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WEIGHT REDUCTION FOR COHOMOLOGICAL MOD p MODULAR FORMS OVER IMAGINARY QUADRATIC FIELDS

by

Adam Mohamed

Abstract. — Let F be an imaginary quadratic field and $\mathcal{O}$ its ring of integers. Let $\mathfrak{n} \subset \mathcal{O}$ be a non-zero ideal and let $p > 5$ be a rational inert prime in F and coprime with $\mathfrak{n}$. Let V be an irreducible finite dimensional representation of $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{p^2})]$. We establish that a system of Hecke eigenvalues appearing in the cohomology with coefficients in V already lives in the cohomology with coefficients in $\overline{\mathbb{F}}_p \otimes det^e$ for some $e \geq 0$; except possibly in some few cases.

Résumé. — Soient F un corps quadratique imaginaire et $\mathcal{O}$ son anneau d'entiers. Soient $\mathfrak{n} \subset \mathcal{O}$ un idéal non nul et $p > 5$ un nombre premier inerte dans F copremier avec $\mathfrak{n}$. Soit V une représentation irréductible de dimension finie de $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{p^2})]$. Nous établissons qu'un système de valeurs propres de Hecke appartenant au groupe de cohomologie coefficients dans V appartient aussi au groupe de cohomologie coefficients dans $\overline{\mathbb{F}}_p \otimes det^e$ pour $e \geq 0$ à l'exception, éventuellement, de quelques cas.

1. Introduction

Let F be an imaginary quadratic field with $\mathcal{O}$ as its ring of integers. The class number of F is denoted as h. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$. Let σ be the non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$. We consider the representations of $\mathrm{GL}_2(\mathcal{O})$ defined as $V_{r,s}^{a,b}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes det^a \otimes (\mathrm{Sym}^s(\mathcal{O}^2))^\sigma \otimes (det^b)^\sigma$ where a, b, r, s are positive integers. For an $\mathcal{O}$-algebra A, we define $V_{r,s}^{a,b}(A) := V_{r,s}^{a,b}(\mathcal{O}) \otimes_{\mathcal{O}} A$. A cohomological modular form of level Γ and weight $V_{r,s}^{a,b}(A)$ over F is a class in $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$. As in the classical setting, the space $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$ can be endowed with a structure of Hecke module. The Hecke algebra acting on $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$ is commutative and has its elements indexed over the integral ideals of F. So, one can consider eigenclasses or eigenforms which are eigenvectors for all the Hecke operators $T_{\mathfrak{a}}$. Hence to such an eigenform corresponds a system of Hecke eigenvalues.

Integral systems of eigenvalues when reduced modulo a prime p are believed to be related to mod p representations of Galois groups as conjectured by Ash et al. in [**4**]. One instance

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of this correspondence being the theorem of Deligne constructing l-adic representations of the absolute Galois group of $\mathbb{Q}$, $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, via systems of Hecke eigenvalues arising from modular forms over $\mathbb{Q}$. Let N be a positive integer and $\Gamma_0(N)$ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Take V to be the $\mathrm{SL}_2(\mathbb{Z})$-module given as $V := Sym^{k-2}(\mathbb{Z}^2) = \mathbb{Z}[X,Y]_{k-2}$, the space of homogeneous polynomials of degree $k-2$ over $\mathbb{Z}$ in two variables and with k even. The converse of Deligne's theorem, Serre's modularity conjecture, which is now a theorem of Khare and Wintenberger, has been formulated in the language of group cohomology in [**5**] and the standard conjecture in there relates mod p Galois representations of $G_{\mathbb{Q}}$ to systems of Hecke eigenvalues on $\mathrm{H}^1(\Gamma_0(N), V \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p)$.
Next let N and n be positive integers. In [**2**], it was shown that a system of Hecke eigenvalues occurring in the cohomology of $\Gamma_1(N)$ with coefficients in some $\mathrm{GL}_n(\mathbb{F}_p)$-module also occurs in the cohomology with coefficients in some irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-module. This fact has some interesting features. In fact it allows one to obtain a cohomological avatar of the so-called Hasse invariant, see [**9**]. That is, one can produce congruences between weight two and higher weight modular forms using cohomological methods.
As for the case of an imaginary quadratic field F of class number one, then when p splits in F and is coprime with $\mathfrak{n}$, in [**12**], it is established that a Hecke system of eigenvalues occurring in the first cohomology with non-trivial coefficients can be realized in the first cohomology with trivial coefficients. This should also hold when the class number of F is greater than one. Let p be a rational prime coprime to $\mathfrak{n}$ and inert in F. Let E be a finite dimensional representation of $\mathrm{GL}_2(\mathbb{F}_{p^2})$ over $\overline{\mathbb{F}}_p$. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$. Then a cohomological mod p modular form of level Γ and weight E is defined to be a class in $\mathrm{H}^1(\Gamma, \mathrm{E})$. As in the classical setting there is a Hecke algebra action on the space $\mathrm{H}^1(\Gamma, \mathrm{E})$ and one can consider systems of Hecke eigenvalues for the space $\mathrm{H}^1(\Gamma, \mathrm{E})$. Our aim will be to say something more precise about systems of Hecke eigenvalues in this setting. We will prove that a system of Hecke eigenvalues living in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$ where M is an irreducible $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{\mathrm{p}^2})]$-module also occurs in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^e)$ for some $e \geq 0$ depending on M; except possibly for some cases. See Theorem 4.11 for the precise statement. Here $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ are some congruence subgroups defined in Section 3. The strategy for proving Theorem 4.11 was initiated by Ash and Stevens in [**2**], and it was also adapted in [**12**] where a reduction to weight 2 statement is proved.
There is an application of Theorem 4.11 related to Serre type questions about mod p Galois representations of the absolute Galois group of F. When we are dealing with cohomological modular forms mod p with trivial coefficients $\overline{\mathbb{F}}_p$, we shall say that we are in *weight two*. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ and let be given

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

an irreducible mod p Galois representation of conductor $\mathfrak{n}$. Let Tr denotes the trace of a matrix. Then the following questions arise:

(a) Does there exist a cohomological Hecke eigenform of some weight V and level $\mathfrak{n}$ with eigenvalues $\psi(T_\lambda)$ such that $Tr(\rho(Frob_\lambda)) = \psi(T_\lambda)$ for all unramified prime ideals $\lambda \nmid \mathfrak{pn}$?

(b) Does there exist a cohomological Hecke eigenform of weight 2 ($V = \overline{\mathbb{F}}_p \otimes det^e$ for some $e \geq 0$) and level $\mathfrak{pn}$ with $Tr(\rho(Frob_\lambda)) = \psi(T_\lambda)$ for all unramified prime ideals $\lambda \nmid \mathfrak{pn}$?

As a consequence of Theorem 4.11, we shall see that the two questions above are equivalent. See Proposition 4.12 for the precise statement. Proposition 4.12 proves that when investigating Serre type questions as above, it is enough to work in weight two. For example, in [**11**], some computational investigations of Serre's conjecture over imaginary quadratic fields were carried out and the principle illustrated by Proposition 4.12 was assumed to hold.
Here is our outline. We shall first recall Hecke theory in our context. This is the content of Section 2. In Section 3, we shall compare some modules. The main result is proved in Section 4.

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2. Hecke operators

We set some of the necessary notation and recall briefly how Hecke operators are defined in our setting. As in the classical setting, we define Hecke operators via Hecke correspondences on hyperbolic 3-manifolds. This section is mainly notational as what we shall recall is very well explained in for example the works of Hida or Shimura.
So, some of the notation are as follows. Let F be an imaginary quadratic field of class number $h \geq 1$. Denote by $\mathcal{O}$ its ring of integer and let $\mathfrak{n}$ be an ideal of $\mathcal{O}$. The class group of F is denoted by Cl and we fix a rational prime p inert in F and $\mathfrak{p} = p\mathcal{O}$. We also assume that $\mathfrak{p}$ is coprime with $\mathfrak{n}$. Let $\hat{\mathcal{O}}$ be the profinite completion of $\mathcal{O} : \hat{\mathcal{O}} = \prod_{\mathfrak{q}\neq 0} \mathcal{O}_{\mathfrak{q}}$. We will denote the adeles of F by $\mathbb{A}$, and $\mathbb{A}_f$, $\mathbb{A}_\infty$ stand for the finite part and the infinite part of $\mathbb{A}$. We write $G := \mathrm{GL}_2$, so that, $G(\mathbb{A}), G(F), G(\mathbb{A}_f)$ are the usual linear algebraic groups of 2×2 matrices with entries in $\mathbb{A}, F, \mathbb{A}_f$, respectively. Let $\mathbb{H}_3 := G(\mathbb{C})/\mathbb{C}^*\mathrm{U}_2 \cong \mathbb{C} \times \mathbb{R}_{>0}$, the three dimensional equivalent of the classical Poincaré upper half plane $\mathbb{H}_2 = G(\mathbb{R})/\mathbb{R}^*\mathrm{O}_2$. Here U_2 is the unitary subgroup of $G(\mathbb{C})$. Let K be an open compact subgroup of $G(\hat{\mathcal{O}})$ such that the determinant homomorphism

$$det : K \to \hat{\mathcal{O}}^*$$

is surjective. We define the following homogeneous space

$$\begin{aligned} Y_K &:= G(F)\backslash(\mathbb{H}_3 \times G(\mathbb{A}_f)/K) \\ &= G(F)\backslash(G(\mathbb{C})/\mathbb{C}^*\mathrm{U}_2 \times G(\mathbb{A}_f)/K) \\ &= G(F)\backslash G(\mathbb{A})/K.U_2.\mathbb{C}^*. \end{aligned}$$

By the determinant map we have

$$Y_K \twoheadrightarrow F^*\backslash\mathbb{A}^*/\hat{\mathcal{O}}^*\mathbb{C}^* \cong F^*\backslash\mathbb{A}_f^*/\hat{\mathcal{O}}^* \cong Cl.$$

2.1. Hecke correspondences and Hecke operators.

— Let σ be the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Let

$$V_{\mathcal{O}} = V_{r,s}^{a,b}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes det^a \otimes (\mathrm{Sym}^s(\mathcal{O}^2))^\sigma \otimes (det^b)^\sigma$$

be an $\mathcal{O}[G(\mathcal{O})]$-module endowed with the discrete topology. We define $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p) := V_{\mathcal{O}} \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$. This space is also endowed with the discrete topology.
Let $X = G(\mathbb{A})/U_2\mathbb{C}^* \cong \mathbb{H}_3 \times G(\mathbb{A}_f)$. Under the assumption that K acts freely on $X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$, one has a topological cover

$$\pi_1 : G(F)\backslash(X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))/K \to G(F)\backslash X/K \cong Y_K.$$

We consider the locally constant sheaf $\mathcal{V}_{\overline{\mathbb{F}}_p}$ on Y_K given by the sections of π_1 : for an open subset U of Y_K, we have

$$\mathcal{V}_{\overline{\mathbb{F}}_p}(U) = \{s : U \to G(F)\backslash(X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))/K;\ \pi_1 \circ s = id\}.$$

We take $g \in Mat_2(\hat{\mathcal{O}})_{\neq 0}$ (by which we mean the 2×2 matrices with entries in $\hat{\mathcal{O}}$ and non-zero determinant) be such that all its local factors $g_{\mathfrak{q}}$ at almost all the finite places $\mathfrak{q}$ including those dividing $\mathfrak{pn}$ are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and otherwise $g_{\mathfrak{q}}$ are of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \pi_{\mathfrak{q}}^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\pi_{\mathfrak{q}}$ a uniformizer of $\mathcal{O}_{\mathfrak{q}}$.

Remark 2.1. — Often one takes $g \in Mat_2(\hat{\mathcal{O}})$ with the component at only one finite place $\mathfrak{q}$ away from $\mathfrak{pn}$ $g_{\mathfrak{q}}$ being of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and all the remaining components are the identity matrices.

