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FUNDAMENTAL UNITS FOR ORDERS GENERATED BY A UNIT

by

Stéphane R. Louboutin

Abstract. — Let ε be an algebraic unit for which the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1. Assume that ε is not a complex root of unity. It is natural to wonder whether ε is a fundamental unit of this order. It turns out that the answer is in general positive, and that a fundamental unit of this order can be explicitly given (as an explicit polynomial in ε) in the rare cases when the answer is negative. This paper is a self-contained exposition of the solution to this problem, solution which was up to now scattered in many papers in the literature. We also include the state of the art in the case that the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is greater than 1 when now one wonders whether the set $\{\varepsilon\}$ can be completed in a system of fundamental units of the order $\mathbb{Z}[\varepsilon]$.

Résumé. — Soit ε une unité algébrique pour laquelle le rang du groupe des unités de l'ordre $\mathbb{Z}[\varepsilon]$ est égal à 1. Supposons que ε ne soit pas une racine complexe de l'unité. Il est alors naturel de se demander si ε est une unité fondamentale de cet ordre. Nous montrons que la réponse est en général positive et que, dans les rares cas où elle ne l'est pas, une unité fondamentale de cet ordre peut être explicitement donnée (comme polynôme en ε). Nous présentons ici une exposition complète de la solution à ce problème, solution jusqu'à présent dispersée dans plusieurs articles. Nous incluons l'état de l'art de ce problème dans le cas où la rang du groupe des unités de l'ordre $\mathbb{Z}[\varepsilon]$ est strictement plus grand que 1, où la question naturelle est maintenant de savoir si on peut adjoindre à ε d'autres unités de l'ordre $\mathbb{Z}[\varepsilon]$ pour obtenir un système fondamental d'unités de cet ordre.

Contents

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1. Introduction and Notation

Let ε be an algebraic unit for which the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1. Then either (i) ε is a totally real quadratic unit, or (ii) ε is a non-totally real cubic unit or (iii) ε is a totally imaginary quartic unit. In these three situations, if we assume that ε is not a complex root of unity, it is natural to ask whether ε is a fundamental unit of the order $\mathbb{Z}[\varepsilon]$. And in case it is not, it is natural to construct one from it. In Section 2, in the very simple case of the real quadratic units, we introduce the method that we will also use in Sections 3 and 4 to answer to this question in the more difficult remaining cases of cubic units of negative discriminants and totally imaginary quartic units. Now, assume that the the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is greater than 1. The natural question is now to wonder whether the set $\{\varepsilon\}$ can be completed in a system of fundamental units of the order $\mathbb{Z}[\varepsilon]$. In Section 5, we will answer to this question in the case of totally real cubic units, the only situation where to date this question has been answered. We conclude this paper by giving in Section 6 a conjecture for the case of quartic units of negative discriminants and by showing in Section 7 that the solution to this problem for units of degree greater than 4 is bound to be more complicated.

In order to put the previously published elements of solution to our natural question in a general framework that might lead to solutions in presently unsolved cases, we introduce heights for polynomials and algebraic units. They will enable us to formulate our mains results in (8), (14), (18), (20) and (21) in a clear and uniform way. Let $\Pi_\alpha(X) = X^n - a_{n-1}X^{n-1} + \cdots + (-1)^{n-1}a_1X + (-1)^n a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of an algebraic integer α. It is monic and $\mathbb{Q}$-irreducible. Let $d_\alpha > 0$ be the absolute value of its discriminant $D_\alpha \neq 0$. Let $\beta \in \mathbb{Z}[\alpha]$ and assume that $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$, i.e. that $\deg \Pi_\beta(X) = \deg \Pi_\alpha(X)$. Then

$$D_\beta = (\mathbb{Z}[\alpha] : \mathbb{Z}[\beta])^2 D_\alpha \text{ and hence } d_\beta = (\mathbb{Z}[\alpha] : \mathbb{Z}[\beta])^2 d_\alpha. \tag{1}$$

In particular, $\mathbb{Z}[\beta] = \mathbb{Z}[\alpha]$ if and only if $d_\beta = d_\alpha$.

Now, assume that α is an algebraic unit, i.e. assume that $a_0 = \pm 1$. For $p > 0$ a real number and $p = \infty$ we define the **heights** of α and of its minimal polynomial $\Pi_\alpha(X)$ by

$$H_p(\alpha) = H_p(\Pi_\alpha(X)) := \max_{1\leq k\leq n} (|\alpha_k|^{n_k p} + |\alpha_k|^{-n_k p})^{1/p} > 1$$

and

$$H_\infty(\alpha) = H_\infty(\Pi_\alpha(X)) := \max_{1\leq k\leq n} \max(|\alpha_k|^{n_k}, |\alpha_k|^{-n_k}) \geq 1,$$

where $\alpha_1, \cdots, \alpha_n$ are the complex roots of $\Pi_\alpha(X)$ and $n_k = 1$ if α_k is real and $n_k = 2$ is α_k is not real. Notice that $p \mapsto H_p(P(X))$ is a decreasing function of $p > 0$ such that $\lim_{p\to\infty} H_p(P(X)) = H_\infty(P(X))$. Notice also that $H_\infty(\alpha) > 1$ as soon as α is not a complex root of unity (e.g. see [Was, Lemma 1.6]). The most useful property of our height, as compared to the usual height $H(P(X)) = \max_{0\leq i\leq n-1} |a_i|$ and the Malher measure $M(P(X)) = \prod_{1\leq i\leq n} \max(1, |\alpha_i|)$ of $P(X) = \prod_{i=1}^n (X - \alpha_i) = X^n - a_{n-1}X^{n-1} + \cdots + (-1)^{n-1}a_1X + (-1)^n a_0 \in \mathbb{C}[X]$, is that for any algebraic unit α and any $0 \neq m \in \mathbb{Z}$ we have

$$H_\infty(\alpha^m) = H_\infty(\alpha)^{|m|}, \tag{2}$$

provided that $\mathbb{Q}(\alpha^m) = \mathbb{Q}(\alpha)$, a property akin to the one satisfied by the canonical height on an elliptic curve.

Assume that we have proved that for all algebraic units α of a given degree $n > 1$ we have

$$C_a H_\infty(\alpha)^a \leq d_\alpha \leq C_b H_\infty(\alpha)^b \tag{3}$$

where $0 < a < b$ and $C_a, C_b > 0$ depend only on the numbers of real and complex conjugates of α (see Lemma 3, Theorems 9, 24, 33 and Conjecture 39). Since there are only finitely many algebraic units of a given degree of a bounded height, there exists a unit η_0 of degree n such that $H_\infty(\alpha) \geq H_\infty(\eta_0) > 1$ for all algebraic units α of degree n that are not a complex root of unity. Let N be the least rational integer greater than b/a. If an algebraic unit ε of degree n that is not a complex root of unity is such that $\varepsilon = \pm\eta^m$ for some $\eta \in \mathbb{Z}[\varepsilon]$ and some $m \geq N$, then $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $D_\eta = D_\varepsilon$ (Lemma 2). In particular, ε and η are of the same degree and have the same numbers of real and complex conjugates. Using (2) and (3), we obtain

$$C_a H_\infty(\eta)^{Na} \leq C_a H_\infty(\eta)^{ma} = C_a H_\infty(\varepsilon)^a \leq d_\varepsilon = d_\eta \leq C_b H_\infty(\eta)^b.$$

Hence, $H_\infty(\eta) \leq (C_b/C_a)^{1/(Na-b)}$ is bounded and there are only finitely many such η's. Moreover, the inequalities

$$H_\infty(\eta_0)^{ma} \leq H_\infty(\eta)^{ma} \leq (C_b/C_a) H_\infty(\eta)^b \leq (C_b/C_a)^{Na/(Na-b)}$$

show that m is bounded and there are only finitely many such ε's. In conclusion, assuming that (3) holds true, we obtain that if $\varepsilon = \pm\eta^m$ for some $\eta \in \mathbb{Z}[\varepsilon]$ and some $0 \neq m \in \mathbb{Z}$ then $|m| \leq b/a$, appart from finitely many sporadic algebraic units ε's. This line of reasoning is clearly a key step toward solving our general question. Indeed, as it is rather easy to settle the case $m = 2$ (see Lemmas 13, 20 and 35), our problem is almost solved in the situations where $b/a < 3$. However, solving the problem $\varepsilon = \pm\eta^3$ is not that easy. In fact, we do not even know how to solve it for totally imaginary quartic units of negative discriminant (see Conjecture 38). Hence our problem is probably hard to solve in the situations where the best exponents in (3) turn out to satisfy $b/a \geq 3$.

Anyway, clearly our main tool to tackle our problem will thus to obtain lower and upper bounds of the form (3) for the absolute discriminants d_α of algebraic units α. We could have

restrained ourselves to the use of the single height H_∞, but the proof of Corollary 12 makes it clear that using the height H_1 yields better bounds.
Notice that if ε is an algebraic unit, then $\mathbb{Z}[\varepsilon] = \mathbb{Z}[-\varepsilon] = \mathbb{Z}[1/\varepsilon] = \mathbb{Z}[-1/\varepsilon]$, $D_\varepsilon = D_{-\varepsilon} = D_{1/\varepsilon} = D_{-1/\varepsilon}$ and $H_p(\varepsilon) = H_p(-\varepsilon) = H_p(1/\varepsilon) = H_p(-1/\varepsilon)$ for $p > 0$ or $p = \infty$. The four monic polynomials $\Pi_\varepsilon(X) = X^n - a_{n-1}X^{n-1} + \cdots + (-1)^{n-1}a_1X + (-1)^n a_0 \in \mathbb{Z}[X]$, with $a_0 \in \{\pm 1\}$, $\Pi_{-\varepsilon}(X) = (-1)^n\Pi_\alpha(-X)$, $\Pi_{1/\varepsilon}(X) = (-1)^n a_0 X^n \Pi_\varepsilon(1/X)$ and $\Pi_{-1/\varepsilon}(X) = (-1)^n a_0 X^n \Pi_\varepsilon(-1/X)$ are called **equivalent**. They have the same discriminant and the same height. At least one of them is such that $|a_1| \leq a_{n-1}$. We call it **reduced**. By an appropriate choice of the root of $\Pi_\varepsilon(X)$, we may also assume that $|\varepsilon| \geq 1$. If ε is real, we may instead assume that $\varepsilon > 1$.
For example, take $\Pi_\varepsilon(X) = X^4+3X^3+6X^2+4X+1$. Since $\Pi_\varepsilon(X)$ is not reduced, we set $\varepsilon' = -1/\varepsilon = \varepsilon^3+3\varepsilon^2+6\varepsilon+4 = P(\varepsilon)$, for which $\Pi_{\varepsilon'}(X) = X^4\Pi_\varepsilon(-1/X) = X^4-4X^3+6X^2-3X+1$ is reduced. By Point 2(c)iii of Theorem 18, $\eta = \varepsilon'^3 - 3\varepsilon'^2 + 3\varepsilon' = \varepsilon^3 + 2\varepsilon^2 + 4\varepsilon + 1$ is a fundamental unit of $\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon]$ and $\varepsilon' = -1/\eta^4$ yields $\varepsilon = -1/\varepsilon' = \eta^4$. Moreover, $\Pi_\eta(X) = X^4 - X^3 + 1$ is reduced, $d_\varepsilon = d_\eta = 229$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$.

2. The real quadratic case

We introduce in this very simple situation the three tools and method we will use to solve the more difficult cubic and quartic cases.
Let ε be an algebraic quadratic unit which is not a complex root of unity and for which the rank of the group of units of the quadratic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is totally real. We may and we will assume that $\varepsilon > 1$.

Lemma 1. — *The smallest quadratic unit greater than 1 is $\eta_0 := (1+\sqrt{5})/2$.*

Lemma 2. — *Let ε be an algebraic integer. If $\varepsilon = \pm\eta^n$ with $n \in \mathbb{Z}$ and $\eta \in \mathbb{Z}[\varepsilon]$, then $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$. Hence, $D_\eta = D_\varepsilon$ and $d_\eta = d_\varepsilon$.*

Proof. — Notice that $\mathbb{Z}[\eta] \subseteq \mathbb{Z}[\varepsilon] = \mathbb{Z}[\pm\eta^n] \subseteq \mathbb{Z}[\eta]$ and use (1). □

Lemma 3. — *Let α be a real quadratic unit. Then*

$$\left(|\alpha| - |\alpha|^{-1}\right)^2 \leq d_\alpha \leq \left(|\alpha| + |\alpha|^{-1}\right)^2. \tag{4}$$

Proof. — We have $d_\alpha = (\alpha - \alpha')^2$, where $\alpha' = \pm 1/\alpha$ be the conjugate of α. □

Theorem 4. — *A real quadratic unit $\varepsilon > 1$ is always the fundamental unit of the quadratic order $\mathbb{Z}[\varepsilon]$, except if $\varepsilon = (3+\sqrt{5})/2$, in which case $\varepsilon = \eta^2$, where $1 < \eta = (1+\sqrt{5})/2 = \varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ is the fundamental unit of $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$, and $d_\varepsilon = d_\eta = 5$.*

Proof. — Assume that $\varepsilon > 1$ is not the fundamental unit of $\mathbb{Z}[\varepsilon]$. Then there exist a quadratic unit $1 < \eta \in \mathbb{Z}[\varepsilon]$ and $n \geq 2$ such that $\varepsilon = \eta^n$, which yields $d_\varepsilon = d_\eta$ (Lemma 2). Using (4) we obtain

$$0 < \eta - \eta^{-1} = \frac{\eta^2 - \eta^{-2}}{\eta + \eta^{-1}} \leq \frac{\eta^n - \eta^{-n}}{\eta + \eta^{-1}} = \frac{\varepsilon - \varepsilon^{-1}}{\eta + \eta^{-1}} \leq \sqrt{d_\varepsilon/d_\eta} = 1.$$

But $0 < \eta - \eta^{-1} \leq 1$ implies $1 < \eta \leq (1+\sqrt{5})/2$ and $\eta = \eta_0$, by Lemma 1. We now obtain $1 = \eta_0 - \eta_0^{-1} = (\eta_0^2 - \eta_0^{-2})/(\eta_0 + \eta_0^{-1}) \leq (\eta_0^n - \eta_0^{-n})/(\eta_0 + \eta_0^{-1}) \leq 1$, which yields $n = 2$. □

3. The non-totally real cubic case

The aim of this Section 3 is to prove Theorem 8. It was first proved in [Nag]. However, while working on class numbers of some cubic number fields, we come up in [Lou06] with a completely different proof of Nagell's result. Our proof was based on lower bounds on absolute discriminants of non-totally real algebraic cubic units (see Theorem 9), proof then simplified in [Lou10].

