U,W}$ on U determined by

$$A \subset U \text{ open} \Rightarrow \text{m}_{U,W}(A) = \int_W \alpha_{U,W,A}(w) \, d\mu^{cs}_{x_0}(w).$$

The key property is the following.

Proposition 3.1. *If W' is another vertical slice of U then $\text{m}_{U,W} = \text{m}_{U,W'}$.*

Proof. Without loss of generality, assume $W' \subset W^{cs}(x'_0)$, and consider $h = \text{hol}^u_{w_0, w'_0} : W \to W'$ the Poincaré map. For $A \subset U$ open and $w \in U$ denote $A_w = A \cap W^u(w, U)$;

then

$$m_{U,W'}(A) = \int_{W'} \nu^u_{w'}(A_{w'}) \, d\mu^{cs}_{w'_0}(w') = \int_W \nu^u_{h(w)}(A_{h(w)}) \mathrm{Jac}^u_{w_0,w'_0}(w) \, d\mu^{cs}_{w_0}(w)$$

$$= \int_W \left(\nu^u_{h(w)}(A_w) \prod_{k=1}^{+\infty} \frac{e^{\varphi \circ f^{-k} h(w)}}{e^{\varphi \circ f^{-k}(w)}} \right) d\mu^{cs}_{w_0}(w)$$

$$= \int_W \nu^u_w(A_w) \, d\mu^{cs}_{w_0}(w) = m_{U,W}(A). \qquad \square$$

We write $m_U = m_{U,W}$ where $W \subset U$ is any vertical slice. As a consequence, we have that if U, U', U'' are foliation boxes of $\mathscr{W}^u$ with $U \cup U' \subset U''$ then $m_U = m_{U'}$. We can thus fix a finite covering $\{U_i\}_{i=1}^r$ of M by foliation boxes of $\mathscr{W}^u$ and define a Borel probability measure m_φ on M such that for every i it holds $m_\varphi|U_i = c \cdot m_{U_i}$, where $c > 0$ is a normalization constant.

It is not hard to realize that m_φ is f-invariant (compare (1) and (2) in Theorem 2.1). The following is also true.

Theorem 3.1.

(1) m_φ is the unique equilibrium state for the potential φ and $P = P_{\mathrm{top}}(f, \varphi)$; in particular m_φ is ergodic.

(2) If U is a sufficiently small foliation box centered at $x \in M$, then $m_\varphi|U$ is equivalent to $\mu^u_x \times \mu^{cs}_x$.

(3) There exists $K \geq 0$ only depending on φ such that for every $\epsilon > 0$ there exists $c(\epsilon) > 0$ satisfying for every $x \in M, n \geq 0$

$$c(\epsilon)^{-1} e^{-Kn} \leq \frac{m_\varphi(D(x, \epsilon, n))}{e^{S_n \varphi(x) - n P_{\mathrm{top}}(\varphi)}} \leq e^{Kn} c(\epsilon).$$

If the potential is constant along leaves of $\mathscr{W}^c$ then one can take $K = 0$.

Uniqueness implies ergodicity, by standard arguments. The last part is a generalization of the so-called Gibbs property, a concept of central importance in statistical physics, see [32].

4. Characterization of equilibrium states in terms of conditional measures

In hyperbolic systems (say, taking $E^c = 0$ in Definition 2.5), a particularly relevant measure is the SRB measure. This is an f-invariant measure μ that can be characterized by one of the two following equivalent conditions:

(1) μ is an equilibrium state for the the potential $\varphi = -\log \det Df|E^u$;

(2) the conditionals of μ induced in leaves of $\mathscr{W}^u$ are absolutely continuous with respect the induced Lebesgue measure in the leaf.

This notion was introduced first by Sinai in [35] and developed by Ruelle and Bowen [31, 4, 6]. It is difficult to overestimate the importance of SRB measures in ergodic theory; the survey article [38] remains as an excellent introduction to the subject, and the reader can also check [2] for more recent developments.

Let us explain the second condition. Given a Borel probability m on M and a family of measures $\eta^u = \{\eta^u_x\}_{x \in M}$ on the leaves of $\mathscr{W}^u$ one says that m has conditionals absolutely continuous/equivalent to η^u if for some[1] measurable m-partition ξ subordinated to the partition by leaves of $\mathscr{W}^u$ the induced conditional measures m^ξ_x are absolutely continuous/equivalent with respect to η^u_x, for m-a.e.; in this part we use freely the theory of conditional measures as developed, for example, in [30]. Going back to (2), for

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[1] Equivalently, for every measurable partition.

SRB measures the family η^u is given by the corresponding induced Lebesgue measures on leaves of $\mathcal{W}^u$. Let us also point out that this characterization of SRB measures for (non-hyperbolic) systems is consequence of the very general results of Ledrappier and Young [21, 22], a culmination of several other important contributions that for the sake of keeping this presentation short, we refer to the above cited articles for the references.