We introduce $K'_{g^{-1}} = K \cap g^{-1}Kg$ and $K'_g = gKg^{-1} \cap K$. The group isomorphism

$$K'_{g^{-1}} \cong K'_g;\ \lambda \mapsto g\lambda g^{-1}$$

induces the isomorphism $g^* : Y_{K'_{g^{-1}}} \cong Y_{K'_g}; y \mapsto gy$, that allows us to form the *Hecke correspondence* diagram

$$\begin{array}{ccc} Y_{K'_{g^{-1}}} & \xrightarrow{g^*} & Y_{K'_g} \\ \downarrow{\scriptstyle s_g} & & \downarrow{\scriptstyle \tilde{s}_g} \\ Y_K & & Y_K, \end{array}$$

where s_g and $\tilde{s}_g$ are the natural projections.
The Hecke operator T_g acting on the $\overline{\mathbb{F}}_p$-vector spaces $\mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p})$ is defined by the following diagram:

$$\begin{array}{ccc} \mathrm{H}^i(Y_{K'_{g^{-1}}}, s_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) & \xrightarrow{conj_g^*} & \mathrm{H}^i(Y_{K'_g}, \tilde{s}_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) \\ \uparrow{\scriptstyle res} & & \downarrow{\scriptstyle cor} \\ \mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}) & & \mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}). \end{array}$$

So we have $T_g = cor \circ conj_g^* \circ res$. Here $conj_g^*$ is the isomorphism induced by the conjugation map, cor, res are the corestriction and the restriction maps. It is also known that T_g is independent of the choice of the uniformizers $\pi_{\mathfrak{q}}$ but in fact depends only on the double coset KgK.

2.2. Explicit formulas for the Hecke action. — Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}$. For our purposes, we choose the following representatives of the class group Cl of F. By the Chebotarev density theorem, we can choose representatives of the class group $[\mathfrak{b}_1] = [\mathcal{O}], [\mathfrak{b}_2], \cdots, [\mathfrak{b}_h]$, where for $i > 1$, the $\mathfrak{b}_i$ are prime ideals coprime with $\mathfrak{pn}$. Thus we denote the class group as $Cl = \{[\mathfrak{b}_1], \cdots, [\mathfrak{b}_h]\}$. Let $\pi_{\mathfrak{b}_i}$ be a uniformizer of the local ring $\mathcal{O}_{\mathfrak{b}_i}$. We define $t_1 := (1, \cdots, 1, 1, 1, \cdots, 1, \cdots)$, and for $i > 1$, $t_i := (1, \cdots, 1, \pi_{\mathfrak{b}_i}, 1, \cdots, 1, \cdots) \in \mathbb{A}_f^*$, i.e, t_i is the idele having 1 at all places expect at the place $\mathfrak{b}_i$ where we have $\pi_{\mathfrak{b}_i}$. Via the group homomorphism

$$\begin{array}{ccc} \mathbb{A}_f^* & \to & Cl \\ (\cdots x_{\mathfrak{q}} \cdots) & \mapsto & [\prod_{\mathfrak{q} \neq \infty} \mathfrak{q}^{v_{\mathfrak{q}}(x_{\mathfrak{q}})}], \end{array}$$

where $v_{\mathfrak{q}}$ is the normalized valuation of $\mathcal{O}_{\mathfrak{q}}$, we see that t_i corresponds to $\mathfrak{b}_i$. We define $g_i := \left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, i.e, $(g_i)_{\mathfrak{q}} = \begin{pmatrix} (t_i)_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix}$. Similarly g_i corresponds to the class $[\mathfrak{b}_i]$ via the determinant map.

From the strong approximation theorem, the topological space Y_K decomposes into the disjoint union of its connected components as:

$$Y_K = \amalg_{i=1}^{h} \Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3,$$

where $\Gamma_{[\mathfrak{b}_i]} := G(F) \cap g_i K g_i^{-1}$. This is an arithmetic subgroup of $G(F)$. We next recall the definition of neatness for subgroups of $G(\mathbb{A}_f)$.

Let us take K to be neat so that the groups $\Gamma_{[\mathfrak{b}_i]}$ are torsion free. To achieve this, if $K = K_1(\mathfrak{n})$, the open compact subgroup of level $\mathfrak{n}$ defined below, where the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3, then $\Gamma_{[\mathfrak{b}_i]}$ are torsion free. This is Lemma 2.3.1 from [**15**].

From a general comparison theorem it is known that an isomorphism $\mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ holds, see [**7**] for details. Hence we can write

$$\mathrm{H}^r(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)).$$

Let us further specialize the open compact subgroup K. We define the open compact subgroup of level $\mathfrak{n}$

$$K_1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \prod_{\mathfrak{q} \nmid \infty} G(\mathcal{O}_{\mathfrak{q}}) : c, d-1 \in \mathfrak{n}\hat{\mathcal{O}} \right\}.$$

This is an open compact subgroup which surjects on $\hat{\mathcal{O}}^*$ by the determinant map. The corresponding congruence subgroups $G(F) \cap g_i K_1(\mathfrak{n}) g_i^{-1}$ are denoted as $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. As already alluded to, the Hecke operators T_g do not act componentwise on the $\overline{\mathbb{F}}_p$-vector space $\oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. By this we mean that in general T_g permutes the components when acting on an element from $\oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ as we will soon see.

2.2.1. Prelude to the formulas. — Let $\mathfrak{q}$ be an integral ideal away from $\mathfrak{pn}$. We consider the following subset of $Mat_2(\hat{\mathcal{O}})$. Define

$$\Delta_1^{\mathfrak{q}}(\mathfrak{n}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\hat{\mathcal{O}}) : (ad - bc)\hat{\mathcal{O}} = \mathfrak{q}\hat{\mathcal{O}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\mathfrak{n}} \}.$$

The open compact subgroup $K_1(\mathfrak{n})$ acts on $\Delta_1^{\mathfrak{q}}(\mathfrak{n})$ via multiplication: for $g \in K_1(\mathfrak{n})$ and $\delta \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$ we have $g\delta \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$. We have that $\Delta_1^{\mathfrak{q}}(\mathfrak{n}) K_1(\mathfrak{n}) = K_1(\mathfrak{n}) \Delta_1^{\mathfrak{q}}(\mathfrak{n}) = \Delta_1^{\mathfrak{q}}(\mathfrak{n})$. For $\delta \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$ we define the subgroup

$$K'_{1,\delta}(\mathfrak{n}) = \delta K_1(\mathfrak{n}) \delta^{-1} \cap K_1(\mathfrak{n})$$

of $K_1(\mathfrak{n})$. The subsets $\Delta_1^{\mathfrak{q}}(\mathfrak{n})$ act on any left $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{p^2})]$-module via reduction modulo $\mathfrak{p}$. There is the following fact that is worth mentioning.

Lemma 2.2. — *Let $\delta \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$. Then there is a bijection between the coset space $K_1(\mathfrak{n})/K'_{1,\delta}(\mathfrak{n})$ and the orbit space $K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n})$ given as*

$$\begin{array}{rcl} K_1(\mathfrak{n})/K'_{1,\delta}(\mathfrak{n}) & \to & K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n}) \\ \lambda K'_{1,\delta}(\mathfrak{n}) & \mapsto & \lambda\delta K_1(\mathfrak{n}). \end{array}$$

Proof. — There is a surjective map $K_1(\mathfrak{n}) \to K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n})$ which sends $\lambda K'_{1,\delta}(\mathfrak{n})$ to $\lambda\delta K_1(\mathfrak{n})$. Two distinct elements λ and λ' map to the same orbit if and only of they lie in the same class modulo $K'_{1,\delta}(\mathfrak{n})$. □

For $\delta \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$, there are finitely many $\gamma_j \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$ such that the double coset $K_1(\mathfrak{n})\delta K_1(\mathfrak{n})$ decomposes as

$$K_1(\mathfrak{n})\delta K_1(\mathfrak{n}) = \amalg_j \gamma_j K_1(\mathfrak{n}).$$

Let $g \in Mat_2(\hat{\mathcal{O}})$ be such that its components at a finite number of finite places $\mathfrak{q}$ away from $\mathfrak{p}\mathfrak{n}$ are of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \pi_{\mathfrak{q}}^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\pi_{\mathfrak{q}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{q}}$ and are the identity otherwise. When we denote $\mathfrak{c} = (det(g))$ the ideal corresponding to g, then $g \in \Delta_1^{\mathfrak{c}}(\mathfrak{n})$.

Lemma 2.3. — *Let $g \in \Delta_1^{\mathfrak{c}}(\mathfrak{n})$ as above. Let g_i corresponding to $[\mathfrak{b}_i]$ and $K_1(\mathfrak{n})$ as above. Then, for each i there exist a unique index j_i, $1 \leq j_i \leq h$, matrices $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in g_i K_1(\mathfrak{n}) g_i^{-1}$ and $\beta_i := g_{j_i} g g_i^{-1} k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$ such that $K_1(\mathfrak{n}) g K_1(\mathfrak{n}) = K_1(\mathfrak{n}) g_{j_i}^{-1} \beta_i g_i K_1(\mathfrak{n})$.*

Proof. — For each i let j_i be the unique index such that the ideal $(det(g_{j_i} g g_i^{-1}))$ is principal. Set then $\alpha_i := g_{j_i} g g_i^{-1} = \left(\begin{smallmatrix} det(\alpha_i) & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The ideal $(det(\alpha_i))$ being principal means that $det(\alpha_i) = x_i y_i$ with $y_i \in F^*$ and $x_i \in \hat{\mathcal{O}}^*$. Set $u_i = x_i^{-1}$ and define $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in K_1(\mathfrak{n})$. Then $k_i \in g_i K_1(\mathfrak{n}) g_i^{-1}$ and $\beta_i := \alpha_i k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$. Hence for each i there exists a matrix

$$\beta_i \in g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) K_1(\mathfrak{n}) g_i^{-1} \cap G(F) = g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) g_i^{-1} \cap G(F)$$

such that $K_1(\mathfrak{n}) g K_1(\mathfrak{n}) = K_1(\mathfrak{n}) g_{j_i}^{-1} \beta_i g_i K_1(\mathfrak{n})$. Indeed, $g_{j_i}^{-1} \beta_i g_i = g_{j_i}^{-1} \alpha_i k_i g_i = g g_i^{-1} k_i g_i$, and we observe that we have $g_i^{-1} k_i g_i \in K_1(\mathfrak{n})$. □

For $1 \leq i \leq h$, let j_i and β_i as given in the above lemma. Let $\mathfrak{f}_i := (det(\beta_i)) = \mathfrak{b}_{j_i} \mathfrak{b}_i^{-1} \mathfrak{c}$. Define $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{f}_i}(\mathfrak{n}) := g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) g_i^{-1} \cap G(F)$. Explicitly this is the set

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(F) : a \in \mathfrak{b}_{j_i} \mathfrak{b}_i^{-1}, b \in \mathfrak{b}_{j_i}, c \in \mathfrak{b}_i^{-1}, d - 1 \in \mathfrak{n}\mathcal{O}; (ad - bc)\mathcal{O} = \mathfrak{f}_i \right\}.$$

We set $j := j_i$. Let $\alpha \in \Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{f}_i}(\mathfrak{n})$ (we have in mind β_i). We consider the following double coset $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. This double coset defines a Hecke operator T_α mapping

$$\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \text{ to } \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$$

as follows. Firstly one needs to introduce the following subgroups

1. $\Gamma_{1,[\mathfrak{b}_i]}^{\prime,\alpha^{-1}}(\mathfrak{n}) := \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \alpha^{-1} \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \alpha$
2. $\Gamma_{1,[\mathfrak{b}_j]}^{\prime\prime,\alpha}(\mathfrak{n}) := \alpha \Gamma'_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \alpha^{-1} = \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \alpha^{-1} \cap \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$.