Definition 5. — *A cubic polynomial of type (T) is a monic cubic polynomial $P(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ which is $\mathbb{Q}$-irreducible ($\Leftrightarrow b \neq a$ and $b \neq -a-2$), of negative discriminant $D_{P(X)} < 0$ and whose only real root ε_P satisfies $\varepsilon_P > 1$ ($\Leftrightarrow P(1) < 0 \Leftrightarrow b \leq a-1$). In that situation, $H_p(P(X)) = (\varepsilon_P^p + \varepsilon_P^{-p})^{1/p}$ and $H_\infty(P(X)) = \varepsilon_P$.*

Let ε be an algebraic cubic unit for which the rank of the group of units of the cubic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is not totally real. We may and we will assume that ε is real and that $\varepsilon > 1$, i.e. that $\Pi_\varepsilon(X)$ is a cubic polynomial of type (T) (notice that if ε is of type (T) and $\varepsilon = \eta^n$ for some odd $n \geq 3$ and some $\eta \in \mathbb{Z}[\varepsilon]$, then η is clearly also of type (T), whereas if ε is reduced then it is not clear whether η is also necessarily reduced):

Lemma 6. — *Let $\varepsilon_P > 1$ be the only real root of a cubic polynomial $P(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ of type (T). Then*

$$0 \leq a < \varepsilon_P + 2 \text{ and } |b| \leq \sqrt{4a+4}. \tag{5}$$

Proof. — Let $\varepsilon_P^{-1/2}e^{i\phi}$ and $\varepsilon_P^{-1/2}e^{-i\phi}$ be the non-real complex roots of $P(X)$. Then $a = \varepsilon_P + 2\varepsilon_P^{-1/2}\cos\phi \geq \varepsilon_P - 2 > -1$ and $b = 2\varepsilon_P^{1/2}\cos\phi + \varepsilon_P^{-1}$. Hence,

$$4a - b^2 = 4\varepsilon_P \sin^2\phi + 4\varepsilon_P^{-1/2}\cos\phi - \varepsilon_P^{-2} > -5 \tag{6}$$

and (5) holds true. □

Notice that (5) makes it easy to list all the cubic polynomials of type (T) whose real roots are less than or equal to a given upper bound B. Taking $B = 2$, we obtain:

Lemma 7. — *The real root $\eta_0 = 1.32471\cdots$ of $\Pi(X) = X^3 - X - 1$ is the smallest real but non-totally real cubic unit greater than 1.*

3.1. Statement of the result for the cubic case. —

Theorem 8. — *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant $D_\varepsilon = -d_\varepsilon < 0$. Let $\eta > 1$ be the fundamental unit of the cubic order $\mathbb{Z}[\varepsilon]$. Then $\varepsilon = \eta$, except in the following cases:*

1. *The infinite family of exceptions for which $\Pi_\varepsilon(X) = X^3 - M^2X^2 - 2MX - 1$, $M \geq 1$, in which case $\varepsilon = \eta^2$ where $\eta = \varepsilon^2 - M^2\varepsilon - M \in \mathbb{Z}[\varepsilon]$ is the real root of $X^3 - MX^2 - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta = 4M^3 + 27$.*
2. *The 8 following sporadic exceptions:*
   (a) (i) $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$.
   (ii) $\Pi_\varepsilon(X) = X^3 - 3X^2 + 2X - 1$, *in which case* $\varepsilon = \eta^3$ *where* $\eta = \varepsilon - 1 \in \mathbb{Z}[\varepsilon]$.
   (iii) $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$, *in which case* $\varepsilon = \eta^4$ *where* $\eta = \varepsilon^2 - 2\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$,

(iv) $\Pi_\varepsilon(X) = X^3 - 5X^2 + 4X - 1$, *in which case* $\varepsilon = \eta^5$ *where* $\eta = \varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.
(v) $\Pi_\varepsilon(X) = X^3 - 12X^2 - 7X - 1$, *in which case* $\varepsilon = \eta^9$ *where* $\eta = -3\varepsilon^2 + 37\varepsilon + 10 \in \mathbb{Z}[\varepsilon]$.

In these five cases, $\eta > 1$ *is the real root of* $\Pi_\eta(X) = X^3 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 23$.

(b) (i) $\Pi_\varepsilon(X) = X^3 - 4X^2 + 3X - 1$, *in which case* $\varepsilon = \eta^3$ *where* $\eta = \varepsilon^2 - 3\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.
(ii) $\Pi_\varepsilon(X) = X^3 - 6X^2 - 5X - 1$, *in which case* $\varepsilon = \eta^5$ *where* $\eta = -2\varepsilon^2 + 13\varepsilon + 5 \in \mathbb{Z}[\varepsilon]$.

In these two cases, $\eta > 1$ *is the real root of* $\Pi_\eta(X) = X^3 - X^2 - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 31$.

(c) $\Pi_\varepsilon(X) = X^3 - 7X^2 + 5X - 1$, *in which case* $\varepsilon = \eta^3$, *where* $1 < \eta = -\varepsilon^2 + 7\varepsilon - 3 \in \mathbb{Z}[\varepsilon]$ *is the real root of* $\Pi_\eta(X) = X^3 - X^2 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 44$.

3.2. Sketch of proof. — Let $\varepsilon > 1$ be a real but non-totally real cubic unit. Then ε is not a fundamental unit of the order $\mathbb{Z}[\varepsilon]$ if and only if there exists $p \geq 2$ a prime and $\eta \in \mathbb{Z}[\varepsilon]$ such that $\varepsilon = \eta^p$ (the main feature that makes it easier to deal with this cubic case than with the totally imaginary quartic case dealt with below is that -1 and $+1$ are the only complex roots of unity in $\mathbb{Z}[\varepsilon]$). Now, if $\varepsilon = \eta^n$ for some $\eta \in \mathbb{Z}[\varepsilon]$, then $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta$, by Lemma 2.

1. Assume that $\varepsilon = \eta^n$ for some non-totally real cubic unit $1 < \eta \in \mathbb{Z}[\varepsilon]$ and some $n \geq 3$. Using $\eta \geq \eta_0 = 1.32471\cdots$ (Lemma 7) and a double bound (7) for d_ε similar to (4), we will obtain in Corollary 12 that $1 < \eta \leq 4.5$ and $n \leq 10$.

2. In contrast with the quadratic case, this double bound (7) does not prevent ε from being infinitely many often a square in $\mathbb{Z}[\varepsilon]$. Hence, we characterize in Lemma 13 when this is indeed the case.

3. Finally, to determine all the $1 < \varepsilon$'s that admit a p-th root in $\mathbb{Z}[\varepsilon]$ for some $p \in \{3, 5, 7\}$, and to determine this p-th root $\eta > 1$, we make a list of all the cubic polynomials $\Pi_\eta(X) = X^3 - AX^2 + BX - 1 \in \mathbb{Z}[X]$ of type (T) with $0 \leq A \leq 6 < 4.5 + 2$, by (5), for which there exist $p \in \{3, 5, 7\}$ such that $\mathbb{Z}[\eta] = \mathbb{Z}[\eta^p]$, i.e. such that $D_\eta = D_{\eta^p}$, where $\Pi_{\eta^p}(X) = X^3 - aX^2 + bX - 1$ is computed as the resultant of $\Pi_\eta(Y)$ and $X - Y^n$, considered as polynomials of the variable Y. A Maple Program 1 settling this step is given below. We found 6 such occurrences, of discriminants -23, -31 or -44. Taking also into account the points 2 and 3 of Lemma 13, both of discriminant -23, and singling out the only case of point 1 of Lemma 13 of discriminant in $\{-23, -31, -44\}$, namely the case $M = 1$ of discriminant -31, we obtain Table 1, which completes the proof of Theorem 8:

p	$\Pi_\eta(X)$	$\Pi_{\eta^p}(X)$	$D_\eta = D_{\eta^p}$
2	X^3-X-1	$\mathbf{X^3-2X^2+X-1}$	-23
2	$\mathbf{X^3-2X^2+X-1}$	X^3-2X^2-3X-1	-23
3	X^3-X-1	$\mathbf{X^3-3X^2+2X-1}$	-23
3	$\mathbf{X^3-3X^2+2X-1}$	X^3-12X^2-7X-1	-23
5	X^3-X-1	X^3-5X^2+4X-1	-23
2	X^3-X^2-1	X^3-X^2-2X-1	-31
3	X^3-X^2-1	X^3-4X^2+3X-1	-31
5	X^3-X^2-1	X^3-6X^2-5X-1	-31
3	X^3-X^2-X-1	X^3-7X^2+5X-1	-44

TABLE 1.

Program 1:
for A **from** 0 **to** 6 **by** 1 **do**
$borneB := isqrt(4A+4)$:
for B **from** $-borneB$ **to** $\min(borneB, A-1)$ **by** 1 **do**
$p := x^3 - A \cdot x^2 + B \cdot x - 1$;
if $irreduc(p)$ **then**
$Dp := discrim(p, x)$;
if $Dp < 0$ **then**
for n **in** [3,5,7] do
$q := resultant(subs(x = y, p), x - y^n, y)$;
$Dq := discrim(q, x)$;
if $Dq = Dp$ **then** $print(n, sort(p, x), sort(q, x), Dp)$ **end if**
end do
end if
end if
end do
end do:

3.3. Bounds on discriminants. —

Theorem 9. — *Let α be a real cubic algebraic unit of negative discriminant. Then*

$$\max(|\alpha|^{3/2}, |\alpha|^{-3/2})/2 \leq d_\alpha \leq 4(|\alpha| + |\alpha|^{-1})^3 \leq 32\max(|\alpha|^3, |\alpha|^{-3}). \tag{7}$$

Hence, if $P(X) = X^3 - aX^2 + bX - c \in \mathbb{Z}[X]$, $c \in \{\pm 1\}$, is $\mathbb{Q}$-irreducible and of negative discriminant $D_{P(X)} < 0$, then

$$H_\infty(P(X))^{3/2}/2 \leq |D_{P(X)}| \leq 4H_1(P(X))^3 \leq 32H_\infty(P(X))^3. \tag{8}$$

Proof. — Clearly, (8) follows from (7): if α (real), β and $\bar\beta$ (not real) are the roots of $P(X)$, then $|\alpha||\beta|^2 = 1$, hence $H_p(P(X)) = (|\alpha|^p + |\alpha|^{-p})^{1/p} = (|\beta|^{2p} + |\beta|^{-2p})^{1/p}$ and $H_\infty(P(X)) = \max(|\alpha|, |\alpha|^{-1}) = \max(|\beta|^2, |\beta|^{-2})$.
Since (7) remains unchanged if we change α into $-\alpha$, $1/\alpha$ and $-1/\alpha$, we may assume that $\alpha > 1$, i.e. that $\Pi_\alpha(X)$ is of type (T). Let $\beta = \alpha^{-1/2}e^{i\phi}$ and $\bar\beta = \alpha^{-1/2}e^{-i\phi}$ be the non-real complex roots of $\Pi_\alpha(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$. Then

$$d_\alpha = -(\alpha - \beta)^2(\alpha - \bar\beta)^2(\beta - \bar\beta)^2 = 4(\alpha^{3/2} - 2\cos\phi + \alpha^{-3/2})^2 \sin^2\phi. \tag{9}$$

Hence, setting $X_\alpha = \alpha^{3/2} + \alpha^{-3/2} \geq 2$, we have

$$d_\alpha = X_\alpha^2 + 4 - (X_\alpha \cos\phi + 2\sin^2\phi)^2 - 4\cos^2\phi \leq X_\alpha^2 + 4 = \alpha^3 + \alpha^{-3} + 6 \leq (\alpha + \alpha^{-1})^3.$$

Let us now prove the lower bound on d_α. By (9) we have

$$d_\alpha \geq 4(\alpha^{3/4} - \alpha^{-3/4})^4 \sin^2\phi.$$

Assume that $\alpha > 16.2$.
First, assume that $\sin^2\phi \geq 2\alpha^{-3/2}$. Then we obtain $d_\alpha \geq 7\alpha^{3/2}$.
Secondly, assume that $\sin^2\phi < 2\alpha^{-3/2}$. By (6), we have

$$-1 < -4\alpha^{-1/2} - \alpha^{-2} \leq 4a - b^2 = 4\alpha\sin^2\phi + 4\alpha^{-1/2}\cos\phi - \alpha^{-2} < 12\alpha^{-1/2} < 3.$$

Since $4a - b^2 \equiv 0$ or $3 \pmod 4$, we obtain $4a = b^2$ and $\cos\phi < 0$ (otherwise $4a - b^2 \geq 4\alpha^{-1/2}\sin^2\phi + 4\alpha^{-1/2}\cos^2\phi - \alpha^{-2} > 0$). Hence, $0 = 4a - b^2 = 4\alpha\sin^2\phi - 4\alpha^{-1/2}\sqrt{1-\sin^2\phi} - \alpha^{-2}$. Therefore, $\sin^2\phi = \alpha^{-3/2} - \alpha^{-3}/4$, $\cos\phi = -1 + \alpha^{-3/2}/2$ (hence $b = 2\alpha^{1/2}\cos\phi + \alpha^{-1} = -2(\alpha^{1/2} - \alpha^{-1}) < 0$ and $\Pi_\alpha(X) = X^3 - B^2X^2 - 2BX - 1$ for some $B \geq 1$) and (9) yields

$$d_\alpha = 4(\alpha^{3/2} + 2)^2(\alpha^{-3/2} - \alpha^{-3}/4) > 4\alpha^{3/2}.$$

Therefore, $d_\alpha \geq 4\alpha^{3/2}$ for $\alpha > 16.2$.
Finally, if $1 < \alpha \leq 16.2$, then $\Pi_\alpha(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ is of type (T) with $0 \leq a \leq 18$, by (5). Using (5) we obtain that there are 211 such cubic polynomials. By computing approximations to the real root $\alpha > 1$ of each of these 211 cubic polynomials, we check that the lower bound on d_α given in (7) holds true for each of these 211 cubic polynomials. □

Remark 10. — *The exponents 3/2 and 3 in (7) are optimal. Indeed,*
if $\Pi_\alpha(X) = X^3 - M^2X^2 - 2MX - 1$, $M \geq 1$, then $M^2 < \alpha < M^2 + 1$, and $d_\alpha = 4M^3 + 27$ is asymptotic to $4\alpha^{3/2}$.
If $\Pi_\alpha(X) = X^3 - MX^2 - 1$, $M \geq 1$, then $M < \alpha < M + 1$, and $d_\alpha = 4M^3 + 27$ is asymptotic to $4\alpha^3$.

Remark 11. — *We can reformulate the lower bound on d_α in (7) as follows: let γ be a non-real cubic algebraic unit of negative discriminant satisfying $|\gamma| > 1$. Then $|\Im(\gamma)| \gg |\gamma|^{-1/2}$ (explicitly). We wish we understood beforehand why such a lower bound must hold true.*

Corollary 12. — *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant. If $\varepsilon = \eta^n$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ and some $n \geq 3$, then $\eta \leq 4.5$ and $n \leq 10$. In particular, by (5), if $\Pi_\eta(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ is of type (T), then $0 \leq a \leq 6$ and $|b| \leq \sqrt{4a+4}$.*

Proof. — By (7) we have

$$\eta^{9/2}/2 \leq \eta^{3n/2}/2 = \varepsilon^{3/2}/2 \leq d_\varepsilon = d_\eta \leq 4\left(\eta + \eta^{-1}\right)^3,$$

that implies $\eta \leq 4.5$. Moreover, we have $\eta \geq \eta_0 = 1.32471\cdots$ (Lemma 7). Hence, by (7), we have

$$1 = d_\eta/d_\varepsilon \leq \frac{4\left(\eta + \eta^{-1}\right)^3}{\varepsilon^{3/2}/2} = 8\left(\frac{\eta + \eta^{-1}}{\eta^{n/2}}\right)^3 \leq 8\left(\frac{\eta_0 + \eta_0^{-1}}{\eta_0^{n/2}}\right)^3,$$

that implies $n < 11$. □

3.4. Being a square. —

Lemma 13. — *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant. Then $\varepsilon = \eta^2$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ if and only if we are in one of the following three cases:*

1. $\Pi_\varepsilon(X) = X^3 - M^2X^2 - 2MX - 1$ *with* $M \geq 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = \varepsilon^2 - M^2\varepsilon - M \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - MX^2 - 1$ *and* $d_\eta = d_\varepsilon = 4M^3 + 27$.

2. $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^2 + 3\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - 2X^2 + X - 1$ *and* $d_\eta = d_\varepsilon = 23$.

3. $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - X - 1$ *and* $d_\eta = d_\varepsilon = 23$.

