FAMILIES OF CURVES OVER ANY FINITE FIELD ATTAINING THE GENERALIZED DRINFELD-VLADUT BOUND

by

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Abstract. — We prove the existence of asymptotically exact sequences of algebraic function fields defined over any finite field $\mathbb{F}_q$ having an asymptotically maximum number of places of a degree r where r is an integer ≥ 1. It turns out that these particular families have an asymptotic class number widely greater than all the Lachaud - Martin-Deschamps bounds when $r > 1$. We emphasize that we exhibit explicit asymptotically exact sequences of algebraic function fields over any finite field $\mathbb{F}_q$, in particular when q is not a square, with $r = 2$. We explicitly construct an example for $q = 2$ and $r = 4$. We deduce of this study that for any finite field $\mathbb{F}_q$ there exists a sequence of algebraic function fields defined over any finite field $\mathbb{F}_q$ reaching the generalized Drinfeld - Vladut bound.

Résumé. — Nous prouvons l'existence de familles asymptotiquement exactes de corps de fonctions algébriques définis sur un corps fini $\mathbb{F}_q$ qui ont un nombre maximum de places de degré r où r est un entier ≥ 1. Il s'avère que pour ces familles particulières le nombre de classes est asymptotiquement beaucoup plus grand que la borne générale de Lachaud - Martin-Deschamps quand $r > 1$. Pour $r = 2$ nous construisons explicitement des suites de corps de fonctions algébriques sur tout corps fini $\mathbb{F}_q$ qui sont asymptotiquement exactes et ceci même lorsque q n'est pas un carré. Nous construisons un exemple pour $r = 4$ dans le cas où $q = 2$. Nous déduisons de cette étude que pour tout corps fini $\mathbb{F}_q$ il existe une suite de corps de fonctions algébriques sur $\mathbb{F}_q$ qui atteint la borne de Drinfeld - Vladut généralisée.

1. Introduction

1.1. General context and main result. — When, for a given finite ground field, the sequence of the genus of a sequence of algebraic function fields tends to infinity, there exist asymptotic *formulae* for different numerical invariants. In [**10**], Tsfasman generalizes some results on the number of rational points on the curves (due to Drinfeld-Vladut [**13**], Ihara [**5**], and Serre [**8**]) and on its Jacobian (due to Vladut [**12**], Rosembloom and Tsfasman [**7**]). He gives a formula for the asymptotic number of divisors, and some estimates for the number of points in the Poincaré filtration.

Key words and phrases. — Finite field, function field, asymptotically exact sequence, class number, tower of function fields.

For this purpose, he introduced the notion of asymptotically exact family of curves defined over a finite field. It turns out that this notion is very fruitful. Indeed, for such sequences of curves, we can evaluate enough precisely the asymptotic behavior of certain numerical invariants, in particular the asymptotic number of effective divisors and the asymptotic class number. Moreover, when these sequences are maximal (cf. the basic inequality (1)) they have an asymptotically large class number. In particular, it is the case when they have a maximal number of places of a given degree. Unfortunately the existence of such sequences is not known for any finite field $\mathbb{F}_q$, in particular when q is not a square (cf. Remark 5.2 in [**11**]). In this paper, we mainly explicitly construct examples of asymptotically exact sequences of algebraic function fields for any finite field, so proving the existence of maximal asymptotically exact sequences of curves defined over any finite field $\mathbb{F}_q$ where q is not a square. So, we answer in Corollary 3.4 the question asked by Tsfasman and Vladut in [**11**, Remark 5.2].

1.2. Notation and detailed questions. — Let us recall the notion of asymptotically exact family of curves defined over a finite field in the language of algebraic function fields.

