MathematicS

MathSInA.

In Action

Luc Paquet, Raouf El Cheikh, Dominique Lochegnies & Norbert Siedow

Radiative Heating of a Glass Plate

Volume 5 (2012), p. 1-30.

<http://msia.cedram.org/item?id=MSIA_2012__5_1_1_0>

MathematicS In Action
Vol. 5, 1-30 (2012)

Radiative Heating of a Glass Plate

Luc Paquet *
Raouf El Cheikh **
Dominique Lochegnies **
Norbert Siedow * * *

* Univ. Lille Nord de France UVHC-ISTV, LAMAV-EDP FR n° 2956, 59313 Valenciennes, France (Author to whom all correspondence should be addressed)
E-mail addresses: luc.paquet@univ-valenciennes.fr, paquet.luc@wanadoo.fr
** Univ. Lille Nord de France UVHC-ISTV, TEMPO, 59313 Valenciennes, France
E-mail address: raouf_cheikh@hotmail.com
E-mail address: Dominique.Lochegnies@univ-valenciennes.fr
* * * Fraunhofer Institute for Industrial Mathematics, ITWM, 67663 Kaiserlautern, Germany
E-mail address: norbert.siedow@itwm.fraunhofer.de.

Abstract

This paper aims to prove existence and uniqueness of a solution to the coupling of a nonlinear heat equation with nonlinear boundary conditions with the exact radiative transfer equation, assuming the absorption coefficient $\kappa(\lambda)$ to be piecewise constant and null for small values of the wavelength λ as in the paper of N. Siedow, T. Grosan, D. Lochegnies, E. Romero, "Application of a New Method for Radiative Heat Tranfer to Flat Glass Tempering", *J. Am. Ceram. Soc.,* **88**(8):2181-2187 (2005). An important observation is that for a fixed value of the wavelength λ, Planck function is a Lipschitz function with respect to the temperature. Using this fact, we deduce that the solution is at most unique. To prove existence of a solution, we define a fixed point problem related to our initial boundary value problem to which we apply Schauder theorem in a closed convex subset of the Banach separable space $L^2(0,t_f;C([0,l]))$. We use also Stampacchia truncation method to derive lower and upper bounds on the solution.

1. Introduction and statement of the problem

We consider an infinite plane horizontal glass plate of width l, laid down on its lower face $x_g = 0$, on a black sheet-metal maintained at absolute ambient temperature $T = T_a$. The x-axis is directed upward orthogonally to the glass plate so that the upper (resp. lower) face of the glass-plate has $x_g = l$ (resp. $x_g = 0$) for equation. An infinite plane black sheet metal S, at absolute temperature $T_S(t)$ at time t, placed above the glass plate emits radiation i.e. thermal rays in every direction. For a thermal ray, we denote by θ the angle of its unit directing vector $\vec{s}$ with the unit vector $\vec{e}_x$ of the $x-$axis and by $\mu := \cos(\theta)$. After refraction at the interface $x_g = l$ between air and glass, some part of the radiative energy emitted by the source S, will be absorbed i.e. converted into heat in the glass producing in such a way an increase of the temperature $T(x,t)$ in the glass plate. We have assumed independency of the temperature field in the glass plate with respect to the coordinates y and z. To describe the energetic flux associated to a radiation, one introduces in Photometry the notion of spectral radiative intensity which is defined in the following way. Let us consider a small oriented surface $\overrightarrow{dA} = \vec{n}\ dA$ through a point P with normal $\vec{n}$ at P. The radiative energy dE with wavelength in the interval $[\lambda, \lambda + d\lambda]$ which flows through $\overrightarrow{dA}$, during the time interval $[t, t + dt]$, in directions confined to a narrow cone of solid

Keywords: elementary pencil of rays, Planck function, radiative transfer equation, glass plate, nonlinear heat-conduction equation, Stampacchia truncation method, Schauder theorem, Vitali theorem.
Math. classification: 35K20, 35K55, 35K58, 35K90, 35Q20, 35Q60, 35Q80.

angle $d\Omega$ whose mean axis $\vec{s}$ makes an angle φ with $\vec{n}$, is given by the formula ([10], p. 7) ([23], p. 13):

$$dE = I(P,t,\vec{s},\lambda)\ \cos(\varphi)dA\ dt\ d\Omega\ d\lambda. \tag{1.1}$$

The coefficient of proportionality $I(P,t,\vec{s},\lambda)$ is called the spectral radiative intensity at point P and time t, in the direction $\vec{s}$, and at wavelength λ. As usual in Thermics, λ means the wavelength of the wave in vacuum (or dry air), the corresponding wavelength in glass being then $\frac{\lambda}{n_g}$, where n_g denotes the refractive index of the material (for glass: $n_g \approx 1.46$) (to avoid confusion, the wavelength in vacuum is denoted λ_0 in [20] p.8). We assume that the spectral radiative intensity $I(P,t,\vec{s},\lambda)$ is independent with respect to the y and z coordinates of the point P and of the azimuthal angle of the direction $\vec{s}$. Consequently: $I(P,t,\vec{s},\lambda) = I(x,t,\mu,\lambda)$. Concerning the radiative intensity of the radiation emitted by the black source (sheet metal) S, it is given at any point P of the air gap between S and the glass plate and in any direction, by the famous Planck function:

$$B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)} \text{ with } T = T_S(t), \tag{1.2}$$

where $C_1 = hc_0^2 = 0.595531\ 10^{-16}$ W.m^2/sr and $C_2 = \frac{hc_0}{k_B} = 1.438786\ 10^{-2}$ m.°K ([5], p.98). Let us note that $B(T,\lambda)$ depends only on the absolute temperature T and on the wavelength λ. We also denote by $B_g(T,\lambda) := n_g^2 B(T,\lambda)$ Planck function in glass.
Following [25], [15], [3], we assume that the absorption coefficient $\kappa(\lambda)$ is piecewise constant:

$$\kappa(\lambda) = \kappa_k \in \mathbb{R}_+^* \text{ for } \lambda \in [\lambda_k, \lambda_{k+1}[,\ k = 1,\dots,M, \tag{1.3}$$

where the M intervals $[\lambda_k, \lambda_{k+1}[,\ k = 1,\dots,M$, form a partition of the glass' electromagnetic wave spectrum in the semi-tranparent region (as explained a few lines above, the genuine wavelength in glass is $\frac{\lambda}{n_g}$). Like in [25], we introduce the "mean" radiative intensity $I^k(x,t,\mu)$ on each interval $[\lambda_k, \lambda_{k+1}[,\ \ (k = 1,\dots,M)$ of the partition (1.3) of the glass' electromagnetic wave spectrum in the semi-transparent region, by:

$$I^k(x,t,\mu) := \int_{\lambda_k}^{\lambda_{k+1}} I(x,t,\mu,\lambda)\ d\lambda.$$

Similarly for a given absolute temperature T, we define:

$$B^k(T) := \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)\ d\lambda,\ k = 1,\dots,M,$$

where $B(T,\lambda)$ denotes the Planck function defined by formula (1.2). Let us denote by $B_g^k(T) := n_g^2 B^k(T)$ $(k = 1,\dots,M)$. In the glass plate, the spectral radiative intensity is governed by the radiative transfer equation, which assuming no scattering in the glass plate ([10], p.343, p.9), reduces to:

$$\begin{aligned}\mu\frac{dI(x,t,\mu,\lambda)}{dx} + \kappa(\lambda)I(x,t,\mu,\lambda) &= \kappa(\lambda)B_g(T(x,t),\lambda),\\ &(0 \le x \le l,\ 0 \le t \le t_f,\ -1 < \mu < 1,\ \lambda > 0).\end{aligned} \tag{1.4}$$

Integrating both sides of (1.4) and using (1.3), we obtain the following system of M differential equations in the M dependent variables $I^1(x,t,\mu),\dots,I^M(x,t,\mu)$, only coupled by the temperature $T(x,t)$:

$$\left\{\begin{aligned}\mu\frac{dI^k(x,t,\mu)}{dx} + \kappa_k I^k(x,t,\mu) &= \kappa_k B_g^k(T(x,t)),\\ &(0 \le x \le l,\ 0 \le t \le t_f,\ -1 < \mu < 1),\ k = 1,\dots,M.\end{aligned}\right. \tag{1.5}$$

Now let us derive the boundary conditions for the differential equations (1.5). Considering an elementary pencil of thermal rays ([23], p.18, second paragraph) emitted by the black source S

of radiative intensity $B(T_S(t),\lambda)$ in (dry) air, a balance of energy shows that after refraction in the "direction" μ $(-1<\mu<0)$ at the interface $x_g=l$ between air and glass, its radiative intensity is

$$(1-\rho_g(\mu))n_g^2 B(T_S(t),\lambda)=(1-\rho_g(\mu))B_g(T_S(t),\lambda), \tag{1.6}$$

where $\rho_g(\mu)$ denotes the reflectivity coefficient given by Fresnel's relation ([20], formula (2.96) p. 47). The factor n_g^2 appearing in the left-hand side of equation (1.6) is due to the conservation of the optical outspread of an elementary pencil of thermal rays after refraction (see [10] p.8 or [5] (29) p.35 or [17], pp.56-59). Making once again a balance of energy, in an elementary pencil of thermal rays diverging in directions forming a narrow cone of solid angle $d\Omega$, whose mean axis makes an angle $\theta = Arc\cos(\mu)$ $(-1<\mu<0)$ with the x-axis, from a small area dA ([23], p.18, second paragraph) contained in the interface $x_g=l$, we obtain the following boundary condition on the surface $x_g=l$ of the glass plate (somewhat similar to [23], p.33-34):

$$\begin{aligned} I(l,t,\mu,\lambda) = \rho_g(\mu) I(l,t,-\mu,\lambda) + (1-\rho_g(\mu))B_g(T_S(t),\lambda),& \\ \text{for } -1<\mu<0,\ 0\le t\le t_f\ ,\ \lambda>0.& \end{aligned}$$

After integration of both sides of this equation with respect to wavelength λ from λ_k to λ_{k+1}, we obtain the following boundary condition on the surface $x_g=l$ of the glass plate for equation (1.5):

$$I^k(l,t,\mu)=\rho_g(\mu)I^k(l,t,-\mu)+(1-\rho_g(\mu))B_g^k(T_S(t)),\ -1<\mu<0,\ 0\le t\le t_f\ , \tag{1.7}$$

for $k=1,\ldots,M$. On the lower face $x_g=0$ of the glass plate in contact with the black-sheet metal maintained at ambient absolute temperature $T=T_a$, we have the simple boundary condition for the "integrated" radiative intensity $I^k(x,t,\mu)$:

$$I^k(0,t,\mu)=B_g^k(T_a),\ 0<\mu<1,\ 0\le t\le t_f\ . \tag{1.8}$$

Thus $I^k(x,t,\mu)$ may be seen as the solution of the first-order differential equation (1.5) in the x variable with parameter μ, and boundary condition (1.8) at $x=0$ (resp. boundary condition (1.7) at $x=l$) if $0<\mu<1$ (resp. $-1<\mu<0$). The right-hand side $\kappa_k B_g^k(T(x,t))$ of the first-order differential equation (1.5) depending on the distribution of temperature $T(\cdot,t)$ in the glass plate at time t, it follows that $I^k(\cdot,t,\mu)$ depends nonlinearly of $T(\cdot,t)$. When, wanting to underline this fact, we will write in the following $I_T^k(x,t,\mu)$ instead of $I^k(x,t,\mu)$. Now, the heat source in the heat-conduction equation is given by minus times the divergence of the radiative flux ([28], p.221-222), ([27], p.354-355):

$$q(x,t)=2\pi\int_0^{+\infty}\int_{-1}^{+1}\mu\ I(x,t,\mu,\lambda)\ d\mu\ d\lambda.$$

Using the radiative transfer equation (1.4) and our hypothesis (1.3) on the absorption coefficient $\kappa(\lambda)$, we have:

$$\begin{aligned} -\frac{\partial q}{\partial x}(x,t) &= -2\pi\int_0^{+\infty}\int_{-1}^{+1}\mu\frac{\partial I}{\partial x}(x,t,\mu,\lambda)\ d\mu\ d\lambda \\ &= -4\pi\int_0^{+\infty}\kappa(\lambda)B_g(T(x,t),\lambda)\ d\lambda + 2\pi\int_0^{+\infty}\int_{-1}^{+1}\kappa(\lambda)\ I(x,t,\mu,\lambda)\ d\mu\ d\lambda \\ &= -\sum_{k=1}^{k=M}4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{k=M}2\pi\kappa_k\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu. \end{aligned} \tag{1.9}$$

Thus the quantities of interest are $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$ $(k = 1,\dots,M)$ for which we shall give an explicit formula by solving explicitely for each $k \in \{1,\dots,M\}$ equation (1.5) with the boundary condition (1.7) for $-1 < \mu < 0$, respectively (1.8) for $0 < \mu < 1$. Now by (1.9), the heat-conduction equation inside the glass plate is the following:

$$\begin{aligned} c_p m_g \tfrac{\partial T}{\partial t}(x,t) = k_h \tfrac{\partial^2 T}{\partial x^2}(x,t) - \sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T(x,t)) \\ + \sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu,\ 0 < x < l,\ 0 < t < t_f\ , \end{aligned} \tag{1.10}$$

where c_p, m_g, k_h are assumed to be positive constants named respectively heat capacity, mass density and thermal conductivity of the glass [28], [27]. Now, what are the boundary conditions for the heat-conduction equation (1.10). On the lower face $x_g = 0$ of the glass plate which is in contact with a black sheet-metal maintained at absolute ambient temperature $T = T_a$, we have simply the inhomogeneous Dirichlet boundary condition:

$$T(0,t) = T_a,\ \forall t \in]0,t_f[. \tag{1.11}$$

On the upper face $x_g = l$ of the glass plate, due to radiative emission and absorption very near the boundary for wavelengths λ belonging to the glass opaque region $[\lambda_0, +\infty[$ ($\lambda_0 \approx 5\mu m$ for glass, [26], p.70) in the electromagnetic wave spectrum, we have the following nonlinear boundary condition expressing the continuity of the density of heat flux:

$$\begin{aligned} -k_h \tfrac{\partial T}{\partial x}(l,t) = h_c(T(l,t) - T_a) + \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda [B(T(l,t),\lambda) - B(T_S(t),\lambda)]\ d\lambda, \\ \forall t \in]0,t_f[. \end{aligned} \tag{1.12}$$

In the boundary condition (1.12), ϵ_λ is a positive constant called the spectral hemispherical emittance ([20], pp.62-63); like in ([26], p. 70), we have supposed that the spectral hemispherical absorptance is equal to the spectral hemispherical emittance for wavelength λ belonging to the glass opaque region in the electromagnetic wave spectrum, and independent of the temperature. Boundary condition (1.12) is the same as boundary condition (3) in [3] or (3) in [15]. In the boundary condition (1.12), $h_c > 0$ denotes the convective heat transfer coefficient, the term $h_c(T(l,t) - T_a)$ representing the conducto-convective flux density at the infinite surface $x_g = l$ of the glass plate according to Newton's law ([28], p.16) ([27], p.13-16). Finally, to close our system of equations, we need an initial condition for the temperature:

$$T(x,0) = T_0(x),\ \forall x \in [0,l]. \tag{1.13}$$

We assume that the initial condition $T_0(\cdot)$ is a continuous strictly positive function on the closed interval $[0,l]$, as an absolute temperature is always positive in classical physics and $T_0(\cdot)$ is a datum. For the same reasons, we also assume that $T_a > 0$ and that $T_S(t) > 0$, $\forall t \in [0,t_f]$. We suppose that the compatibility condition $T_0(0) = T_a$ between the initial condition and the inhomogeneous Dirichlet boundary condition (1.11) on the lower face $x_g = 0$ of the glass plate is verified and that T_S is bounded.