Proof. — Assume that $\varepsilon = \eta^2$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ with $\Pi_\eta(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ of type (T). Then $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$ and $d_\varepsilon = d_\eta$ (Lemma 2). Clearly, the index $(\mathbb{Z}[\eta] : \mathbb{Z}[\eta^2])$ is equal to $|ab - 1|$, where $\Pi_\eta(X) = X^3 - aX^2 + bX - 1$. Hence, we must have $|ab - 1| = 1$, and we will have $\eta = (\varepsilon^2 - (a^2 - b)\varepsilon - a)/(1 - ab)$ and $\Pi_\varepsilon(X) = \Pi_{\eta^2}(X) = X^3 - (a^2 - 2b)X^2 + (b^2 - 2a)X - 1$. First, assume that $ab = 2$. Then $a = 2$ and $b = 1$ (for $a \geq 0$ and $b \leq a - 1$), $\Pi_\eta(X) = X^3 - 2X^2 + X - 1$ and $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$.
Secondly, assume that $ab = 0$. If $a = 0$, then $b \leq a - 1 = -1$ and $d_\eta = 4b^3 + 27 > 0$ yields $b = -1$, $\Pi_\eta(X) = X^3 - X - 1$, and $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$. If $a \neq 0$, then $b = 0$, $\Pi_\eta(X) = X^3 - aX^2 - 1$, $d_\eta = 4a^3 + 27$ and $\Pi_\varepsilon(X) = X^3 - a^2X^2 - 2aX - 1$. ☐

3.5. Computation of explicit nth roots. — Let θ be an algebraic integer of degree n. Suppose that $\theta = \alpha^m$ for some $\alpha \in \mathbb{Q}(\theta)$ and some $m \geq 2$. Let us explain how to compute the coordinates of α in the canonical $\mathbb{Q}$-basis of $\mathbb{Q}(\theta)$, provided that $\Pi_\alpha(X)$ is known. First, we compute the matrix $P := [p_{i,j}]_{1 \leq i,j \leq m} \in M_n(\mathbb{Z})$ such that

$$\theta^{j-1} = \alpha^{m(j-1)} = \sum_{i=1}^{n} p_{i,j}\alpha^{i-1} \qquad (1 \leq j \leq n).$$

Since $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\theta) = \mathbb{Q}(\alpha^n) \subseteq \mathbb{Q}(\alpha)$, we have $\mathbb{Q}(\alpha) = \mathbb{Q}(\theta)$. Therefore, $\det P \neq 0$ and we can compute $P^{-1} = [q_{i,j}]_{1 \leq i,j \leq n} \in M_n(\mathbb{Q})$. Clearly, we have $\alpha = \sum_{i=1}^{n} q_{i,2}\alpha^{m(i-1)} = \sum_{i=1}^{n} q_{i,2}\theta^{i-1}$. For example, if $\Pi_\alpha(X) = X^3 - uX^2 + vX - w \in \mathbb{Z}[X]$ is the minimal polynomial of a cubic algebraic number α, then $\theta = \alpha^2$ is also a cubic algebraic number, $\Pi_{\alpha^2}(X) = X^3 - (u^2 - 2v)X^2 + (v^2 - 2uw)X - w^2 \in \mathbb{Z}[X]$ and $\alpha = (\theta^2 + (v - u^2)\theta - uw)/(w - uv)$.

4. The totally imaginary quartic case

The aim of this Section 4 is to prove Theorem 18. Indeed, after having found a completely different proof of Nagell's result we thought it should now be possible to settle this third case where the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1. In [Lou08a] we partially solved this problem and conjectured Theorem 18. We could not prove it because we could not come up with lower bounds on discriminants of totally imaginary quartic algebraic units (see Theorem 24). Such a lower bound was then obtained in [PL] and their proof was simplified in [Lou10].

Let ε be an algebraic quartic unit which is not a complex root of unity, and for which the rank of the group of units of the quadratic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is totally imaginary and $|\varepsilon| \neq 1$ (use [Was, Lemma 1.6]). Notice that if ε_1, $\varepsilon_2 = \overline{\varepsilon_1}$, ε_3 and $\varepsilon_4 = \overline{\varepsilon_3}$ are the four complex conjugates of ε, then $1 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = |\varepsilon_1|^2|\varepsilon_3|^2$). By changing ε into $-\varepsilon$, $1/\varepsilon$, or $-1/\varepsilon$ if necessary, we may and we will assume that its minimal polynomial $\Pi_\varepsilon(X)$ is of type (T):

Definition 14. — *A quartic polynomial of type (T) is a $\mathbb{Q}$-irreducible monic quartic polynomial $P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ which satisfies $|c| \leq a$ and which has no real root (see Lemma 15 for a characterization). It is of positive discriminant $D_{P(X)}$.*

Lemma 15. — *Let ε_P be any complex root of a quartic polynomial $P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ of type (T). Then*

$$-1 \leq b \leq |\varepsilon_P|^2 + 1/|\varepsilon_P|^2 + 4 \text{ and } |c| \leq a \leq \sqrt{4b+5}. \tag{10}$$

Proof. — Let $\rho e^{i\phi}$, $\rho e^{-i\phi}$, $\rho^{-1}e^{i\psi}$ and $\rho^{-1}e^{-i\psi}$ be the four complex roots of $P(X)$. Then $a = 2\rho\cos\phi + 2\rho^{-1}\cos\psi$ and

$$b = \rho^2 + \rho^{-2} + 4(\cos\phi)(\cos\psi). \tag{11}$$

Hence,

$$4b - a^2 = 4(\sin\phi)^2\rho^2 + 4(\sin\psi)^2\rho^{-2} + 8(\cos\phi)(\cos\psi) > -8. \tag{12}$$

Since $4b - a^2 \equiv 0$ or $3 \pmod 4$, we have $4b - a^2 \geq -5$. □

Lemma 16. — *Let $P(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Q}[X]$ be $\mathbb{Q}$-irreducible. Then $P(X)$ has no real root if and only if $D_{P(X)} > 0$ and either $A := 3a^2 - 8b < 0$ or $B := 3a^4 - 16a^2b + 16ac + 16b^2 - 64d < 0$.*
Assume moreover that $d = 1$ and let $\eta = \rho e^{i\alpha}$, $\bar\eta$, $\eta' = \rho^{-1}e^{i\beta}$ and $\bar\eta'$ be these four non-real roots. Then $\rho^2 + 1/\rho^2 \geq 2$, $2\cos(\alpha+\beta)$ and $2\cos(\alpha-\beta)$ are the roots of

$$R(X) = X^3 - bX^2 + (ac-4)X - (a^2 - 4b + c^2) \in \mathbb{Q}[X],$$

of positive discriminant $D_{R(X)} = D_{P(X)} = d_\eta$.

Proof. — Write $P(X) = (X-\alpha_1)(X-\alpha_2)(X-\alpha_3)(X-\alpha_4)$ in $\mathbb{C}[X]$.
1. Set $\beta_1 = (\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2$, $\beta_2 = (\alpha_1-\alpha_2+\alpha_3-\alpha_4)^2$ and $\beta_3 = (\alpha_1-\alpha_2-\alpha_3+\alpha_4)^2$. Then $Q(X) := (X-\beta_1)(X-\beta_2)(X-\beta_3) = X^3 - AX^2 + BX - C$, where A and B are given in the statement of Lemma 16 and $C = (a^3 - 4ab + 8c)^2$. Moreover, $D_{Q(X)} = 2^{12}D_{P(X)}$.
(i) If $P(X)$ has two real roots, say α_1 and α_2, and two non-real roots, say α_3 and $\alpha_4 = \bar\alpha_3$, then $\beta_1 > 0$, whereas β_2 and $\beta_3 = \bar\beta_2$ are non-real. Hence $D_{Q(X)} < 0$. (ii). If $P(X)$ has four non-real roots, say α_1, $\alpha_2 = \bar\alpha_1$, α_3 and $\alpha_4 = \bar\alpha_3$, then $Q(X)$ has three real roots $\beta_1 > 0$, $\beta_2 < 0$ and $\beta_3 < 0$. Hence $D_{Q(X)} > 0$ and $Q'(X) = 3X^2 - 2AX + B$ has a negative real root $\gamma \in (\beta_2, \beta_3)$, which implies $A < 0$ or $B < 0$. (iii). If $P(X)$ has four real roots, say α_1, α_2, α_3 and α_4, then $Q(X)$ has three real roots $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$. Hence $D_{Q(X)} > 0$ and $Q'(X) = 3X^2 - 2AX + B$ has two positive real roots γ_1 and γ_2, which implies $A \geq 0$ and $B \geq 0$.
The proof of the first part is complete.
2. Set $\gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$ and $\gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Then $R(X) := (X - \gamma_1)(X-\gamma_2)(X-\gamma_3) = X^3 - bX^2 + (ac-4d)X - (a^2d - 4bd + c^2)$, and $D_{R(X)} = D_{P(X)}$. In

our situation, $d = 1$ and we may assume that $\alpha_1 = \eta$, $\alpha_2 = \bar{\eta}$, $\alpha_3 = \eta'$ and $\alpha_4 = \bar{\eta}'$. Hence, $\gamma_1 = |\eta|^2 + 1/|\eta|^2 \geq 2$, $\gamma_2 = 2\Re(\eta\eta') = 2\cos(\alpha+\beta)$ and $\gamma_3 = 2\Re(\eta\bar{\eta}') = 2\cos(\alpha-\beta)$. □

Notice that (10) and the first part of Lemma 16 make it easy to list of all the quartic polynomials of type (T) whose roots are of absolute values less than or equal to a given upper bound B. Using the second part of Lemma 16 we can compute these absolute values. Taking $B = 2$, we obtain:

Lemma 17. — *Let η be a totally imaginary quartic unit. If $|\eta| > 1$ then $|\eta| \geq |\eta_0| = 1.18375\cdots$, where $\Pi_{\eta_0}(X) = X^4 - X^3 + 1$.*

4.1. Statement of the result for the quartic case. —

Theorem 18. — *Let ε be a totally imaginary quartic unit with $\Pi_\varepsilon(X)$ of type (T). Assume that ε is not a complex root of unity. Let η be a fundamental unit of $\mathbb{Z}[\varepsilon]$. We can choose $\eta = \varepsilon$, except in the following cases:*

1. *The* **infinite family of exceptions** *for which $\Pi_\varepsilon(X) = X^4 - 2bX^3 + (b^2+2)X^2 - (2b-1)X + 1$, $b \geq 3$, in which cases $\varepsilon = -1/\eta^2$ where $\eta = \varepsilon^3 - 2b\varepsilon^2 + (b^2+1)\varepsilon - (b-1) \in \mathbb{Z}[\varepsilon]$ is a root of $\Pi_\eta(X) = X^4 - X^3 + bX^2 + 1$ of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 16b^4 - 4b^3 - 128b^2 + 144b + 229$.*

2. *The* 14 **following sporadic exceptions**
   - (a) (i) *$\Pi_\varepsilon(X) = X^4 - 3X^3 + 2X^2 + 1$, in which case $\varepsilon = -\eta^{-2}$ where $\eta = -\varepsilon^2 + \varepsilon + 1 \in \mathbf{Z}[\varepsilon]$ is a root of $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta = 117$.*
     - (ii) *$\Pi_\varepsilon(X) = X^4 - 3X^3 + 5X^2 - 3X + 1$, in which case $\varepsilon = \eta^2$ where $\eta = -\varepsilon^3 + 2\varepsilon^2 - 2\varepsilon \in \mathbf{Z}[\varepsilon]$ is a root of $\Pi_\eta(X) = X^4 - X^3 - X^2 + X + 1$ of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta = 117$.*
     - (iii) *$\Pi_\varepsilon(X) = X^4 - 5X^3 + 8X^2 - 4X + 1$, in which case $\zeta_3 = \varepsilon^3 - 4\varepsilon^2 + 5\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$ and $\varepsilon = \zeta_3\eta^3$ where $\eta = -\varepsilon^2 + 3\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ is a root of $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ is of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 117$.*
   - (b) *$\Pi_\varepsilon(X) = X^4 - 5X^3 + 9X^2 - 5X + 1$, in which case $\varepsilon = -\eta^2$ where $\eta = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ is a root of $\Pi_\eta(X) = X^4 - X^3 + 3X^2 - X + 1$ of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 189$.*
   - (c) (i) *$\Pi_\varepsilon(X) = X^4 - X^3 + 2X^2 + 1$, in which case $\varepsilon = \eta^2$ where $\eta = -\varepsilon^3 + \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$.*
     - (ii) *$\Pi_\varepsilon(X) = X^4 - 3X^3 + 3X^2 - X + 1$, in which case $\varepsilon = 1/\eta^3$ where $\eta = -\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.*
     - (iii) *$\Pi_\varepsilon(X) = X^4 - 4X^3 + 6X^2 - 3X + 1$, in which case $\varepsilon = -1/\eta^4$ where $\eta = \varepsilon^3 - 3\varepsilon^2 + 3\varepsilon \in \mathbb{Z}[\varepsilon]$.*
     - (iv) *$\Pi_\varepsilon(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$, in which case $\varepsilon = -\eta^6$ where $\eta = \varepsilon^2 - 2\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$.*
     - (v) *$\Pi_\varepsilon(X) = X^4 - 7X^3 + 14X^2 - 6X + 1$, in which case $\varepsilon = -1/\eta^7$ where $\eta = \varepsilon^2 - 4\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$.*

     In these five cases, $\Pi_\eta(X) = X^4 - X^3 + 1$ is of type (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 229$.
   - (d) (i) *$\Pi_\varepsilon(X) = X^4 - 2X^3 + 3X^2 - X + 1$, in which case $\varepsilon = -1/\eta^2$ where $\eta = \varepsilon^3 - 2\varepsilon^2 + 2\varepsilon \in \mathbb{Z}[\varepsilon]$.*

(ii) $\Pi_\varepsilon(X) = X^4 - 3X^3 + X^2 + 2X + 1$, *in which case* $\varepsilon = 1/\eta^3$ *where* $\eta = \varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$.
(iii) $\Pi_\varepsilon(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$, *in which case* $\varepsilon = -\eta^4$ *where* $\eta = \varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$.

In these three cases, $\Pi_\eta(X) = X^4 - X^3 + X^2 + 1$ *is of type (T),* $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 257$.

(e) $\Pi_\varepsilon(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$, *in which case* $\zeta_4 = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ *and* $\varepsilon = \zeta_4\eta^2$, *where* $\eta = -\varepsilon^3 + 3\varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + X^2 + 1$ *of type (T),* $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 272$.

(f) $\Pi_\varepsilon(X) = X^4 - 13X^3 + 43X^2 - 5X + 1$, *in which case* $\varepsilon = -\eta^3$ *where* $\eta = -\varepsilon^3 + 6\varepsilon^2 + 3\varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 4X^2 - X + 1$ *of type (T),* $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 1229$.

4.2. Sketch of proof. — Let ε be a totally imaginary quartic unit which is not a complex root of unity. Compared with the non-totally real cubic case, this quartic case is more tricky. The first problem is that the totally imaginary quartic order $\mathbb{Z}[\varepsilon]$ may contain complex roots of unity. Let $\mu(\varepsilon)$ denote the cyclic group of order $2N \geq 2$ of complex roots of unity contained in $\mathbb{Z}[\varepsilon]$. Since the cyclotomic field of conductor $2N$ and degree $\phi(2N)$ is contained in the quartic field $\mathbb{Q}(\varepsilon)$, we obtain that $\phi(2N)$ divides 4, hence that $2N \in \{2, 4, 6, 8, 10, 12\}$.
We will devote Section 4.4 to the case that $2N \in \{8, 10, 12\}$ and will settle our problem in this situation.
Hence we may and we now assume that $2N \in \{2, 4, 6\}$.
We want to determine when $\varepsilon = \zeta\eta^p$ for some $\zeta \in \mu(\varepsilon)$, some $\eta \in \mathbb{Z}[\varepsilon]$ and some prime $p \geq 2$. We may and we will assume that η is also a totally imaginary quartic unit which is not a complex root of unity (if η is not totally imaginary then it is a real quadratic unit and $\zeta \neq \pm 1$, and we have $\varepsilon = \zeta'\eta'^p$ with $\eta' = \zeta\eta \in \mathbb{Z}[\varepsilon]$ a totally imaginary quartic unit and $\zeta' = \zeta^{1-p} \in \mu(\varepsilon)$).
Clearly there are three subcases.