The operator T_α is defined as the composition of the following maps:

$$\begin{tikzcd}
\mathrm{H}^r(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[r, "conj_\alpha"] & \mathrm{H}^r(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[d, "cor"] \\
\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[u, "res"] & \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)).
\end{tikzcd}$$

Here res is the restriction map, $conj_\alpha$ is the isomorphism induced by the compatible maps:

$$\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \cong \Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})$$
$$\omega \mapsto \alpha^{-1}\omega\alpha$$

and

$$V^{a,b}_{r,s}(\overline{\mathbb{F}}_p) \to V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)$$
$$v \mapsto \alpha.v.$$

Here cor is the corestriction homomorphism. We explicitly describe T_α in degree zero and one. In degree zero T_α is given as

$$\begin{tikzcd}
\mathrm{H}^0(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[r, "v\mapsto\alpha v"] & \mathrm{H}^0(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[d, "v\mapsto\sum_h hv"] \\
\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \arrow[u, "v\mapsto v"] & \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)).
\end{tikzcd}$$

where the sum is over a set of cosets representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. Hence one obtains that:

$$\begin{aligned} T_\alpha : \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) &\to \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ v &\mapsto \sum_{\lambda\in\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})} (\lambda\alpha).v. \end{aligned}$$

It is worthwhile observing that the decomposition $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_r \lambda_r \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$ is equivalent to the decomposition of the double cosets $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_r \lambda_r\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$.

2.2.2. Formula on degree one. — We now give the formula of T_α on degree one cohomology. To this end, we need to recall the formulas describing the isomorphism $conj_\alpha$ and the corestriction in terms of non-homogeneous cocycles. The conjugation isomorphism is described by the formula

$$\begin{aligned} conj_\alpha : \mathrm{H}^1(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) &\to \mathrm{H}^1(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c &\mapsto (\omega \mapsto \alpha.c(\alpha^{-1}\omega\alpha)). \end{aligned}$$

For the corestriction homomorphism, let $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_n \gamma_n \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. For $\omega \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$, let s_n be the unique index such that $\gamma_n^{-1}\omega\gamma_{s_n} \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. Then the corestriction homomorphism is given as

$$\begin{array}{rcl} cor : \mathrm{H}^1(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & \to & \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c & \mapsto & (\omega \mapsto \sum_n \gamma_n.c(\gamma_n^{-1}\omega\gamma_{s_n})). \end{array}$$

The formula of T_α on degree one cohomology is thus

$$\begin{array}{rcl} T_\alpha : \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & \to & \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c & \mapsto & (\omega \mapsto \sum_n \gamma_n\alpha.c((\gamma_n\alpha)^{-1}\omega\gamma_{s_n}\alpha)). \end{array}$$

Indeed with the given formulas we have

$$\begin{aligned} (cor(conj_\alpha(c)))(w) &= \sum_{\gamma_n \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma'_{1,[\mathfrak{b}_j]}(\mathfrak{n})} \gamma_n.(conj_\alpha(c)(\gamma_n^{-1}w\gamma_{s_n})) \\ &= \sum_{\gamma_n \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma'_{1,[\mathfrak{b}_j]}(\mathfrak{n})} \gamma_n\alpha.c(\alpha^{-1}\gamma_n^{-1}w\gamma_{s_n}\alpha). \end{aligned}$$

Let λ_i be another set of representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$, and $\sigma_i \in \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}$ such that $\lambda_i = \gamma_i\sigma_i$. With this we have

$$(cor(c))(w) = \gamma_i\sigma.c(\sigma_i^{-1}\gamma_i^{-1}w\gamma_{j_i}\sigma_i).$$

Because taking conjugation by an element from $\Gamma'^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$ gives cohomologous cocycle, we deduce that the corestriction map does not depend on the choice of representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. This means that T_α does not depend on the choice of set of representatives and so only depends on the double coset $\Gamma_{1,[\mathfrak{b}_j]}\alpha\Gamma_{1,[\mathfrak{b}_i]}$ since we know that

$$\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_n \gamma_n \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]} \Longleftrightarrow \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_n \gamma_n\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

2.2.3. Action of T_g on $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$. — Now that we have recalled the formulas of the Hecke operators on group cohomology, let us say how Hecke operators act on the $\overline{\mathbb{F}}_p$-vector spaces $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$. Let g be as in Lemma 2.3 and consider β_i and j_i provided by the lemma loc. cit. Let T_{β_i} the Hecke operator corresponding to the double coset $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Then T_{β_i} sends an element from $\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$ to $\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$. It was proved by Shimura, see [**14**], that for $(c_1, \cdots, c_h) \in \oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$, the Hecke action of T_g is

$$T_g.(c_1, \cdots, c_h) = (d_1, \cdots .d_h),$$

where $d_{j_i} = T_{\beta_i}.c_i$.

Remark 2.4. — In the idyllic situation where the ideal $(det(g))$ is principal, then, the Hecke operator T_g does not permute the summands in $\oplus_{l=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_l]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p))$. Indeed $(det(g_{j_i}gg_i^{-1})) = (det(g))$, so $j_i = i$ in Lemma 2.3. Therefore $T_g.(c_1, \cdots, c_h) = (d_1, \cdots, d_h)$ where $d_i = T_{\beta_i}.c_i$.

Remark 2.5. — Let g be as in Lemma 2.3. Let us denote the ideal $(det(g))$ as $\mathfrak{c}$. Then T_g maps the $\overline{\mathbb{F}}_p$-vector spaces $\oplus_{l=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_l]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ to $\oplus_{l=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{c}^{-1}\mathfrak{b}_l]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. To see this, one needs to just recall that T_g maps

$$\oplus_{i=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \text{ to } \oplus_{i=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$$

where j_i is such that $(det(g_{j_i} g g_i^{-1}))$ is principal. In terms of ideals this means that $[\mathfrak{c}^{-1}\mathfrak{b}_i] = [\mathfrak{b}_{j_i}]$.

Remark 2.6. — (Diamond action) There is an action of Diamond operators inducing an action of the group $(\mathcal{O}/\mathfrak{n})^*$ on $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. Let $\chi : (\mathcal{O}/\mathfrak{n})^* \to \overline{\mathbb{F}}_p^*$ be a character. As a representation of the abelian group $(\mathcal{O}/\mathfrak{n})^*$, then when $p \nmid \sharp(\mathcal{O}/\mathfrak{n})^*$, the space $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ decomposes as a direct sum of χ-eigenspaces. So by denoting the spaces $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ and a χ-eigenspace as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}, \chi)$, then we have

$$\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}) = \oplus_\chi \mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}, \chi).$$

Let us turn next to the definition of the Hecke algebra.

2.2.4. Hecke algebra. — We start by defining first what we call mod p cohomological modular forms over F. Recall that we have denoted $p\mathcal{O}$ as $\mathfrak{p}$ and we are assuming that p is inert in F. The residue field is then $\mathbb{F}_{p^2}$. The congruence subgroups $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ act on $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ via reduction modulo p.

Definition 2.7. — *A cohomological mod p modular form of weight $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ and level $\mathfrak{n}$ over F is a class in*

$$\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)).$$

We have denoted this $\overline{\mathbb{F}}_p$-vector spaces as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$.

We next define the Hecke algebra of interest for our purposes.

***Definition 2.8* (Hecke algebra)**. —
1. *The abstract Hecke algebra $\mathcal{H}$ is the polynomial algebra $\mathbb{Z}[T_\mathfrak{q}, S_\mathfrak{q} | \ \mathfrak{q} \nmid \mathfrak{pn}$ maximal ideal $\subset \mathcal{O}]$.*
2. *The Hecke algebra $\mathcal{H}(\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}))$ acting on $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ is the homomorphic image of: $\mathcal{H} \to End_{\overline{\mathbb{F}}_p}(\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})); T_\mathfrak{q}, S_\mathfrak{q} \mapsto T_\mathfrak{q}, S_\mathfrak{q}$.*

As we said an eigenform for all the Hecke operator $T_\mathfrak{q}$ for $\mathfrak{q}$ away from $\mathfrak{pn}$ gives rise to a system of Hecke eigenvalues. Here is a formal definition of a system of Hecke eigenvalues with values in $\overline{\mathbb{F}}_p$.

Definition 2.9. — *A system of Hecke eigenvalues with values in $\overline{\mathbb{F}}_p$ is a ring homomorphism $\psi : \mathcal{H} \to \overline{\mathbb{F}}_p$. We say that it occurs in $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ if there is a non-zero $f \in \mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ such that $Tf = \psi(T)f$ for all $T \in \mathcal{H}$.*

In the next section we shall relate the induced modules $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ to some irreducible $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathcal{O})]$-modules of the form $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$.

3. The relevant induced modules

We recall that by assumption we have fixed a rational inert prime p and $\mathfrak{p} = p\mathcal{O}$ does not divide an integral ideal $\mathfrak{n}$ which was also fixed. Here we will be concerned with the induced modules $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. We shall derive a more explicit decomposition of the latter. Let $\tilde{G} = \mathrm{GL}_2(\mathbb{F}_{p^2})$ and $\tilde{S} = \mathrm{SL}_2(\mathbb{F}_{p^2})$.
Define the following congruence subgroups of $\mathrm{GL}_2(F)$:

$$\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) := g_i K_1(\mathfrak{n}) g_i^{-1} \cap \mathrm{SL}_2(F).$$

Because $\left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\begin{pmatrix} t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & t_i b \\ t_i^{-1} c & d \end{pmatrix}$, one obtains that

$$\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F) : a-1, d-1 \in \mathfrak{n}; b \in \mathfrak{b}_i, c \in \mathfrak{b}_i^{-1}\mathfrak{n} \right\}.$$

In particular with our assumptions one has that

$$\Gamma^1_{1,[\mathfrak{b}_1]}(\mathfrak{n}) = \Gamma_1(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : a-1,\ d-1,\ c \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

Furthermore, let

$$\Gamma(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : a-1,\ d-1,\ b,\ c \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

Lemma 3.1. — *Let $\tilde{S} = \mathrm{SL}_2(\mathbb{F}_{p^2})$. Then, we have an exact sequence*

$$1 \to \Gamma(\mathfrak{p}) \cap \Gamma_1(\mathfrak{n}) \to \Gamma_1(\mathfrak{n}) \to \tilde{S} \to 1$$

where the third arrow is reduction modulo p.