Our main purpose in this paper is to give a rigorous proof that the coupled problem formed by equations (1.5) with the boundary conditions (1.7) (resp. (1.8)) for $-1 < \mu < 0$ (resp. for $0 < \mu < 1$) and equation (1.10) completed by the boundary conditions (1.11), (1.12) and the initial condition (1.13) has a unique bounded weak solution

$$T \in \{T \in L^2(0,t_f; H^1(]0,l[); \frac{dT}{dt} \in L^2(0,t_f; \left[H_L^1(]0,l[)\right]^*)\}, \tag{1.14}$$

which is also continuous on $[0,l]\times[0,t_f]$. In (1.14), $H^1_L(]0,l[)$ denotes the subspace of $H^1(]0,l[)$ formed by those functions of $H^1(]0,l[)$ which vanish at the left extremity $x_g=0$ of the interval $]0,l[$. The proof of the existence will be achieved by defining a fixed point problem to which we will apply Schauder's theorem. Using Stampacchia's trunction method [2], we will also prove that the solution T is positive, lower bounded by T_a if $T_S(t)\geq T_a$, $\forall t\in[0,t_f]$ and $T_0(x)\geq T_a$, $\forall x\in[0,l]$ and upper bounded by

$$\max(\|T_0\|_{\infty,[0,l]}\,,\|T_S\|_{\infty,[0,t_f]}\,,T_a).$$

Let us conclude by mentioning, that though an absolute temperature is always positive in classical physics, for mathematical purposes only, we extend the definition of the Planck's function (1.2) to negative real numbers T by setting $B(T,\lambda)=0$, if $T\leq 0$; in that way for fixed $\lambda>0$, the function $T\mapsto B(T,\lambda)$ is defined on the whole real line and is Lipschitz with constant $\frac{2C_1}{C_2\lambda^4}$ (see Lemma 3.6).

To close this introductory section, let us situate our paper among existing works in the litterature. Assuming the grey property of the material (i.e. that the absorption coefficients of the material are independent of the wavelength λ), the existence and uniqueness of the solution of the SP_1-approximation to the radiative heat tranfer equation, coupled with the heat conduction equation assuming Robin-type boundary conditions has been established by R. Pinnau in [22]. This SP_1-approximation is sufficiently accurate only if the optical thickness of the plate is large which is certainly not the case with a glass plate of 6 mm thickness. In the present work, we have considered the exact radiative transfer equation, we do not assume the grey hypothesis and we consider the exact nonlinear boundary condition (1.12) on the upper face $\{x=l\}$ of the glass plate for the heat conduction equation. In [14], the authors assume the grey hypothesis and consider the nonlinearity arising from the well-known Stefan-Boltzmann law, making the resulting heat equation non-monotone but pseudo-monotone. In M. Laitinen's thesis [13], only grey materials are considered. The paper of P.-E. Druet [6] is concerned by proving the existence of a solution to a time-dependent heat equation modelizing the heating of several opaque bodies contained in an inclosure and separated from each other by a transparent medium. He has mathematically implemented the well-known inverse square law in heat radiation's theory for two boundary points of the opaque bodies in each other's range of vision (see e.g. [20], pp.133-136). In [19], the problem of optimizing the temperature gradient in the gas phase by directly controlling the heat source in the solid phase is considered in a crucible. The problem is described by the stationary heat equation with a nonlocal radiation interface between the solid and the gas phase and a local radiative boundary condition. In particular, these authors show the boundedness of weak solutions of the state equation.

One of the main characteristic of the model of radiative heating of a glass plate studied mathematically here, defined by the equations (1.5), (1.10) completed with the boundary conditions (1.7), (1.8), (1.11), (1.12) and the initial condition (1.13), is that the short wavelengths of the radiation emitted by the black source S are neglected, which seems reasonable. One possible justification of that attitude of mind is that the emissive power of the black source S for radiations of wavelength $\lambda\in[0,\lambda_{\text{inf}}]$:

$$\pi\int_0^{\lambda_{\text{inf}}} B(T_S,\lambda)\;d\lambda$$

([20], pp. 6-11) may be made as small as desired with respect to the total emissive power

$$\pi\int_0^{+\infty} B(T_S,\lambda)\;d\lambda$$

of the black source S, equal to σT_S^4, where σ denotes the Stefan-Boltzmann constant equal to $5.670\;10^{-8}$ W/m^2.°K^4, by choosing λ_{inf} sufficiently small.

Remark 1.1. As pointed out to us by one of the referees, we could extend the following existence and uniqueness theory to non piecewise constant absorption coefficient in the glass semi-transparent region, only assuming it measurable and positively lower and upper bounded. Our purpose was merely to study from the mathematical point of view the extension of the model introduced by N. Siedow et al. in [25] to radiative heating, i.e. to prove existence and uniqueness of the solution. Thus, we prefer to remain in the setting of [25].

2. Computation of the integrals $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu\ (k=1,\ldots,M)$

Let us recall, that we write $I_T^k(x,t,\mu)$ instead of more simply $I^k(x,t,\mu)$, to emphasize the dependence of $I^k(x,t,\mu)$ with respect to the temperature $T(\cdot,\cdot)$, as shown by formula (1.5). Firstly, we are going to compute the explicit solution of equation (1.5) with the boundary condition (1.7) for $-1<\mu<0$, respectively (1.8) for $0<\mu<1$, from which we will derive explicit expressions for the integrals $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu\ (k=1,\ldots,M)$. From differential equation (1.5), it follows that for $-1<\mu<1$:

$$I_T^k(x,t,\mu)=I_T^k(0,t,\mu)\ e^{-\kappa_k\frac{x}{\mu}}+\frac{\kappa_k}{\mu}e^{-\kappa_k\frac{x}{\mu}}\int_0^x e^{\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx' \tag{2.1}$$

or:

$$I_T^k(x,t,\mu)=I_T^k(l,t,\mu)\ e^{\kappa_k\frac{l-x}{\mu}}-\frac{\kappa_k}{\mu}e^{-\kappa_k\frac{x}{\mu}}\int_x^l e^{\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'. \tag{2.2}$$

Suppose $0<\mu<1$. By boundary condition (1.8), we have:

$$I_T^k(0,t,\mu)=B_g^k(T_a),\ \text{for}\ 0<\mu<1. \tag{2.3}$$

Thus by equation (2.1) it follows that

$$\begin{gathered}I_T^k(x,t,\mu)=B_g^k(T_a)\ e^{-\kappa_k\frac{x}{\mu}}+\tfrac{\kappa_k}{\mu}e^{-\kappa_k\frac{x}{\mu}}\int_0^x e^{\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx',\\ \text{for}\ 0<\mu<1.\end{gathered} \tag{2.4}$$

Suppose now $-1<\mu<0$. By boundary condition (1.7) and equation (2.1), we obtain:

$$\begin{gathered}I_T^k(l,t,\mu)=\rho_g(\mu)\ I_T^k(0,t,-\mu)e^{\kappa_k\frac{l}{\mu}}-\rho_g(\mu)\tfrac{\kappa_k}{\mu}e^{\kappa_k\frac{l}{\mu}}\int_0^l e^{-\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\\ +(1-\rho_g(\mu))B_g^k(T_S(t)),\ \text{for}\ -1<\mu<0.\end{gathered} \tag{2.5}$$

For $-1<\mu<0$: $0<-\mu<1$; thus using (2.3), (2.5) becomes:

$$\begin{gathered}I_T^k(l,t,\mu)=\rho_g(\mu)\ B_g^k(T_a)e^{\kappa_k\frac{l}{\mu}}-\rho_g(\mu)\tfrac{\kappa_k}{\mu}\int_0^l e^{-\kappa_k\frac{x'-l}{\mu}}B_g^k(T(x',t))\ dx'\\ +(1-\rho_g(\mu))B_g^k(T_S(t)),\ \text{for}\ -1<\mu<0.\end{gathered} \tag{2.6}$$

Now using formula (2.2), we obtain for $-1 < \mu < 0$:

$$\begin{aligned} I_T^k(x,t,\mu) &= \rho_g(\mu)\ B_g^k(T_a) e^{\kappa_k \frac{2l-x}{\mu}} + (1-\rho_g(\mu)) B_g^k(T_S(t)) e^{\kappa_k \frac{l-x}{\mu}} \\ -\rho_g(\mu) \tfrac{\kappa_k}{\mu} \int_0^l e^{-\kappa_k \frac{x'+x-2l}{\mu}} & B_g^k(T(x',t))\ dx' - \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \int_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx'. \end{aligned} \tag{2.7}$$

Thus the explicit solution of equation (1.5) with the boundary condition (1.7) for $-1 < \mu < 0$, respectively (1.8) for $0 < \mu < 1$, is given by formula (2.4) for $0 < \mu < 1$ and by formula (2.7) for $-1 < \mu < 0$. Now, these two formulas allow us to compute $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$:

$$\begin{aligned} \int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu &= \int_0^1 I_T^k(x,t,\mu)\ d\mu + \int_{-1}^0 I_T^k(x,t,\mu)\ d\mu \\ = B_g^k(T_a) \int_0^1 e^{-\kappa_k \frac{x}{\mu}}\ d\mu &+ \int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right] d\mu \\ + B_g^k(T_a) \int_{-1}^0 \rho_g(\mu)\ e^{\kappa_k \frac{2l-x}{\mu}}\ d\mu &+ B_g^k(T_S(t)) \int_{-1}^0 (1-\rho_g(\mu)) e^{\kappa_k \frac{l-x}{\mu}}\ d\mu \\ - \int_{-1}^0 \rho_g(\mu) \tfrac{\kappa_k}{\mu} e^{\kappa_k \frac{2l-x}{\mu}} & \left[\int_0^l e^{-\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right] d\mu \\ - \int_{-1}^0 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} & \left[\int_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right] d\mu\ . \end{aligned} \tag{2.8}$$

Let us now examine the different terms in the right-hand side of formula (2.8). For the first term it is rather immediate:

$$B_g^k(T_a) \int_0^1 e^{-\kappa_k \frac{x}{\mu}}\ d\mu = B_g^k(T_a) \int_1^{+\infty} e^{-\kappa_k x \zeta}\ \frac{d\zeta}{\zeta^2} = B_g^k(T_a)\ E_2(\kappa_k x), \tag{2.9}$$

where

$$E_2 : \mathbb{R}_+^* \to \mathbb{R} : y \mapsto \int_1^{+\infty} e^{-y\zeta}\ \frac{d\zeta}{\zeta^2} = \int_0^1 e^{-\frac{y}{\mu}} d\mu$$

denotes the integro-exponential function of order 2 ([28], p.244-245) ([20], p.779-781). Let us now inspect the second term in the right-hand side of formula (2.8):

$$\begin{aligned} &\int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right] d\mu \\ &= \int_0^x \left[\int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x-x'}{\mu}}\ d\mu \right] B_g^k(T(x',t))\ dx' \\ &= \int_0^x \left[\kappa_k \int_1^{+\infty} e^{-\kappa_k (x-x')\zeta}\ \tfrac{d\zeta}{\zeta} \right] B_g^k(T(x',t))\ dx' \\ = \int_0^x \kappa_k\ E_1(\kappa_k(x-x'))\ & B_g^k(T(x',t))\ dx' = \int_0^x G_k(x,x')\ B_g^k(T(x',t))\ dx', \end{aligned} \tag{2.10}$$

where

$$E_1 : \mathbb{R}_+^* \to \mathbb{R} : y \mapsto \int_1^{+\infty} e^{-y\zeta} \, \frac{d\zeta}{\zeta} = \int_0^1 e^{-\frac{y}{\mu}} \frac{d\mu}{\mu}$$

denotes the integro-exponential function of order 1 ([28], p.244-245) ([20], p.779-781), and where we have set:

$$G_k(x, x') := \kappa_k E_1(\kappa_k \, |x - x'|), \ \forall \, (x, x') \in [0, l]^2. \tag{2.11}$$

Similarly for the sixth term in the right-hand side of formula (2.8):

$$\begin{aligned}
&-\int_{-1}^{0} \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x', t)) \, dx' \right] \, d\mu \\
&= \int_0^1 \tfrac{\kappa_k}{\mu} e^{\kappa_k \frac{x}{\mu}} \left[\int_x^l e^{-\kappa_k \frac{x'}{\mu}} B_g^k(T(x', t)) \, dx' \right] \, d\mu \\
&= \int_x^l \left[\int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x' - x}{\mu}} \, d\mu \right] B_g^k(T(x', t)) \, dx' \\
&= \int_x^l \left[\int_1^{+\infty} \kappa_k \; e^{-\kappa_k (x' - x)\zeta} \; \tfrac{d\zeta}{\zeta} \right] B_g^k(T(x', t)) \, dx' \\
&= \int_x^l \kappa_k \; E_1(\kappa_k(x' - x)) \; B_g^k(T(x', t)) \, dx' \\
= \int_x^l \kappa_k \; E_1(\kappa_k \, |x - x'|) \; B_g^k(T(x', t)) \, dx' &= \int_x^l G_k(x, x') \; B_g^k(T(x', t)) \, dx'.
\end{aligned} \tag{2.12}$$

These two terms can be gathered in $\int_0^l G_k(x, x') \; B_g^k(T(x', t)) \, dx'$. To write the third and fourth terms in the right-hand side of formula (2.8) each containing $\rho_g(\mu)$ under the integral sign in a compact fashion, let us introduce the function

$$\Phi_2 : \mathbb{R}_+^* \to \mathbb{R}_+ : y \mapsto \Phi_2(y) := \int_0^1 \rho_g(\mu) e^{-\frac{y}{\mu}} d\mu \ . \tag{2.13}$$

Let us note that $\rho_g(\mu) = 1$ for $\mu \in [0, \sqrt{\frac{n_g^2 - 1}{n_g^2}}]$, as for such grazing incidence “angles”, elementary pencils of rays are completely reflected. Thus $\Phi_2(\cdot)$ is somewhat similar to the integro-exponential function $E_2(\cdot)$, but it takes into account the reflectivity coefficient. Using the function $\Phi_2(\cdot)$ the third term $B_g^k(T_a) \int_{-1}^{0} \rho_g(\mu) \; e^{\kappa_k \frac{2l - x}{\mu}} \; d\mu$ in (2.8) may now be rewritten:

$$\begin{aligned}
B_g^k(T_a) \int_{-1}^{0} \rho_g(\mu) \; e^{\kappa_k \frac{2l - x}{\mu}} \; d\mu &= B_g^k(T_a) \int_0^1 \rho_g(\mu) \; e^{-\kappa_k \frac{2l - x}{\mu}} \; d\mu \\
&= B_g^k(T_a) \; \Phi_2(\kappa_k(2l - x)).
\end{aligned} \tag{2.14}$$