1. For $p = 2$ we determine when $\varepsilon = \pm\eta^2$ (Lemma 20), and when $\varepsilon = \zeta_4\eta^2$ where ζ_4 is a complex root of unity of order 4 in $\mathbb{Z}[\varepsilon]$ (Lemma 22).

2. For $p = 3$ we determine when $\varepsilon = \eta^3$ (next subcase), and when $\varepsilon = \zeta_3\eta^3$ where ζ_3 is a complex root of unity of order 3 in $\mathbb{Z}[\varepsilon]$ (Lemma 22).

3. For $p \geq 3$ we determine when $\varepsilon = \eta^p$ for some $\eta \in \mathbb{Z}[\varepsilon]$. Using $|\eta| \geq |\eta_0| = 1.18375\cdots$ (Lemma 17) and a double bound (13) for d_ε similar to (4) and (7), we will obtain in Corollary 29 that $p \in \{3, 5, 7\}$.

In the cubic case, if $\varepsilon = \eta^p > 1$ and $\Pi_\varepsilon(X)$ is of type (T), then so is $\Pi_\eta(X)$. This is no longer true in the present quartic case. For example, if $\Pi_\varepsilon(X) = X^4 - 3X^3 + X^2 + 2X + 1$, of type (T), and $\eta = -\varepsilon^2 + \varepsilon + 1 \in \mathbb{Z}[\varepsilon]$, then $\varepsilon = \eta^3$ and $\Pi_\eta(X) = X^4 + X^2 - X + 1$ is not of type (T). Since we want η to be of type (T) in all our statements, we may have to present our results in using $-\eta$, $1/\eta$ or $-1/\eta$ instead, as in Lemma 20.
Putting together the results of Lemma 20, Lemma 22 and Corollary 29, we obtain Table 2 (similar to Table 1 of section 3.2, where we single out the cases $b = 1$ and $b = 2$ of point 1 of Lemma 20, of discriminants $D = 257$ and $D = 229$), which completes the proof of Theorem 18:

p	$\Pi_\eta(X)$	$Q(X)$	D
2	$X^4-X^3-X^2+X+1$	$\Pi_{\eta^2}(X)=X^4-3X^3+5X^2-3X+1$	117
2	$X^4-2X^3+2X^2-X+1$	$\Pi_{-1/\eta^2}(X)=X^4-3X^3+2X^2+1$	117
3	$X^4-2X^3+2X^2-X+1$	$\Pi_{\zeta_3\eta^3}(X)=X^4-5X^3+8X^2-4X+1$	117
2	$X^4-X^3+3X^2-X+1$	$\Pi_{-\eta^2}(X)=X^4-5X^3+9X^2-5X+1$	189
3	X^4-X^3+1	$\Pi_{1/\eta^3}(X)=\mathbf{X^4-3X^3+3X^2-X+1}$	229
2	$\mathbf{X^4-3X^3+3X^2-X+1}$	$\Pi_{-1/\eta^2}(X)=X^4-5X^3+5X^2+3X+1$	229
2	X^4-X^3+1	$\Pi_{\eta^2}(X)=\mathbf{X^4-X^3+2X^2+1}$	229
2	$\mathbf{X^4-X^3+2X^2+1}$	$\Pi_{-1/\eta^2}(X)=X^4-4X^3+6X^2-3X+1$	229
3	$\mathbf{X^4-X^3+2X^2+1}$	$\Pi_{-\eta^3}(X)=X^4-5X^3+5X^2+3X+1$	229
7	X^4-X^3+1	$\Pi_{-1/\eta^7}(X)=X^4-7X^3+14X^2-6X+1$	229
2	$X^4-X^3+X^2+1$	$\Pi_{-1/\eta^2}(X)=\mathbf{X^4-2X^3+3X^2-X+1}$	257
2	$\mathbf{X^4-2X^3+3X^2-X+1}$	$\Pi_{-1/\eta^2}(X)=X^4-5X^3+7X^2-2X+1$	257
3	$X^4-X^3+X^2+1$	$\Pi_{1/\eta^3}(X)=X^4-3X^3+X^2+2X+1$	257
2	$X^4-2X^3+X^2+1$	$\Pi_{\zeta_4\eta^2}(X)=X^4-4X^3+7X^2-4X+1$	272
3	$X^4-2X^3+4X^2-X+1$	$\Pi_{-\eta^3}(X)=X^4-13X^3+43X^2-5X+1$	1229

TABLE 2.

Remark 19. — *Contrary to the non-totally real cubic case, here $\varepsilon=\zeta\eta^n$, with $\eta\in\mathbb{Z}[\varepsilon]$ and $\zeta\in\mu(\varepsilon)$, does not always imply $\mathbb{Z}[\varepsilon]=\mathbb{Z}[\eta]$. For example for $\varepsilon=\zeta_5\eta=1+\zeta_5^2$ with $\eta=\zeta_5+\zeta_5^4\in\mathbb{Q}(\sqrt{5})$ we have $\mathbb{Z}[\varepsilon]=\mathbb{Z}[\zeta_5]\neq\mathbb{Z}[\eta]=\mathbb{Z}[(1+\sqrt{5})/2]$ (notice that $\zeta_5=(\varepsilon-1)^3\in\mathbb{Z}[\varepsilon]$ and $\eta=\zeta_5+\zeta_5^4\in\mathbb{Z}[\zeta_5]\subseteq\mathbb{Z}[\varepsilon]$). Indeed, for this conclusion to hold true we need to have $\mathbb{Q}(\varepsilon)=\mathbb{Q}(\eta)$, i.e. η must also be a totally imaginary quartic unit. But it is not quite clear to us whether this necessary condition for the conclusion to hold true is also sufficient. But in the case that η is also a totally imaginary quartic unit, then $\mathbb{Z}[\eta]\subseteq\mathbb{Z}[\varepsilon]$ implies that d_ε divides d_η.*

4.3. Being a square. —

Lemma 20. — *Let ε be a totally imaginary quartic algebraic unit which is not a complex root of unity, with $\Pi_\varepsilon(X)$ of type (T). Then $\pm\varepsilon$ is a square in $\mathbf{Z}[\varepsilon]$ if and only if we are in one of the seven following cases:*

1. $\Pi_\varepsilon(X)=X^4-2bX^3+(b^2+2)X^2-(2b-1)X+1$, $b\geq 1$, *in which cases* $\varepsilon=-1/\eta^2$ *where* $\eta=\varepsilon^3-2b\varepsilon^2+(b^2+1)\varepsilon-(b-1)\in\mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X)=X^4-X^3+bX^2+1$ *of type (T), and* $d_\varepsilon=d_\eta=16b^4-4b^3-128b^2+114b+229$.

2. $\Pi_\varepsilon(X)=X^4-X^3+2X^2+1$, *in which case* $\varepsilon=\eta^2$ *where* $\eta=-\varepsilon^3+\varepsilon^2-\varepsilon\in\mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X)=X^4-X^3+1$ *of type (T), and* $d_\varepsilon=d_\eta=229$.

3. $\Pi_\varepsilon(X)=X^4-3X^3+2X^2+1$, *in which case* $\varepsilon=-\eta^{-2}$ *where* $\eta=-\varepsilon^2+\varepsilon+1\in\mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X)=X^4-2X^3+2X^2-X+1$ *of type (T), and* $d_\varepsilon=d_\eta=117$.

4. $\Pi_\varepsilon(X)=X^4-3X^3+5X^2-3X+1$, *in which case* $\varepsilon=\eta^2$ *where* $\eta=-\varepsilon^3+2\varepsilon^2-2\varepsilon\in\mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X)=X^4-X^3-X^2+X+1$ *of type (T), and* $d_\varepsilon=d_\eta=117$.

5. $\Pi_\varepsilon(X)=X^4-5X^3+9X^2-5X+1$, *in which case* $\varepsilon=-\eta^2$ *where* $\eta=-\varepsilon^3+4\varepsilon^2-6\varepsilon+2\in\mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X)=X^4-X^3+3X^2-X+1$ *of type (T), and* $d_\varepsilon=d_\eta=189$.

6. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = -\varepsilon^2 + 2\varepsilon + 2 \in \mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 3X^3 + 3X^2 - X + 1$ *of type (T),* $d_\eta = d_\varepsilon = 229$ *.*

7. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = \varepsilon^3 - 4\varepsilon^2 + 4\varepsilon \in \mathbf{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 3X^2 - X + 1$ *of type (T), and* $d_\eta = d_\varepsilon = 257$*.*

Proof. — Assume that $\varepsilon = \pm\eta^2$ or $\pm\eta^{-2}$ for some $\eta \in \mathbf{Z}[\varepsilon]$, with $\Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ of type (T). Hence,

$$-1 \leq b \text{ and } |c| \leq a \leq \sqrt{4b+5}.$$

The index $(\mathbf{Z}[\eta] : \mathbf{Z}[\eta^2])$ is equal to $|a^2 + c^2 - abc|$. Hence, we must have

$$|a^2 + c^2 - abc| = 1,$$

and we will have $\Pi_{\eta^2}(X) = X^4 - AX^3 + BX^2 - CX + 1$, where $A = a^2 - 2b$, $B = b^2 - 2ac + 2$ and $C = c^2 - 2b$.
Assume that $c = 0$. Then $1 = |a^2 + c^2 - abc| = a^2$, hence $a = 1$ and we are in the first or the second case.
Assume that $c \neq 0$. Then $a \geq 1$ and $a^2 + c^2 - abc = \pm 1$ yield $|b| \leq f(|c|)$, where $f(x) = \frac{a}{x} + \frac{x}{a} + \frac{1}{ax}$ is convex. Hence,

$$|b| \leq g(a) := \max(f(1), f(a)) = \max(a + 2/a, 2 + 1/a^2) = a + 2/a.$$

Since g is convex we obtain $|b| \leq \max(g(1), g(\sqrt{4b+5})) = \max(3, \frac{4b+7}{\sqrt{4b+5}})$. Hence, $b \leq 5$. There are 9 triplets (a, b, c) satisfying $-1 \leq b \leq 5$ and $1 \leq |c| \leq a \leq \sqrt{4b+5}$ for which $|a^2 + c^2 - abc| = 1$. Getting rid of the three of them for which $\Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1$ is of negative discriminant and of the one of them for which η is a 5th complex root of unity, namely $(a, b, c) = (1, 1, 1)$, we fall in one of the five remaining last cases.
Finally, by choosing between the four units $\varepsilon = \pm\eta^2$ or $\varepsilon = \pm\eta^{-2}$ the ones for which $\Pi_\varepsilon(X) = X^4 - AX^3 + BX^2 - CX + 1$ satisfies $|C| \leq A$, we complete the proof of this Lemma. □

4.4. The case that $\mu(\varepsilon)$ is of order 8, 10 or 12. —

Lemma 21. — *Let* ε *be a totally imaginary quartic algebraic unit which is not a complex root of unity. If* $\zeta_{2N} \in \mathbb{Z}[\varepsilon]$*, with* $2N \in \{8, 10, 12\}$*, then* ε *is a fundamental unit of the order* $\mathbb{Z}[\varepsilon]$*.*

Proof. — We have $\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\zeta_{2N}]$ ($\mathbb{Z}[\zeta_{2N}]$ is the ring of algebraic integers of $\mathbb{Q}(\zeta_{2N})$). Hence, $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\zeta_{2N}]$ and $d_\varepsilon = d_{\zeta_{2N}}$. Let η_{2N} be a fundamental unit of $\mathbb{Z}[\zeta_{2N}]$. Then $\eta_{2N} \in \mathbb{Z}[\varepsilon]$ and $\varepsilon = \zeta_{2N}^m \eta_{2N}^n$, with $m \in \mathbb{Z}$ and $0 \neq n \in \mathbb{Z}$. We want to prove that $|n| = 1$. We prove that $|n| \geq 2$ implies $d_\varepsilon > d_{\zeta_{2N}}$. Set $\rho = |\eta_{2N}|$. Let σ_t be the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\zeta_{2N})$ such that $\sigma_t(\zeta_{2N}) = \zeta_{2N}^t$, where $\gcd(t, 2N) = 1$ and $t \not\equiv \pm 1 \pmod{2N}$, i.e. σ_t is neither the identity nor the complex conjugation. Then $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \sigma_t(\varepsilon)$, $\varepsilon_3 = \overline{\varepsilon_1}$ and $\varepsilon_4 = \overline{\varepsilon_2}$ are the four complex conjugates of ε. Since $|\varepsilon_1|^2 |\varepsilon_2|^2 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = N_{\mathbb{Q}(\zeta_{2N})/\mathbb{Q}}(\varepsilon) = 1$, we have $|\varepsilon_2| = 1/|\varepsilon_1| = 1/\rho^n$ and

$$\begin{aligned} d_\varepsilon &= ((\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4))^2 \\ &= 16\Im^2(\varepsilon_1)\Im^2(\varepsilon_2)|\varepsilon_1 - \varepsilon_2|^4 |\varepsilon_1 - \overline{\varepsilon_2}|^4 \geq 16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon))|\rho^n - 1/\rho^n|^8. \end{aligned}$$

Notice that if η_{2N} is real, then $16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon)) = 16\sin^2(\frac{\pi m}{N})\sin^2(\frac{t\pi m}{N})$.