Proof. — It is clear that $\Gamma(\mathfrak{p}) \cap \Gamma_1(\mathfrak{n})$ is the kernel of the reduction modulo $\mathfrak{p}$ of $\Gamma_1(\mathfrak{n})$. So, we are left to see the surjectivity of the third arrow. To this end let $a, b, c, d \in \mathcal{O}$ with $ad - bc \equiv 1 \pmod{\mathfrak{p}}$. We need to find $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ such that $\alpha\delta - \beta\gamma = 1$ with the congruences:

$$\begin{aligned} \alpha &\equiv a \pmod{\mathfrak{p}} \\ \alpha &\equiv 1 \pmod{\mathfrak{n}} \\ \beta &\equiv b \pmod{\mathfrak{p}} \\ \gamma &\equiv c \pmod{\mathfrak{p}} \\ \gamma &\equiv 0 \pmod{\mathfrak{n}} \\ \delta &\equiv d \pmod{\mathfrak{p}} \\ \delta &\equiv 1 \pmod{\mathfrak{n}}. \end{aligned}$$

It is readily seen that if $0 \neq c \in \mathfrak{n}$ and is coprime with $\mathfrak{p}$ then the Chinese Remainder Theorem permits to conclude. Indeed, set $\gamma = c$, there exist $\alpha, \delta \in \mathcal{O}$ with $\alpha \equiv a \pmod{\mathfrak{p}}$, $\alpha \equiv 1 \pmod{\gamma}$, $\delta \equiv d \pmod{\mathfrak{p}}$, $\delta \equiv 1 \pmod{\gamma}$. This gives $\alpha\delta \equiv 1 \pmod{\gamma}$, and so there exists $\beta \in \mathcal{O}$ such that $\alpha\delta - \beta\gamma = 1$ and $\beta \equiv b \pmod{\mathfrak{p}}$. So we need to see that we can always reduce to this case. To this end as $\mathfrak{n}$ is coprime with $\mathfrak{p}$, we can find $n \in \mathfrak{n}$, $k \in \mathfrak{p}$, and $r, s \in \mathcal{O}$ such that $nr - ks = 1$. The image of the matrix $\left(\begin{smallmatrix} r & 0 \\ n & n \end{smallmatrix}\right)$ belongs to $\tilde{S}$ and can be lifted by the previous arguments. Then $\left(\begin{smallmatrix} r & 0 \\ n & n \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} ra & rb \\ n(c+a) & n(d+b) \end{pmatrix}$ is a matrix in $\tilde{S}$ whose bottom line has entries in $\mathfrak{n}$. Then if $n(c+a) \neq 0$ we are done, otherwise we just have to multiply from the right by $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ for the condition to hold. $\square$

Corollary 3.2. — *For $i \geq 1$, the congruence subgroup $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ surjects onto $\tilde{S}$ via reduction modulo p.*

Proof. — From Lemma 3.1, we have that $\Gamma_1(\mathfrak{n})$ surjects onto $\tilde{S}$. So let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{S}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1(\mathfrak{n})$ be a lift of M. For each $i > 1$ take $\lambda_i \in \mathfrak{b}_i$ such that $\lambda_i \equiv 1 \pmod{\mathfrak{p}}$ (this is possible since $\mathfrak{b}_i$ is coprime with $\mathfrak{p}$). Then the matrix $\left(\begin{smallmatrix} \alpha & \lambda_i\beta \\ \lambda_i^{-1}\gamma & \delta \end{smallmatrix}\right)$ belongs to $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and its reduction is M. $\square$

From the fact that $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \subset \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$, we deduce that the reduction modulo p of $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ contains $\tilde{S}$. Now suppose we are given two subgroups H_1, H_2 of $\tilde{G} = \mathrm{GL}_2(\mathbb{F}_{p^2})$ containing $\tilde{S}$ and such that their images by the determinant map are the same: $det(H_1) = det(H_2) < \mathbb{F}^*_{p^2}$. The fact $det(H_1 \cap H_2) = det(H_1) \cap det(H_2)$ implies the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{S} & \longrightarrow & H_1 \cap H_2 & \xrightarrow{det} & det(H_1) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \tilde{S} & \longrightarrow & H_1 & \xrightarrow{det} & det(H_1) & \longrightarrow & 1. \end{array}$$

Therefore one has $H_1 = H_2$, and we have established that any subgroup H of $\tilde{G}$ containing $\tilde{S}$ is uniquely determined by the image of the determinant map $H \xrightarrow{det} \mathbb{F}^*_{p^2}$. From this fact we derive that $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ reduces to

$$\mathcal{T}_1(\mathfrak{n}) := \left\{ g \in \tilde{G} : det(g) \in Im(\mathcal{O}^* \xrightarrow{reduction} \mathbb{F}^*_{p^2}) \right\}.$$

We also derive that $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})$ reduces to

$$\mathcal{T}_1(\mathfrak{p}\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G} : a \in Im(\mathcal{O}^* \xrightarrow{reduction} \mathbb{F}^*_{p^2}) \right\}.$$

In summary, reduction mod p gives the following bijection:

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}) \backslash \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \to \mathcal{T}_1(\mathfrak{p}\mathfrak{n}) \backslash \mathcal{T}_1(\mathfrak{n}).$$

Define $\tilde{U} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G} \right\}$. Next we have the following bijection

$$\begin{array}{ccc} \mathcal{T}_1(\mathfrak{p}\mathfrak{n}) \backslash \mathcal{T}_1(\mathfrak{n}) & \longleftrightarrow & \tilde{U} \backslash \tilde{G} \\ \mathcal{T}_1(\mathfrak{p}\mathfrak{n}) g & \mapsto & \tilde{U} g. \end{array}$$

Indeed the map is surjective and two elements from $\mathcal{T}_1(\mathfrak{n})$ are sent to the same class modulo $\tilde{U}$ if and only they belong to the same class modulo $\mathcal{T}_1(\mathfrak{p}\mathfrak{n})$ because we have $\tilde{U} \cap \mathcal{T}_1(\mathfrak{n}) = \mathcal{T}_1(\mathfrak{p}\mathfrak{n})$. Composing these two bijections, we obtain the bijection

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}) \backslash \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow \tilde{U} \backslash \tilde{G}.$$

3.1. Induced modules. — Let H be a group and $J < H$ a subgroup of finite index. For a left J-module M the induced module, and a twisted induced module are defined as follows.

Definition 3.3. — *1.* $\mathrm{Ind}_J^H(M) = \{f : H \to M : f(gh) = gf(h)\ \forall\ g \in J, h \in H\}$.

2. *Given a character $\chi : J \to \overline{\mathbb{F}}_p^*$, we define a twisted induced module as*

$$\mathrm{Ind}_J^H(\overline{\mathbb{F}}_p^{\chi}) = \{f : H \to \overline{\mathbb{F}}_p : f(gh) = \chi(g)f(h) \ \forall\ g \in J, h \in H\}.$$

Recall how a left action of H on $\mathrm{Ind}_J^H(M)$ can be defined: for $g \in H$ and $f \in \mathrm{Ind}_J^H(M)$ we have $(g.f)(h) := f(hg)$.
Let $\tilde{B} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \in \tilde{G} \right\}$ be the Borel subgroup of $\tilde{G}$ and define the character χ of $\tilde{B}$ by

$$\begin{array}{rcl} \chi : \tilde{B} & \to & \mathbb{F}_{p^2}^* \\ \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) & \mapsto & e. \end{array}$$

For an integer d, we also set $\chi^d(.) = (\chi(.))^d$. The homomorphism χ induces a group isomorphism

$$\tilde{U}\backslash\tilde{B} \cong \mathbb{F}_{p^2}^*.$$

From this isomorphism we obtain the following isomorphism of $\tilde{B}$-modules

$$\mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p).$$

The isomorphism is defined as follows:

$$\begin{array}{rcl} \Phi : \mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p) & \to & \mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p) \\ f & \mapsto & \left(\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \mapsto f(e) \right). \end{array}$$

The representation $\mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p)$ is the regular representation of $\mathbb{F}_{p^2}^*$. This is a (p^2-1)-dimensional representation of an abelian group of order prime to p and hence it admits a decomposition into a direct sum of one-dimensional representations of $\mathbb{F}_{p^2}^*$. By a slight abuse of notation, the summands are the $\mathbb{F}_{p^2}^*$-modules $\overline{\mathbb{F}}_p^{\chi^d}$, where for $x \in \mathbb{F}_{p^2}^*, y \in \overline{\mathbb{F}}_p$, we have $x.y := x^d y$ with $0 \le d \le p^2 - 2$.

Proposition 3.4. — *For all i, there is the following isomorphism of left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules and left $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules respectively:*

1. $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \oplus_{d=0}^{p^2-2} \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d})$;
2. $\mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \oplus_{d=0}^{p^2-2} \mathrm{Ind}_{\tilde{B}\cap\tilde{S}}^{\tilde{S}}(\overline{\mathbb{F}}_p^{\chi^d})$.

Proof. — Because of the bijection $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})\backslash\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow \tilde{U}\backslash\tilde{G}$ given by reducing modulo p, the transitivity of Ind, and the observation above, we have the following identifications of left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules:

$$\begin{array}{rcl} \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) & \cong & \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) \\ & \cong & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p)) \\ & \cong & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\oplus_{d=0}^{p^2-2}(\overline{\mathbb{F}}_p^{\chi^d})) \\ & \cong & \oplus_{d=0}^{p^2-2}\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}). \end{array}$$

For the second item, one uses the bijection $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})\backslash\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow (\tilde{U}\cap\tilde{S})\backslash\tilde{S}$. $\square$

We shall need a more explicit version of $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d})$. For $0\le d\le p^2-2$, we define the following $\tilde{G}$-module which we denote by $\mathrm{U}_d(\overline{\mathbb{F}}_p)$:

$$\mathrm{U}_d(\overline{\mathbb{F}}_p) = \{f:\mathbb{F}_{p^2}^2 \to \overline{\mathbb{F}}_p : f(xa,xb)=x^d f(a,b)\,\forall\, x\in\mathbb{F}_{p^2}^*\}.$$

Next we define the following homomorphism

$$\begin{array}{rcl}\varphi:\mathrm{U}_d(\overline{\mathbb{F}}_p) & \to & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}) \\ F & \mapsto & ((\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})\mapsto F(c,e)).\end{array}$$

We shall show that it is an isomorphism of $\tilde{G}$-modules. It is well defined since

$$\varphi(F)((\begin{smallmatrix} x & y\\ 0 & z\end{smallmatrix})(\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})) = F(zc,ze) = z^dF(c,e)=\chi^d((\begin{smallmatrix} x & y\\ 0 & z\end{smallmatrix}))\varphi(F)((\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})).$$

It is also easy to see that φ is an $\tilde{G}$-homomorphism. In order to conclude that φ is an isomorphism, one can define the inverse ψ of φ as follows. We first note that for $c,\ e\in\mathbb{F}_{p^2}$ not both zero we can find $a,b\in\mathbb{F}_{p^2}$ such that $ae-bc\neq 0$. Hence an element $(c,e)\neq(0,0)$ gives rise to a matrix $(\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})$ in $\tilde{G}$. Another choice of a',b' with $a'e-b'c\neq 0$ amounts to multiply $(\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})$ from the left by a matrix of the form $(\begin{smallmatrix} 1 & *\\ 0 & 1\end{smallmatrix})$ which acts trivially on $\overline{\mathbb{F}}_p^{\chi^d}$. This implies that the map

$$\begin{array}{rcl}\psi:\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}) & \to & \mathrm{U}_d(\overline{\mathbb{F}}_p) \\ f & \mapsto & ((c,e)\mapsto f((\begin{smallmatrix} a & b\\ c & e\end{smallmatrix})),\end{array}$$

is well defined, that is, to mean that any choice of $a,b\in\mathbb{F}_{p^2}$ with $ae-bc\neq 0$ will do. Furthermore it is easy to verify that it is an $\tilde{G}$-homomorphism and it is the inverse of φ.
In the next remark, there is another proof of the isomorphism of $\tilde{G}$-modules : $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d})\cong \mathrm{U}_d(\overline{\mathbb{F}}_p)$.