The fourth term $B_g^k(T_S(t))\int\limits_{-1}^{0}(1-\rho_g(\mu))e^{\kappa_k\frac{l-x}{\mu}}\ d\mu$ in the right-hand side of formula (2.8) may be rewritten:

$$\begin{aligned}B_g^k(T_S(t))\int\limits_{-1}^{0}(1-\rho_g(\mu))e^{\kappa_k\frac{l-x}{\mu}}\ d\mu &= B_g^k(T_S(t))\int\limits_{0}^{1}(1-\rho_g(\mu))e^{-\kappa_k\frac{l-x}{\mu}}\ d\mu\\ &= B_g^k(T_S(t))E_2(\kappa_k(l-x)) - B_g^k(T_S(t))\int\limits_{0}^{1}\rho_g(\mu)\ e^{-\kappa_k\frac{l-x}{\mu}}\ d\mu\\ &= B_g^k(T_S(t))[E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))].\end{aligned}\tag{2.15}$$

To write the fifth term in the right-hand side of formula (2.8) containing also $\rho_g(\mu)$ under the integral sign in a compact fashion, let us introduce the integro-exponential function $\Phi_1(\cdot)$ somewhat similar to $E_1(\cdot)$

$$\Phi_1 : \mathbb{R}_+^* \to \mathbb{R}_+ : y \mapsto \Phi_1(y) := \int\limits_{0}^{1}\rho_g(\mu)e^{-\frac{y}{\mu}}\frac{d\mu}{\mu}.\tag{2.16}$$

Using the function Φ_1, the next to last term in (2.8) may now be rewritten:

$$\begin{aligned}&-\int\limits_{-1}^{0}\rho_g(\mu)\tfrac{\kappa_k}{\mu}e^{\kappa_k\frac{2l-x}{\mu}}\left[\int\limits_{0}^{l}e^{-\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\right]\ d\mu\\ &= \int\limits_{0}^{1}\rho_g(\mu)\tfrac{\kappa_k}{\mu}e^{-\kappa_k\frac{2l-x}{\mu}}\left[\int\limits_{0}^{l}e^{\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\right]\ d\mu\\ &= \kappa_k\int\limits_{0}^{l}\left[\int\limits_{0}^{1}\rho_g(\mu)\ e^{-\kappa_k\frac{2l-x-x'}{\mu}}\ \frac{d\mu}{\mu}\right]B_g^k(T(x',t))\ dx'\\ &= \kappa_k\int\limits_{0}^{l}\Phi_1(\kappa_k(2l-x-x'))B_g^k(T(x',t))\ dx'.\end{aligned}\tag{2.17}$$

Thus formula (2.8) may be rewritten:

$$\begin{aligned}\int\limits_{-1}^{+1}I_T^k(x,t,\mu)\ d\mu &= \int\limits_{0}^{l}G_k(x,x')\ B_g^k(T(x',t))\ dx' + B_g^k(T_a)\ E_2(\kappa_k x)\\ &+B_g^k(T_a)\ \Phi_2(\kappa_k(2l-x)) + B_g^k(T_S(t))[E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))]\\ &+\kappa_k\int\limits_{0}^{l}\Phi_1(\kappa_k(2l-x-x'))B_g^k(T(x',t))\ dx'.\end{aligned}\tag{2.18}$$

3. Weak formulation of the nonlinear initial boundary value problem (1.10)-(1.13).

Let us set:

$$h_T(x,t) := \sum_{k=1}^{k=M}2\pi\kappa_k\int\limits_{-1}^{+1}I_T^k(x,t,\mu)\ d\mu \text{ and } \psi(T(x,t)) := -\sum_{k=1}^{k=M}4\pi\kappa_k B_g^k(T(x,t))\tag{3.1}$$

for $(x,t)\in]0,l[\times]0,t_f[$. Firstly, we want to prove that if $T\in L_+^2(]0,l[\times]0,t_f[)$ and $T_S\in L_+^2(]0,t_f[)$, then h_T and $\psi\circ T$ belong to $L^2(]0,l[\times]0,t_f[)$. We will need several lemmas.

Lemma 3.1. *$B_g^k(T) \leq cT$ for every $T \in \mathbb{R}_+^*$, where c denotes some positive constant depending on k.*

Proof. $B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)} \leq \frac{2C_1}{\lambda^5(\frac{C_2}{\lambda T})} \leq 2\frac{C_1}{C_2}\,\frac{T}{\lambda^4}$. Thus $B^k(T) = \int\limits_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)\,d\lambda \leq 2\frac{C_1}{C_2}\,T \int\limits_{\lambda_k}^{\lambda_{k+1}} \frac{d\lambda}{\lambda^4} \leq cT$. As $B_g^k(T) = n_g^2 B^k(T)$, the previous inequality implies that: $B_g^k(T) \leq c n_g^2 T$. □

Remark 3.2. To simplify the notations, in the following, we shall occasionally use the symbol $\lesssim$ to mean that the left-hand side is bounded by a constant times the right-hand side. Also in the sequel, the notation $\forall'$ means "for almost every…". We shall also use the notation a.e. to mean "almost everywhere".

Corollary 3.3. *If $T \in L^2(]0,l[\times]0,t_f[)$ and $T_S \in L^2_+(]0,t_f[)$, then h_T and $\psi \circ T$ belong to $L^2(]0,l[\times]0,t_f[)$.*

Proof. Firstly, a little thought shows that we can reduce us to the case $T \in L^2_+(]0,l[\times]0,t_f[)$. Due to the preceding lemma, the function:

$$B_g^k \circ T :]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto B_g^k(T(x,t))$$

belongs to $L^2(]0,l[\times]0,t_f[)$. Thus $\psi \circ T$ which is a linear combination of the functions $B_g^k \circ T$ belongs to $L^2(]0,l[\times]0,t_f[)$. To prove that h_T belongs to $L^2(]0,l[\times]0,t_f[)$, it suffices to prove that the functions

$$]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto \int\limits_{-1}^{+1} I_T^k(x,t,\mu)\,d\mu$$

belong to $L^2(]0,l[\times]0,t_f[)$, $\forall k = 1,\ldots,M$. Firstly, in view of formula (2.18), we have to prove that the function

$$\begin{aligned}]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto & \int\limits_0^l G_k(x,x')\, B_g^k(T(x',t))\,dx' \\ &= \kappa_k \int\limits_0^l E_1(\kappa_k\,|x-x'|)\, B_g^k(T(x',t))\,dx' \end{aligned} \tag{3.2}$$

belongs to $L^2(]0,l[\times]0,t_f[)$, $\forall k = 1,\ldots,M$. By Lemma 3.1

$$\int\limits_0^l E_1(\kappa_k\,|x-x'|)\, B_g^k(T(x',t))\,dx' \leq c\int\limits_0^l E_1(\kappa_k\,|x-x'|)\, T(x',t)\,dx'$$

which, by using Cauchy-Schwarz inequality, implies

$$\left(\int\limits_0^l E_1(\kappa_k\,|x-x'|)\, B_g^k(T(x',t))\,dx'\right)^2 \leq c^2\int\limits_0^l E_1(\kappa_k\,|x-x'|)^2\,dx' \cdot \int\limits_0^l T(x',t)^2\,dx'.$$

Integrating both sides with respect to x from 0 to l and with respect to t from 0 to t_f, we obtain:

$$\int_0^{t_f}\int_0^l \left(\int_0^l E_1(\kappa_k\,|x-x'|)\; B_g^k(T(x',t))\;dx'\right)^2 dx\;\otimes dt$$

$$\leq c^2\int_0^{t_f}\int_0^l \left[\int_0^l E_1(\kappa_k\,|x-x'|)^2\;dx'\cdot\int_0^l T(x',t)^2\;dx'\right] dx\otimes\;dt$$

$$\leq c^2\int_0^l\int_0^l E_1(\kappa_k\,|x-x'|)^2\;dx'\otimes dx\cdot\int_0^{t_f}\int_0^l T(x',t)^2\;dx'\otimes\;dt.$$

Now, for $0<y\leq 1$:

$$E_1(y)=\int_1^{+\infty}\tfrac{e^{-yt}}{t}dt=\int_y^{+\infty}\tfrac{e^{-u}}{u}du\leq\int_y^1\tfrac{e^{-u}}{u}du+\int_1^{+\infty}\tfrac{e^{-u}}{u}du$$

$$\leq\int_y^1\tfrac{1}{u}du+\int_1^{+\infty}\tfrac{1}{u(1+u+\cdots)}du\leq|\ln y|+1$$

inequality which is a fortiori true for $y>1$. This inequality implies the bound $E_1(y)^2\leq 2+2\left(\ln y\;\right)^2$, from which follows easily that

$$\int_0^l\int_0^l E_1(\kappa_k\,|x-x'|)^2\;dx'\otimes dx \text{ is finite.}$$

Thus

$$\int_0^{t_f}\int_0^l \left(\int_0^l E_1(\kappa_k\,|x-x'|)\; B_g^k(T(x',t))\;dx'\right)^2 dx\otimes dt\lesssim\int_0^{t_f}\int_0^l T(x',t)^2\;dx^{'}\otimes dt<+\infty.$$

We have thus proven (3.2). The proof that the function

$$(x,t)\mapsto\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\;B_g^k(T(x',t))\;dx'$$

belongs also to $L^2(]0,l[\times]0,t_f[)$ is similar to the preceding one, the only point we have to check being that

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))^2\;dx'\otimes dx=\int_0^l\int_0^l \Phi_1(\kappa_k(x+x^{'}))^2\;dx'\otimes dx$$

is finite. But, by the definition (2.16) of the function Φ_1, $\Phi_1(y)\leq E_1(y)$, $\forall y\in\mathbb{R}_+^*$ and thus $\Phi_1(\kappa_k(x+x^{'}))^2\leq E_1(\kappa_k(x+x^{'}))^2\leq 2+4\left(\ln\kappa_k\right)^2+4\left(\ln\left(x+x^{'}\right)\;\right)^2$. Thus

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))^2\;dx'\otimes dx<+\infty.$$

To conclude that h_T belongs to $L^2(]0,l[\times]0,t_f[)$, in view of the definition of $h_T(.,.)$ and (2.18), it remains to prove that the three functions of (x,t):
$B_g^k(T_a)\;E_2(\kappa_k x)$, $B_g^k(T_a)\;\Phi_2(\kappa_k(2l-x))$, $B_g^k(T_S(t))[1-\Phi_2(\kappa_k(l-x))]$ belong to $L^2(]0,l[\times]0,t_f[)$. Since $E_2(y)\leq 1$, $\forall y\in\mathbb{R}_+^*$, it is clear that the first term is square integrable. Also from the definition (2.13) of Φ_2, it follows that $\Phi_2(y)\leq 1$, $\forall y\in\mathbb{R}_+^*$, so that it is also clear that the second

term belongs to $L^2(]0,l[\times]0,t_f[)$. Let us now turn to the third term. It results immediately from the hypothesis $T_S \in L^2_+(]0,t_f[)$, which combined with Lemma 3.1, implies that $B_g^k(T_S(\cdot)) \in L^2(]0,t_f[)$. □

Corollary 3.4. *Let us consider the mapping*

$$\Theta : \mathbb{R} \to \mathbb{R} : T \mapsto \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T,\lambda)\ d\lambda.$$

Then, Θ is an increasing function and for every $\hat{T} \in L^2(]0,t_f[)$, the function $\Theta \circ \hat{T}$ also denoted $\Theta(\hat{T})$ belongs to $L^2_+(]0,t_f[)$.

Proof. It is obvious that Θ is an increasing function. As we have seen in the proof of lemma 3.1:

$$B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)} \leq \frac{2C_1}{\lambda^5(\frac{C_2}{\lambda T})} \leq 2\frac{C_1}{C_2}\frac{T}{\lambda^4},\ \forall\lambda > 0,\ \forall T > 0.$$

Let us recall that we have set $B(T,\lambda) = 0$ if $T \leq 0$. Taking $T = \hat{T}(t)$, it follows that

$$\int_{\lambda_0}^{+\infty} \epsilon_\lambda B(\hat{T}(t),\lambda)\ d\lambda \leq 2\frac{C_1}{C_2}\hat{T}_+(t)\int_{\lambda_0}^{+\infty} \epsilon_\lambda \frac{d\lambda}{\lambda^4} \leq 2\frac{C_1}{C_2}\hat{T}_+(t)\int_{\lambda_0}^{+\infty} \frac{d\lambda}{\lambda^4} \leq \frac{2C_1}{3C_2\lambda_0^3}\hat{T}_+(t)$$

as $\forall\lambda > 0$: $\epsilon_\lambda \in [0,1]$. Thus:

$$\int_0^{t_f} \Theta(\hat{T}(t))^2 dt \leq \pi^2 \int_0^{t_f} \left[\int_{\lambda_0}^{+\infty} \epsilon_\lambda B(\hat{T}(t),\lambda)\ d\lambda\right]^2 dt \lesssim \int_0^{t_f} \hat{T}_+(t)^2 dt < +\infty.$$

Thus $\Theta \circ \hat{T}$ belongs to $L^2(]0,t_f[)$. □

In terms of the functions $h_T(\cdot,\cdot)$, $\psi(T(\cdot,\cdot))$ and $\Theta(T_S)(.)$, the initial boundary value problem (1.10), (1.11), (1.12), (1.13) can be rewritten:

$$\left\{\begin{array}{c} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \frac{\partial^2 T}{\partial x^2}(x,t) + \psi(T(x,t)) + h_T(x,t),\ \forall'(x,t) \in]0,l[\times]0,t_f[, \\ T(0,t) = T_a,\ \forall' t \in]0,t_f[, \\ -k_h \frac{\partial T}{\partial x}(l,t) = h_c(T(l,t) - T_a) + \Theta(T(l,t)) - \Theta(T_S)(t),\ \forall' t \in]0,t_f[, \\ T(x,0) = T_0(x),\ \forall' x \in [0,l]. \end{array}\right. \tag{3.3}$$

(3.3) has sense if we suppose that

$$T \in L^2(0,t_f;H^2(]0,l[) \text{ and } \frac{dT}{dt} \in L^2(0,t_f;L^2(]0,l[)).$$

Also, a priori, we do not know if the solution (which could even a priori not be unique) is positive, which is a natural property for an absolute temperature in classical physics. Thus to give sense to (3.3) and also to the equations which will follow, we have set $B(T,\lambda) = 0$ if $T \leq 0$ as we have said at the end of the introduction. We shall now define what is a weak solution to the initial boundary value problem (3.3). Let us consider any function $\varphi \in H^1_L(]0,l[) := \{\varphi \in H^1(]0,l[);\ \varphi(0) = 0\}$. Multiplying both members of equation $(3.3)_{(i)}$ by $\varphi(x)$ and integrating by parts from 0 to l using the boundary conditions $(3.3)_{(iii)}$ and $(3.3)_{(ii)}$, we obtain for $\forall' t \in]0, t_f[$:

$$\begin{aligned} c_p m_g \int_0^l \tfrac{\partial T}{\partial t}(x,t)\varphi(x)dx &= k_h \int_0^l \tfrac{\partial^2 T}{\partial x^2}(x,t)\varphi(x)dx + \int_0^l \psi(T(x,t))\varphi(x)dx \\ + \int_0^l h_T(x,t)\varphi(x)dx &= -h_c(T(l,t)-T_a)\varphi(l) - \Theta(T(l,t))\cdot\varphi(l) + \Theta(T_S(t)) \\ \cdot\, \varphi(l) - k_h \int_0^l \tfrac{\partial T}{\partial x}(x,t)\varphi'(x)dx &+ \int_0^l \psi(T(x,t))\varphi(x)dx + \int_0^l h_T(x,t)\varphi(x)dx. \end{aligned} \tag{3.4}$$

To give sense to (3.4) under the weak assumption that $T \in L^2(0,t_f;H^1(]0,l[))$ and that $\frac{dT}{dt} \in L^2(0,t_f;[H^1_L(]0,l[)]^*)$, we have to replace in the left-hand side of (3.4) the integration on $]0,l[$ by a duality bracket: $\forall' t \in]0,t_f[$:

$$\begin{aligned} c_p m_g \left\langle \tfrac{dT}{dt}(\cdot,t), \varphi \right\rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} &= -k_h \int_0^l \tfrac{\partial T}{\partial x}(x,t)\varphi'(x)dx \\ + \int_0^l \psi(T(x,t))\varphi(x)dx &+ \int_0^l h_T(x,t)\varphi(x)dx \\ + \left[\Theta(T_S(t)) - \Theta(T(l,t))\right]\cdot\varphi(l) &+ \left[h_c\left(T_a - T(l,t)\right)\right]\varphi(l). \end{aligned} \tag{3.5}$$

As $T \in L^2(0,t_f;H^1(]0,l[))$ and $\dot{T} \in L^2(0,t_f;[H^1_L(]0,l[)]^*)$, it follows that $T \in C([0,t_f];L^2(]0,l[))$ ([9], p.40) which gives sense to the initial condition $T(\cdot,0) = T_0(\cdot)$. As $T \in L^2(0,t_f;H^1(]0,l[))$ and $H^1(]0,l[) \hookrightarrow C([0,l])$, it follows that $T(0,\cdot) \in L^2(]0,t_f[)$ which gives sense to the boundary condition $T(0,\cdot) = T_a$. We can now define what is a weak solution of the initial boundary value problem (3.3):

Definition 3.5. We shall say that

$$T \in L^2(0,t_f;H^1(]0,l[)) \text{ such that } \frac{dT}{dt} \in L^2(0,t_f;\left[H^1_L(]0,l[)\right]^*)$$

is a weak solution of the initial boundary value problem (3.3) iff (3.5) is satisfied $\forall \varphi \in H^1_L(]0,l[)$, $T(\cdot,0) = T_0(\cdot)$ and $T(0,t) = T_a$, $\forall' t \in]0,t_f[$.