1. If $2N = 8$, we may take $\eta_8 = 1 + \sqrt{2} = \rho$ and $t = 3$ and we obtain

$$d_\varepsilon \geq 16 \sin^2(\frac{\pi m}{4}) \sin^2(\frac{3\pi m}{4})(\rho^n - \rho^{-n})^8 \geq 4(\rho^2 - \rho^{-2})^8 = 2^{22} > 256 = d_{\zeta_8}$$

(since ε is totally imaginary, we have $m \not\equiv 0 \pmod 4$).
2. If $2N = 10$, we may take $\eta_{10} = (1 + \sqrt{5})/2 = \rho$ and $t = 3$ and we obtain

$$d_\varepsilon \geq 16 \sin^2(\frac{\pi m}{5}) \sin^2(\frac{3\pi m}{5})(\rho^n - \rho^{-n})^8 \geq 5(\rho^2 - \rho^{-2})^8 = 5^5 > 125 = d_{\zeta_{10}}$$

(since ε is totally imaginary, we have $m \not\equiv 0 \pmod 5$).
3. If $2N = 12$, then $\varepsilon_0 = 2 + \sqrt{3} = |1 + \zeta_{12}|^2 = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\sqrt{3})}(1 + \zeta_{12})$ is the fundamental unit of $\mathbb{Z}[\sqrt{3}]$. Hence, we may take $\eta_{12} = 1 + \zeta_{12}$, $\rho = \varepsilon_0^{1/2}$ and $t = 5$. Noticing that $\eta_{12} = \zeta_{24}(\zeta_{24} + \zeta_{24}^{-1}) = \zeta_{24}\rho$ and $\sigma(\eta_{12}) = 1 + \zeta_{12}^5 = \frac{\zeta_{12}^3}{1+\zeta_{12}} = \zeta_{12}^3/\eta_{12} = \zeta_{24}^5/\rho$, we obtain $16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon)) = 16 \sin^2(\frac{(2m+n)\pi}{12}) \sin^2(\frac{5(2m+n)\pi}{12}) \in \{1, 4, 9, 16\}$ and

$$d_\varepsilon \geq (\varepsilon_0^{n/2} - \varepsilon_0^{-n/2})^8 \geq (\varepsilon_0 - 1/\varepsilon_0)^8 = 144^2 > 144 = d_{\zeta_{12}}$$

(since ε is totally imaginary, we have $2m + n \not\equiv 0 \pmod{12}$). □

4.5. The case that $\varepsilon = \zeta_4\eta^2$ or $\varepsilon = \zeta_3\eta^3$. —

Lemma 22. — *Let ε be a totally imaginary quartic unit which is not a complex root of unity such that $\Pi_\varepsilon(X) = X^4 - aX^3 + bX^2 - cX + 1$ satisfies $|c| \leq a$.*

1. *Assume that $\varepsilon = \zeta_3\eta^3$ for some totally imaginary quartic unit $\eta \in \mathbb{Z}[\varepsilon]$ and some complex root of unity $\zeta_3 \in \mathbb{Z}[\varepsilon]$ of order 3. Then $\Pi_\varepsilon(X) = X^4 - 5X^3 + 8X^2 - 4X + 1$, in which case $\varepsilon = \zeta_3\eta^3$, where $\eta = -\varepsilon^2 + 3\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ and $\zeta_3 = \varepsilon^3 - 4\varepsilon^2 + 5\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$. Moreover, $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ is of type (T) and $d_\varepsilon = d_\eta = 117$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$.*

2. *Assume that $\varepsilon = \zeta_4\eta^2$ for some totally imaginary quartic unit $\eta \in \mathbb{Z}[\varepsilon]$ and some complex root of unity $\zeta_4 \in \mathbb{Z}[\varepsilon]$ of order 4. Then $\Pi_\varepsilon(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$, in which case $\varepsilon = \zeta_4\eta^2$, where $\eta = -\varepsilon^3 + 3\varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ and $\zeta_4 = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$. Moreover, $\Pi_\eta(X) = X^4 - 2X^3 + X^2 + 1$ is of type (T), $d_\varepsilon = d_\eta = 272$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$.*

Proof. — Set $\mathbb{K} := \mathbb{Q}(\zeta_3)$ and $\mathbb{A} := \mathbb{Z}[\zeta_3]$. Since η is quadratic over $\mathbb{K}$, there exist α and β in $\mathbb{A}$ such that $\eta^2 - \alpha\eta + \beta = 0$. Clearly, $\beta \in \mathbb{A}^*$ and $\alpha \neq 0$. Moreover, $\mathbb{A}[\eta] \subseteq \mathbb{A}[\varepsilon] = \mathbb{A}[\zeta_3\eta^3] = \mathbb{A}[\eta^3] \subseteq \mathbb{A}[\eta]$ yields $\mathbb{A}[\eta] = \mathbb{A}[\eta^3]$. Since, $\eta^3 = (\alpha^2 - \beta)\eta - \alpha\beta$ and since $\eta^3 \notin \mathbb{A}$ (otherwise η would be a complex root of unity), we obtain $\alpha^2 - \beta \in \mathbb{A}^*$. Now, $1 \leq |\alpha|^2 \leq |\alpha^2 - \beta| + |\beta| = 2$ yields $|\alpha|^2 = 1$ (there is no element of norm 2 in $\mathbb{A}$) and $\alpha \in \mathbb{A}^*$. Hence, α, β and $\alpha^2 - \beta$ are in $\mathbb{A}^*$. Setting $\beta = -\alpha^2\gamma$ with $\gamma \in \mathbb{A}^*$, we have $1 + \gamma \in \mathbb{A}^* = \{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$. Hence, $\gamma \in \{\zeta_3, \zeta_3^2\}$ and $\eta^2 - \alpha\eta - \alpha^2\gamma = 0$ yields that $\varepsilon = \zeta_3\eta^3$ is a root of $P(X) = X^2 - \delta\zeta_3(3\gamma + 1)X - \zeta_3^2 \in \mathbb{K}[X]$ (use $\alpha^6 = \gamma^3 = 1$), where $\delta = \alpha^3 \in \{\pm 1\}$. Hence,

$$\Pi_\varepsilon(X) = P(X)\overline{P(X)} = \begin{cases} X^4 + 4\delta X^3 + 8X^2 + 5\delta X + 1 & \text{if } \gamma = \zeta_3, \\ X^4 - 5\delta X^3 + 8X^2 - 4\delta X + 1 & \text{if } \gamma = \zeta_3^2. \end{cases}$$

Set $\mathbb{K} := \mathbb{Q}(\zeta_4)$ and $\mathbb{A} := \mathbb{Z}[\zeta_4]$. Since η is quadratic over $\mathbb{K}$, there exist α and β in $\mathbb{A}$ such that $\eta^2 - \alpha\eta + \beta = 0$. Clearly, $\beta \in \mathbb{A}^*$ and $\alpha \neq 0$. Moreover, $\mathbb{A}[\eta] \subseteq \mathbb{A}[\varepsilon] = \mathbb{A}[\zeta_4\eta^2] = \mathbb{A}[\eta^2] \subseteq \mathbb{A}[\eta]$ yields $\mathbb{A}[\eta] = \mathbb{A}[\eta^2]$. Since, $\eta^2 = \alpha\eta - \beta$ and since $\eta^2 \notin \mathbb{A}$ we obtain $\alpha \in \mathbb{A}^*$. Hence, α and β

are in $\mathbb{A}^*$. Setting $\beta = -\alpha^2\gamma$ with $\gamma \in \mathbb{A}^*$, we obtain $\eta^2 - \alpha\eta - \alpha^2\gamma = 0$, with $\alpha, \gamma \in \mathbb{A}^* = \{\pm 1, \pm\zeta_4\}$. It follows that $\varepsilon = \zeta_4\eta^2$ is a root of $P(X) = X^2 - \delta\zeta_4(1+2\gamma)X - \gamma^2 \in \mathbb{K}[X]$ (use $\alpha^4 = 1$), where $\delta = \alpha^2 \in \{\pm 1\}$. Hence,

$$\Pi_\varepsilon(X) = P(X)\overline{P(X)} = \begin{cases} X^4 - X^2 + 1 & \text{if } \gamma = -1, \\ X^4 + 4\delta X^3 + 7X^2 + 4\delta X + 1 & \text{if } \gamma = \zeta_4, \\ X^4 - 4\delta X^3 + 7X^2 - 4\delta X + 1 & \text{if } \gamma = -\zeta_4, \\ X^4 + 7X^2 + 1 & \text{if } \gamma = 1. \end{cases}$$

If $\gamma = -1$ then ε is a complex root of unity of order 12, a contradiction. If $\gamma = 1$ then $\zeta_4 \in \{\pm(\varepsilon^3 + 8\varepsilon)/3\}$ is not in $\mathbb{Z}[\varepsilon]$. □

Remark 23. — *Whereas there are only finitely many cases for which the quartic order $\mathbb{Z}[\varepsilon]$ contains a complex root of unity of order 8, 10 or 12, by Lemma 21, it happens infinitely often that it contains a complex root of unity of order 3 or 4. For example, if $\Pi_\varepsilon(X) = X^4 - 2AX^3 + (A^2-1)X^2 + AX + 1$, $A \geq 0$, then $\zeta_3 = -\varepsilon^2 + A\varepsilon \in \mathbb{Z}[\varepsilon]$. If $\Pi_\varepsilon(X) = X^4 - 2AX^3 + A^2X^2 + 1$, $A \geq 0$, then $\zeta_4 = \varepsilon^2 - A\varepsilon \in \mathbb{Z}[\varepsilon]$.*

4.6. Bounds on discriminants. —

Theorem 24. — *Let α be a totally imaginary quartic algebraic unit. Then*

$$7\max(|\alpha|^4, |\alpha|^{-4})/10 \leq d_\alpha \leq 16(|\alpha|^2 + |\alpha|^{-2})^4 \leq 256\max(|\alpha|^8, |\alpha|^{-8}). \tag{13}$$

Hence, if $P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ is $\mathbb{Q}$-irreducible of positive discriminant $D_{P(X)} > 0$ and with no real root, then

$$7H_\infty(P(X))^2/10 \leq D_{P(X)} \leq 16H_1(P(X))^4 \leq 256H_\infty(P(X))^4. \tag{14}$$

Proof. — Clearly, (14) follows from (13): if α, $\bar{\alpha}$, β and $\bar{\beta}$ are the (non-real) roots of $P(X)$, then $|\alpha|^2|\beta|^2 = 1$, hence $H_p(P) = (|\alpha|^{2p} + |\alpha|^{-2p})^{1/p} = (|\beta|^{2p} + |\beta|^{-2p})^{1/p}$ and $H_\infty(P(X)) = \max(|\alpha|^2, |\alpha|^{-2}) = \max(|\beta|^2, |\beta|^{-2})$.
Since both terms of (13) remain unchanged if we change α into $-\alpha$, $1/\alpha$ and $-1/\alpha$, we may assume that $\Pi_\alpha(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ satisfies $|c| \leq a$. By taking an appropriate choice of the root of $\Pi_\alpha(X)$, we may also assume that $\rho := |\alpha| \geq 1$.
Let $\alpha = \rho e^{i\phi}$, $\bar{\alpha}$, $\alpha' = \rho^{-1}e^{i\psi}$ and $\bar{\alpha}'$ be the four complex roots of $\Pi_\alpha(X)$ (use $|\alpha|^2|\alpha'|^2 = 1$). Then

$$\begin{aligned} d_\alpha &= \left((\alpha - \bar{\alpha})(\alpha - \alpha')(\alpha - \bar{\alpha}')(\bar{\alpha} - \alpha')(\bar{\alpha} - \bar{\alpha}')(\alpha' - \bar{\alpha}')\right)^2 \\ &= 16(\sin\phi)^2(\sin\psi)^2|\rho - \rho^{-1}e^{i(\psi-\phi)}|^4|\rho - \rho^{-1}e^{i(\psi+\phi)}|^4. \end{aligned}$$

Setting $X = \rho^2 + \rho^{-2}$, we have $d_\alpha = 4\left(F(\cos(\psi - \phi), \cos(\psi + \phi))\right)^2 \leq 16X^4$, by Lemma 25 below.
(Notice that $d_\alpha = 256 = 16(|\alpha|^2 + |\alpha|^{-2})^4 = 256\max(|\alpha|^8, |\alpha|^{-8})$ if $\alpha = \zeta_8$).
Asssume that $\rho \geq \sqrt{3}$ and $a \geq 37$.
Then $|2\rho^{-1}\cos\phi + 2\rho\cos\psi| = |c| \leq a = 2\rho\cos\phi + 2\rho^{-1}\cos\psi$ implies $\cos\phi \geq |\cos\psi|$ and

$$d_\alpha \geq \left(4(\rho - \rho^{-1})^4\sin^2\phi\right)^2.$$

First, if $\sin^2\phi \geq \frac{3}{4}\rho^{-2}$, then $d_\alpha \geq \left(3(1 - \rho^{-2})\right)^2\rho^4 \geq 4\rho^4$ (use $\rho \geq \sqrt{3}$).

Secondly, assume that $\sin^2\phi < \frac{3}{4}\rho^{-2}$. Since $\rho \geq 1$, we have

$$-8 < 4b - a^2 < 3 + 4\sin^2\psi + 8\sqrt{1-\sin^2\psi} \leq 11,$$

by (12) (α is totally imaginary, hence $\sin\phi \neq 0$ and $\sin\psi \neq 0$, and $\sqrt{1-t} \leq 1-t/2$ for $0 \leq t = \sin^2\psi \leq 1$). Since $4b-a^2 \equiv 0$ or $3 \pmod 4$, we obtain $J := 4b-a^2 \in \{-5,-4,-1,0,3,4,7,8\}$ and we are in the situation of Lemma 26. By Lemma 27 we have $d_\alpha \geq 4\max(|\alpha|^4, |\alpha|^{-4}) = 4\rho^4$. **Finally**, since $|c| \leq a \leq \sqrt{4b+5}$ and $-1 \leq b \leq \rho^2 + \rho^{-2} + 4$, it is easy to list all the possible polynomials $\Pi_\alpha(X)$ for which $1 \leq \rho \leq \sqrt{3}$ or for which $a \leq 36$ and $b = (a^2+J)/4$ with $J \in \{-5,-4,-1,0,3,4,7,8\}$ and also to check that the lower bound for d_α in (13) holds true for these polynomials, by using Lemma 16. ☐

Lemma 25. — *For $X \geq 2$ we have $\sup_{|x|,|y|\leq 1} F(x,y) \leq 2X^2$, where $F(x,y) := |x-y|(X-2x)(X-2y)$.*

Proof. — First, for $X \geq 2$ we have

$$\begin{aligned} S_1 := \sup_{\substack{|x|,|y|\leq 1\\ xy\geq 0}} F(x,y) &= \sup_{0\leq x\leq y\leq 1} (y-x)(X+2x)(X+2y) \\ &= \sup_{0\leq x\leq 1} (1-x)(X+2x)(X+2) = X(X+2) \leq 2X^2. \end{aligned}$$

Secondly, for $X \geq 2$ we have

$$\begin{aligned} S_2 &:= \sup_{\substack{|x|,|y|\leq 1\\ xy\leq 0}} F(x,y) \\ &= \sup_{0\leq x,y\leq 1} (x+y)(X-2x)(X+2y) = \sup_{0\leq x\leq 1} (x+1)(X-2x)(X+2). \end{aligned}$$

If $X \geq 6$ then $S_2 = 2(X-2)(X+2) \leq 2X^2$. If $2 \leq X \leq 6$ then $f(x) := (x+1)(X-2x)(X+2) \leq f((X-2)/4) = (X+2)^3/8 \leq 2X^2$. ☐

Lemma 26. — *Fix $J \in \{-5,-4,-1,0,3,4,7,8\}$. For $a \in \mathbb{Z}$ with $a^2 \equiv -J \pmod 4$, i.e. with a even if $J \equiv 0 \pmod 4$ and a odd if $J \equiv 3 \pmod 4$, set*

$$\Pi_J(X) = X^4 - aX^3 + \frac{a^2+J}{4}X^2 - cX + 1 \in \mathbb{Z}[X].$$

Assume $|c| \leq a$, that $a \geq 15$ and that $\Pi_J(X)$ is $\mathbb{Q}$-irreducible with no real roots, hence is of positive discriminant $D(a,J,c)$ (a quartic polynomial in c). Set

$$B := \begin{cases} a-1 & \text{if } J = 8 \\ Ja/8 & \text{if } J \in \{-4,0,4\} \\ (Ja-1)/8 & \text{if } J \in \{-5,-1,3,7\}. \end{cases}$$

Then $-a \leq c \leq B$ and $D(a,J,c) \geq F(a,J) := \min(D(a,J,-a), D(a,J,B))$.

Proof. — Assume that $a = 2A$ is even. Then $J = 4j$ and $\Pi_J(A) = jA^2 - cA + 1 > 0$ (for $\Pi_J(X)$ has no real root) yields $c \leq jA = Ja/8$. Moreover, since $X^4 - 2AX^3 + (A^2+2)X^2 - 2AX + 1 = (X^2 - AX + 1)^2$ is not irreducible, we obtain that $c \neq a = Ja/8$ for $J = 8$. Hence, $c \leq B$. Now, assume that a is odd. Then $0 < 16\Pi_J(a/2) = Ja^2 - 8ac + 16 \equiv 3 \pmod 4$ yields $Ja^2 - 8ac + 16 \geq 3$. Hence, $8c \leq Ja + 13/a < Ja + 1$ for $a \geq 15$. Hence, $c \leq (Ja-1)/8 = B$.