Remark 3.5. — We start with the identification of $\overline{\mathbb{F}}_p$-vector spaces

$$\overline{\mathbb{F}}_p[X,Y]/(X^{p^2}-X,Y^{p^2}-Y)\cong\{f:\mathbb{F}_{p^2}^2\to\overline{\mathbb{F}}_p\}$$

where $P(X,Y)$ maps to the function $(a,b)\mapsto P(a,b)$. To see this we observe that the spaces on both sides have dimensions p^4 as $\overline{\mathbb{F}}_p$-vector spaces. So, we just have to prove injectivity. To this end for any $x\in\mathbb{F}_{p^2}$ if the polynomial $f_x(Y)=P(x,Y)\in\overline{\mathbb{F}}_p[Y]$ vanishes for all $y\in\mathbb{F}_{p^2}$, then this means that Y and $Y^{p^2-1}-1$ divide $f_x(Y)$ for all $x\in\mathbb{F}_{p^2}$. Because x is arbitrarily chosen we deduce that $Y^{p^2}-Y$ divides $P(X,Y)$. As the role of X and Y are symmetric, one obtains that $P(X,Y)$ lies in the ideal $(X^{p^2}-X,Y^{p^2}-Y)$. In fact this is an isomorphism of $\overline{\mathbb{F}}_p[\tilde{G}]$-modules. Let $\mathcal{W}(p,\overline{\mathbb{F}}_p) := \{f\in\overline{\mathbb{F}}_p[X,Y]/(X^{p^2}-X,Y^{p^2}-Y) : f((0,0))=0\}$. This module can be identified with $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ as $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module, see [**16**] for details. Then $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ is the subspace of homogeneous polynomial classes of degree d in $\overline{\mathbb{F}}_p[X,Y]/(X^{p^2}-X,Y^{p^2}-Y)$ with $f((0,0))=0$. And as a graded $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module $\mathcal{W}(p,\overline{\mathbb{F}}_p)$ decomposes as follows:

$$\mathcal{W}(p,\overline{\mathbb{F}}_p)=\oplus_{d=0}^{p^2-2}\mathrm{U}_d(\overline{\mathbb{F}}_p).$$

The isomorphism in Proposition 3.4 will permit us to obtain a better understanding of the non-semisimple $\tilde{G}$-module $\mathrm{U}_d(\overline{\mathbb{F}}_p)$. We shall turn to this among other things.

4. Irreducible $\tilde{G}$-modules

We keep the same notation as in the previous sections. We will prove here the main results. For an irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-module W, this is done by embedding a cohomology group with coefficients in W into another cohomology group with trivial coefficients roughly speaking. First of all, we shall see how the irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules can be embedded in a twist of $\mathrm{U}_d(\overline{\mathbb{F}}_p)$. Let τ be the non-trivial automorphism of $\mathbb{F}_{p^2}$. For $0 \leq r, s \leq p-1, 0 \leq l, t, \leq p-1$, recall that when l and t are not both equal to $p-1$, the representations

$$V_{r,s}^{l,t}(\overline{\mathbb{F}}_p) := \mathrm{Sym}^r(\overline{\mathbb{F}}_p^2) \otimes_{\mathbb{F}_{p^2}} det^l \otimes_{\mathbb{F}_{p^2}} \mathrm{Sym}^s(\overline{\mathbb{F}}_p^2)^\tau \otimes_{\mathbb{F}_{p^2}} (det^t)^\tau,$$

exhaust all the irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules. Here, we identify $\mathrm{Sym}^r(\overline{\mathbb{F}}_p^2)$ with the homogeneous polynomials in the variables X, Y over $\overline{\mathbb{F}}_p$ of degree r which we denote by $\overline{\mathbb{F}}_p[X,Y]_r$. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{G}$ acts from the left on $V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)$ as follows: on the first factor $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).X^iY^j := (aX+bY)^i(cX+dY)^j$ followed by multiplication by $(ad-bc)^l$ and on the second factor we apply first τ on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and proceed as for the first factor followed by multiplication by $(ad-bc)^{pt}$, e.g,

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).X^iY^j \otimes X^{i'}Y^{j'} := (ad-bc)^{l+pt}(aX+bY)^i(cX+dY)^j \otimes (a^pX+b^pY)^{i'}(c^pX+d^pY)^{j'}.$$

For $e \geq 0$, we write $\mathrm{U}_d^e(\overline{\mathbb{F}}_p)$ to mean the $\overline{\mathbb{F}}_p[\tilde{G}]$-module $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ with the natural action of $\tilde{G}$ followed by multiplication by det^e, e.g, $\mathrm{U}_d^e(\overline{\mathbb{F}}_p) = \mathrm{U}_d(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_{p^2}} det^e$.

Lemma 4.1. — *We have the following embedding of left $\overline{\mathbb{F}}_p[\tilde{G}]$-modules*

$$\begin{array}{rcl} \Psi : V_{r,s}^{q,t}(\overline{\mathbb{F}}_p) & \to & \mathrm{U}_{r+ps}^{q+pt}(\overline{\mathbb{F}}_p) \\ f \otimes g & \mapsto & ((a,b) \mapsto f(a,b)g(a^p,b^p)). \end{array}$$

Proof. — In polynomial terms we can write $\Psi(f(X,Y) \otimes g(X,Y)) = f(X,Y)g(X^p,Y^p)$. By definition of Ψ we have $\Psi(\sum f_i \otimes g_i) = \sum \Psi(f_i \otimes g_i)$. Now let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{G}$, we need to check that $\Psi(M.f \otimes M^\tau.g) = M.\Psi(f \otimes g)$. For $f(X,Y) = \sum_{n+m=r} a_{n,m}X^nY^m$, $g(X,Y) = \sum_{l+k=s} b_{l,k}X^lY^k$, then $M.f = (ad-bc)^q \sum_{n+m=r} a_{n,m}(aX+bY)^n(cX+dY)^m$ and $M^\tau.g = (ad-bc)^{pt}\sum_{l+k=s} b_{l,k}(a^pX+b^pY)^l(c^pX+d^pY)^k$. We set $\alpha = (ad-bc)^{q+pt}$. Hence by denoting $\Psi(M.(f \otimes g))$ as $(*)$, we have

$$\begin{aligned} (*) &= \alpha \sum_{n+m=r} \sum_{l+k=s} a_{n,m}(aX+bY)^n(cX+dY)^m b_{l,k}(a^pX^p+b^pY^p)^l(c^pX^p+d^pY^p)^k \\ &= \alpha \sum_{n+m=r} \sum_{l+k=s} a_{n,m}(aX+bY)^n(cX+dY)^m b_{l,k}(aX+bY)^{pl}(cX+dY)^{pk} \\ &= M.\Psi(f \otimes g). \end{aligned}$$

□

One would then like to have a more concrete description of the cokernel of Ψ. In other words, one has to compute the Jordan-Hölder series of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$.

Remark 4.2. — In the special case $s=0$, the semi-simplification of $\mathrm{U}^e_r(k)$ for k a finite field can be obtained by immediate generalization of the case $k=\mathbb{F}_p$ which is treated for instance in [**16**]. But as it seems that this method does not apply when $s>0$, we will naturally follow the Brauer character theory approach which gives the semisimplification of $\mathrm{U}^e_d(k)$ in complete generality.

For our purpose we shall next see that

$$(\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p))^{ss} = V^{0,0}_{r,s}(\overline{\mathbb{F}}_p) \oplus V^{r,s}_{p-r-1,p-1-s}(\overline{\mathbb{F}}_p) \oplus V^{0,s+1}_{r-1,p-2-s}(\overline{\mathbb{F}}_p) \oplus V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p).$$

From this we deduce the semisimplification of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$ by twisting.

4.1. The constituents of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$. — Let k be a finite field and $\mathfrak{G}=\mathrm{GL}_2(k)$, and $\mathfrak{B}$ its Borel subgroup of upper triangular matrices. For a character ϕ of $\mathfrak{B}$ with values in $\overline{\mathbb{F}}_p$, we consider $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(\overline{\mathbb{F}}_p^{\phi^d})$ where $0 \leq \ d \leq \sharp k-2$. The semisimplication of $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(\overline{\mathbb{F}}_p^{\phi^d})$ is computed in [**8**] via Brauer character theory. Given two homomorphisms $\chi_1,\ \chi_2 : k^* \to \overline{\mathbb{Q}}^*$(or $\overline{\mathbb{Q}}_p^*$), one obtains a character of $\mathfrak{B}$ induced by $\chi_1,\ \chi_2$ as

$$\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(e).$$

Furthermore for $V=\overline{\mathbb{Q}}$ (or $\overline{\mathbb{Q}}_p$) let $\mathrm{I}(\chi_1,\chi_2) := \mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(V^{\chi_1,\chi_2})$ where

$$\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(V^{\chi_1,\chi_2}) = \{f : \mathfrak{G} \to V : f(\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) g) = \chi_1(a)\chi_2(e)f(g)\ \forall \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \in \mathfrak{B},\ g \in \mathfrak{G}\}.$$

This is a $(q+1)$-dimensional representation of $\tilde{G}$ where $q=\sharp k$. It is known as a principal series representation of $\mathfrak{G}$. Next let E be the set of embeddings $k \to \overline{\mathbb{F}}_p$. Then the complete list of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathfrak{G}$ is given by:

$$R_{\overrightarrow{m},\overrightarrow{n}} = \otimes_{\tau\in E}(\mathrm{Sym}^{n_\tau-1}(k^2)))^{\tau} \otimes (det^{m_\tau})^{\tau} \otimes \overline{\mathbb{F}}_p;$$

for integers $0 \leq m_\tau \leq p-1$ and $1 \leq n_\tau \leq p$ associated with each $\tau \in E$, and some n_τ is less than $p-1$. Here one makes the convention $\mathrm{Sym}^{-1}(k^2)=\{0\}$, the null module. Before we go further, note that in our notation we have

$$V^{l,t}_{r,s}(\overline{\mathbb{F}}_p) = R_{(l,t),(r+1,s+1)}$$

as irreducible $\mathfrak{G}$-modules.

To obtain the semisimplification of an $\overline{\mathbb{F}}_p$-representation of $\mathfrak{G}$, the approach is via Brauer character theory. One starts with a $\overline{\mathbb{Q}}_p$-representation W of $\mathfrak{G}$, and reduction modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ yields an $\overline{\mathbb{F}}_p$-representation of $\mathfrak{G}$. More precisely for such a W, we know that there exists a $\overline{\mathbb{Z}}_p$-lattice L inside W invariant under the action of $\mathfrak{G}$. Then reduction of L modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ gives rise to an $\overline{\mathbb{F}}_p$-representation whose Brauer character is the restriction of the character of W to the p-regular classes of $\mathfrak{G}$. In this way the semisimplification thus obtained is independent of the lattice L.

Any group homomorphism $\varphi : k^* \to \overline{\mathbb{Q}}_p^*$ can be written as $\varphi = \prod_\tau \tilde{\tau}^{a_\tau}$ with $0 \leq a_\tau \leq p-1$ and $\tilde{\tau}$ the Teichmüller lift of τ. Then the reduction of φ is $\bar{\varphi} = \prod_\tau \tau^{a_\tau}$. By a twist it suffices to consider the irreducible representation of the form $\mathrm{I}(1,\chi)$. Then it was shown in [**8**] that

***Proposition 4.3* (Diamond).** — *Let $M = \mathrm{I}(1, \prod_\tau \tilde{\tau}^{a_\tau})$ with $0 \leq a_\tau \leq p-1$ for each $\tau \in E$. Let $Frob$ be the Frobenius in E. Then the semisimplification of the reduction $\overline{M}$ of M is $\overline{M} \cong \oplus_{J\subset S} M_J$ with $M_J = R_{\vec{m}_J, \vec{n}_J}$, where*

$$m_{J,\tau} = \begin{cases} 0 & if \;\; \tau \in J \\ a_\tau + \delta_J(\tau) & otherwise; \end{cases} \qquad n_{J,\tau} = \begin{cases} a_\tau + \delta_J(\tau) & if \;\; \tau \in J \\ p - a_\tau - \delta_J(\tau) & otherwise; \end{cases}$$

with δ_J the characteristic function of $J^{(p)} = \{\tau \circ Frob : \tau \in J\}$.