We begin by proving uniqueness of the solution of the initial boundary value problem (3.5). We will need the following lemmas:

Lemma 3.6. *Let us fix $\lambda > 0$. Then the function $\mathbb{R} \to \mathbb{R} : \hat{T} \longmapsto B(\hat{T},\lambda)$ is Lipschitz with constant $\frac{2C_1}{C_2\lambda^4}$.*

Proof. Let us recall that Planck's function (1.2) is defined by $B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)}$.

First case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 > 0$ and $\hat{T}_2 \leq 0$ (or $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 \leq 0$ and $\hat{T}_2 > 0$). Then:

$$\left|B(\hat{T}_1,\lambda) - B(\hat{T}_2,\lambda)\right| = B(\hat{T}_1,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda \hat{T}_1}}-1)} \leq \frac{2C_1}{C_2\lambda^4}\hat{T}_1 \leq \frac{2C_1}{C_2\lambda^4}\left|\hat{T}_1 - \hat{T}_2\right|.$$

Second case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 \leq 0$ and $\hat{T}_2 \leq 0$. Then $B(\hat{T}_1,\lambda) = B(\hat{T}_2,\lambda) = 0$ so that the inequality

$$\left|B(\hat{T}_1,\lambda) - B(\hat{T}_2,\lambda)\right| \leq \frac{2C_1}{C_2\lambda^4}\left|\hat{T}_1 - \hat{T}_2\right| \tag{3.6}$$

is obvious.

Third case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 > 0$ and $\hat{T}_2 > 0$. Then by Lagrange's theorem on finite increments:

$$B(\hat{T}_1, \lambda) - B(\hat{T}_2, \lambda) = (\hat{T}_1 - \hat{T}_2)\frac{\partial B}{\partial T}(\check{T}, \lambda)$$

where $\check{T} \in \mathbb{R}$ is some intermediate point between $\hat{T}_1$ and $\hat{T}_2$.
$\frac{\partial B}{\partial T}(\check{T}, \lambda) = 2C_1C_2\frac{e^{C_2/\lambda\check{T}}}{\lambda^6\check{T}^{\ 2}\left(e^{C_2/\lambda\check{T}}-1\right)^2} = \frac{2C_1}{C_2\lambda^4}\frac{\left(\frac{C_2}{\check{\mu}}\right)^2 e^{C_2/\check{\mu}}}{\left(e^{C_2/\check{\mu}}-1\right)^2}$ where we have set $\check{\mu} := \lambda\check{T}$. This formula shows that $\frac{\partial B}{\partial T}(\check{T}, \lambda) > 0$.
Also: $\forall s \in \mathbb{R}_+^* : \frac{s^2e^s}{(e^s-1)^2} = \frac{s^2}{\left(e^{\frac{s}{2}}-e^{-\frac{s}{2}}\right)^2} = \frac{s^2}{4\sinh(\frac{s}{2})^2} = \left(\frac{\frac{s}{2}}{\sinh(\frac{s}{2})}\right)^2 \leq 1$. Thus: $\frac{\partial B}{\partial T}(\check{T}, \lambda) \leq \frac{2C_1}{C_2\lambda^4}$. So inequality (3.6) is still true. □

Corollary 3.7. *Let* $T_1, T_2 \in L^2(]0, l[)$. *Then:*

$$\left\|B_g^k(T_1(\cdot)) - B_g^k(T_2(\cdot))\right\|_{L^2([0,l])} \leq 2n_g^2\frac{C_1}{C_2}\frac{\lambda_{k+1}^3 - \lambda_k^3}{3\lambda_k^3\ \lambda_{k+1}^3}\left\|T_1(\cdot) - T_2(\cdot)\right\|_{L^2([0,l])}. \tag{3.7}$$

Proof. Let us recall that $B_g^k(T) := n_g^2\int\limits_{\lambda_k}^{\lambda_{k+1}} B(T, \lambda)d\lambda$. By lemma 3.6, we have:

$$\left|B_g^k(T_1(x)) - B_g^k(T_2(x))\right| \leq \left(n_g^2\frac{2C_1}{C_2}\int\limits_{\lambda_k}^{\lambda_{k+1}}\frac{d\lambda}{\lambda^4}\right)\left|T_1(x) - T_2(x)\right|,$$

from which follows (3.7). □

Proposition 3.8. *There is at most one weak solution* T *belonging to*

$$\{T \in L^2(0, t_f; H^1(]0, l[)); \frac{dT}{dt} \in L^2(0, t_f; \left[H_L^1(]0, l[)\right]^*)\} \cap C([0, l] \times [0, t_f])$$

of the initial boundary value problem (3.3).

Proof. Let $T_1, T_2 \in \{T \in L^2(0, t_f; H^1(]0, l[)); \dot{T} \in L^2(0, t_f; [H_L^1(]0, l[)]^*)\} \cap C([0, l] \times [0, t_f])$ be two weak solutions of the initial boundary value problem (3.5). We are going to prove that $T_1 = T_2$. $T := T_1 - T_2$ is solution of

$$\begin{aligned}
c_pm_g\left\langle\tfrac{dT}{dt}(\cdot, t), T(\cdot, t)\right\rangle_{H^1(]0,l[)^*, H^1(]0,l[)} &= -k_h\int\limits_0^l\left[\tfrac{\partial T}{\partial x}(x, t)\right]^2dx \\
&+\int\limits_0^l\left[\psi(T_1(x, t)) - \psi(T_2(x, t))\right]T(x, t)dx \\
&-h_cT(l, t)^2 - \left[\Theta(T_1(l, t)) - \Theta(T_2(l, t))\right]\cdot T(l, t) \\
&+\int\limits_0^l\left[h_{T_1}(x, t) - h_{T_2}(x, t)\right]T(x, t)dx,\ \ \forall' t \in]0, t_f[.
\end{aligned}$$

This last equation may be rewritten:

$$\frac{c_p m_g}{2}\frac{d}{dt}\int_0^l T(x,t)^2 dx + k_h\int_0^l \left[\frac{\partial T}{\partial x}(x,t)\right]^2 dx + h_c T(l,t)^2 + [\Theta(T_1(l,t)) - \Theta(T_2(l,t))$$
$$]\cdot T(l,t) + \sum_{k=1}^{k=M} 4\pi\kappa_k \int_0^l \left[B_g^k(T_1(x,t)) - B_g^k(T_2(x,t))\right](T_1(x,t)) - T_2(x,t))\, dx \tag{3.8}$$
$$= \sum_{k=1}^{k=M} 2\pi\kappa_k \int_0^l [\int_{-1}^{+1} I_{T_1}^k(x,t,\mu)\, d\mu - \int_{-1}^{+1} I_{T_2}^k(x,t,\mu)\, d\mu](T_1(x,t)) - T_2(x,t))\, dx.$$

As for $T,\ \hat{T}\in\mathbb{R}$, $T\leq\hat{T}$ implies $B(T,\lambda)\leq B(\hat{T},\lambda),\ \forall\lambda>0$ ([20], p.8), the last two terms in the left-hand side of (3.8) are positive. Thus by (2.18):

$$\frac{c_p m_g}{2}\frac{d}{dt}\int_0^l T(x,t)^2 dx \leq \sum_{k=1}^{k=M} 2\pi\kappa_k \cdot$$
$$\int_0^l [\int_{-1}^{+1} I_{T_1}^k(x,t,\mu)\, d\mu - \int_{-1}^{+1} I_{T_2}^k(x,t,\mu)\, d\mu](T_1(x,t)) - T_2(x,t))\, dx \leq \sum_{k=1}^{k=M} 2\pi\kappa_k$$
$$\cdot\int_0^l \left[\int_0^l G_k(x,x')\ (B_g^k(T_1(x',t)) - B_g^k(T_2(x',t))\ dx'\right](T_1(x,t) - T_2(x,t))\ dx$$
$$+\sum_{k=1}^{k=M} 2\pi\kappa_k^2 \int_0^l \left[\int_0^l \Phi_1(\kappa_k(2l-x-x'))\ (B_g^k(T_1(x',t)) - B_g^k(T_2(x',t))\ dx'\right]$$
$$\cdot\ (T_1(x,t)) - T_2(x,t))\ dx$$
$$\leq \sum_{k=1}^{k=M} 2\pi\kappa_k \left\|G_k(\cdot,\cdot)\right\|_{L^2([0,l]^2)} \left\|B_g^k(T_1(\cdot,t)) - B_g^k(T_2(\cdot,t))\right\|_{L^2([0,l])}$$
$$\cdot\left\|T_1(\cdot,t) - T_2(\cdot,t)\right\|_{L^2([0,l])} + \sum_{k=1}^{k=M} 2\pi\kappa_k^2 \left\|\Phi_1(\kappa_k(2l-\cdot-\cdot))\right\|_{L^2([0,l]^2)}$$
$$\cdot\ \left\|B_g^k(T_1(\cdot,t)) - B_g^k(T_2(\cdot,t))\right\|_{L^2([0,l])} \left\|T_1(\cdot,t) - T_2(\cdot,t)\right\|_{L^2([0,l])}.$$

Applying corollary 3.7 to $T_1(\cdot,t)$ and $T_2(\cdot,t)$, it follows from the preceding inequality that for some positive constant C:

$$\frac{d}{dt}\int_0^l T(x,t)^2 dx \leq C\int_0^l T(x,t)^2 dx.$$

Thus $\frac{d}{dt}\left[e^{-Ct}\int_0^l T(x,t)^2 dx\right] = e^{-Ct}\left[\frac{d}{dt}\int_0^l T(x,t)^2 dx - C\int_0^l T(x,t)^2 dx\right]\leq 0$. Thus the function

$$\mathbb{R}_+^* \to \mathbb{R}_+ : t\mapsto e^{-Ct}\int_0^l T(x,t)^2 dx$$

is a decreasing positive function and being 0 at time $t=0$, is identically 0. Thus $T(\cdot,\cdot)=0$ i.e. $T_1(\cdot,\cdot)=T_2(\cdot,\cdot)$. □

Now, we are going to prove the existence of a weak solution (3.5) to the initial boundary value problem (3.3). The nonlinear integral term h_T in (3.3) defines a Lipschitz mapping

$$L^2(Q) \to L^2(Q) : T \mapsto h_T.$$

However, we have also an inhomogeneous nonlinear boundary condition on the face $\{x = l\}$ of the glass plate so that, it does not seem possible to use perturbation results of the semi-groups theory. Rather, to circumvent this difficulty, we define the following "fixed point problem": find $\tilde{T} \in \{\tilde{T} \in L^2(0, t_f; H^1(]0, l[)); \frac{d\tilde{T}}{dt} \in L^2(0, t_f; [H^1_L(]0, l[)]^*)\}$ such that

$$\begin{cases} c_p m_g \frac{\partial \tilde{T}}{\partial t}(x,t) = k_h \frac{\partial^2 \tilde{T}}{\partial x^2}(x,t) + \psi(\tilde{T}(x,t)) + h_T(x,t), \ \forall'(x,t) \in]0,l[\times]0,t_f[, \\ \tilde{T}(0,t) = T_a, \ \forall' t \in]0, t_f[, \\ -k_h \frac{\partial \tilde{T}}{\partial x}(l,t) = h_c(\tilde{T}(l,t) - T_a) + \Theta(\tilde{T}(l,t)) - \Theta(T_S(t)), \ \forall' t \in]0,t_f[, \\ \tilde{T}(x,0) = T_0(x), \ \forall' x \in [0,l]. \end{cases} \tag{3.9}$$

By a weak solution of the initial boundary value problem (3.9), we mean a function

$$\tilde{T} \in \{\tilde{T} \in L^2(0,t_f; H^1(]0,l[)); \frac{d\tilde{T}}{dt} \in L^2(0,t_f; \left[H^1_L(]0,l[)\right]^*)\}$$

such that $\forall' t \in]0, t_f[$:

$$\begin{cases} c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), \varphi \right\rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} = -k_h \int_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)dx + \\ \int_0^l \psi(\tilde{T}(x,t))\varphi(x)dx + h_c\left(T_a - \tilde{T}(l,t)\right)\varphi(l) + \left[\Theta(T_S(t)) - \Theta(\tilde{T}(l,t))\right] \\ \cdot\, \varphi(l) + \int_0^l h_T(x,t)\varphi(x)dx, \ \forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0) = 0, \\ \tilde{T}(0,t) = T_a, \\ \tilde{T}(x,0) = T_0(x), \ \forall' x \in [0,l]. \end{cases} \tag{3.10}$$