Numerical investigations suggest that $D(a,J,c)$ as a function of c has four real roots close to $-a$, $Ja/8$, a and $a^3/54+Ja/24$. Set $e=0$ if J is even and $e=1$ if J is odd, and

$$\Delta(a,J,c)=-27(c+a)\left(c-\frac{Ja-e}{8}\right)\left(c-(a-1+\frac{e}{8})\right)\left(c-\frac{a^3}{54}-\frac{Ja}{24}-1\right).$$

Using any software of symbolic computation, we check that $D(a,J,c)=\Delta(a,J,c)+P(a,J,c)$, where $P(a,J,c)=\alpha(a,J)c^2+\beta(a,J)c+\gamma(a,J)$ is a quadratic polynomial in c whose leading coefficient

$$\alpha(a,J)=\begin{cases}-\frac{a^3}{2}+\left(3+\frac{J^2}{64}\right)a^2+\left(27-\frac{9J}{8}\right)a-\frac{J^3}{16}+36J-27 & J \text{ even}\\ -\frac{a^3}{2}+\left(3+\frac{J^2}{64}\right)a^2+\left(\frac{189}{8}-\frac{45J}{64}\right)a-\frac{J^3}{16}+36J-\frac{1539}{64} & J \text{ odd}\end{cases}$$

is less than or equal to 0 for $J\in\{-5,-4,-1,0,3,4,7,8\}$ and $a\geq 14$.
Clearly, we have $\Delta(a,J,c)\geq 0$ for $-a\leq c\leq B$ (notice that $a^3/54-Ja/24-1\geq a^3/54-a/3-1\geq a-1\geq B$ for $a\geq 9$). Hence, $D(a,J,c)\geq P(a,J,c)$. Since $\alpha(a,J)\leq 0$, we have

$$P(a,J,c)\geq \min(P(a,J,-a),P(a,J,B)) \text{ for } -a\leq c\leq B.$$

Finally, we have $\Delta(a,J,-a)=\Delta(a,J,B)=0$. So, $D(a,J,-a)=P(a,J,-a)$ and $D(a,J,B)=P(a,J,B)$. The desired result follows. □

Lemma 27. — *Assume that $\Pi_\alpha(X)=X^4-aX^3+\frac{a^2+J}{4}X^2-cX+1\in\mathbb{Z}[X]$ is a quartic polynomial of type (T) with $J\in\{-5,-4,-1,0,3,4,7,8\}$, $a\in\mathbb{Z}$ and $a\equiv J \pmod 2$. Assume that $a\geq 37$. Then $d_\alpha\geq 4\max(|\alpha|^4,|\alpha|^{-4})$.*

Proof. — Set $M=\max(|\alpha|,|\alpha|^{-1})$. Using (11), we obtain

$$(a^2+J-16)/8\leq M^2\leq (a^2+J+16)/4. \tag{15}$$

By Lemma 26, for $J\in\{-4,0,4,8\}$ we have

$$\begin{aligned} D(a,J,c) &\geq \min(D(a,J,-a),D(a,J,B))\\ &\geq D(a,4,a/2)=9((a^2-8)^2+192)/16\end{aligned}$$

((i) check the quartic polynomials with positive leading coefficient $D(a,J,-a)-D(a,4,a/2)=\frac{(J+5)(J+11)}{16}a^4+\cdots$ are non-negative for $J\in\{-4,0,4,8\}$ and $a\geq 1$, (ii) check that $D(a,8,a-1)-D(a,4,a/2)\geq 0$ for $a\geq 0$ and (iii) check that the quartic polynomials with non-negative leading coefficient $D(a,J,Ja/8)-D(a,4,a/2)=\frac{(16-J^2)(112-J^2)}{4096}a^4+\cdots$ are non-negative for $J\in\{-4,0,4\}$ and $a\in\mathbb{Z}$).
Using (15), we have $M^2\geq (a^2-20)/8>8$ and $a^2-8\geq 4(M^2-8)>0$. Hence, $d_\alpha\geq 9((M^2-8)^2+12)\geq 4M^4$.
In the same way, for $J\in\{-5,-1,3,7\}$ we have

$$\begin{aligned} D(a,J,c) &\geq \min(D(a,J,-a),D(a,J,(Ja-1)/8))\\ &\geq D(a,-5,-a)=9((a^2+19)^2-192)/16\end{aligned}$$

(check (i) that the quartic polynomials with non-negative leading coefficient $D(a,J,-a)-D(a,-5,-a)=\frac{(J+5)(J+11)}{16}a^4+\cdots$ are non-negative for $J\in\{-5,-1,3,7\}$ and $a\geq 1$ and (ii) that the quintic polynomials with positive leading coefficient $D(a,J,(Ja-1)/8)-D(a,-5,-a)=\left(1-\frac{J^2}{64}\right)\frac{a^5}{16}+\cdots$ are positive for $J\in\{-5,-1,3,7\}$ and $a\geq 36$).

Using (15), we have $M^2 \geq (a^2-21)/8 > 1$ and $4(M^2-1) \leq a^2+19$. Hence, $d_\alpha \geq 9((M^2-1)^2-12) \geq 4M^4$. □

Remark 28. — *The exponents 4 and 8 in (13) are optimal. Indeed, if $\Pi_\alpha(X) = X^4-2bX^3+(b^2+2)X^2-(2b-1)X+1$, $b \geq 3$, then $|\alpha|^2+1/|\alpha|^2$ is the root greater than 2 of $R(X) = X^3-(b^2+2)X^2+(4b^2-2b-4)X-4b^2+4b+7$ (Lemma 16), hence is asymptotic to b^2. Hence, $\max(|\alpha|, |\alpha|^{-1})$ is asymptotic to b and d_α is asymptotic to $16b^4$, hence to $16\max(|\alpha|^4, |\alpha|^{-4})$. If $\Pi_\beta(X) = X^4-X^3+bX^2+1$, $b \geq 3$, then $\beta^2 = \alpha$ and $d_\beta = d_\alpha$. Hence, d_β is asymptotic to $16\max(|\beta|^8, |\beta|^{-8})$.*

Corollary 29. — *Let ε be a totally imaginary quartic algebraic unit which is not a complex root of unity. If $\varepsilon = \eta^n$ for some $\eta \in \mathbb{Z}[\varepsilon]$ and some $n \in \mathbb{Z}$, then $\max(|\eta|, |\eta|^{-1}) < 2.27$ and $|n| \leq 9$. In particular, by (10), if $\Pi_\eta(X) = X^4-aX^3+bX^2-cX+1 \in \mathbb{Z}[X]$ is of type (T), then $-1 \leq b \leq 9$ and $|c| \leq a \leq \sqrt{4b+5}$.*

Proof. — We may assume that $|\varepsilon| \geq 1$, that $|\eta| \geq 1$ and that $n \geq 3$. Notice that η is necessarily a totally imaginary quartic algebraic unit which is not a complex root of unity. By (13), we have

$$7|\eta|^{12}/10 \leq 7|\eta|^{4n}/10 = 7|\varepsilon|^4/10 \leq d_\varepsilon = d_\eta \leq 16\left(|\eta|^2+|\eta|^{-2}\right)^4,$$

that implies $|\eta| \leq 2.27$. Moreover, we have $|\eta| \geq |\eta_0| = 1.18375\cdots$ (Lemma 17). Hence, by (13), we have

$$1 = d_\eta/d_\varepsilon \leq \frac{16\left(|\eta|^2+|\eta|^{-2}\right)^4}{7|\varepsilon|^4/10} = \frac{160}{7}\left(\frac{|\eta|^2+|\eta|^{-2}}{|\eta|^n}\right)^4 \leq \frac{160}{7}\left(\frac{|\eta_0|^2+|\eta_0|^{-2}}{|\eta_0|^n}\right)^4,$$

that implies $n < 10$. □

5. The totally real cubic case

The aim of this Section 5 is to prove Theorem 36. After having settled in Sections 2, 3 and 4 the problem of the determination of a fundamental unit of any order $\mathbf{Z}[\varepsilon]$ generated by a unit ε whenever the rank of the group of units is equal to 1 we attacked the general situation in [Lou12] . We solved this problem in the case that ε is a totally real cubic algebraic unit, in which case the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 2. We managed to prove the best result one could expect, namely that there should exist a second unit $\eta \in \mathbb{Z}[\varepsilon]$ such that $\{\varepsilon, \eta\}$ is a system of fundamental units of the cubic order $\mathbb{Z}[\varepsilon]$ (and unbeknown to us this problem was simultaneously solved in [BHMMS] and [MS]). We present here a streamlined proof of this result, once again based on lower bounds for discriminants of totally real algebraic cubic units (Theorem 33).
Let ε be a totally real cubic unit of $\mathbb{Q}$-irreducible minimal polynomial $\Pi_\varepsilon(X) = X^3-aX^2+bX-c \in \mathbb{Z}[X]$, $c \in \{\pm 1\}$, of positive discriminant $D_\varepsilon = d_\varepsilon = D_{\Pi_\varepsilon(X)}$. Since the rank of the group of units of the cubic order $\mathbb{Z}[\varepsilon]$ is equal to 2, the natural question becomes: does there exist a unit $\eta \in \mathbb{Z}[\varepsilon]$ such that $\{\varepsilon, \eta\}$ is a system of fundamental units of $\mathbb{Z}[\varepsilon]$? Clearly, there exists $\eta \in \mathbb{Z}[\varepsilon]$ such that $\{\varepsilon, \eta\}$ is a system of fundamental units of $\mathbb{Z}[\varepsilon]$ if and only if $\pm\varepsilon$ are not an nth power in $\mathbb{Z}[\varepsilon]$ for any $n \geq 2$ (e.g., see [Lou12, end of proof of Lemma 7]).

5.1. Cubic units of type (T+). — We may assume that $|\varepsilon| \geq |\varepsilon'| \geq |\varepsilon''| > 0$, where ε, ε' and ε'' are the three conjugates of ε, i.e. the three real roots of $\Pi_\varepsilon(X)$. By changing ε into $1/\varepsilon''$, we may assume that $|\varepsilon| > 1 > |\varepsilon'| \geq |\varepsilon''| > 0$ (use $\varepsilon\varepsilon'\varepsilon'' = c = \pm 1$). Finally, by changing ε into $-\varepsilon$, we obtain that at least one of the four polynomials equivalent to $\Pi_\varepsilon(X)$ is of type (T+) (notice that if ε is of type (T+) and $\varepsilon = \eta^n$ for some odd $n \geq 3$ and some $\eta \in \mathbb{Z}[\varepsilon]$, then η is clearly also of type (T+), whereas if ε is reduced then it is not clear whether η is also necessarily reduced):

Definition 30. — *A cubic polynomial of type (T+) is a monic cubic polynomial $P(X) = X^3 - aX^2 + bX - c \in \mathbb{Z}[X]$, $c \in \{\pm 1\}$, which is $\mathbb{Q}$-irreducible ($\Leftrightarrow b \neq a+c-1$ and $b \neq -a-c-1$), of positive discriminant $D_{P(X)} > 0$ and whose three real roots ε_P, ε'_P and ε''_P can be sorted so as to satisfy*

$$\varepsilon_P > 1 > |\varepsilon'_P| \geq |\varepsilon''_P| > 0. \tag{16}$$

In that situation, $H_p(P(X)) = (\varepsilon_P^p + \varepsilon_P^{-p})^{1/p}$ and $H_\infty(P(X)) = \varepsilon_P$.

Lemma 31. — *(See* [Lou12, Lemma 3]*). $P(X) = X^3 - aX^2 + bX - c \in \mathbf{Z}[X]$ with $c \in \{\pm 1\}$ is of type (T+) if and only if (i) $D_{P(X)} > 0$ and (ii) $-a - c \leq b \leq a + c - 2$. It implies $0 \leq a \leq H_\infty(P(X)) + 2$.*

Lemma 31 makes it easy to list of all the cubic polynomials of type (T+) whose real roots are less than or equal to a given upper bound B. Taking $B = 3$, we obtain:

Lemma 32. — *The real root $\eta_0 = 2.24697\cdots$ greater than one of $\Pi(X) = X^3 - 2X^2 - X + 1$ of type (T+) is the smallest totally real cubic unit greater than 1 that satisfies* (16)*.*

5.2. Bounds on discriminants. —

Theorem 33. — *Let α be a totally real cubic algebraic unit satisfying (16). Then*

$$\alpha^{3/2}/2 \leq D_\alpha \leq 4(\alpha + \alpha^{-1})^4 \leq 64\alpha^4. \tag{17}$$

Hence, if $P(X) = X^3 - aX^2 + bX - c \in \mathbb{Z}[X]$, $c \in \{\pm 1\}$, is $\mathbb{Q}$-irreducible and of positive discriminant $D_{P(X)} > 0$, then

$$H_\infty(P(X))^{3/2}/2 \leq D_{P(X)} \leq 4H_1(P(X))^4 \leq 64H_\infty(P(X))^4. \tag{18}$$

Proof. — Write $\alpha' = t/\sqrt{\alpha}$ and $\alpha'' = c/t\sqrt{\alpha}$, with $1 \leq |t| \leq \sqrt{\alpha}$. Then $a = \alpha + (t + c/t)/\sqrt{\alpha}$, $b = (t + c/t)\sqrt{\alpha} + c/\alpha$ and

$$b^2 - 4ac = (t - c/t)^2\alpha - 2c(t + c/t)/\sqrt{\alpha} + 1/\alpha^2. \tag{19}$$

Let us prove the upper bound. We have

$$D_\alpha = (\alpha - \alpha')^2(\alpha - \alpha'')^2(\alpha' - \alpha'')^2.$$

Hence, writing $x = \sqrt{\alpha} > 1$, and $1 \leq y = |t| \leq x$, we have

$$\begin{aligned}
\sqrt{D_\alpha} &\leq (x^2 + \frac{y}{x})(x^2 + \frac{1}{xy})(\frac{y}{x} + \frac{1}{xy}) \\
&= (x^3 + y + 1/y + 1/x^3)(y + 1/y) \\
&\leq (x^3 + x + 1/x + 1/x^3)(x + 1/x) \\
&= (x^2 + 1/x^2)(x + 1/x)^2 \leq 2(x^2 + x^{-2})^2 = 2H_1(P(X))^2.
\end{aligned}$$

Let us now prove the lower bound. We have

$$D_\alpha \geq (\alpha - 1)^4(\alpha' - \alpha'')^2 = (\alpha - 1)^4(t - c/t)^2/\alpha.$$

Assume that $\alpha \geq 44$.
First, assume that $|\, t - c/t|^2 \geq 1/(3\alpha)$. Then $D_\alpha \geq (\alpha - 1)^4/(3\alpha^2) \geq 2\alpha^{3/2}$.
Secondly, assume that $|t - c/t|^2 < 1/(3\alpha)$. Then $|t - c/t| < 2$. Hence $c = 1$, $|t| \geq 1$ and $|t - 1/t| \leq 5/6$, hence $1 \leq |t| \leq 3/2$ and $2|t + c/t| = 2|t + 1/t| \leq 13/3$ and (19) yields

$$-1 < -\frac{13/3}{\sqrt{\alpha}} < b^2 - 4a < \frac{1}{3} + \frac{13/3}{\sqrt{\alpha}} + \frac{1}{\alpha^2} < 1.$$