We shall next specialize to our setting. We take $\mathfrak{G} = \tilde{G}$ and $\mathfrak{B} = \tilde{B}$. Let $\overline{\psi} : \mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$ be a fixed injection. We also denote by $\overline{\psi}$ the character obtained by restriction to $\mathbb{F}_{p^2}^*$. Now, let $\psi : \mathbb{F}_{p^2}^* \to \overline{\mathbb{Q}}_p^*$ be the Teichmüller lift of $\overline{\psi}$. The homomorphism ψ induces the character of $\tilde{B}$:

$$\tilde{B} \to \overline{\mathbb{Q}}_p^*; \;\; \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mapsto \psi(z).$$

Via ψ, $\overline{\mathbb{Q}}_p$ is endowed with a structure of $\tilde{B}$-module which we denote $\overline{\mathbb{Q}}_p^{\psi}$: for $h \in \tilde{B}, x \in \overline{\mathbb{Q}}_p$ we have $h.x = \psi(h)x$. The $\overline{\mathbb{Q}}_p$-representation

$$\mathrm{I}(1,\psi) = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{Q}}_p^{\psi}) = \{f : \tilde{G} \to \overline{\mathbb{Q}}_p : f(hg) = \psi(h)f(g) \; \forall h \in \tilde{B}, \, g \in \tilde{G}\}$$

of $\tilde{G}$ has reduction the $\overline{\mathbb{F}}_p$-representation $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}})$ where by abuse of notation $\overline{\psi}$ is the character of $\tilde{B}$ induced by $\overline{\psi}$:

$$\tilde{B} \to \overline{\mathbb{F}}_p^*; \;\; \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mapsto \overline{\psi(z)}.$$

For $0 \leq d \leq p^2 - 2$, we consider the $\overline{\mathbb{F}}_p$-representations of $\tilde{G}$: $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}^d})$. We write $d = r + ps$ with $0 \leq r, s \leq p-1$, $E = \{id, \tau\}$ so that $\overline{\psi}^d = id^r \tau^s$. From Proposition 4.3, we have

$$(\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p))^{ss} = V_{r,s}^{0,0}(\overline{\mathbb{F}}_p) \oplus V_{p-r-1,p-1-s}^{r,s}(\overline{\mathbb{F}}_p) \oplus V_{r-1,p-2-s}^{0,s+1}(\overline{\mathbb{F}}_p) \oplus V_{p-r-2,s-1}^{r+1,0}(\overline{\mathbb{F}}_p);$$

where we have used the identification of $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ with $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}^d})$ as $\overline{\mathbb{F}}_p[\tilde{G}]$-modules from page 57. We define the representation $W_{r,s}^{l,t}$ by the exact sequence

$$0 \to V_{r,s}^{l,t}(\overline{\mathbb{F}}_p) \to \mathrm{U}_{r+ps}^{l+pt}(\overline{\mathbb{F}}_p) \to W_{r,s}^{l,t} \to 0.$$

Thus the semisimplification of $W_{r,s}^{l,t}$ is

$$(W_{r,s}^{l,t})^{ss} = V_{p-r-1,p-s-1}^{r+l,s+t}(\overline{\mathbb{F}}_p) \oplus V_{r-1,p-s-2}^{l,s+1+t}(\overline{\mathbb{F}}_p) \oplus V_{p-r-2,s-1}^{r+l+1,t}(\overline{\mathbb{F}}_p).$$

4.2. Some invariants. — Let $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ be the congruence subgroups of $G(F)$ defined in Section 2. We view $\overline{\mathbb{F}}_p$ as a trivial left $\tilde{G}$-module. We need to remind us once more how Hecke operators act on the degree zero group cohomology. Let $g \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$ where $\Delta_1^{\mathfrak{q}}(\mathfrak{n})$ is the subset of $Mat_2(\hat{\mathcal{O}})_{\neq 0}$ defined in Section 2. From Lemma 2.3, we have that for each i, $1 \leq i \leq h$, there are a unique index j_i and a matrix $\beta_i \in \Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$ such that $K_1(\mathfrak{n})gK_1(\mathfrak{n}) = K_1(\mathfrak{n})g_{j_i}\beta_i g_i^{-1}K_1(\mathfrak{n})$ with g_i the matrices corresponding to the ideal classes as defined in Subsection 2.2.
Let M be a finite dimensional left $\overline{\mathbb{F}}_p[\tilde{G}]$-module. We have seen that the Hecke operator corresponding to the double coset $K_1(\mathfrak{n})gK_1(\mathfrak{n})$ which we have denoted as T_g sends $(m_1, \cdots, m_h) \in \oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$ to $(n_1, \cdots, n_h)$ with $n_{j_i} = T_{\beta_i}.m_i$. Here T_{β_i} is the

Hecke operator corresponding to the double coset $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Explicitly one defines $\Gamma''^{,\beta_i}_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) = \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})$, then we have

$$\begin{array}{rcl} T_{\beta_i} : \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M) & \to & \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}), M) \\ m & \mapsto & \displaystyle\sum_{\lambda\in\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})/\Gamma''^{,\beta_i}_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})} (\lambda\beta_i).m. \end{array}$$

This being said here are the $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-invariants for $\mathrm{U}^e_d(\overline{\mathbb{F}}_p)$, $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)$, $(W^{l,t}_{r,s})^{ss}$ and $W^{l,t}_{r,s}$.

Lemma 4.4. — *Let d and n be integers greater than or equal to zero. Then one has*

1. *for all $n \geq 0$, one has*
$$\oplus_{i=1}^h \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}^n_d(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus_{i=1}^h \overline{\mathbb{F}}_p & \text{if } d \equiv 0 \pmod{p^2-1} \\ 0 & \text{otherwise} \end{cases}$$
as $\overline{\mathbb{F}}_p$-vector spaces.
2. $\oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}^n_d(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus_{i=1}^h \overline{\mathbb{F}}_p & \text{if } d \equiv 0 \pmod{p^2-1} \text{ and } (\mathcal{O}^*)^n = 1 \\ 0 & \text{otherwise} \end{cases}$
as $\overline{\mathbb{F}}_p$-vector spaces.
3. *the Hecke operator T_g acts on $(m_1, \cdots, m_h) \in \oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}^n_d(\overline{\mathbb{F}}_p)) = \oplus_{i=1}^h \overline{\mathbb{F}}_p$, where $d \equiv 0 \pmod{p^2-1}$ and $(\mathcal{O}^*)^n = 1$, by sending m_i to n_{j_i} with*
$$n_{j_i} = [\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})]m_i.$$

Proof. — As for the first item, because $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ reduces modulo $\mathfrak{p}$ to $\tilde{S}$ and since we can identify $\mathrm{U}^n_d(\overline{\mathbb{F}}_p)$ with $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ as $\tilde{S}$-module, we see that the invariants do not depend on the values of n. Having this, it is suitable to view $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ as the set of $\overline{\mathbb{F}}_p$-valued functions on $\mathbb{F}^2_{p^2}$ and homogeneous of degree d. Observe first that a non-null constant function belongs to $(\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if $d \equiv 0 \pmod{p^2-1}$. Any nonzero $f \in (\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ is a constant function. Indeed let $(0,0) \neq (a,b), (a',b') \in \mathbb{F}^2_{p^2}$ and suppose that $f(a,b) = x \neq f(a',b') = y$. Now since (a,b), $(a',b') \neq (0,0)$ there are $c, e, c', d' \in \mathbb{F}_{p^2}$ such that $\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \tilde{S}$. Then $(a',b') = (a,b)\left(\begin{smallmatrix} e & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Therefore $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ acts transitively on $\mathbb{F}^2_{p^2} - \{(0,0)\}$. Indeed from Lemma 3.1, reduction modulo $\mathfrak{p}$ is a surjective homomorphism $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \twoheadrightarrow \tilde{S}$. So we have $y = f(a',b') = f((a,b)\gamma) = \gamma f((a,b)) = f(a,b) = x$, contradicting the hypothesis $x \neq y$. Hence $f \in (\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if f is constant.
For the second item one firstly observes that a non-null constant function belongs to $(\mathrm{U}^n_d(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if $d \equiv 0 \pmod{p^2-1}$ and $(\mathcal{O}^*)^n = 1$. From here the same proof as the one given for the first item applies.
For the third item, let $f_x \in (\mathrm{U}^n_d(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ with $f(a,b) = x$ for all $(a,b) \in \mathbb{F}^2_{p^2} - \{(0,0)\}$ and let given $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_k \delta_k\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Then

$$T_{\beta_i}.f_x(a,b) = \sum \delta_k\beta_i.f_x(a,b) = [\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})]x.$$

From this what we have claimed follows. $\square$

We also have the following

Lemma 4.5. — *Let $0 \leq r, s \leq p-1$, and let l, t be integers greater than or equal to zero. Then one has*

1. $\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if } r = s = 0 \text{ and for all } l, t \\ 0 & \text{otherwise} \end{cases}$
   as $\overline{\mathbb{F}}_p$-vector spaces.
2. $\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if } r = s = 0 \text{ and } (\mathcal{O}^*)^{l+pt} = 1 \\ 0 & \text{otherwise} \end{cases}$
   as $\overline{\mathbb{F}}_p$-vector spaces.
3. *the Hecke operator T_g acts on $(m_1, \cdots, m_h)$ from $\oplus_{i=1}^{h} \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{0,0}(\overline{\mathbb{F}}_p))$ which is equal to $\oplus_{i=1}^{h} \overline{\mathbb{F}}_p$ when $(\mathcal{O}^*)^{l+pt} = 1$, by sending m_i to n_{j_i} with $n_{j_i} = [\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})] m_i$.*

Proof. — Firstly when $r = s = 0$, and for all l, t then $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$ as $\tilde{S}$-module. By definition we have $\overline{\mathbb{F}}_p^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})} = \overline{\mathbb{F}}_p$. Otherwise use the fact that $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)$ is irreducible as $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module. The second item is proved similarly. The statement about the Hecke action is verified similarly as in the proof of Lemma 4.4. $\square$

Lemma 4.6. — *Let $0 \leq r, s \leq p-1, e := l + tp, e_1 := e + p(p-1), e_2 := e + p - 1 \geq 0$ and let f be the order of $\mathcal{O}^*$. The following isomorphisms of $\overline{\mathbb{F}}_p$-vectors spaces hold:*

1. $\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{r,s})^{ss}) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if} \begin{cases} r = s = p-1 \text{ or} \\ r = 1, s = p-2 \text{ or} \\ r = p-2, s = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$
2. *suppose that $(r \neq 1 \text{ or } s \neq p-2)$ and $(r \neq p-2 \text{ or } s \neq 1)$; then we have*
$$\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if } r = s = p-1 \\ 0 & \text{otherwise} \end{cases}$$
3. $\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W^{l,t}_{r,s})^{ss}) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if} \begin{cases} r = s = p-1 \text{ and } f \mid e \text{ or} \\ r = 1, s = p-2 \text{ and } f \mid e_1 \text{ or} \\ r = p-2, s = 1 \text{ and } f \mid e_2 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$
4. *suppose that $(r \neq 1 \text{ or } s \neq p-2 \quad \text{or } f \nmid e_1)$ and $(r \neq p-2 \text{ or } s \neq 1 \quad \text{or } f \nmid e_2)$; then we have*
$$\oplus_{1=i}^{h} \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \begin{cases} \oplus_{1=i}^{h} \overline{\mathbb{F}}_p & \text{if } r = s = p-1 \text{ and } f \mid e \\ 0 & \text{otherwise} \end{cases}$$

Lastly, the Hecke action on these spaces is as in the previous lemmas.