Let us assume that the function T in the definition of $h_T(\cdot,\cdot)$ which appears in the right-hand side of equation $(3.9)_{(\mathrm{i})}$ or (3.10) is given and belongs to $L^2(]0,l[\times]0,t_f[)$. Corollary 3.3 tells us that h_T belongs to $L^2(]0,l[\times]0,t_f[)$. Firstly, we want to prove that the initial boundary value problem (3.9) possesses one and only one weak solution $\tilde{T} \in L^2(0,t_f; H^1(]0,l[)) \cap C([0,l] \times [0,t_f])$ such that $\frac{d\tilde{T}}{dt} \in L^2(0,t_f; [H^1(]0,l[)]^*)$ i.e. that $\tilde{T}$ verifies (3.10). To prove this, we will use Theorem 1.40 page 49 of [9] on semilinear parabolic equations, but we have to check the hypotheses of that theorem. We know already that h_T belongs to $L^2(]0,l[\times]0,t_f[)$ and thus a fortiori to $L^r(]0,l[\times]0,t_f[)$ with $r > \frac{3}{2}$. The function $g :]0,t_f[\to \mathbb{R} : t \mapsto \frac{h_c T_a}{k_h} + \frac{\Theta(T_S(t))}{k_h} = \frac{h_c T_a}{k_h} + \frac{\pi}{k_h}\int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T_S(t),\lambda)\, d\lambda$ belongs to $L^{s^*}(]0,t_f[)$ for some $s^* > 2$ if we suppose that $T_S \in L^{s^*}(]0,t_f[)$ as follows from the inequality $B(\hat{T},\lambda) \lesssim \frac{\hat{T}}{\lambda^4}$, $\forall \hat{T} \in \mathbb{R}^*_+$. In particular, this will be the case if $T_S \in H^1(]0,t_f[)$. Now, the nonlinear term in equation $(3.9)_{(\mathrm{i})}$ is given by $-\psi(\tilde{T}(x,t) := \sum_{k=1}^{k=M} 4\pi\kappa_k B^k_g(\tilde{T}(x,t))$ for $(x,t) \in]0,l[\times]0,t_f[$. We must verify that the nonlinear mapping

$$\mathbb{R} \to \mathbb{R} : \hat{T} \mapsto \sum_{k=1}^{k=M} 4\pi\kappa_k B^k_g(\hat{T}) \tag{3.11}$$

is monotone increasing and Lipschitz continuous (let us recall that we have set $B_g^k(\hat{T}) = 0$ in case $\hat{T}$ would be negative). It is obvious that it is monotone increasing. By Corollary 3.7:

$$\left|B_g^k(\hat{T}_1) - B_g^k(\hat{T}_2)\right| \leq 2n_g^2 \frac{C_1}{C_2} \frac{\lambda_{k+1}^3 - \lambda_k^3}{3\lambda_k^3\ \lambda_{k+1}^3} \left|\hat{T}_1 - \hat{T}_2\right|.$$

This proves that the nonlinear mapping (3.11) is Lipschitz continuous. The nonlinear term in the boundary condition on the surface $x_g = l$ of the glass plate is given by $\Theta(\tilde{T}(l,\ t)) = \pi \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda [B(\tilde{T}(l,t),\lambda)] d\lambda$ for $t \in]0, t_f[$. As already said in Corollary 3.4 the nonlinear mapping

$$\mathbb{R} \to \mathbb{R} : \hat{T} \mapsto \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda [B(\hat{T},\lambda)] d\lambda \tag{3.12}$$

is obviously monotone increasing. By an immediate adaptation of the proof of Corollary 3.7:

$$\left| \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda [B(\hat{T}_1,\lambda)] d\lambda - \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda [B(\hat{T}_2,\lambda)] d\lambda \right| \leq \frac{2C_1}{C_2} \frac{1}{3\lambda_0^3} \left|\hat{T}_1 - \hat{T}_2\right|. \tag{3.13}$$

Thus the nonlinear mapping Θ is also Lipschitz continuous. All the hypotheses of Theorem 1.40 page 49 of [9] on semilinear parabolic equations being verified (see also [12]) we have:

Theorem 3.9. *Let us assume that function T that appears in the definition (3.1)-(2.18) of $h_T(\cdot,\cdot)$ which appears in the right-hand side of equation $(3.9)_{(i)}$ or (3.10) is given and belongs to $L^2(]0,l[\times]0,t_f[)$. We assume that the initial condition $T_0 \in C([0,l])$ and verifies the compatibility condition $T_0(0) = T_a$ with the boundary condition on the surface $x_g = 0$ of the glass plate and that the absolute temperature of the black source $T_S(\cdot)$ belongs to $L^{s^*}(]0,t_f[)$ for some $s^* > 2$. Then, the initial boundary value problem (3.9) possesses one and only one weak solution $\tilde{T} \in L^2(0,t_f;H^1(]0,l[)) \cap C([0,l]\times[0,t_f])$ such that $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H_L^1(]0,l[)\right]^*)$ i.e. $\tilde{T}$ verifies equation (3.10). Moreover, we have the following estimate:*

$$\begin{aligned} &\left\|\tilde{T}\right\|_{L^2(0,t_f;H^1(]0,l[))} + \left\|\tfrac{d\tilde{T}}{dt}\right\|_{L^2(0,t_f;\left[H_L^1(]0,l[)\right]^*)} + \left\|\tilde{T}\right\|_{C([0,l]\times[0,t_f])} \\ &\leq C_1(\left\| \sum_{k=1}^{k=M} 2\pi\kappa_k \int\limits_{-1}^{+1} I_T^k(\cdot,\cdot,\mu)\ d\mu \right\|_{L^2([0,l]\times[0,t_f])} + T_a \\ &\quad + \left\|\Theta(T_S(\cdot))\right\|_{L^{s^*}(]0,t_f[)} + \left\|T_0\right\|_{C([0,l])}) \\ &\leq C_2(\left\|T(\cdot,\cdot)\right\|_{L^2([0,l]\times[0,t_f])} + T_a + \left\|T_S(\cdot)\right\|_{L^{s^*}(]0,t_f[)} + \left\|T_0\right\|_{C([0,l])}). \end{aligned} \tag{3.14}$$

Remark 3.10. For uniqueness, the requirement $\tilde{T} \in C([0,l]\times[0,t_f])$ is not necessary as may be seen by adapting to this new initial boundary value problem the proof of Proposition 3.8 which simplifies greatly, the right-hand side of equation (3.8) being 0 in the present case.

We want now to prove under certain hypotheses on T, T_0, T_S that the solution of the initial boundary value problem (3.9) $\tilde{T}$ is lower bounded by T_a. We will use Stampacchia's truncation method [2]. Before, we need the following lemmas:

Lemma 3.11. *1°) If $\varphi \in H^1(]0,l[)$ and $\theta \in H^1(]0,l[)$, then $\varphi\theta \in H^1(]0,l[)$.*
2°) If $\varphi \in H^1(]0,l[)$ (resp. $H_L^1(]0,l[)$) and $\psi \in \left[H^1(]0,l[)\right]^*$ (resp. $\left[H_L^1(]0,l[)\right]^*$), then $\varphi\psi$ defined by $\langle\varphi\psi,\theta\rangle := \langle\psi,\varphi\theta\rangle, \forall\theta \in H^1(]0,l[)$ belongs to $\left[H^1(]0,l[)\right]^*$ and

$$\left\|\varphi\psi\right\|_{[H^1(]0,l[)]^*} \lesssim \left\|\varphi\right\|_{H^1(]0,l[)}\ \left\|\psi\right\|_{[H^1(]0,l[)]^*} \tag{3.15}$$

respectively

$$\|\varphi\psi\|_{[H^1(]0,l[)]^*} \lesssim \|\varphi\|_{H^1_L(]0,l[)} \ \|\psi\|_{[H^1_L(]0,l[)]^*}\,. \tag{3.16}$$

Proof. 1°) This is the well known fact that in dimension 1, the space $H^1(]0,l[)$ is a normed algebra [2].
2°) $\varphi\psi$ is defined by $\langle\varphi\psi,\theta\rangle_{[H^1]^*,\ H^1} := \langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}$, $\forall\theta \in H^1(]0,l[)$ (for short, we have denoted $H^1(]0,l[)$ by H^1). By the previous point $\varphi\theta \in H^1(]0,l[)$ and we have the inequality

$$\left|\langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}\right| \lesssim \|\psi\|_{[H^1(]0,l[)]^*}\,\|\varphi\|_{H^1(]0,l[)}\,\|\theta\|_{H^1(]0,l[)} \lesssim \|\theta\|_{H^1(]0,l[)}\,,$$

$\forall\theta \in H^1(]0,l[)$. Thus the mapping $\theta \mapsto \langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}$ is a continuous linear form on $H^1(]0,l[)$, i.e. an element of $\left[H^1(]0,l[)\right]^*$ and inequality (3.15) holds. The proof of inequality (3.16) is similar. □

Lemma 3.12. *We have the following equality:* $\forall k = 1,\ldots,M$:

$$\int_0^l G_k(x,x')\ dx' + E_2(\kappa_k x) + \Phi_2(\kappa_k(2l-x)) + [E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))]$$
$$+\kappa_k\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' = 2.$$

Proof. Firstly:

$$\begin{aligned}\int_0^l G_k(x,x')\ dx' &= \int_0^x G_k(x,x')\ dx' + \int_x^l G_k(x,x')\ dx' = \kappa_k\int_0^x E_1(\kappa_k(x-x'))\ dx'\\
&+\kappa_k\int_x^l E_1(\kappa_k(x'-x))dx' = \kappa_k\int_0^x E_1(\kappa_k y)dy + \kappa_k\int_0^{l-x} E_1(\kappa_k y)dy = \int_0^{\kappa_k x} E_1(z)dz\\
&+\int_0^{\kappa_k(l-x)} E_1(z)\ dz = [-E_2(z)]_{z=0}^{z=\kappa_k x} + [-E_2(z)]_{z=0}^{z=\kappa_k(l-x)} \text{ as } E_2' = -E_1\\
&= 1 - E_2(\kappa_k x) + 1 - E_2(\kappa_k(l-x)) = 2 - E_2(\kappa_k x) - E_2(\kappa_k(l-x)).\end{aligned}$$

Thus $\int_0^l G_k(x,x')\ dx' + E_2(\kappa_k x) + E_2(\kappa_k(l-x)) = 2.$
Secondly:

$$\begin{aligned}\kappa_k\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' &= \int_0^l \left[\int_0^1 \tfrac{\kappa_k}{\mu}\rho_g(\mu)e^{-\kappa_k\frac{2l-x-x^{'}}{\mu}}d\mu\right]\ dx'\\
= \int_0^1 \rho_g(\mu)e^{-\kappa_k\frac{2l-x}{\mu}}\left[\int_0^l \tfrac{\kappa_k}{\mu}e^{\frac{\kappa_k x^{'}}{\mu}}\ dx'\right]d\mu &= \int_0^1 \rho_g(\mu)e^{-\kappa_k\frac{2l-x}{\mu}}\left[e^{\frac{\kappa_k l}{\mu}} - 1\right]d\mu\\
&= \int_0^1 \rho_g(\mu)e^{-\kappa_k\frac{l-x}{\mu}}d\mu - \int_0^1 \rho_g(\mu)e^{-\kappa_k\frac{2l-x}{\mu}}d\mu.\end{aligned}$$

Thus: $\kappa_k\int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' - \Phi_2(\kappa_k(l-x)) + \Phi_2(\kappa_k(2l-x)) = 0.$
The result follows from these two points. □

In the following, to alleviate the notations, we will allow us to write $L^p(X)$ instead of $L^p(0,t_f;X)$, for $1 \le p \le +\infty$ and X a Banach space. Also, in the following, to shorten the notations, we will write sometimes H^1 (resp. H^1_L) instead of $H^1(]0,l[)$ (resp. $H^1_L(]0,l[)$) and $(H^1)^*$ (resp. $(H^1_L)^*$) instead of $H^1(]0,l[)^*$ (resp. $H^1_L(]0,l[)^*$).

Proposition 3.13. *We keep the hypotheses of theorem 3.9. Moreover, we assume that $T(\cdot,\cdot) \ge T_a$ a.e. on $]0,l[\times]0,t_f[$, that $T_S(\cdot) \ge T_a$ a.e. on $]0,t_f[$, and that the initial condition $T_0(.) \ge T_a$. Then the weak solution $\tilde{T}$ of the initial boundary value problem (3.9) satisfies the lower bound $\tilde{T}(\cdot,\cdot) \ge T_a$ on $[0,l]\times[0,t_f]$.*

Proof. Let us introduce the function

$$H:\mathbb{R}\to\mathbb{R}: y \mapsto \begin{cases} \frac{y^2}{2} \text{ if } y<0,\\ 0 \text{ if } y \ge 0,\end{cases} \tag{3.17}$$

and let us set

$$\tilde{\varphi}(t) := c_p m_g \int_0^l H(\tilde{T}(x,t)-T_a)dx = c_p m_g \left\langle H(\tilde{T}(\cdot,t)-T_a), \mathbf{1}_{]0,l[}\right\rangle,\ \forall t\in[0,t_f].$$

As the mapping from $L^2(]0,l[)$ into $L^2(]0,l[)$ which sends a function onto its negative part is lipschitzian according to Lemma 4.1 and thus continuous, $\tilde{\varphi}:[0,t_f]\to\mathbb{R}$ is a continuous function. Moreover $\tilde{\varphi}(0)=0$ because $\tilde{T}(\cdot,0)-T_a = T_0(.)-T_a \ge 0$. Using the density of $\mathcal{D}([0,t_f];H^1_L(]0,l[)$ in the space ([4], p.571)

$$W(0,t_f;H^1_L(]0,l[) := \{v\in L^2(0,t_f;H^1_L(]0,l[)); \frac{dv}{dt}\in L^2(0,t_f;\left[H^1_L(]0,l[)\right]^*)\}$$

endowed with its natural norm, one can prove that

$$\frac{d}{dt}H(\tilde{T}-T_a) = -(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt}, \tag{3.18}$$

in the sense of distributions and belongs to the space $L^1(0,t_f;[H^1_L(]0,l[)]^*$. Formula (3.18) is first proved for regular functions $\theta_n\in\mathcal{D}([0,t_f];H^1_L(]0,l[)$ approaching $\tilde{T}-T_a$ in the norm of the space $W(0,t_f;H^1_L(]0,l[)$ as $n\to+\infty$. Proposition 4.3 and the remark which follows implies that $(\theta_n)_-$ tends to $(\tilde{T}-T_a)_-$ in $L^2(0,t_f;H^1_L(]0,l[))$ as $n\to+\infty$. Then, the second point of Lemma 3.11 implies that the sequence of mappings $t\mapsto(\theta_n(\cdot,t))_-\frac{d\theta_n}{dt}(\cdot,t)$ converges as $n\to+\infty$ to $t\mapsto(\tilde{T}(\cdot,t)-T_a)_-\frac{d\tilde{T}}{dt}(\cdot,t)$ in the space $L^1(0,t_f;[H^1_L(]0,l[)]^*)$. In particular, this proves (3.18). Moreover, as $\mathbf{1}_{]0,l[}\in H^1(]0,l[)$, it follows that $\left\langle(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt},\mathbf{1}_{]0,l[}\right\rangle\in L^1(]0,t_f[)$. By the definition of the function $\tilde{\varphi}$ and (3.18):

$$\frac{d\tilde{\varphi}}{dt}(t) = -c_p m_g\left\langle(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt},\mathbf{1}_{]0,l[}\right\rangle,\ \forall' t\in]0,t_f[.$$

Thus $\frac{d\tilde{\varphi}}{dt}\in L^1(]0,t_f[)$ which implies that the function $\tilde{\varphi}$ is absolutely continuous on the interval $[0,t_f]$. By the definition of the product of an element of $[H^1_L(]0,l[)]^*$ by an element of $H^1_L(]0,l[)$ stated in lemma 3.11:

$$\left\langle(\tilde{T}(\cdot,t)-T_a)_-\frac{d\tilde{T}}{dt}(\cdot,t),\mathbf{1}_{]0,l[}\right\rangle = \left\langle\frac{d\tilde{T}}{dt}(\cdot,t),(\tilde{T}(\cdot,t)-T_a)_-\right\rangle,\ \forall' t\in]0,t_f[.$$