Definition 1.1 **(Asymptotically Exact Sequence)**. — Let $\mathcal{F}/\mathbb{F}_q=(F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields $F_k/\mathbb{F}_q$ defined over a finite field $\mathbb{F}_q$ of genus $g_k=g(F_k/\mathbb{F}_q)$. We suppose that the sequence of the genus g_k is an increasing sequence growing to infinity. The sequence $\mathcal{F}/\mathbb{F}_q$ is said to be asymptotically exact if for all $m\geq 1$ the following limit exists:

$$\beta_m(\mathcal{F}/\mathbb{F}_q)=\lim_{k\rightarrow+\infty}\frac{B_m(F_k/\mathbb{F}_q)}{g_k}$$

where $B_m(F_k/\mathbb{F}_q)$ is the number of places of degree m on $F_k/\mathbb{F}_q$.
The sequence $\beta=(\beta_1,\beta_2,....,\beta_m,...)$ is called the type of the asymptotically exact sequence $\mathcal{F}/\mathbb{F}_q$.

Definition 1.2 **(Generalized Drinfeld-Vladut Bound)**. — Let $\mathcal{F}/\mathbb{F}_q=(F_k/\mathbb{F}_q)_{k\geq 1}$ be a asymptotically exact sequence of algebraic function fields $F_k/\mathbb{F}_q$ defined over a finite field $\mathbb{F}_q$ of genus $g_k=g(F_k/\mathbb{F}_q)$ and of type β. The sequence β (respectively the sequence $\mathcal{F}/\mathbb{F}_q$) is said *maximal* (of maximal type) when the following basic inequality, called *Generalized Drinfeld-Vladut bound*,

$$\sum_{m=1}^{\infty}\frac{m\beta_m}{q^{m/2}-1}\leq 1 \tag{1}$$

is reached.

Definition 1.3 **(Drinfeld-Vladut Bound of order r)**. — Let

$$B_r(q,g)=\max\{B_r(F/\mathbb{F}_q)\mid F/\mathbb{F}_q \text{ is a function field over } \mathbb{F}_q \text{ of genus } g\}.$$

Let us set

$$A_r(q)=\limsup_{g\rightarrow+\infty}\frac{B_r(q,g)}{g}.$$

We have the Drinfeld-Vladut Bound of order r :

$$A_r(q) \leq \frac{1}{r}(q^{\frac{r}{2}} - 1).$$

Remark 1.4. — Note that if for a family $\mathcal{F}/\mathbb{F}_q$ of algebraic function fields, there exists an integer r such that the Drinfeld-Vladut Bound of order r is reached, then this family is a maximal asymptotically exact sequence namely attaining the Generalized Drinfeld-Vladut bound. Moreover, its type is

$$\beta = (0, 0, ..., 0, \beta_r = \frac{1}{r}(q^{\frac{r}{2}} - 1), 0, ...).$$

Tsfasman and Vladut in [**11**] made use of these notions to obtain new general results on the asymptotic properties of zeta functions of curves.
Note that a simple diagonal argument proves that each sequence of algebraic function fields of growing genus, defined over a finite field admits an asymptotically exact subsequence. However until now, we do not know if there exists an asymptotically exact sequence $\mathcal{F}/\mathbb{F}_q$ with a maximal β sequence when q is not a square. Moreover, the diagonal extraction method is not really suitable for the two following reasons. First, in general we do not obtain by this process an explicit asymptotically exact sequence of algebraic function fields defined over an arbitrary finite field, in particular when q is not a square. Moreover, we have no control on the growing of the genus in the extracted sequence of algebraic function fields defined over an arbitrary finite field. More precisely, let us define the notion of density of a family of algebraic function fields defined over a finite field of growing genus:

Definition 1.5. — Let $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields $F_k/\mathbb{F}_q$ of genus $g_k = g(F_k/\mathbb{F}_q)$, defined over $\mathbb{F}_q$. We suppose that the sequence of genus g_k is an increasing sequence growing to infinity. Then, the density of the sequence $\mathcal{F}/\mathbb{F}_q$ is

$$d(\mathcal{F}/\mathbb{F}_q) = \liminf_{k\to+\infty} \frac{g_k}{g_{k+1}}.$$

A high density can be a useful property in some applications of sequences or towers of function fields. Until now, no explicit examples of dense asymptotically exact sequences $\mathcal{F}/\mathbb{F}_q$ have been pointed out unless for the case q square and type $\beta = (\sqrt{q} - 1, 0, \cdots)$.