Proof. — We have $(W_{r,s})^{\mathrm{ss}} = V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p) \oplus V^{0,s+1}_{r-1,p-s-2}(\overline{\mathbb{F}}_p) \oplus V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p)$. From the above lemma we know that $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p))$ is non zero only when $r = s = p-1$. Indeed $V^{r,s}_{p-1-r,p-s-1}(\overline{\mathbb{F}}_p) \cong \overline{\mathbb{F}}_p$ as $\tilde{S}$-modules if and only if $r = p-1, s = p-1$. In this case we have $(W_{r,s})^{\mathrm{ss}} = V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p) \cong \overline{\mathbb{F}}_p$. Therefore we obtain that

$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{r,s})^{\mathrm{ss}}) = \overline{\mathbb{F}}_p.$$

From the same lemma $V^{0,s+1}_{r-1,p-s-2}(\overline{\mathbb{F}}_p)$ has non zero invariants only when $r = 1, s = p-2$. In this case we have

$$(W_{1,p-2})^{\mathrm{ss}} = V^{1,p-2}_{p-2,1}(\overline{\mathbb{F}}_p) \oplus V^{0,p-1}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{2,0}_{p-3,p-3}(\overline{\mathbb{F}}_p).$$

From this one has

$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{1,p-2})^{\mathrm{ss}}) = \overline{\mathbb{F}}_p.$$

From the same lemma the invariants of $V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p)$ are non zero if and only if $r = p-2, s = 1$. Similarly we obtain that

$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{p-2,1})^{\mathrm{ss}}) = \overline{\mathbb{F}}_p.$$

As for the second item, when $r = s = p-1$, then $(W_{r,s})^{\mathrm{ss}} = W_{r,s}$. Otherwise, from the fact that $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{r,s})^{ss}) = 0$, one deduces that $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s})$.
The remaining items are proved in a similar fashion. □

In the cases ($r = 1, s = p-2$ and $f \mid e_1$) or ($r = p-2, s = 1$ and $f \mid e_2$), further analysis is needed.
We shall next discuss the case $r = 1, s = p-2$ and $f \mid e_1$ in detail as it is symmetric to the remaining one. We suppose in addition that $p > 5$. So the representation $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$ has dimension $2(p-1)$ and we identify it with its image in $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$. Inside $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$ lies the submodule M generated by the homogeneous monomials of degree $(p-1)^2$. The dimension of M is $(p-1)^2+1$ and it contains $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$ as submodule. By dimensional consideration (it is here that we need to have $p > 5$ to avoid discussing many cases), one deduces an exact sequence of $\overline{\mathbb{F}}_p[\tilde{G}]$-modules

$$0 \to V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p) \to M \to V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p) \to 0.$$

Indeed from

$$(W^{l,t}_{1,p-2})^{\mathrm{ss}} = V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p) \oplus V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p),$$

we know that the constituents of any submodule of $W^{l,t}_{1,p-2}$ are among the representations $V^{l,p-1}_{0,0}(\overline{\mathbb{F}}_p), V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ and $V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p)$. From the equality $(p-1)^2+1 = (p-2)^2+2(p-1)$, it follows that

$$M/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p) \cong V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p).$$

Therefore $V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ is a submodule of $\mathrm{U}^{l+pt}_{(p-1)^2}/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p) = (W^{l,t}_{1,p-2})^{\mathrm{ss}}$.
Next we can realize the module $V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p)$ as submodule of $(W^{l,t}_{1,p-2})^{\mathrm{ss}}$ by sending 1 to the

class $X^{p(p-1)}Y^{p(p-1)} + V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$. To see this we define

$$\begin{array}{rcl}\varphi : V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p) & \to & \mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)\\ 1 & \mapsto & X^{p(p-1)}Y^{p(p-1)} + V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p).\end{array}$$

Then for $g = \left(\begin{smallmatrix} a & b \\ c & e\end{smallmatrix}\right) \in \tilde{G}$, we need to check that $\varphi(1) = g.\varphi(1)$. We have

$$g.X^{p(p-1)}Y^{p(p-1)} = (ae-bc)^{l+pt}(a^pX^p + b^pY^p)^{p-1}(c^pX^p + e^pY^p)^{p-1}.$$

The latter polynomial is a linear combination of the monomials $X^{2p^2-2p-i}Y^i$ with $p|i$. For all multiples i of p less or equal to $2p(p-1)$ except $p(p-1)$ the monomials $X^{2p^2-2p-i}Y^i$ belong to $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$. Indeed let $i = pk$, we recall the relations $X^{p^2} = X, Y^{p^2} = Y$ in $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$, then we have

$$X^{2p^2-2p-pk}Y^{pk} = X^{p^2-2p-pk}X^{p^2}Y^{pk} = X^{(p-1)^2-pk}Y^{pk} \in V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p).$$

Therefore $g.\varphi(1) \equiv \varphi(1) \pmod{V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)}$. This implies that the direct sum $V^{l,p-1}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{1+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ is a submodule of $W^{l,t}_{1,p-2}$. Thus we get an exact sequence

$$0 \to V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p) \to (W^{l,t}_{1,p-2}) \to V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p) \to 0.$$

Hence for ($r = 1, s = p-2$ and $l + pt \equiv p-1 \pmod{p^2-1}$), we obtain that

$$\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p.$$

For ($r = p-2, s = 1$) and $l + pt \equiv 1-p \pmod{p^2-1}$, similar arguments yield

$$\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p.$$

In summary we have the following

Lemma 4.7. — *Let $p > 5$ and $e := l + pt$. Then we have*

1. $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) = \overline{\mathbb{F}}_p$ *if* $\begin{cases} r = 1, s = p-2 \text{ or} \\ r = p-2, s = 1\end{cases}$
2. $\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p$ *if* $\begin{cases} r = 1, s = p-2 \text{ and } f \mid p(p-1)+e \text{ or} \\ r = p-2, s = 1 \text{ and } f \mid e+p-1.\end{cases}$

4.2.1. Some indexes. — Let $\mathfrak{t}$ be a finite place coprime with $\mathfrak{pn}$. The matrix $g \in Mat_2(\hat{\mathcal{O}})$ which has at the $\mathfrak{t}$-th place the matrix $\left(\begin{smallmatrix} \pi_{\mathfrak{t}} & 0 \\ 0 & 1\end{smallmatrix}\right)$ where $\pi_{\mathfrak{t}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{t}}$ and in all the remaining places has the identity matrix, belongs to $\Delta^{\mathfrak{t}}_1(\mathfrak{n})$. We shall fix in this subsection such a g. Because

$$\begin{pmatrix} \pi_{\mathfrak{t}}^{-1} & 0 \\ 0 & 1\end{pmatrix}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} \pi_{\mathfrak{t}} & 0 \\ 0 & 1\end{smallmatrix}\right) = \begin{pmatrix} a & \pi_{\mathfrak{t}}^{-1}b \\ \pi_{\mathfrak{t}}c & d\end{pmatrix};$$

we deduce that

$$K'_{1,g^{-1}}(\mathfrak{n}) := g^{-1}K_1(\mathfrak{n})g \cap K_1(\mathfrak{n}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in K_1(\mathfrak{n}) : \pi_{\mathfrak{t}} \mid c_{\mathfrak{t}}\right\}.$$

Consider also the subgroup

$$K^1_1(\mathfrak{n}) = \{\alpha \in K_1(\mathfrak{n}) : det(\alpha) = 1\}.$$

Similarly as in Lemma 3.1, one can prove that reduction modulo $\mathfrak{t}$ provides us with a surjective homomorphism

$$K_1^1(\mathfrak{n}) \twoheadrightarrow \mathrm{SL}_2(\mathcal{O}/\mathfrak{t}).$$

From this we deduce that we have a surjective map

$$\begin{array}{rcl} K_1(\mathfrak{n}) & \twoheadrightarrow & \mathbb{P}^1(\mathcal{O}/\mathfrak{t}) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & (c:d). \end{array}$$

This map is surjective because there is a surjective map $\mathrm{SL}_2(\mathcal{O}/\mathfrak{t}) \twoheadrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{t})$; $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c:d)$ and also a surjective map $K_1(\mathfrak{n}) \supset K_1^1(\mathfrak{n}) \twoheadrightarrow \mathrm{SL}_2(\mathcal{O}/\mathfrak{t})$. Since the subgroup $K'_{1,g^{-1}}(\mathfrak{n})$ is the subset of all elements that are mapped to $(0:1)$, we deduce that we have a bijection

$$K'_{1,g^{-1}}(\mathfrak{n})\backslash K_1(\mathfrak{n}) \longleftrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{t}).$$

Therefore we obtain the index $[K_1(\mathfrak{n}) : K'_{1,g^{-1}}(\mathfrak{n})] = N(\mathfrak{t}) + 1$. Recall the definition $\Gamma^{\prime,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}) := \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \beta_i^{-1}\Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i$. Similarly as $Y_{K_1(\mathfrak{n})}$ decomposes into disjoint union of its connected component $\amalg_{i=1}^h \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3$, $Y_{K'_{1,g^{-1}}(\mathfrak{n})}$ decomposes as follows. We have $Y_{K'_{1,g^{-1}}(\mathfrak{n})} = \amalg_{i=1}^h \Gamma^{\prime,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3$. Indeed we know that the connected components of $Y_{K'_{1,g^{-1}}(\mathfrak{n})}$ are $\Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3$ where $\Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = g_iK'_{1,g^{-1}}(\mathfrak{n})g_i^{-1} \cap G(F) = g_ig^{-1}K_1(\mathfrak{n})gg_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Recall that $\beta_i = g_{j_i}gg_i^{-1}k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$ with $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in g_iK_1(\mathfrak{n})g_i^{-1}$. For $\sigma \in \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) = g_{j_i}K_1(\mathfrak{n})g_{j_i}^{-1} \cap G(F)$ we have

$$\begin{array}{rcl} \beta_i^{-1}\sigma\beta_i & \in & \beta_i^{-1}g_{j_i}K_1(\mathfrak{n})g_{j_i}^{-1}\beta_i \cap G(F) \\ & = & k_i^{-1}g_ig^{-1}g_{j_i}^{-1}g_{j_i}K_1(\mathfrak{n})g_{j_i}g_{j_i}^{-1}gg_i^{-1}k_i \cap G(F) \\ & = & g_ig^{-1}K_1(\mathfrak{n})gg_i^{-1} \cap G(F) \end{array}$$

where we have used the facts that g_i, g, k_i commute and $k_i, k_i^{-1} \in g_iK_1(\mathfrak{n})g_i^{-1}$. This means that $\beta_i^{-1}\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i = \Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Therefore we deduce that

$$\Gamma^{\prime,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \beta_i^{-1}\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i = \Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

So we have the following projection map

$$\begin{array}{c} Y_{K'_{1,g^{-1}}(\mathfrak{n})} = \amalg_{i=1}^h \Gamma^{\prime,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3 \\ \Big\downarrow{\scriptstyle s_g} \\ Y_{K_1(\mathfrak{n})} = \amalg_{i=1}^h \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3. \end{array}$$

The map s_g is of degree $[K_1(\mathfrak{n}) : K'_{1,g^{-1}}(\mathfrak{n})] = N(\mathfrak{t}) + 1$. The maps induced by s_g on the connected components are also of degree $N(\mathfrak{t}) + 1$. The discussion we just made implies that for β_i corresponding to g, that is to mean $K_1(\mathfrak{n})gK_1(\mathfrak{n}) = K_1(\mathfrak{n})g_{j_i}^{-1}\beta_ig_iK_1(\mathfrak{n})$ where j_i is the unique index such that the ideal $(det(g_{j_i}gg_i^{-1}))$ is principal, the following holds.

Lemma 4.8. — *Keeping the same assumptions as above, then for any ideal* $\mathfrak{n}$ *coprime with* $\mathfrak{t} = (det(g))$*, we have*

$$[\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})] = N(\mathfrak{t}) + 1.$$

Therefore, the Hecke eigenvalue corresponding to the action of $T_\mathfrak{t}$ on the $\overline{\mathbb{F}}_p$-vector space $\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ where $\mathfrak{t}$ is a prime ideal coprime with $\mathfrak{pn}$ is $N(\mathfrak{t})+1$. Hence eigenvalue systems coming from the $\overline{\mathbb{F}}_p$-vector space $\oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ are Eisenstein because the semisimplification of the conjecturally attached Galois representations is the direct sum of the cyclotomic character and the trivial character.
As we shall make use of Shapiro's isomorphism, we need to verify that it is compatible with the Hecke action on group cohomology. From the above discussion, we deduce that we can choose identical coset representatives for the double cosets

$$\Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}) \text{ and for } \Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

This will be used for the compatibility of the Hecke action with the Shapiro's isomorphism.