$\forall' t\in]0,t_f[$: $(\tilde{T}(\cdot,t)-T_a)_-\in H^1(]0,l[)$ and vanishes at the point $x=0$ of the interval $[0,l]$. Thus by the definition (3.10) of what is a weak solution of the initial boundary value problem (3.9):

$\forall' t \in]0, t_f[$:

$$\begin{gathered}
\frac{d\tilde{\varphi}}{dt}(t) = -c_p m_g \left\langle (\tilde{T}(\cdot,t) - T_a)_- \frac{d\tilde{T}}{dt}(\cdot,t), \mathbf{1}_{]0,l[} \right\rangle_{[H^1(]0,l[)]^*, H^1(]0,l[)} = \\
-c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), (\tilde{T}(\cdot,t) - T_a)_- \right\rangle_{(H^1_L)^*, H^1_L} = -k_h \int_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)^2 \mathbf{1}_{\{\tilde{T}(\cdot,t) < T_a\}}(x) dx \\
-\int_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - T_a)_- dx - \int_0^l h_T(x,t)(\tilde{T}(x,t) - T_a)_- dx + h_c \cdot \\
\left(\tilde{T}(l,t) - T_a\right)(\tilde{T}(l,t) - T_a)_- + \left[\Theta(\tilde{T}(l,t)) - \Theta(T_S(t))\right] \cdot (\tilde{T}(l,t) - T_a)_-.
\end{gathered} \tag{3.19}$$

In (3.19), we have used the fact that

$$\forall' t \in]0, t_f[: \frac{\partial}{\partial x}(\tilde{T}(\cdot,t) - T_a)_- = -\frac{\partial \tilde{T}}{\partial x}(\cdot,t)\mathbf{1}_{\{\tilde{T}(\cdot,t) < T_a\}}$$

by ([11], pp.50-54). Now let us look carefully at each term in the right-hand side of equality (3.19) in order to see that $\frac{d\tilde{\varphi}}{dt}(t)$ is negative for almost every $t \in]0, t_f[$. The first term in the right-hand side of equality (3.19) is obviously negative. By the explicit expression of $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$ given by formula (2.18) and the hypotheses $T(\cdot,\cdot) \geq T_a$ a.e. on $]0,l[\times]0,t_f[$, and $T_S(\cdot) \geq T_a$ a.e. on $]0,t_f[$, it follows by lemma 3.12 that

$$\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \geq 2B_g^k(T_a), \text{ a.e. on }]0,l[\times]0,t_f[.$$

Thus $h_T(x,t) := \sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \geq \sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T_a)$. But

$$\begin{aligned}
-\int_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - T_a)_- dx &= \sum_{k=1}^{k=M} 4\pi\kappa_k \int_0^l B_g^k(\tilde{T}(x,t))(\tilde{T}(x,t) - T_a)_- dx \\
&\leq \sum_{k=1}^{k=M} 4\pi\kappa_k \int_0^l B_g^k(T_a)(\tilde{T}(x,t) - T_a)_- dx \leq \int_0^l h_T(x,t)(\tilde{T}(x,t) - T_a)_- dx
\end{aligned}$$

because if $(\tilde{T}(x,t) - T_a)_- \neq 0$, then $\tilde{T}(x,t) < T_a$ which implies $B_g^k(\tilde{T}(x,t)) < B_g^k(T_a)$. Thus

$$-\int_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - T_a)_- dx - \int_0^l h_T(x,t)(\tilde{T}(x,t) - T_a)_- dx \leq 0. \tag{3.20}$$

Obviously: $h_c\left(\tilde{T}(l,t) - T_a\right)(\tilde{T}(l,t) - T_a)_- \leq 0$. Also

$$\left[\Theta(\tilde{T}(l,t)) - \Theta(T_S(t))\right] \cdot (\tilde{T}(l,t) - T_a)_-$$

is negative because if $(\tilde{T}(l,t) - T_a)_- \neq 0$, then $\tilde{T}(l,t) < T_a$ implies $\Theta(\tilde{T}(l,t)) \leq \Theta(T_a) \leq \Theta(T_S(t))$ as $T_S(t) \geq T_a$ by hypothesis, and thus $\Theta(\tilde{T}(l,t)) - \Theta(T_S(t)) \leq 0$. Thus

$$\begin{gathered}
-\int_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - T_a)_- dx - \int_0^l h_T(x,t)(\tilde{T}(x,t) - T_a)_- dx \\
+h_c\left(\tilde{T}(l,t) - T_a\right)(\tilde{T}(l,t) - T_a)_- + \left[\Theta(\tilde{T}(l,t)) - \Theta(T_S(t))\right] \cdot (\tilde{T}(l,t) - T_a)_-
\end{gathered} \tag{3.21}$$

is negative. From equation (3.19) follows that $\frac{d\tilde{\varphi}}{dt}(t) \leq 0,\ \forall' t \in]0, t_f[$. $\tilde{\varphi}$ being an absolutely continuous and positive function, null at $t = 0$ due to our hypothesis $T_0(\cdot) \geq T_a$ on the initial condition, it follows that $\tilde{\varphi}(t) = 0,\ \forall t \in [0, t_f[$. Thus $\forall t \in [0, t_f]$: $\tilde{T}(x, t) \geq T_a$, $\forall x \in [0, l]$ as $\tilde{T} \in C([0, l] \times [0, t_f])$. □

Now, we want also to prove under certain hypotheses that the solution $\tilde{T}$ of the initial boundary value problem (3.9) is upper bounded. The proof is more or less similar to the proof of the lower bound (3.13). Still, we use Stampacchia's truncation method.

Proposition 3.14. *We keep the hypotheses of theorem 3.9. Moreover, we assume that $\bar{T}$ denotes any positive real number such that $\bar{T} \geq T_a > 0$, $\bar{T} \geq T_S(t) > 0$, $\forall' t \in]0, t_f[$ and $\bar{T} \geq T_0(x) > 0$, $\forall x \in [0, l]$. We suppose that $T(\cdot, \cdot) \leq \bar{T}$ a.e. on $]0, l[\times]0, t_f[$. Then the weak solution $\tilde{T}$ of the initial boundary value problem (3.9) satisfies the upper bound $\tilde{T}(\cdot, \cdot) \leq \bar{T}$ on $]0, l[\times]0, t_f[$.*

Proof. The proof is very similar to the previous one. We introduce this time the function

$$\check{H} : \mathbb{R} \to \mathbb{R} : y \mapsto \begin{cases} \frac{y^2}{2} \text{ if } y > 0, \\ 0 \text{ if } y \leq 0, \end{cases}$$

and we set

$$\tilde{\varphi}(t) := c_p m_g \int_0^l \check{H}(\tilde{T}(x, t) - \bar{T}) dx = c_p m_g \left\langle \check{H}(\tilde{T}(\cdot, t) - \bar{T}), \mathbf{1}_{]0,l[} \right\rangle,\ \forall t \in [0, t_f].$$

As in the proof of the previous proposition, we prove that $\frac{d\tilde{\varphi}}{dt}(t) \leq 0,\ \forall' t \in]0, t_f[$. $\tilde{\varphi}$ being an absolutely continuous and positive function null for $t = 0$, it follows that $\tilde{\varphi}(t) = 0,\ \forall t \in [0, t_f]$ i.e. that $\forall t \in [0, t_f]$: $\tilde{T}(x, t) \leq \bar{T}$, $\forall x \in [0, l]$ as $\tilde{T} \in C([0, l] \times [0, t_f])$. □

In the following, we will assume at almost every time $t \in]0, t_f[$, that the absolute temperature $T_S(t)$ of the black source S satisfies

$$T_a \leq T_S(t) \leq \bar{T}. \tag{3.22}$$

To prove that the initial boundary value problem (3.3) possesses a solution, we will apply Schauder's fixed point theorem to prove that the initial boundary value problem (3.9) possesses a fixed point. We will apply the version of Schauder's fixed point theorem stated in A.Friedman's book ([8], p.171):

Theorem 3.15. *Let S be a closed convex set in a Banach space Y and let Φ be a continuous operator from S into Y such that $\Phi(S)$ is contained in S and such that the closure of $\Phi(S)$ is compact. Then Φ has a fixed point.*

This version of Schauder's fixed point theorem follows from the classical statement of Schauder's theorem ([7], p.502) applied to the closed convex hull of $\Phi(S)$: $\overline{co}(\Phi(S)) = \overline{co}(\overline{\Phi(S)})$ which is compact by Mazur's theorem, and to $\Phi_{|\overline{co}(\Phi(S))}$.
As a Banach space Y, we choose $L^2(0, t_f; C([0, l]))$. We consider two positive real numbers $0 < T_a < \bar{T}$ such that

$$\forall x \in [0, l] :\ T_a \leq T_0(x) \leq \bar{T} \text{ and } \forall' t \in]0, t_f[:\ T_a \leq T_S(t) \leq \bar{T}.$$

Now, for S we consider:

$$S := \{T \in L^2(0, t_f; C([0, l]));\ \forall' t \in]0, t_f[:\ T_a \leq T(., t) \leq \bar{T}\}, \tag{3.23}$$

and for Φ, the mapping which sends $T \in S$ onto $\tilde{T}$ solution of the initial boundary value problem (3.9) which still belongs to S due to proposition 3.13 and proposition 3.14. That S is closed, results from the fact every convergent sequence in the space $L^2(0, t_f; C([0, l]))$ possesses an almost everywhere convergent subsequence (see e.g. [18], p.192 for L^2 spaces of square integrable functions with values in a Banach space). Firstly, we are going to prove that the mapping Φ is continuous in several steps. To shorten the notations, we set $Q :=]0,l[\times]0,t_f[$.

Proposition 3.16. *The mapping from $L^2(Q)$ into $L^2(Q)$ which associates T to $B_g^k(T)$ is lipschitzian and thus a fortiori continuous.*

Proof. This is an immediate consequence of Corollary 3.7. □

Similarly:

Proposition 3.17. *The mapping from $L^2(]0,t_f[)$ into $L^2(]0,t_f[)$ which associates $\xi \in L^2(]0,t_f[)$ to*

$$]0,t_f[\to \mathbb{R} : t \mapsto \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(\xi(t),\lambda)\ d\lambda$$

is lipschitzian and thus a fortiori continuous.

Proof. This follows from inequality (3.13). □

We want now to deduce using in particular Proposition 3.16, that the mapping $L^2(Q) \to L^2(Q) : T \mapsto h_T$ is lipschitzian and thus a fortiori continuous. This amounts to prove by (3.1), that for every $k \in \{1,\ldots,M\}$, that the mapping

$$L^2(Q) \to L^2(Q) : T \mapsto \int_{-1}^{+1} I_T^k(\cdot,\cdot,\mu)\ d\mu$$

is lipschitzian. In view of the explicit formula (2.18) for $\int_{-1}^{+1} I_T^k(\cdot,\cdot,\mu)\ d\mu$, we are reduced to the following lemmas. The first one is a variant of a classical result about integral operator of the Hilbert-Schmidt type ([29], pp.197-198):

Lemma 3.18. *Let $(x,x') \mapsto K(x,x')$ be an almost everywhere defined real-valued measurable function on the square $]0,l[^2$such that $\int_0^l\int_0^l K(x,x')^2 dx\otimes dx' < +\infty$. For every $\varphi \in L^2(Q)$, let us set:*

$$\mathcal{K}\varphi(x,t) = \int_0^l K(x,x')\varphi(x',t)dx',\ \forall'(x,t) \in Q. \tag{3.24}$$

Then, $\mathcal{K}$ so defined, is a linear continuous operator in $L^2(Q)$ whose operator norm satisfies the inequality:

$$\|\mathcal{K}\| \leq \|K\|_{L^2(]0,l[^2)} = \left(\int_0^l\int_0^l K(x,x')^2 dx\otimes dx'\right)^{\frac{1}{2}}. \tag{3.25}$$

Proof. By Cauchy-Schwarz inequality and Fubini theorem:

$$\begin{aligned}
\int_0^{t_f}\int_0^l (\mathcal{K}\varphi(x,t))^2 dx\otimes dt &= \int_0^{t_f}\int_0^l (\int_0^l K(x,x')\varphi(x',t)dx')^2 dx\otimes dt \\
&\leq \int_0^{t_f}\int_0^l \left(\int_0^l K(x,x')^2dx' \cdot \int_0^l \varphi(x',t)^2 dx'\right) dx\otimes dt \\
\leq \int_0^l\int_0^l K(x,x')^2 dx\otimes dx' \int_0^{t_f}\int_0^l \varphi(x',t)^2 dx'\otimes dt &= \|K\|^2_{L^2(]0,l[^2)} \|\varphi\|^2_{L^2(Q)}.
\end{aligned}$$

Thus $\|\mathcal{K}\varphi\|_{L^2(Q)} \le \|K\|_{L^2(]0,l[^2)} \|\varphi\|_{L^2(Q)}$, $\forall\varphi \in L^2(Q)$, so that:

$$\|\mathcal{K}\| := \sup_{\|\varphi\|_{L^2(Q)}\le 1} \|\mathcal{K}\varphi\|_{L^2(Q)} \le \|K\|_{L^2(]0,l[^2)}\,.$$

$\square$

Corollary 3.19. *Let $K(x,x') := G_k(x,x') := \kappa_k E_1(\kappa_k |x-x'|)$ (cfr.(2.11)). For every $\varphi \in L^2(Q)$, let us set:*

$$\mathcal{G}_k\varphi(x,t) = \int_0^l G_k(x,x')\varphi(x',t)dx',\ \forall'(x,t) \in Q. \tag{3.26}$$

Then, $\mathcal{G}_k$ so defined, is a linear continuous operator in $L^2(Q)$ whose operator norm satisfies the inequality:

$$\|\mathcal{G}_k\| \le \sqrt{2}\ \kappa_k \left(l^2 + \int_0^l\int_0^l \ln(\kappa_k |x-x'|)^2 dx\otimes dx'\right)^{\frac{1}{2}}. \tag{3.27}$$

Proof. In view of lemma 3.18, it suffices to verify that $\int_0^l\int_0^l G_k(x,x')^2 dx\otimes dx' < +\infty$ i.e. that

$$\int_0^l\int_0^l E_1(\kappa_k |x-x'|)^2 dx\otimes dx' < +\infty. \tag{3.28}$$

But $E_1(y)^2 \le 2 + 2(\ln y)^2$ and thus $E_1(\kappa_k |x-x'|)^2 \le 2\left[1+\ln(\kappa_k |x-x'|)^2\right]$, so that (3.28) is trivial. (3.27) follows from (3.25) and the bound $G_k(x,x')^2 \le 2\kappa_k^2\left[1+\ln(\kappa_k |x-x'|)^2\right]$. $\square$

Corollary 3.20. *Let $K(x,x') := \Phi_1(\kappa_k(2l-x-x'))$. For every $\varphi \in L^2(Q)$, let us set:*

$$\mathcal{U}_k\varphi(x,t) = \int_0^l \Phi_1(\kappa_k(2l-x-x'))\ \varphi(x',t)dx',\ \forall'(x,t) \in Q. \tag{3.29}$$

Then, $\mathcal{U}_k$ so defined, is a linear continuous operator in $L^2(Q)$ whose operator norm

$$\|\mathcal{U}_k\| \le \sqrt{2}\ \left(l^2 + \int_0^l\int_0^l \left[\ln(\kappa_k(x+x'))\right]^2 dx\otimes dx'\right)^{\frac{1}{2}}. \tag{3.30}$$

Proof. In view of lemma 3.18, it suffices to verify that $\int_0^l\int_0^l K(x,x')^2 dx\otimes dx' < +\infty$ i.e. that