1.3. New results. — In this paper, we answer the questions mentioned above. More precisely, for any prime power q we contruct examples of asymptotically exact sequences attaining the Drinfeld-Vladut Bound of order 2. We deduce that for any prime power q (in particular when q is not a square), there exists a maximal asymptotically exact sequences of algebraic functions fields namely attaining the Generalized Drinfeld-Vladut bound. We also construct for $q = 2$ and $r = 4$ an example of maximal asymptotically exact sequence attaining the Drinfeld-Vladut Bound of order 4. Then, we show that the asymptotically exact sequences of considered maximal type have an asymptotically large class number with respect to the Lachaud - Martin-Deschamps bounds. The paper is organized as follows. In section 2, we study the general families of asymptotically exact sequences of algebraic function fields defined over an arbitrary finite field $\mathbb{F}_q$ of maximal type $(0,...,\frac{1}{r}(q^{\frac{r}{2}}-1),0,...,0,...)$ where r is an integer ≥ 1, under the assumption of their existence. In particular, we study for these general families the behavior of the asymptotic class number h_k, and we compare our estimation to the general known bounds of Lachaud - Martin-Deschamps (cf. [**6**]). More precisely, we show that for such families, if they exist, the asymptotic class number is widely greater than the general bounds of Lachaud - Martin-Deschamps when $r>1$. Next, in section 3, we constructively prove the existence of asymptotically exact sequences of algebraic function fields defined over any arbitrary finite field $\mathbb{F}_q$ of maximal type $\beta=(0,...,\frac{1}{r}(q^{\frac{r}{2}}-1),0,...,0,...)$ with $r=2$. Moreover, we exhibit the example of an asymptotically exact sequence of algebraic function fields defined over $\mathbb{F}_q$ of maximal type $\beta=(0,...,\frac{1}{r}(q^{\frac{r}{2}}-1),0,...,0,...)$ with $q=2$ and $r=4$. For this purpose, we use towers of algebraic function fields coming from the descent of the densified towers of Garcia-Stichtenoth (cf. [**4**] and [**1**]). Note that all these examples are explicit and consist on very dense asymptotically exact towers algebraic function fields (maximally dense tower for $q=2$ and $r=4$).

2. General results

2.1. Particular families of asymptotically exact sequences. — First, let us recall certain asymptotic results. Let us first give the following result obtained by Tsfasman in [**10**]:

Proposition 2.1. — *Let $\mathcal{F}/\mathbb{F}_q=(F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields of increasing genus g_k growing to infinity. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $f(g_k)=o(log(g_k))$. Then*

$$\limsup_{g_k\to+\infty}\frac{1}{g_k}\sum_{m=1}^{f(g_k)}\frac{mB_m(F_k)}{q^{m/2}-1}\leq 1. \tag{2}$$

Now, we can easily obtain the following result as immediate consequence of Proposition 2.1:

Theorem 2.2. — *Let r be an integer ≥ 1 and $\mathcal{F}/\mathbb{F}_q=(F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields of increasing genus defined over $\mathbb{F}_q$ such that $\beta_r(\mathcal{F}/\mathbb{F}_q)=\frac{1}{r}(q^{\frac{r}{2}}-1)$. Then $\beta_m(\mathcal{F}/\mathbb{F}_q)=0$ for any integer $m\neq r$. In particular, the sequence $\mathcal{F}/\mathbb{F}_q$ is asymptotically exact.*

Proof. — Let us fix $m\neq r$ and let us prove that $\beta_m(\mathcal{F}/\mathbb{F}_q)=0$. We use Proposition 2.1 with the constant function $f(g)=s=\max(m,r)$. Then we get

$$\limsup_{k\to+\infty}\frac{1}{g_k}\sum_{j=1}^{s}\frac{jB_j(F_k)}{q^j-1}\leq 1.$$

But by hypothesis

$$\limsup_{k\to+\infty}\frac{rB_r(F_k)}{g_k(q^{\frac{r}{2}}-1)}=\lim_{k\to+\infty}\frac{rB_r(F_k)}{g_k(q^{\frac{r}{2}}-1)}=1.$$