While trying to obtain a similar characterization for other equilibrium states, the difficulty is the absence of families of measures to compare with. Here we solve this problem, that even for classical hyperbolic system remained unknown until now (observe that if $E^c = 0$ then any potential is automatically constant along center leaves).

Theorem 4.1. *Let* $m \in \mathcal{P}r_f(M)$ *and assume that with respect to some[2] partition ξ we have that* $m_x \ll \mu_x^u$ *for* m*-a.e.*(x). *Then* m *is an equilibrium estate (a posteriori,* $m = m_\varphi$*).*

5. Bernoulli property of the equilibrium state

We can say more about the statistical properties of the system (f, m_φ) besides its ergodicity. A much stronger property is the so-called Kolmogorov property, that can be characterized as follows.

Definition 5.1. For $m \in \mathcal{P}r_f(M)$ we say that (f, m) has the Kolmogorov property if its Pinsker σ-algebra $\mathrm{Pin}(f, m)$ is trivial, where $\mathrm{Pin}(f, m)$ is generated by

$$\left\{ \xi \text{ countable measurable partition} : H_m(\xi) < \infty, h_m(f; \xi) = 0 \right\}.$$

Above $h_m(f; \xi)$ corresponds to the entropy of f in the partition ξ; see [37].

There is a useful characterization of $\mathrm{Pin}(f, m)$. Given $X \subset M$ we say that it is

- s-saturated (u-saturated) if $x \in X \Rightarrow W^s(x) \subset X$ (resp. $W^u(x) \subset X$);
- bi-saturated if it is both s and u saturated;
- m-essentially s-saturated (u-saturated) if there exists a s-saturated (u-saturated) Borel set X_0 so that $m(X \Delta X_0) = 0$.

Theorem 5.1 (Ledrappier-Young [21]). *If* $X \in \mathrm{Pin}(f, m)$ *then* X *is both* m-*essentially s-saturated and* m-*essentially u-saturated.*

Our argument to show the Kolmogorov property is based on the program to establish stable ergodicity for conservative systems, developed originally by Grayson, Pugh, and Shub [16] and extended by Pugh and Shub [27] and several other authors, in particular Burns and Wilkinson [8] and Hertz, Hertz, and Ures [18]. These methods are very geometrical but rely on the properties of the Lebesgue measure (in particular, its conditionals on leaves of the associated invariant foliations); in our case the corresponding measures μ^u, μ^{cs} are more difficult to deal with, so a non-trivial adaptation is necessary.

Fix then X that is m-essentially s-saturated: we will show that it coincides with a s-saturated set $D(X)$ that corresponds to the density points of some dynamically defined differentiation basis.

Choose small numbers $0 < \varepsilon, \sigma < 1$ and define the $*$-juliennes as

$$J_n^u(x) := f^{-n}(W^u(f^n x; \varepsilon))$$
$$J_n^s(x) := f^n(W^s(f^{-n} x; \varepsilon))$$
$$B_n^c(x) := W^c(x; \sigma^n)$$
$$J_n^{cu}(x) := \bigcup_{y \in B_n^c(x)} J_n^u(y)$$
$$J_n^{scu}(x) := \bigcup_{y \in J_n^{cu}(x)} J_n^s(y).$$

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[2]Of the type considered in [21].

By a careful control of the sizes of the juliennes we are able to deduce that $J^{scu} = \left\{ J_n^{scu}(x) : x \in M, n \in \mathbb{N} \right\}$ is a Vitali differentiation basis (cf. [17]) and also the following.

Theorem 5.2. *For every Borel set $X \subset M$, the set $D(X)$ of its density points with respect to the basis J^{scu} is coincides m_φ-a.e. with X, and is s-saturated. Therefore, any m_φ-essentially s-saturated set coincides m_φ-a.e. with a s-saturated set.*

The above theorem implies that (f, m_φ) has the Kolmogorov property due to the fact that every leaf of W^s is dense (minimality of $\mathscr{W}^s$): given $X \in \mathrm{Pin}(f, \mathrm{m}_\varphi)$ the sets $D(X), D(X^c)$ are s-saturated, therefore if $0 < \mathrm{m}_\varphi(X) < 1$ we will have density points of X and X^c arbitrarily close, contradicting their definition.

After establishing the Kolmogorov property, we improve the result and show that in fact the system (f, m_φ) is isomorphic to a Bernoulli scheme.

Definition 5.2. Let $\mathrm{m} \in \mathscr{P}r_f(M)$. The system (f, m_φ) is Bernoulli if its induced stochastic process is isomorphic to the Bernoulli process with finite marginal distribution.

Equivalently, (f, m_φ) is measure theoretically isomorphic to a map $\sigma : \{1, \ldots, N\}^{\mathbb{Z}} \to \{1, \ldots, N\}^{\mathbb{Z}}$, $\sigma(\{x_n\}_n) = \{x_{n+1}\}_n$, where the σ-invariant measure is the product measure induced by some finite distribution on $\{1, \ldots, N\}$.