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x'))^2 dx\otimes dx' < +\infty.$$

But by the definition of Φ_1 (see 2.16) and due to $0 \le \rho_g(\cdot) \le 1$: $\Phi_1(\kappa_k(2l-x-x')) \le E_1(\kappa_k(2l-x-x'))$. Thus

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x'))^2 dx\otimes dx' \le \int_0^l\int_0^l \left[E_1(\kappa_k((l-x)+(l-x')))\right]^2 dx\otimes dx'$$
$$= \int_0^l\int_0^l \left[E_1(\kappa_k\,(y+y'))\right]^2 dy\otimes dy' \le 2l^2 + 2\int_0^l\int_0^l \left[\ln(\kappa_k\,(y+y'))\right]^2 dy\otimes dy' < +\infty.$$

$\square$

Proposition 3.21. *The nonlinear mapping $L^2(Q) \to L^2(Q) : T \mapsto h_T$ is lipschitzian and thus a fortiori continuous.*

Proof. This follows from formulas (2.18), (3.1), Proposition 3.16, Corollary 3.19 and Corollary 3.20. $\square$

We are now in a position to prove the continuity of the mapping Φ:

Theorem 3.22. *The mapping Φ which sends $T \in S$ defined by (3.23) onto $\tilde{T} \in S$, the unique weak solution of the initial boundary value problem (3.9), i.e. the unique $\tilde{T} \in L^2(0, t_f; H^1(]0, l[))$ such that $\frac{d\tilde{T}}{dt} \in L^2(0, t_f; \left[H^1_L(]0, l[)\right]^*)$ verifying equation (3.10), is continuous.*

Proof. So, let us consider a sequence of functions $(T_n)_{n\in\mathbb{N}}$ belonging to S converging to some $T \in S$ in the sense of the norm of $L^2(0, t_f; C([0, l])$. Let $\tilde{T}_n := \Phi(T_n),\ \forall n \in \mathbb{N}$ and $(\tilde{T}_n)_{n\in\mathbb{N}}$ the corresponding sequence. By the estimate (3.14) of Theorem 3.9, the sequence $(\tilde{T}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0, t_f; H^1(]0, l[))$ and $(\frac{d\tilde{T}_n}{dt})_{n\in\mathbb{N}}$ is bounded in $L^2(0, t_f; H^1_L(]0, l[)^*)$. Thus some subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is weakly convergent in $L^2(0, t_f; H^1(]0, l[))$ and $(\frac{d\tilde{T}_{n_k}}{dt})_{k\in\mathbb{N}}$ is weakly convergent in $L^2(0, t_f; H^1_L(]0, l[)^*)$. Let us call $\tilde{T}$ the weak-limit of the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ in the space $L^2(0, t_f; H^1(]0, l[))$. Using the definition of the weak time derivative ([9], p.39-40), it is easy to see that the weak-limit of the subsequence $(\frac{d\tilde{T}_{n_k}}{dt})_{k\in\mathbb{N}}$ in the space $L^2(0, t_f; H^1(]0, l[)^*)$ is $\frac{d\tilde{T}}{dt}$. The injection from the space $H^1(]0, l[) \hookrightarrow C([0, l])$ being compact [2] and $C([0, l]) \hookrightarrow H^1_L(]0, l[)^*$, by the compactness Theorem 5.1 p.58 of [16], the continuous embedding from the space $\{\check{T} \in L^2(0, t_f; H^1(]0, l[)); \frac{d\check{T}}{dt} \in L^2(0, t_f; H^1(]0, l[)^*)\}$ endowed with its natural norm into $L^2(0, t_f; C([0, l]))$, is also compact. Thus the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also strongly convergent in the space $L^2(0, t_f; C([0, l]))$ ([24], p. 199) to $\tilde{T}$. We have to prove that $\tilde{T} = \Phi(T)$. For every $k \in \mathbb{N}$: $\tilde{T}_{n_k}(x = 0, t) = T_a,\ \forall' t \in]0, t_f[$. Thus we have also $\tilde{T}(x = 0, t) = T_a,\ \forall' t \in]0, t_f[$ by the strong convergence of $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ to $\tilde{T}$ in $L^2(0, t_f; C([0, l]))$. Also by Theorem 1.32 p.40 of [9], we have a continuous imbedding from the space

$$\{\check{T} \in L^2(0, t_f; H^1_L(]0, l[)); \frac{d\check{T}}{dt} \in L^2(0, t_f; H^1_L(]0, l[)^*)\} \text{ into } C([0, t_f]; L^2(]0, l[)).$$

Thus the subsequence $(\tilde{T}_{n_k} - T_a)_{k\in\mathbb{N}}$ and consequently $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also weakly convergent in the space $C([0, t_f];\ L^2(]0, l[))$. For every $k \in \mathbb{N}$: $\tilde{T}_{n_k}(\cdot, t = 0) = T_0(\cdot)$. Let ψ be an arbitrary function in $L^2(]0, l[)$. The mapping

$$\varphi^* : C([0, t_f]; L^2(]0, l[)) \to \mathbb{R} : v \mapsto \int_0^l v(x, t = 0)\psi(x)dx$$

is a continuous linear form on the space $C([0, t_f]; L^2(]0, l[))$. Thus $\left\langle \varphi^*, \tilde{T}_{n_k} \right\rangle = \int_0^l T_0(x)\psi(x)dx \to \int_0^l \tilde{T}(x, 0)\psi(x)dx,\ \forall \psi \in L^2(]0, l[)$. Thus $\tilde{T}(\cdot, 0) = T_0(\cdot)$ in $L^2(]0, l[)$. Therefore, we know already that $\tilde{T}$ verifies the adequate initial condition and the adequate boundary condition on the face $\{x_g = 0\}$ of the glass plate. Now, taking an arbitrary fonction $\xi(\cdot) \in L^2(]0, t_f[)$, multiplying both sides of equation (3.10) by the function ξ and integrating with respect to time from 0 to

t_f, we obtain:

$$\left\{\begin{array}{c} c_p m_g \int_0^{t_f} \left\langle \frac{\partial \tilde{T}_{n_k}}{\partial t}(\cdot,t),\varphi \right\rangle_{H^1_L(]0,l[)^*,H^1_L(]0,l[)} \xi(t)\,dt = \\ -k_h \int_0^{t_f}\int_0^l \frac{\partial \tilde{T}_{n_k}}{\partial x}(x,t)\varphi'(x)\xi(t)\,dx\otimes dt + \int_0^{t_f}\int_0^l \psi(\tilde{T}_{n_k}(x,t))\varphi(x)\xi(t)\,dx\otimes dt \\ -\int_0^{t_f} \Theta(\tilde{T}_{n_k}(l,t))\xi(t)\,dt\cdot\varphi(l) + \int_0^{t_f}\int_0^l h_{T_{n_k}}(x,t)\varphi(x)\xi(t)\,dx\otimes dt \\ +\int_0^{t_f} \Theta(T_S(t))\xi(t)\,dt\cdot\varphi(l) + \int_0^{t_f} h_c\left(T_a - \tilde{T}_{n_k}(l,t)\right)\xi(t)\,dt\cdot\varphi(l), \\ \forall\varphi\in H^1(]0,l[) \text{ such that } \varphi(0)=0, \qquad \forall\xi\in L^2(]0,t_f[). \end{array}\right. \tag{3.31}$$

We have seen a few lines above that the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also strongly convergent to $\tilde{T}$ in the space $L^2(0,t_f;C([0,l]))$, thus a fortiori in the space $L^2(Q)$ (Q denotes $]0,l[\times]0,t_f[$). Thus by (3.1) and proposition 3.16, $\psi\circ\tilde{T}_{n_k}$ converges to $\psi\circ\tilde{T}$ in $L^2(Q)$. $\tilde{T}_{n_k}(l,\cdot)\to\tilde{T}(l,\cdot)$ in $L^2(]0,t_f[)$ and thus by proposition 3.17:

$$\Theta(\tilde{T}_{n_k}(l,\cdot))\to\Theta(\tilde{T}(l,\cdot)) \text{ in } L^2(]0,t_f[).$$

Using Proposition 3.21 and all the previous convergence properties to pass to the limit in (3.31) as $k\to+\infty$, we obtain:

$$\left\{\begin{array}{c} c_p m_g \int_0^{t_f} \left\langle \frac{\partial \tilde{T}}{\partial t}(\cdot,t),\varphi \right\rangle_{H^1(]0,l[)^*,H^1(]0,l[)} \xi(t)\,dt = \\ -k_h \int_0^{t_f}\int_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)\xi(t)\,dx\otimes dt + \int_0^{t_f}\int_0^l \psi(\tilde{T}(x,t))\varphi(x)\xi(t)\,dx\otimes dt \\ -\int_0^{t_f} \Theta(\tilde{T}(l,t))\xi(t)\,dt\cdot\varphi(l) + \int_0^{t_f}\int_0^l h_T(x,t)\varphi(x)\xi(t)\,dx\otimes dt \\ +\int_0^{t_f} \Theta(T_S(t))\xi(t)\,dt\cdot\varphi(l) + \int_0^{t_f} h_c\left(T_a - \tilde{T}(l,t)\right)\xi(t)\,dt\cdot\varphi(l), \\ \forall\varphi\in H^1(]0,l[) \text{ such that } \varphi(0)=0,\ \forall\xi\in L^2(]0,t_f[). \end{array}\right. \tag{3.32}$$

(3.32) being true $\forall\xi\in L^2(]0,t_f[)$, we have that $\tilde{T}\in\{\check{T}\in L^2(0,t_f;H^1(]0,l[));\ \frac{d\check{T}}{dt}\in L^2(0,t_f;H^1_L(]0,l[)^*)\}$ verifies $\forall' t\in]0,t_f[$:

$$\left\{\begin{array}{c} c_p m_g \left\langle \frac{\partial \tilde{T}}{\partial t}(\cdot,t),\varphi \right\rangle_{H^1(]0,l[)^*,H^1(]0,l[)} = -k_h \int_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)dx \\ +\int_0^l \psi(\tilde{T}(x,t))\varphi(x)dx + \int_0^l h_T(x,t)\varphi(x)dx \\ +\left\{h_c(T_a-\tilde{T}(l,t)) + \left[\Theta(T_S(t))-\Theta(\tilde{T}(l,t))\right]\right\}\varphi(l), \\ \forall\varphi\in H^1(]0,l[) \text{ such that } \varphi(0)=0. \end{array}\right.$$

In conclusion, $\tilde{T} \in \{\check{T} \in L^2(0, t_f; H^1(]0, l[)); \frac{d\check{T}}{dt} \in L^2(0, t_f; H_L^1(]0, l[)^*)\}$ verifies (3.10). By proposition 3.13 and proposition 3.14, $\tilde{T} \in S$. In conclusion $\tilde{T} = \Phi(T)$. We have seen that the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is strongly convergent to $\Phi(T)$ in the space $L^2(0, t_f; C([0, l]))$. A standard argument of general topology allows now to conclude that the sequence $(\tilde{T}_n := \Phi(T_n))_{n\in\mathbb{N}}$ itself is strongly convergent to $\Phi(T)$ in the space $L^2(0, t_f; C([0, l]))$. Thus Φ is continuous from S into S. □

It remains to prove that $\Phi(S)$ is relatively compact in the space $L^2(0, t_f;\ C([0, l]))$ to be allowed to apply Schauder's Theorem 3.15.

Proposition 3.23. *$\Phi(S)$ is relatively compact in the space $L^2(0, t_f; C([0, l]))$.*

Proof. $\|T\|_{L^2(Q)}$ for T running over the closed convex subset S of the space $L^2(0, t_f; C([0, l]))$ is bounded by a constant depending only on the upper bound $\bar{T} \in \mathbb{R}_+^*$ appearing in the definition of S. Thus by the estimate (3.14) of Theorem 3.9, the set $\{\Phi(T); T \in S\}$ is bounded in the space

$$\{\check{T} \in L^2(0, t_f; H^1(]0, l[)); \frac{d\check{T}}{dt} \in L^2(0, t_f; H_L^1(]0, l[)^*)\}$$

endowed with its natural norm. But, by the compactness Theorem 5.1 p.58 of [16], the continuous embedding from the space $\{\check{T} \in L^2(0, t_f; H^1(]0, l[)); \frac{d\check{T}}{dt} \in L^2(0, t_f; H_L^1(]0, l[)^*)\}$ endowed with its natural norm, into $L^2(0, t_f; C([0, l]))$ is compact. Thus, the set $\{\Phi(T); T \in S\}$ is relatively compact in the space $L^2(0, t_f; C([0, l]))$. □

We are now in a position to apply Schauder theorem 3.15 to the mapping $\Phi : S \to S : T \mapsto \tilde{T}$. This mapping has thus at least one fixed point, which gives us the existence of a solution to the initial boundary value problem (3.3). This proves also that the solution of the initial boundary value problem (3.3) which we know to be unique by Proposition 3.8 is lower bounded by T_a and upper bounded by $\bar{T}$. From these bounds on the temperature follows by Lemma 3.12:

$$2B_g^k(T_a) \leq \int_{-1}^{+1} I_T^k(x, t, \mu)\ d\mu \leq 2B_g^k(\bar{T}),\ \forall k = 1, \ldots, M.$$

Remark 3.24. In the preceding proof of our existence result, we have supposed that $T_0(\cdot) \geqslant T_a$ and that $T_S(\cdot) \geqslant T_a$. Although, these assumptions are natural in our physical context, these assumptions are not necessary. It suffices to replace in the definition of our closed convex subset S of $L^2(0, t_f; C([0, l]))$ (*cf.* 3.23), T_a by $T_{\inf} \in [0, T_a]$ such that $T_0(\cdot) \geqslant T_{\inf}$ and $T_S(\cdot) \geqslant T_{\inf}$ a. e. on $]0, t_f[$. In particular, Proposition 3.13 remains true with T_a replaced by $T_{\inf}$ in its statement.

4. **Numerical results**

In this section, we present some numerical results made to test the heating of a glass plate, with different constant temperatures of the black radiative source i.e. the black steel-metal S situated above it. The aim of these tests is to see for what temperature of the black radiative source, one is able to attain a temperature in the glass plate suitable for manufacturing applications such as glass forming. In our case, we consider that the thickness of our glass plate is equal to 6 mm. We have choosen four different values for the temperature T_S of the black radiative source S: $1500°C$, $1750°C$, $2000°C$ and $2250°C$, and we have ploted the evolution of the temperature on the upper face of the glass plate and in mid thickness with respect to time. The results are shown in the figure below: The graphs of the temperature on the upper face of the glass plate (resp. at half thickness) have the same behaviour and show that stationary temperatures are attained very quickly in the glass plate, let us say approximately after one minute. These stationary temperatures e.g. on the upper face of the glass plate, are approximatively respectively $350°C$,

$525°C$, $675°C$, $825°C$ showing a nonlinear dependence with respect to the temperature T_S of the black radiative source.