Hence, $b^2 - 4a = 0$, $T = t + 1/t \notin (-2, 2)$, $\alpha T^2 - 2T/\sqrt{\alpha} + 1/\alpha^2 - 4\alpha = 0$, $T = 2 + 1/\alpha^{3/2}$, $t - 1/t = \sqrt{4/\alpha^{3/2} + 1/\alpha^3}$ and

$$D_\alpha = 4\alpha^{3/2}\left(1 - \frac{2}{\alpha^{5/2}} + \frac{1}{\alpha^3} - \frac{1}{\alpha^4}\right)^2\left(1 + \frac{1}{4\alpha^{3/2}}\right) > 2\alpha^{3/2}$$

(and $b = 2B = (t + 1/t)\sqrt{\alpha} + 1/\alpha = 2\sqrt{\alpha} + 2/\alpha > 2$ is even, $a = B^2$ and $P(X) = X^3 - B^2X^2 + 2BX - 1$, $B \geq 3$).
Therefore, $D_\alpha \geq 2\alpha^{3/2}$ for $\alpha \geq 44$.
Finally, using Lemma 31 to find the 3236 cubic polynomials $P(X) = X^3 - aX^2 + bX - c \in \mathbb{Z}[X]$ of type (T+) with $0 \leq a \leq 45$ (for $1 < \varepsilon_P < 44$ implies $a < 46$), and checking that $D_{P(X)} \geq H_\infty(P(X))^{3/2}/2$ for each of these polynomials (which is equivalent to checking that $P((2D_{P(X)})^{2/3}) > 0$), we end up with the desired result. ☐

Remark 34. — *The exponents $3/2$ and 4 in Theorem 33 are optimal. Indeed, if $P(X) = X^3 - B^2X^2 + 2BX - 1$, $B \geq 3$, (of type (T+) by Lemma 31), then $D_{P(X)} = 4B^3 - 27$ is asymptotic to $4B^3$ and $H_\infty(P(X))$ is asymptotic to B^2, i.e. $D_{P(X)}$ is asymptotic to $4H_\infty(P(X))^{3/2}$. If $P(X) = X^3 - aX^2 - (a+3)X - 1$, $a \geq -1$, (of type (T+) by Lemma 31), then $D_{P(X)} = (a^2 + 3a + 9)^2$ is asymptotic to a^4 and $H_\infty(P(X))$ is asymptotic to a, i.e. $D_{P(X)}$ is asymptotic to $H_\infty(P(X))^4$.*

5.3. Being a square. —

Lemma 35. — *Let ε be a reduced totally real algebraic cubic unit, i.e. such that $\Pi_\varepsilon(X) = X^3 - uX^2 + vX - w \in \mathbb{Z}[X]$ with $D_\varepsilon > 0$, $w \in \{\pm 1\}$ and $|v| \leq u$. Then $\varepsilon = \pm\eta^2$ for some $\eta \in \mathbf{Z}[\varepsilon]$ if and only if we are in one of the two following cases:*

1. *$\Pi_\varepsilon(X) = X^3 - 6X^2 + 5X - 1$, in which case $\varepsilon = \eta^2$, where $\eta = \varepsilon^2 - 5\varepsilon + 2$, $\Pi_\eta(X) = X^3 - 2X^2 - X + 1$ and $D_\varepsilon = D_\eta = 49$.*

2. *$\Pi_\varepsilon(X)=X^3-B^2X^2+2BX-1$ with $B\geq 3$, in which case $\varepsilon=\eta^2$, where $\eta=-\varepsilon^2+B^2\varepsilon-B$, $\Pi_\eta(X)=X^3-BX^2+1$ and $D_\varepsilon=D_\eta=4B^3-27$.*

In both cases, η is reduced and η is not a square in $\mathbb{Z}[\eta]=\mathbb{Z}[\varepsilon]$.

Proof. — Assume that $\varepsilon=\pm\eta^2$ for some $\eta\in\mathbb{Z}[\varepsilon]$. By changing η into $-\eta$ if necessary, we may assume that $\Pi_\eta(X)=X^3-aX^2+bX-1\in\mathbb{Z}[X]$, which gives $\Pi_{\eta^2}(X)=X^3-(a^2-2b)X^2+(b^2-2a)X-1$ and $\Pi_{-\eta^2}(X)=X^3+(a^2-2b)X^2+(b^2-2a)X+1$. Then $\mathbb{Z}[\eta^2]=\mathbb{Z}[\varepsilon]=\mathbb{Z}[\eta]$ (Lemma 2), hence $(\mathbb{Z}[\eta]:\mathbb{Z}[\eta^2])=|ab-1|=1$, and we will have $\eta=(\varepsilon^2\mp(a^2-b)\varepsilon-a)/(1-ab)$.
First, assume that $ab=2$. We are in one of the following eight cases, with X^3-6X^2+5X-1 being the only reduced polynomials of positive discriminant among these eight polynomials:

a	b	$\Pi_{\eta^2}(X)$	$\Pi_{-\eta^2}(X)$
2	1	X^3-2X^2-3X-1	X^3+2X^2-3X+1
1	2	X^3+3X^2+2X-1	X^3-3X^2+2X+1
-2	-1	X^3-6X^2+5X-1	X^3+6X^2+5X+1
-1	-2	X^3-5X^2+6X-1	X^3+5X^2+6X+1

Secondly, assume that $ab=0$.
If $a=0$, then $\Pi_\varepsilon(X)=\Pi_{\eta^2}(X)=X^3+2bX^2+b^2X-1$ with $|b^2|\leq -2b$ implies $b\in\{-2,-1,0\}$, and $\Pi_\varepsilon(X)=\Pi_{-\eta^2}(X)=X^3-2bX^2+b^2X+1$ with $|b^2|\leq 2b$ implies $b\in\{-0,1,2\}$. In these six cases, either $\Pi_\varepsilon(X)$ is reducible, a contradiction, or of negative discriminant, another contradiction.
If $b=0$ and $a\neq 0$, then $\Pi_\varepsilon(X)=\Pi_{-\eta^2}(X)=X^3+a^2X^2+2aX+1$ with $0\neq|2a|\leq -a^2$ is impossible, and $\Pi_\varepsilon(X)=\Pi_{\eta^2}(X)=X^3-a^2X^2-2aX-1$ with $0\neq|2a|\leq a^2$ and $D_\varepsilon=-4a^3-27>0$ implies $a=-B$ with $B\geq 3$. □

5.4. Statement and proof of the result for the totally real cubic case. —

Theorem 36. — *Let ε be a reduced totally real algebraic cubic unit, i.e. such that $\Pi_\varepsilon(X)=X^3-uX^2+vX-w\in\mathbb{Z}[X]$ with $D_\varepsilon>0$, $w\in\{\pm1\}$ and $|v|\leq u$.
If $\Pi_\varepsilon(X)=X^3-6X^2+5X-1$, set $\xi_1=\varepsilon^2-5\varepsilon+2$ for which $\xi_1^2=\varepsilon$.
If $\Pi_\varepsilon(X)=X^3-B^2X^2+2BX-1$ with $B\geq 3$, set $\xi_1=-\varepsilon^2+B^2\varepsilon-B$ for which $\xi_1^2=\varepsilon$.
Otherwise, set $\xi_1=\varepsilon$.
Hence, ξ_1 is always reduced, by Lemma 35.
Then there exists another unit $\xi_2\in\mathbf{Z}[\varepsilon]$ such that $\{\xi_1,\xi_2\}$ is a system of fundamental units of the cubic order $\mathbf{Z}[\varepsilon]$.*

Proof. — By Lemma 35, it suffices to prove that if ξ satisfies (16), then there does not exist any prime $p\geq 3$ and any $\eta\in\mathbb{Z}[\xi]$ such that $\xi=\eta^p$. Notice that such an η must also satisfy (16). We would have $\mathbb{Z}[\xi]=\mathbb{Z}[\eta]$ and $D_\xi=D_\eta$. Theorem 33 would yield

$$\eta^{9/2}/2\leq\eta^{3p/2}/2\leq\xi^{3/2}/2\leq D_\xi=D_\eta\leq 4(\eta+\eta^{-1})^4,$$

that implies $1<\eta\leq 64.2$. If η_0 is as in Lemma 32, we then obtain

$$1=D_\eta/D_\xi\leq\frac{4(\eta+\eta^{-1})^4}{\xi^{3/2}/2}=8\left(\frac{\eta+\eta^{-1}}{\eta^{3p/8}}\right)^4\leq 8\left(\frac{\eta_0+\eta_0^{-1}}{\eta_0^{3p/8}}\right)^4,$$

that implies $p<5$, hence $p=3$, $\xi=\eta^3$ and $D_\eta=D_{\eta^3}$.

It remains to computationally check that $D_{\eta^3} \neq D_\eta$ for the 7271 cubic polynomial $\Pi_\eta(X) = X^3 - aX^2 + bX - c \in \mathbb{Z}[X]$ of type (T+) with $0 \leq a \leq 66$ (for $1 < \varepsilon_P \leq 64.2$ implies $0 \leq a \leq 66$). Noticing that

$$D_{\eta^3}/D_\eta = |a^3c + b^3 - a^2b^2|^2$$

and using Lemma 31, this is a straightforward verification □

Remark 37. — *For several parametrized families of totally real cubic orders, an explicit system of 2 fundamental units is known (e.g., see* [Tho]*). By* [Lou12, Theorem 2]*, if ε is a totally real cubic unit for which (i) the cubic number field $\mathbf{Q}(\varepsilon)$ is Galois and (ii) the order $\mathbb{Z}[\varepsilon]$ is invariant under the action of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$*, then it was reasonable to conjecture that ε and any one of its two other conjugates ε' should form a system of fundamental units of the cubic order $\mathbb{Z}[\varepsilon]$ (allowing for safety a finite number of exceptions). Surprisingly, in* [LL15] *we found that such a cubic order $\mathbb{Z}[\varepsilon]$ is almost never invariant under the action of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$*. And proving that $\{\varepsilon, \varepsilon'\}$ is in general a system of fundamental units of the larger Galois-invariant totally real cubic order $\mathbb{Z}[\varepsilon, \varepsilon']$ is hopeless (see however* [LL14] *for related results). In contrast, we still think that if $\varepsilon > 1$ is a real cubic unit of negative discriminant with complex conjugates η and $\bar{\eta}$, then the group $< -1, \varepsilon, \eta >$ is of bounded index in the group of units of the totally imaginary sextic order $\mathbb{Z}[\varepsilon, \eta, \bar{\eta}]$ (see* [LL14] *and use Theorem 8 for a partial solution to this problem).*

6. A conjecture in a quartic case of unit rank 2

Let ε be a quartic algebraic unit which is neither totally real nor totally imaginary, i.e. a quartic unit of negative discriminant $D_\varepsilon < 0$. Here again, the rank of the group of units of the quartic order $\mathbb{Z}[\varepsilon]$ is equal to 2 and the only complex root of unity in $\mathbb{Z}[\varepsilon]$ are ± 1. As in Section 5 we would like to prove that, in general, there exists a second unit $\eta \in \mathbb{Z}[\varepsilon]$ such that $\{\varepsilon, \eta\}$ is a system of fundamental units of $\mathbb{Z}[\varepsilon]$. Since we can assume that $\Pi_\varepsilon(X)$ is reduced, we would like to prove the very precise statement Conjecture 38:

Conjecture 38. — *Let ε be a reduced quartic unit of negative discriminnant, i.e. such that $\Pi_\varepsilon(X) = X^4 - AX^3 + BX^2 - CX + D \in \mathbb{Z}[X]$ with $D_\varepsilon < 0$, $D \in \{\pm 1\}$ and $|C| \leq A$. Let us define a unit $\xi_1 \in \mathbb{Z}[\varepsilon]$ as follows:*

1. *An infinite family: if $\Pi_\varepsilon(X)$ is one of the four polynomials equivalent to $\Pi_{\xi_1^2}(X) = X^4 - (a^2-2b)X^3 + (b^2-2ac+2d)X^2 - (c^2-2bd)X + 1$, where $\Pi_{\xi_1}(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Z}[X]$ with $d \in \{\pm 1\}$ and $|c| \leq a$ is a $\mathbb{Q}$-irreducible reduced polynomial of negative discriminant different from the ones that appear in the two sporadic cases 3f and 3g below and such that $|a^2d + c^2 - abc| = 1$, then we may assume that $\varepsilon \in \{\pm\xi_1^2, \pm 1/\xi_1^2\}$ and we have*
$$\xi_1 = \frac{-a\xi_1^6 + (a^3 - 2ab + c)\xi_1^4 + (a^2c - ab^2 - ad + bc)\xi_1^2 - (ab - c)d}{a^2d + c^2 - abc} \in \mathbb{Z}[\varepsilon],$$
2. *An infinite family: if $\Pi_\varepsilon(X) = X^4 - a^3X^3 + 3a^2dX^2 - 3aX + d$, $d \in \{\pm 1\}$ and $a \geq 2$, set $\xi_1 = \varepsilon^3 - a^3\varepsilon^2 + 3a^2d\varepsilon - 2a \in \mathbb{Z}[\varepsilon]$, for which $\varepsilon = \xi_1^3$, $\Pi_{\xi_1}(X) = X^4 - aX^3 + d$ and $D_\varepsilon = D_{\xi_1} = -27a^4 + 256d$.*
3. *The following 8 sporadic cases:*

(a) *If* $\Pi_\varepsilon(X) = X^4 - 5X^3 - 9X^2 - 5X - 1$*, set* $\xi_1 = \varepsilon^3 - 6\varepsilon^2 - 3\varepsilon \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = \xi_1^3$*,* $\Pi_{\xi_1}(X) = X^4 - 2X^3 + X - 1$ *and* $D_\varepsilon = D_{\chi_1} = -275$*.*
(b) *If* $\Pi_\varepsilon(X) = X^4 - 3X^3 + 3X^2 - X - 1$*, set* $\xi_1 = -\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ *for which* $\varepsilon = -1/\xi_1^3$*,* $\Pi_{\xi_1}(X) = X^4 - X^3 - 1$ *and* $D_\varepsilon = D_{\chi_1} = -283$*.*
(c) *If* $\Pi_\varepsilon(X) = X^4 - 5X^3 + 7X^2 - X - 1$*, set* $\xi_1 = \varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = -\xi_1^3$*,* $\Pi_{\xi_1}(X) = X^4 - X^3 + X^2 + X - 1$ *and* $D_\varepsilon = D_{\chi_1} = -331$*.*
(d) *If* $\Pi_\varepsilon(X) = X^4 - 11X^3 + 15X^2 - 7X + 1$*, for which* $\varepsilon = \xi_1^3$ *with* $\xi_1 = \varepsilon^3 - 10\varepsilon^2 + 5\varepsilon \in \mathbb{Z}[\varepsilon]$ *and* $\Pi_{\xi_1}(X) = X^4 - 2X^3 - X + 1$*, in which case* $D_\varepsilon = D_{\chi_1} = -643$*.*
(e) *If* $\Pi_\varepsilon(X) = X^4 - 19X^3 + 91X^2 - 7X - 1$*, set* $\xi_1 = -\varepsilon^3 + 10\varepsilon^2 - 5\varepsilon \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = -\xi_1^3$*,* $\Pi_{\xi_1}(X) = X^4 - 2X^3 + 4X^2 + X - 1$ *and* $D_\varepsilon = D_{\chi_1} = -5987$*.*
(f) *If* $\Pi_\varepsilon(X) = X^4 - 5X^3 + 6X^2 - 4X + 1$*, set* $\xi_1 = \varepsilon^3 - 4\varepsilon^2 + 2\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = \xi_1^4$*,* $\Pi_{\xi_1}(X) = X^4 - X^3 - 1$ *and* $D_\varepsilon = D_{\chi_1} = -283$*.*
(g) *If* $\Pi_\varepsilon(X) = X^4 - 15X^3 + 17X^2 - 7X + 1$*, set* $\xi_1 = \varepsilon^3 - 15\varepsilon^2 + 17\varepsilon - 5 \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = \xi_1^4$*,* $\Pi_{\xi_1}(X) = X^4 - X^3 - X^2 - X - 1$ *and* $D_\varepsilon = D_{\chi_1} = -563$*.*
(h) *If* $\Pi_\varepsilon(X) = X^4 - 8X^3 - 14X^2 - 7X - 1$*, set* $\xi_1 = 5\varepsilon^3 - 43\varepsilon^2 - 44\varepsilon - 9 \in \mathbb{Z}[\varepsilon]$*, for which* $\varepsilon = \xi_1^7$*,* $\Pi_{\xi_1}(X) = X^4 - X^3 - 1$ *and* $D_\varepsilon = D_{\chi_1} = -283$*.*