Using the Kolmogorov property, we are able to adapt the arguments of Ornstein and Weiss [25] for the time-one map of the geodesic flow to our setting and prove

Theorem 5.3. (f, m_φ) *is Bernoulli.*

Again, a non-trivial amount of work is required since the Ornstein and Weiss method is adapted for the Lebesgue measure, whereas in our case the control in the conditionals is necessarily more delicate.

6. Applications to the rank-one case: unique ergodicity of quasi-invariant measures

Our results are a generalization of the classical theorems for hyperbolic diffeomorphisms and flows [31], that is, rank-one Anosov actions (of $\mathbb{Z}$ or $\mathbb{R}$). In these cases the results can be obtained using the powerful tool of symbolic dynamics, which permits to reduce the study of the smooth map to a more manageable symbolic model. Regrettably, this technology becomes much more intricate outside hyperbolic systems, and although Sarig made some recent breakthrough in establishing symbolic models for a larger class of systems [33], the tools become more difficult to work and seem to require non-uniform hyperbolicity (which is never satisfied for center isometries).

In spite of the applicability of symbolic dynamics to the study of hyperbolic systems, our geometrical method gives some new information even in this classical setting. We will enunciate one illustrative result.

Suppose that $f : M \to M$ is an hyperbolic diffeomorphism such that $\dim E^u = 1$, which with no loss of generality can be assumed to be oriented. Then E^u is tangent to the orbits of a flow $\Phi^u = \left\{ \Phi_t^u : M \to M \right\}_{t \in \mathbb{R}}$, called the horocyclic flow. This is a prototype of parabolic flow; the dynamics of such systems have several consequences in geometry and number theory; see [13] for a discussion.

The following celebrated theorem is originally due to Furstenberg [15], while the version below is due to Marcus [23].

Theorem 6.1 (Furstenberg). *If f is a hyperbolic map of class $\mathscr{C}^2$ such that every orbit of Φ^u is dense, then Φ^u is uniquely ergodic. That is, there exists only one (probability) measure invariant under Φ^u.*

It follows that every orbit of Φ^u is equidistributed in M.

We remark that no example of hyperbolic map not satisfying the minimality condition is currently known. Above, a Borel measure μ is said to be invariant under flow if its invariant under every Φ_t^u. A more general notion is the following.

Definition 6.1. Given a flow $\Psi = \{\psi_t\}_t$ on a M, a measure μ is conformal for Ψ if there exists a family of positive functions $J = \{J_t : M \to \mathbb{R}_{>0}\}$, called the Jacobian of Ψ with respect to μ such that for every $t \in \mathbb{R}$,

$$(\Phi_{-t}^u)_* \mu = J_t \mu.$$

This definition was given by Patterson in [26] in a different context (limit sets for Fuchsian groups) and has an important role in geometry and ergodic theory. See, for example, [14, 28]. The definition for diffeomorphisms is analogous.

The following is a remarkable existence and uniqueness result for conformal measures.

Theorem 6.2 (Douady and Yoccoz [12]). *Let $f : \mathbb{T} \to \mathbb{T}$ be a $\mathscr{C}^2$ diffeomorphism of the circle with irrational rotation number. Then for every $s \in \mathbb{R}$ there exists a unique conformal measure with Jacobian $s \cdot Df$.*

A particularly natural Jacobian for the flow Φ^u is obtained by taking $J = \mathrm{Jac}^u$ (cf. (2.1)). Using our methods we are able do prove the following theorem.

Theorem 6.3. *In the cited hypotheses, let $\varphi : M \to \mathbb{R}$ be a Hölder function and consider the multiplicative cocycle Jac_{x_0,y_0}^u that it defines. Then there exists a unique quasi-invariant measure for Φ^u with Jacobian Jac^u.*

This generalizes Furstenberg's result and shows some strong rigidity in the possible dynamically relevant measures for the horocyclic flow outside its invariant one. The above theorem has also a version for hyperbolic flows; for the particular case of the geodesic flow in an hyperbolic manifold, the previous result was first established by Babillot and Ledrappier [1]. See also Schapira's article [34].

Comparing with Douady and Yoccoz' result, one gets enough evidence for the existence of some rigidity phenomena in the set of invariant measures of parabolic systems. One can ask the following:

Question. Let Φ be a minimal parabolic flow and J be a non-negative multiplicative cocycle. Does there exist a unique conformal measure with Jacobian J?

A particular instance of parabolic flows are unipotent flows; due to the general results for such type of system obtained by Ratner [29] (and others), a positive answer of the above question in this setting would be very interesting.

7. Concluding remarks

In the present note we gave a resume of some new geometrical methods to study thermodynamic formalism for generalizations of hyperbolic systems. These methods seem to be generalizable to other situations and hopefully will shed some light outside the hyperbolic realm.

All the reported results in this article are in preprint form in [9] and [10].

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