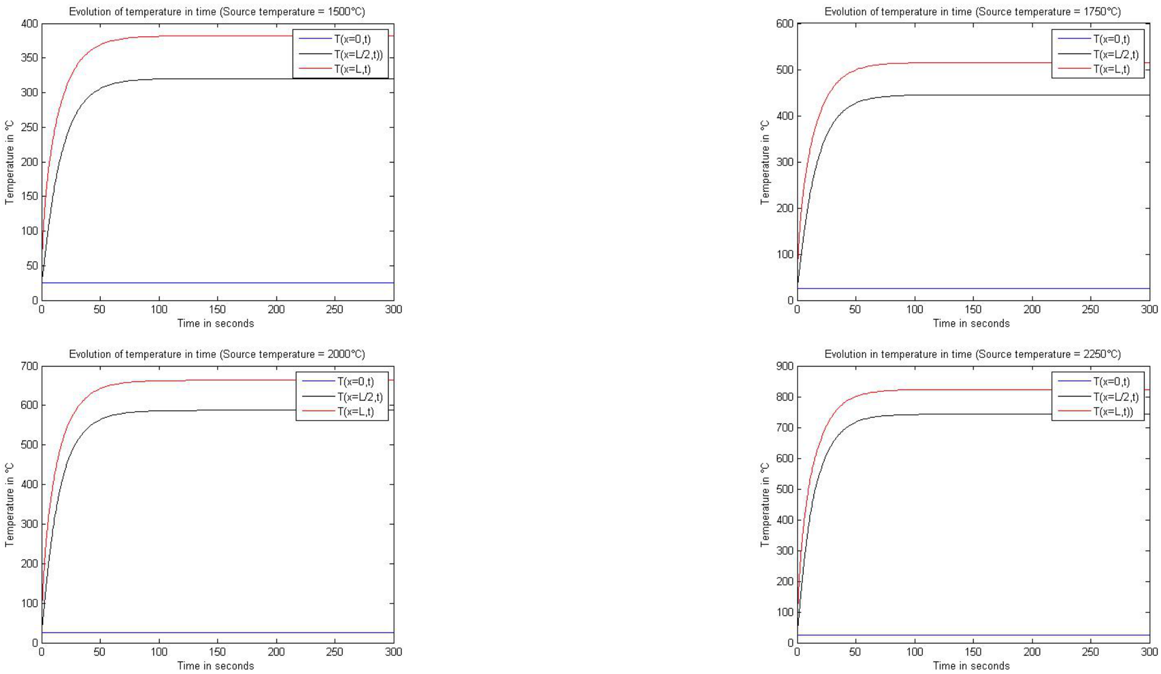

Figure 4.1. Radiative heating of the glass plate: Evolution of the temperature on the surface and on the middle of the glass plate.

To our knowledge, glass forming is usually made at a temperature between 675 and 725 degree Celsius in the glass plate. As one can see from the figure below, to attain this range of temperature in the glass plate, our numerical results suggest that the temperature of the black radiative source should be bigger than 2000 degree Celsius. To conclude, let us pay attention to the fact that in our one-dimensional problem, the distance between the upper face of the glass plate and the radiating black steel-metal S above it, is irrelevant, owing to the fact that a point on the upper face of the glass plate always views the radiating black steel-metal above it on a solid angle equal to 2π, whatever is that distance. This is due to the infinite dimensions in the horizontal directions of the radiating black steel-metal.

Appendix

Lemma 4.1. *If* $u,\ v \in L^2(]0,l[)$*, then* $\|u_- - v_-\|_{L^2(]0,l[)} \leq \|u - v\|_{L^2(]0,l[)}$.

Proof. $\forall x \in]0,l[$: $u_-(x) = \frac{|u(x)|-u(x)}{2}$ and $v_-(x) = \frac{|v(x)|-v(x)}{2}$. Thus:

$$|u_-(x) - v_-(x)| = \left|\frac{|u(x)|-u(x)-|v(x)|+v(x)}{2}\right|$$
$$\leq \frac{1}{2}\left|\ |u(x)| - |v(x)|\ \right| + \frac{1}{2}\left|v(x) - u(x)\right| \leq \left|u(x) - v(x)\right|,\ \forall x \in]0,l[.$$

Therefore:

$$\|u_- - v_-\|_{L^2(]0,l[)} = \sqrt{\int_0^l |u_-(x) - v_-(x)|^2\ dx}$$
$$\leq \sqrt{\int_0^l |u(x) - v(x)|^2\ dx} = \|u - v\|_{L^2(]0,l[)}.$$

□

The following result can be deduced from proposition 4 of [1]; we give only here a direct proof based on Vitali's theorem and lemma A.4 p.53 of [11].

Proposition 4.2. *The nonlinear mapping* $H^1(]0,l[) \to H^1(]0,l[) : \psi \mapsto \psi_-$ *is continuous. Similarly, the nonlinear mapping* $H^1(]0,l[) \to H^1(]0,l[) : \psi \mapsto \psi_+$ *is also continuous.*

Proof. Let $\psi \in H^1(]0,l[)$ and $(\psi_n)_{n\in\mathbb{N}} \subset H^1(]0,l[)$ a sequence tending to ψ in the norm of $H^1(]0,l[)$. A fortiori, $\psi_n \to \psi$ in $L^2(]0,l[)$. Consequently, by the preceding lemma: $\psi_n^- \to \psi_-$ in $L^2(]0,l[)$ (to alleviate the notation, we have denoted the negative part of ψ_n, ψ_n^-). To conclude that $\psi_n^- \to \psi_-$ in $H^1(]0,l[)$, it remains to prove that $\frac{d\psi_n^-}{dx} \to \frac{d\psi_-}{dx}$ in $L^2(]0,l[)$. By ([11], pp.50-54):

$$\frac{d\psi_n^-}{dx} = -\frac{d\psi_n}{dx}\mathbf{1}_{\{\psi_n<0\}} \text{ and } \frac{d\psi_-}{dx} = -\frac{d\psi}{dx}\mathbf{1}_{\{\psi<0\}}. \tag{4.1}$$

As $H^1(]0,l[) \hookrightarrow C([0,l])$, we have also that $\psi_n \to \psi$ in $C([0,l])$. The sequence $\left(\frac{d\psi_n}{dx}\right)_{n\in\mathbb{N}}$ tends to $\frac{d\psi}{dx}$ in $L^2(]0,l[)$, and therefore there exists a subsequence $\left(\frac{d\psi_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ which converges a.e. on $]0,l[$ to $\frac{d\psi}{dx}$. Let us show that the sequence $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$.

First case: $x \in [0,l]$ and $\psi(x) > 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to \frac{d\psi}{dx}(x)$ and let us consider such an x. We have also that $\psi_{n_k}(x) \to \psi(x)$ as $k \to +\infty$, and therefore for k sufficiently large, we will have also $\psi_{n_k}(x) > 0$. Therefore by formula (4.1), we have evidently that $\frac{d\psi_{n_k}^-}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.
Second case: $x \in [0,l]$ such that $\psi(x) < 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to \frac{d\psi}{dx}(x)$ and let us consider such an x. We have also that $\psi_{n_k}(x) \to \psi(x)$ as $k \to +\infty$, and therefore for k sufficiently large, we will have also $\psi_{n_k}(x) < 0$. Therefore by formula (4.1), we have that $\frac{d\psi_{n_k}^-}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.
Third case: $x \in [0,l]$ such that $\psi(x) = 0$. Fortunately as $\psi \in H^1(]0,l[)$ by lemma A.4 p.53 of [11], for almost such x, $\frac{d\psi}{dx}(x) = 0$. Thus let us consider $x \in [0,l]$ such that $\psi(x) = 0$ and $\frac{d\psi}{dx}(x) = 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to \frac{d\psi}{dx}(x)$. By formula (4.1): $\frac{d\psi_-}{dx}(x) = 0$ and $\left|\frac{d\psi_{n_k}^-}{dx}(x)\right| \leq \left|\frac{d\psi_{n_k}}{dx}(x)\right| \to 0$. Thus still $\frac{d\psi_{n_k}^-}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.

We have thus proved that the sequence $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$. To prove that the sequence $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges also to $\frac{d\psi_-}{dx}$ in $L^2(]0,l[)$, we apply Vitali's theorem ([21], p.16). As

$\frac{d\psi_{n_k}}{dx} \to \frac{d\psi}{dx}$ a.e. and in $L^2(]0,l[)$, by the necessary part of Vitali's theorem $\int\limits_A \left|\frac{d\psi_{n_k}}{dx}(x)\right|^2 dx \to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $[0,l]$. But $\left|\frac{d\psi_{n_k}^-}{dx}(x)\right| \leq \left|\frac{d\psi_{n_k}}{dx}(x)\right|$ and thus also $\int\limits_A \left|\frac{d\psi_{n_k}^-}{dx}(x)\right|^2 dx \to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $[0,l]$. As we have shown that $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$, by the sufficient part of Vitali's theorem, $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges also to $\frac{d\psi_-}{dx}$ in $L^2(]0,l[)$. A standard argument of general topology allows now to conclude that the sequence $\left(\frac{d\psi_n^-}{dx}\right)_{n\in\mathbb{N}}$ itself is strongly convergent to $\frac{d\psi_-}{dx}$ in the space $L^2(]0,l[)$. In conclusion, $\psi_n^- \to \psi_-$ in $H^1(]0,l[)$. □

Proposition 4.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(0,t_f;H^1(]0,l[))$ converging to $u \in L^2(0,t_f;H^1(]0,l[))$. Let us set*

$$u^- :]0,t_f[\to H^1(]0,l[) : t \mapsto (u(t))_-$$

and

$$u^+ :]0,t_f[\to H^1(]0,l[) : t \mapsto (u(t))_+.$$

Let us define similarly u_n^- and u_n^+, $\forall n \in \mathbb{N}$. Then the sequence $(u_n^-)_{n\in\mathbb{N}}$ (resp. $(u_n^+)_{n\in\mathbb{N}}$) converges to u^- (resp. u^+) in $L^2(0,t_f;H^1(]0,l[))$.

Proof. We give the proof for $(u_n^-)_{n\in\mathbb{N}}$, the proof for the sequence $(u_n^+)_{n\in\mathbb{N}}$ being similar. By the necessary part of Vitali's theorem ([21], p.16) $\int\limits_A \|u_n(t)\|^2_{H^1(]0,l[)}\, dt \to 0$ uniformly in $n \in \mathbb{N}$ as $meas(A) \to 0$, A arbitrary measurable subset of $]0,t_f[$. Also there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges a.e. on $]0,t_f[$ to u, as vectorvalued functions with values in $H^1(]0,l[)$. By the previous proposition, the subsequence $\left(u_{n_k}^-\right)_{k\in\mathbb{N}}$ converges to u^- a.e. on $]0,t_f[$ .As $\left\|u_{n_k}^-(t)\right\|_{H^1(]0,l[)} \leq \|u_{n_k}(t)\|_{H^1(]0,l[)}$, it follows that a fortiori $\int\limits_A \left\|u_{n_k}^-(t)\right\|^2_{H^1(]0,l[)} dt \to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $]0,t_f[$. Applying the sufficient part of Vitali's theorem ([21], p.16) to the subsequence $\left(u_{n_k}^-\right)_{k\in\mathbb{N}}$, implies that $\left(u_{n_k}^-\right)_{k\in\mathbb{N}}$ converges to u^- in $L^2(0,t_f;H^1(]0,l[))$. A standard argument of general topology shows that the sequence $(u_n^-)_{n\in\mathbb{N}}$ itself, converges to u^- in $L^2(0,t_f;H^1(]0,l[))$. □

Remark 4.4. Propositions 4.2 and 4.3 remain valid if $H^1(]0,l[)$ is replaced by $H^1_L(]0,l[)$ in their respective statements.

Ackowledgements: The authors gratefully acknowledge the supports of the institutions.

References

[1] A. Ancona. *"Continuité des contractions dans les espaces de Dirichlet"*, volume 563. Springer-Verlag, 1976.

[2] H. Brézis. *"Analyse fonctionnelle Théorie et applications"*. Masson, Paris, 1993.

[3] D. Clever and J. Lang. Optimal control of radiative heat transfer in glass cooling with restrictions on the temperature gradient. *Optim. Control Appl. Meth.*, 2011.

[4] L. Dautray and J.-L. Lions. *"Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques Volume 8 Evolution: semi-groupe, variationnel"*. Masson, Paris, 1988.

[5] F. Desvignes. *"Rayonnements optiques Radiométrie-Photométrie"*. Masson, Paris, 1991.

[6] P.-E. Druet. Weak solutions to a time-dependent heat equation with nonlocal radiation boundary condition and arbitrary p-summable right-hand side. *Applications of Mathematics*, 55(2):111–149, 2010.

[7] L.C. Evans. *"Partial Differential Equations"*, volume GSM 19. AMS Providence, Rhode Island, 1999.

[8] A. Friedman. *"Partial Differential Equations"*. Robert E. Krieger Publishing Company, 1976.

[9] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *"Optimization with PDE constraints"*. Springer, 2009.

[10] H.C. Hottel and A.F. Sarofim. *"Radiative Transfer"*. McGraw-Hill Book Company, 1967.

[11] D. Kinderlehrer and G. Stampacchia. *"An Introduction to Variational Inequalities and their Applications"*. Academic Press, 1980.

[12] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Uralceva. *"Linear and Quasilinear Equations of Parabolic Type"*, volume 23. Translations of Mathematical Monographs, AMS Providence, 1968.

[13] L.T. Laitinen. *"Mathematical modelling of conductive-radiative heat transfer"*. PhD thesis, University of Jyväskylä, Finland, 2000.

[14] M.T. Laitinen and T. Tiihonen. Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials. *Math. Methods Appl. Sci.*, 21(5):375–392, 1998.

[15] J. Lang. Adaptive computation for boundary control of radiative heat transfer in glass. *Journal of Computational and Applied Mathematics*, 183:312–326, 2005.

[16] J.-L. Lions. *"Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires"*. Dunod, Paris, 1967.

[17] A. Maréchal. Optique géométrique générale, in encyclopedia of physics, volume xxiv: Fundamentals of optics with 761 figures, edited by S. Flügge. 24:44–170, 1956.

[18] C.-M. Marle. *"Mesures et probabilités"*. Hermann, 1974.

[19] C. Meyer, P. Philip, and F. Tröltzsch. Optimal control of a semilinear PDE with nonlocal radiation interface conditions. *SIAM J. Control Optim.*, 45:699–721, 2006.

[20] M.F. Modest. *"Radiative Heat Transfer"*. Academic Press, second edition, 2003.

[21] D. Pascali and S. Sburlan. *"Non linear mappings of monotone type"*. Editura Academiei, Bucuresti, Romania, Sijthoff & Noordhoff International Publishers, 1978.

[22] R. Pinnau. Analysis of optimal boundary control for radiative heat ransfer modeled by the sp_1-system. *Communications in Mathematical Sciences*, 5,4:951–969, 2007.

[23] M. Planck. *"The theory of Heat Radiation"*. Dover Publications Inc., 1991.

[24] M. Reed and B. Simon. *"Methods of Modern Mathematical Physics I: Functional analysis"*. Academic Press, 1972.

[25] N. Siedow, T. Grosan, D. Lochegnies, and E. Romero. Application of a new method for radiative heat tranfer to flat glass tempering. *J. Am. Ceram. Soc.*, 88(8):2181–2187, 2005.

[26] L. Soudre. *"Etude numérique et expérimentale du thermoformage d'une plaque de verre"*. PhD thesis, Nancy-Université Henri Poincaré, laboratoire LEMTA„ France, 2009.

[27] J. Taine, E. Iacona, and J.-P. Petit. *"Transferts Thermiques Introduction aux transferts d'énergie"*. Dunod, 4th edition, 2008.

[28] J. Taine and J.-P. Petit. *"Heat Transfer"*. Prentice Hall, 1993.

[29] K. Yosida. *"Functional Analysis"*. Springer-Verlag, third edition, 1971.