4. *Otherwise, set* $\xi_1 = \varepsilon$*.*

Hence, ξ_1 *is always reduced.*
Then there exists another unit $\xi_2 \in \mathbb{Z}[\varepsilon]$ *such that* $\{\xi_1, \xi_2\}$ *is a system of fundamental units of the quartic order* $\mathbb{Z}[\varepsilon]$*.*

The main problem is that we have not yet been able to prove useful lower bounds for discriminants of such quartic units, even though we think it reasonable to conjecture that:

Conjecture 39. — *Let* $P(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Z}[X]$*,* $d \in \{\pm 1\}$*, be a* $\mathbb{Q}$*-irreducible quartic polynomial of negative discriminant* $D_{P(X)} < 0$*. Then*

$$H_\infty(P(X))^{4/3} \ll |D_{P(X)}| \ll H_1(P(X))^6, \tag{20}$$

where the implicit contants do not depend on $P(X)$*.*

At least, we know that the exponent on the left hand side of (20) must be less than or equal to $4/3$ (take $n = 3$ in Theorem 44) and we can prove the easiest part of Conjecture 39:

Lemma 40. — *Let* $P(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Z}[X]$*,* $d \in \{\pm 1\}$*, be a* $\mathbb{Q}$*-irreducible quartic polynomial of negative discriminant* $D_{P(X)} < 0$*. Then*

$$|D_{P(X)}| \leq 2^{12} H_1(P(X))^6.$$

This exponent 6 *is optimal: if* $P(X) = X^4 - aX^3 + X^2 - aX + 1$*,* $a \geq 2$*, then* $H_1(P(X)) = (a + \sqrt{a^2+4})/2$ *and* $-D_{P(X)} = (4a^2 - 9)(a^2 + 4)^2 > 0$ *is asymptotic to* $4H_1(P(X))^6$*.*

Proof. — Let ρ_1 and ρ_2 be the absolute values of the two real roots ε_1 and ε_2 of $P(X)$. Let η and $\bar\eta$ be its two non-real roots. Set $\rho = |\eta| = 1/\sqrt{\rho_1\rho_2}$. By changing $P(X)$ into its associate $dX^4P(1/X)$, we may assume that $\rho_1 \geq 1$ and $1/\rho_1 \leq \rho_2 \leq \rho_1$. Then

$$H_1(P(X)) \geq H_\infty(P(X)) = \max(\rho_1, \rho_1\rho_2)$$

and
$$D(X) = \Big((\varepsilon_1-\varepsilon_2)|\varepsilon_1-\eta|^2|\varepsilon_2-\eta|^2(\eta-\bar{\eta})\Big)^2$$
yields
$$|D(X)| \leq \Big((2\rho_1)((2\rho_1)^2(\rho_2+\rho)^2(2\rho))\Big)^2 = 2^8\rho_1^5(\rho_2+1/\sqrt{\rho_1\rho_2})^4/\rho_2.$$
First, if $\rho_2 \leq 1/\sqrt{\rho_1\rho_2}$, then $\rho_2 \leq 1$, $H_\infty(P(X)) = \rho_1$, $\rho_2+1/\sqrt{\rho_1\rho_2} \leq 2/\sqrt{\rho_1\rho_2}$ and $|D(X)| \leq 2^{12}\rho_1^3/\rho_2^3 \leq 2^{12}\rho_1^6 = 2^{12}H_\infty(P(X))^6$.
Secondly, if $1/\sqrt{\rho_1\rho_2} \leq \rho_2 \leq 1$, then $H_\infty(P(X)) = \rho_1$, $\rho_2+1/\sqrt{\rho_1\rho_2} \leq 2\rho_2$ and $|D(X)| \leq 2^{12}\rho_1^5\rho_2^3 \leq 2^{12}\rho_1^5 = 2^{12}H_\infty(P(X))^5$.
Thirdly, if $1 \leq \rho_2 \leq \rho_1$, then then $H_\infty(P(X)) = \rho_1\rho_2$, $\rho_2+1/\sqrt{\rho_1\rho_2} \leq 2\rho_2$ and $|D(X)| \leq 2^{12}\rho_1^5\rho_2^3 \leq 2^{12}(\rho_1\rho_2)^5 = 2^{12}H_\infty(P(X))^5$. □

If Conjecture 39 is true, then ε can be infinitely often an nth-power only for $|n| \in \{2,3,4\}$. Hence a first step towards proving Conjecture 38 would be (i) to prove that $\varepsilon > 1$ cannot be infinitely many often a 4th power, (ii) to prove that ε is infinitely often a 3rd power if and only if we are in case 2 of Conjecture 38 and (iii) to write case 1 of Conjecture 38 much more explicitly, i.e. to give a necessary and sufficient condition easy to check on the coeffcients of $\Pi_\varepsilon(X) = X^4-AX^3+BX^2-CX+D \in \mathbb{Z}[X]$.

Remark 41. — *Regarding Conjecture 39, we also did some numerical computation. For the* 38 413 452 *quartic $\mathbb{Q}$-irreducible polynomials $P(X) = X^4-aX^3+bX^2-cX+d \in \mathbb{Z}[X]$, $d \in \{\pm 1\}$, $|c| \leq a \leq 400$, $|b| \leq 400$, of negative discriminant $D_{P(X)} < 0$ we have*
$$|D_{P(X)}| \geq 15H_\infty(P(X))^{4/3},$$
except for the following 6 *polynomials:*

$P(X)$	$\|D_{P(X)}\|/H_\infty(P(X))^{4/3}$
$X^4-8X^3-14X^2-7X-1$	$13.97761\cdots$
$X^4-8X^3+10X^2+7X+1$	$8.06342\cdots$
$X^4-9X^3+22X^2-8X+1$	$5.81131\cdots$
$X^4-24X^3-26X^2-9X-1$	$3.86062\cdots$
$X^4-95X^3+64X^2-14X+1$	$1.49757\cdots$
$X^4-252X^3+120X^2-19X+1$	$0.30925\cdots$

7. The general situation

The situation for number fields of degree ≥ 5 can be even worse. For example, nothing prevents a quintic algebraic unit ε from being infinitely many often a square, a cube or a fourth power in $\mathbb{Z}[\varepsilon]$:

Proposition 42. — *Let $m > n \geq 2$ be coprime. Assume that*
$$X^m-(aX+1)^n$$
is $\mathbb{Q}$-irreducible. Then any of its complex roots ε is an n-th power in $\mathbb{Z}[\varepsilon]$.

Proof. — If $mu-nv=1$ with $m,n \geq 1$ rational integers, then $\varepsilon = \eta^n$ with $\eta = (a\varepsilon+1)^u/\varepsilon^v \in \mathbb{Z}[\varepsilon]$. Notice that η is a root of X^m-aX^n-1. Hence, $X^m-(aX+1)^n$ is $\mathbb{Q}$-irreducible if and only if X^m-aX^n-1 is $\mathbb{Q}$-irreducible □

To be able to deal with the general situation of algebraic units of any degree, it is not that unreasonable to put forward a conjecture generalizing Lemma 3 and Theorems 9, 24, and 33:

Conjecture 43. — *Let $P(X) = X^n - a_{n-1}X^{n-1} + \cdots + (-1)^{n-1}a_1X + (-1)^n a_0 \in \mathbb{Z}[X]$ range over the monic $\mathbb{Q}$-irreducible polynomials of a given degree $n \geq 3$ with $a_0 \in \{\pm 1\}$. Then for some positive constants $\beta(n) \geq \alpha(n) > 0$ it holds that*

$$H_\infty(P(X))^{\alpha(n)} \ll |D_{P(X)}| \ll H_1(P(X))^{\beta(n)}, \tag{21}$$

where the implicit contants depend on n only.

Clearly, the upper bound in (21) holds true with $\beta(n) = n(n-1)$ (any complex root α of $P(X)$ satisfies $|\alpha| \leq H_1(P(X))$). More interestingly, if Conjecture 43 were true, then we would have $\alpha(n) \leq (n+1)/n$:

Theorem 44. — *Fix $n \geq 2$. Let a range over the rational integers greater than 2. Then the polynomials $R_n(X) = X^{n+1} - (aX+1)^n \in \mathbb{Z}[X]$ are $\mathbf{Q}$-irreducible, of discriminants $D_{R_n(X)} = (-1)^{n(n-1)/2}(n^n a^{n+1} + (n+1)^{n+1})$ of absolute values asymptotic to $n^n a^{n+1}$ and $H_\infty(R_n(X))$ is asymptotic to a^n. Hence, $D_{R_n(X)}$ is asymptotic to $n^n H_\infty(R_n(X))^{(n+1)/n}$. Moreover, any root ε of $R_n(X)$ is an algebraic unit of degree n such that $\varepsilon = \eta^n$ with $\eta = a + \varepsilon^{-1} \in \mathbb{Z}[\varepsilon]$.*

Proof. — By Perron's criterium $P_n(X) = X^{n+1} - aX^n - 1 \in \mathbb{Z}[X]$ is $\mathbb{Q}$-irreducible for $a \geq 3$. If η is a root of $P_n(X)$ then $\varepsilon := \eta^n$ that satisfies $\varepsilon^{n+1} = (\eta^{n+1})^n = (a\eta^n + 1)^n = (a\varepsilon + 1)^n$ is a root of $R_n(X)$. Hence, $R_n(X)$ is $\mathbf{Q}$-irreducible, $D_{R_n(X)} = D_{P_n(X)} = (-1)^{n(n-1)/2}(n^n a^{n+1} + (n+1)^{n+1})$ and $H_\infty(R_n(X)) = H_\infty(P_n(X))^n$ is asymptotic to a^n, by Lemma 45. $\square$

Lemma 45. — *Fix $n \geq 2$, an integer, and a primitive $2n$-th complex root of unity ζ_{2n}. Let $a \geq 3$ range over the integers. Then the $n+1$ complex roots $\rho_a^{(k)}$, $0 \leq k \leq n$, of $P_n(X) = X^{n+1} - aX^n - 1 \in \mathbb{Z}[X]$ can be sorted so as to satisfy $\rho_a^{(0)} = a + O(a^{-n})$ and $\rho_a^{(k)} := \zeta_{2n}^{2k-1} a^{-1/n} + O(a^{-1})$, $1 \leq k \leq n$. Hence, $H_\infty(P_n(X))$ is asymptotic to a.*

Proof. — If $A_n(X) = \sum_{k=0}^{n+1} \alpha_k(a) X^k \in \mathbb{C}[X]$, $a \geq 3$, we write $A_n(X) = O(a^c)$ if $\alpha_k(a) = O(a^c)$ for $0 \leq k \leq n+1$. Set

$$Q_n(X) := (X - a)\prod_{k=0}^{n}(X - \theta_a^{(k)}), \text{ where } \theta_a^{(k)} = \frac{\zeta_{2n}^{2k-1}}{a^{1/n}} + \frac{\zeta_{2n}^{2(2k-1)}}{na^{(n+2)/n}}.$$

We want to prove that the coefficients of $Q_n(X)$ are very close to those of $P_n(X)$ as a goes to infinity, and that it implies that the roots of $P_n(X)$ are close to that of $Q_n(X)$, hence implies the desired result. Set

$$R_n(X) = \prod_{k=1}^{n}\Big(X - \frac{\zeta_{2n}^{2k-1}}{a^{1/n}}\Big) = X^n + a^{-1} \text{ and } S_n(X) := \prod_{k=1}^{n}(X - \theta_a^{(k)}).$$

Clearly, we have

$$S_n(X) = R_n(X) - \sum_{k=1}^{n} \frac{R_n(X)}{X - \frac{\zeta_{2n}^{2k-1}}{a^{1/n}}} \frac{\zeta_{2n}^{2(2k-1)}}{na^{(n+2)/n}} + O(a^{-2(n+2)/n}).$$

Using the partial fraction decomposition over $\mathbb{C}$

$$\frac{X}{R_n(X)} = \frac{X}{X^n + a^{-1}} = -a\sum_{k=1}^{n} \frac{\frac{\zeta_{2n}^{2(2k-1)}}{na^{2/n}}}{X - \frac{\zeta_{2n}^{2k-1}}{a^{1/n}}},$$

we obtain $S_n(X) = X^n + a^{-2}X + a^{-1} + O(a^{-2(n+2)/n})$ and $Q_n(X) - P_n(X) = O(a^{-2})$. Applying [Lou08b, (4) of Section 3] with $\alpha = 2$, $\beta = -1/n$, $\gamma = -1$ and $\delta = -1/n$ we obtain the approximations of the $\rho_a^{(k)}$ for $1 \leq k \leq n$. Using $P_n(a) = -1 < 0$ and $P_n(a + a^{-n}) = (1 + a^{-n-1})^n - 1 > 0$, we obtain the approximations of the $\rho_a^{(0)}$. $\square$

References

[BHMMS] J. Beers, D. Henshaw, C. Mccall, S. Mulay and M. Spindler, *Fundamentality of a cubic unit u for $\mathbb{Z}[u]$*, Math. Comp., **80** (2011), 563–578, Corrigenda and addenda, Math. Comp. **81** (2012), 2383–2387.

[Coh] H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, **138**, Springer-Verlag, Berlin, 1993.

[LL14] J. H. Lee and S. Louboutin, *On the fundamental units of some cubic orders generated by units*, Acta Arith., **165** (2014), 283–299.

[LL15] J. H. Lee and S. Louboutin, *Determination of the orders generated by a cyclic cubic unit that are Galois invariant*, J. Number Theory, **148** (2015), 33–39.

[Lou06] S. Louboutin, *The class-number one problem for some real cubic number fields with negative discriminants*, J. Number Theory, **121**, (2006), 30–39.

[Lou08a] S. Louboutin, *The fundamental unit of some quadratic, cubic or quartic orders*, J. Ramanujan Math. Soc. **23**, No. 2 (2008), 191–210.

[Lou08b] S. Louboutin, *Localization of the complex zeros of parametrized families of polynomials*, J. Symbolic Comput., **43**, (2008), 304–309.

[Lou10] S. Louboutin, *On some cubic or quartic algebraic units*, J. Number Theory, **130**, (2010), 956–960.

[Lou12] S. Louboutin, *On the fundamental units of a totally real cubic order generated by a unit*, Proc. Amer. Math. Soc., **140**, (2012), 429–436.

[MS] S. Mulay and M. Spindler, *The positive discriminant case of Nagell's theorem for certain cubic orders*, J. Number Theory, **131**, (2011), 470–486.

[Nag] T. Nagell, *Zur Theorie der kubischen Irrationalitäten*, Acta Math., **55**, (1930), 33–65.

[PL] S.-M. Park and G.-N. Lee, *The class number one problem for some totally complex quartic number fields*, J. Number Theory, **129**, (2009), 1338–1349.

[Tho] E. Thomas, *Fundamental units for orders in certain cubic number fields* , J. Reine Angew. Math., **310**, (1979), 33-55.

[Was] L. C. Washington, *Introduction to Cyclotomic Fields*, Graduate Texts in Mathematics, **83**, Springer-Verlag, Second Edition, 1997.

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