4.2.2. Compatibility of Shapiro's lemma with the Hecke action. — Recall that when $\Gamma' < \Gamma$ are congruence subgroups and M is a Γ'-module then Shapiro's isomorphism reads as

$$\mathrm{H}^*(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(M)) \cong \mathrm{H}^*(\Gamma', M).$$

It is the isomorphism induced by the restriction $j : \Gamma' \hookrightarrow \Gamma$ and the homomorphism

$$\begin{array}{rcl} \phi : \mathrm{Ind}_{\Gamma'}^{\Gamma}(M) & \to & M \\ f & \mapsto & f(1). \end{array}$$

Therefore in terms of cocycles we have

$$\begin{array}{rcl} Sh : \mathrm{H}^*(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(M)) & \to & \mathrm{H}^*(\Gamma', M) \\ c & \mapsto & \phi \circ c \circ j. \end{array}$$

From Subsection 2.2.1, we know that for $g \in \Delta_1^{\mathfrak{a}}(\mathfrak{pn})$ with $\mathfrak{a}$ coprime with $\mathfrak{pn}$, any set of orbit representatives of the orbit space $K_1(\mathfrak{pn})gK_1(\mathfrak{pn})/K_1(\mathfrak{pn})$ belongs to $\Delta_1^{\mathfrak{a}}(\mathfrak{pn})$ where $\mathfrak{a} = (det(g))\mathcal{O}$. In a similar fashion any set of orbit representatives of $K_1(\mathfrak{n})gK_1(\mathfrak{n})/K_1(\mathfrak{n})$ belongs to $\Delta_1^{\mathfrak{a}}(\mathfrak{n})$. We also obtain that any set of representatives of the orbit space $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ belong to $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$ when β_i is from $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$. For the forthcoming statement we need to recall some important facts. On page 56, we saw that reduction modulo $\mathfrak{p}$ provides us the following isomorphism of $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules:

$$\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$$

where $\tilde{U} = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G}\} \subset \tilde{B}$ with $\tilde{B}$ the Borel subgroup of $\tilde{G}$ and $\overline{\mathbb{F}}_p$ is endowed with the structure of a trivial left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module. The left action of the latter on $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ is as follows: for $\gamma \in \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and $f \in \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ we have $(\gamma.f)(h) := f(h\gamma)$. By definition

$$\mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) := \{f : \tilde{G} \to \overline{\mathbb{F}}_p : f(uh) = uf(h) = f(h), \forall\, u \in \tilde{U}, h \in \tilde{G}\},$$

that is the collection of all the $\tilde{U}$-left invariant maps from $\tilde{G}$ to $\overline{\mathbb{F}}_p$. Because for each $i = 1, \cdots, h$, any element $\lambda \in \Lambda^{\mathfrak{c}}_{1,[\mathfrak{b}_i]}(\mathfrak{pn})$ has its reduction belonging to $\tilde{U}$, we derive that any $f \in \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p)$ satisfies

$$f(\lambda) = f(1).$$

Proposition 4.9. — *Let $\mathfrak{a}$ be a prime ideal coprime with $\mathfrak{pn}$. Let $T_g = T_{\mathfrak{a}}$ be the Hecke operator associated with $g \in \Delta_1^{\mathfrak{a}}(\mathfrak{n})$ where $\mathfrak{a} = det(g)\mathcal{O}$ the ideal corresponding to g. Explicitly g is the matrix with the identity matrix in all finite places expect at the $\mathfrak{a}$-place where there is the matrix $\left(\begin{smallmatrix} \pi_{\mathfrak{a}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Here $\pi_{\mathfrak{a}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{a}}$. Then the following diagram*

$$\begin{array}{ccc}
\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)) & \xrightarrow{Sh} & \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p) \\
\downarrow T_{\beta_i} & & \downarrow T_{\beta_i} \\
\mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)) & \xrightarrow{Sh} & \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p)
\end{array}$$

is well defined and is commutative.

Proof. — For the proof of this proposition, we refer to [**1**, p. 42]. □

One other important fact that tells us that we only have to look at Serre weights, that is to mean irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules for the analysis of Hecke eigenclasses is the following proposition.

Proposition 4.10. — *Let $\mathfrak{n}$ be an integral ideal such that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3. Consider the open compact subgroup $K_1(\mathfrak{n})$ of level $\mathfrak{n}$. Let M be a finite dimensional $\overline{\mathbb{F}}_p[\tilde{G}]$-module. Let Ψ be an eigenvalue system occurring in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$ and taking values in $\overline{\mathbb{F}}_p$. Then, there exists W, an irreducible subquotient of M such that Ψ also occurs in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W)$.*

Proof. — Let W be an irreducible submodule M. Denote the quotient M/W as N. Set $K = K_1(\mathfrak{n})$. This is an open compact subgroup of $G(\hat{\mathcal{O}})$ which is neat and surjects onto $\hat{\mathcal{O}}^*$ via the determinant. Write the following exact sequence of locally constant sheaves on Y_K associated with W, M, N respectively:

$$0 \to \tilde{W} \to \tilde{M} \to \tilde{N} \to 0.$$

From this one obtains the exact sequence in cohomology:

$$\cdots \to \mathrm{H}^1(Y_K, \tilde{W}) \to \mathrm{H}^1(Y_K, \tilde{M}) \to \mathrm{H}^1(Y_K, \tilde{N}) \to \cdots .$$

Let s be a system of Hecke eigenvalues from $\mathrm{H}^1(Y_K, \tilde{M})$. If the image of s is zero, then s occurs in $\mathrm{H}^1(Y_K, \tilde{W})$, and we are done. Otherwise it is arisen from $\mathrm{H}^1(Y_K, \tilde{N})$. We then replace $\tilde{M}$ by $\tilde{N}$ and repeat the argument. □

4.2.3. Statements and proofs of the main results. — The statement about the reduction to weight two is as follows.

Theorem 4.11. — *Let F be an imaginary quadratic field of class number h. Let $\mathfrak{n}$ be an integral ideal in F and let $p > 5$ be a rational prime which is inert in F and coprime with $\mathfrak{n}$. Suppose that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3. Let $0 \leq r, s \leq p-1$ and*

$0 \le l, t \le p-1$, with l, t not both equal to $p-1$. Let ψ be a system of Hecke eigenvalues in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$. Then ψ occurs in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$ except possibly when $(r=1, s=p-2)$ or $(r=p-2, s=1)$. In these potential exceptions, the obstruction is coming from Hecke eigenvalue systems which are Eisenstein.

Proof. — The proof is divided in two parts. Firstly, we show that

$$\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p)$$

as $\overline{\mathbb{F}}_p$-vector spaces except in the exceptional cases named in the statement. Secondly from this, we use an inflation restriction exact sequence and obtain an embedding of Hecke modules $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$.

First part:

The exact sequence

$$0 \to V_{r,s}(\overline{\mathbb{F}}_p) \to \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p) \to W_{r,s} \to 0$$

gives rise to the long exact sequence in cohomology

$$\begin{aligned} 0 \quad \to \quad & \oplus_{i=1}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \oplus_{i=1}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \\ \to \quad & \oplus_{i=1}^{h} \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) \to \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \\ \to \quad & \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) \to \cdots . \end{aligned}$$

This is an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces. If $r = s = p-1$, from Lemmas 4.4, 4.5 and 4.6, we get the exact sequence of $\overline{\mathbb{F}}_p$-vector spaces for each $i = 1, \cdots, h$:

$$0 \to \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

This means that the third arrow is the null map and hence we have an injection

$$\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)).$$

From Lemmas 4.4, 4.5 and 4.6, we see that when ($r \neq 1$ or $s \neq p-2$) and ($r \neq p-2$ or $s \neq 1$), we have an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces

$$0 \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

Therefore in all cases this is an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces. From Proposition 3.4, we know that the representation $\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)$ is a direct summand of $\mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. So, one has an embedding of $\overline{\mathbb{F}}_p$-vector spaces

$$\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{V}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)).$$

By Shapiro's lemma, one concludes that we have an injection of $\overline{\mathbb{F}}_p$-vector spaces

$$\alpha : \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})), \overline{\mathbb{F}}_p).$$

Lastly when $(r = 1, s = p-2)$ or $(r = p-2, s = 1)$, then from Lemmas 4.4, 4.5, 4.6 and 4.7, we have the exact sequence of $\overline{\mathbb{F}}_p$-vector spaces

$$0 \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

Second part:
Consider the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) \xrightarrow{infl} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \xrightarrow{res}$$
$$\xrightarrow{res} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})} \to \mathrm{H}^2(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}).$$

Because of the assumption concerning p we have that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) = \mathrm{H}^2(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) = 0.$$

Then we get the isomorphism of $\overline{\mathbb{F}}_p$-vector spaces induced by the restriction map:

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \xrightarrow{\sim} (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{0,0}_{r,s}(\overline{\mathbb{F}}_p)) \otimes_{\overline{\mathbb{F}}_p} det^{l+pt})^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$$

where we have used the isomorphism

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \backsimeq \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{0,0}_{r,s}(\overline{\mathbb{F}}_p)) \otimes_{\overline{\mathbb{F}}_p} det^{l+pt}.$$

Next notice that for all $1 \le i \le h$, we have isomorphisms of abelian groups

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cong \mathcal{O}^* \cong \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}).$$

From the first part, when we are in the situation ($r \neq 1$ or $s \neq p-2$) and ($r \neq p-2$ or $s \neq 1$), then there is an embedding of $\overline{\mathbb{F}}_p$-vector spaces:

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p).$$

When tensoring with det^{l+pt}, we obtain the embedding

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

We next take $\mathcal{O}^*$-invariants and we get

$$(\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)))^{\mathcal{O}^*} \hookrightarrow (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}))^{\mathcal{O}^*}.$$

This and the isomorphism induced by the inflation restriction exact sequence implies that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}))^{\mathcal{O}^*}.$$

Using once more the inflation restriction exact sequence for the right hand of this embedding, we derive that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

This natural map is compatible with the Hecke action, and so this is an injection of Hecke modules.
Now when the cases ($r = 1, s = p-2$) or ($r = p-2, s = 1$) hold, then the first part provides us with an exact sequence

$$0 \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p).$$

This implies that the following sequences are exact:

$$0 \to \overline{\mathbb{F}}_p \otimes det^{l+pt} \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}),$$

thus

$$0 \to (\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*} \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

Then, when $(\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*} = 0$, the embedding

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$$

holds. Otherwise, we know that the obstruction is coming from $(\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*}$ and is hence Eisenstein as shown by Lemma 4.8. □

Systems of Hecke eigenvalues arising from $\overline{\mathbb{F}}_p \otimes det^e$ are Eisenstein and hence conjecturally correspond to reducible Galois representations. Because of this, the statement about Serre's conjecture is not affected since it only concerns irreducible mod p Galois representations. Now the statement related to Serre type questions is as follows.

Proposition 4.12. — *We keep the same conditions as in Theorem 4.11. A positive answer to Question (a) on page 46 answers positively Question (b) and the reciprocal also holds.*

Proof. — The part $(b) \Rightarrow (a)$ is obtained as follows. By Shapiro's lemma the system is realized in

$$\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)).$$

By Proposition 4.10, this system of Hecke eigenvalues already appears in

$$\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$$

where M is a simple module from the Jordan-Hölder series of $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. This module M is a Serre weight.
The part $(a) \Rightarrow (b)$ follows from Theorem 4.11. □

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