Then

$$\limsup_{g_k \to +\infty} \frac{1}{g_k} \sum_{1 \leq j \leq s; j \neq r} \frac{jB_j(F_k)}{q^j - 1} = \lim_{g_k \to +\infty} \frac{1}{g_k} \sum_{1 \leq j \leq s; j \neq r} \frac{jB_j(F_k)}{q^j - 1} = 0.$$

But

$$\frac{B_m(F_k)}{g_k} \leq \frac{(q^m - 1)}{m} \left(\frac{1}{g_k} \sum_{1 \leq j \leq s; j \neq r} \frac{jB_j(F_k)}{q^j - 1} \right).$$

Hence

$$\lim_{g_k \to +\infty} \frac{B_m(F_k)}{g_k} = 0.$$

□

Note that for any k the following holds:

$$B_1(F_k/\mathbb{F}_{q^r}) = \sum_{i|r} iB_i(F_k/\mathbb{F}_q).$$

Then if $\beta_r(\mathcal{F}/\mathbb{F}_q) = \frac{1}{r}(q^{\frac{r}{2}} - 1)$, by Theorem 2.2 we conclude that $\beta_1(\mathcal{F}/\mathbb{F}_{q^r})$ exists and that

$$\beta_1(\mathcal{F}/\mathbb{F}_{q^r}) = (q^{\frac{r}{2}} - 1).$$

In particular the sequence $\mathcal{F}/\mathbb{F}_{q^r}$ reaches the classical Drinfeld-Vladut bound and consequently q^r is a square.

If $\beta_1(\mathcal{F}/\mathbb{F}_{q^r})$ exists then it does not necessarily imply that $\beta_r(\mathcal{F}/\mathbb{F}_q)$ exists but only that $\lim_{k \to +\infty} \frac{\sum_{m|r} mB_m(F_k/\mathbb{F}_q)}{g_k}$ exists. In fact, this converse depends on the defining equations of the algebraic function fields $F_k/\mathbb{F}_q$.

Now, let us give a simple consequence of Theorem 2.2.

Proposition 2.3. — *Let r and i be integers ≥ 1 such that i divides r. Suppose that*

$$\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k \geq 1}$$

is an asymptotically exact sequence of algebraic function fields defined over $\mathbb{F}_q$ of type $(\beta_1 = 0, \ldots, \beta_{r-1} = 0, \beta_r = \frac{1}{r}(q^{\frac{r}{2}} - 1), \beta_{r+1} = 0, \ldots)$. Then the sequence $\mathcal{F}/\mathbb{F}_{q^i} = (F_k/\mathbb{F}_{q^i})_{k \geq 1}$ of algebraic function field defined over $\mathbb{F}_{q^i}$ is asymptotically exact of type $(\beta_1 = 0, .., \beta_{\frac{r}{i}-1} = 0, \beta_{\frac{r}{i}} = \frac{i}{r}(q^{\frac{r}{2}} - 1), \beta_{\frac{r}{i}+1} = 0, \ldots)$.

Proof. — Let us remark that by [**9**, Lemma V.1.9, p. 163], if P is a place of degree r' of $F/\mathbb{F}_q$, there are $\gcd((r', i))$ places of degree $\frac{r'}{\gcd(r',i)}$ over P in the extension $F/\mathbb{F}_{q^i}$. As we are interested by the places of degree r/i in $F/\mathbb{F}_{q^i}$, let us introduce the set

$$S = \{r'; r \gcd(r', i) = i\, r'\} = \{r'; \mathrm{lcm}\,(r', i) = r\}.$$

Then,

$$B_{r/i}(F/\mathbb{F}_{q^i}) = \sum_{r' \in S} \frac{ir'}{r} B_{r'}(F/\mathbb{F}_q).$$

We know that all the $\beta_j(F/\mathbb{F}_q) = 0$ but $\beta_r(F/\mathbb{F}_q) = \frac{1}{r}(q^{\frac{r}{2}} - 1)$. Then

$$\beta_{r/i}(F/\mathbb{F}_{q^i}) = i\beta_r(F/\mathbb{F}_q),$$

$$\beta_{r/i}(F/\mathbb{F}_{q^i}) = \frac{i}{r}\left(q^{\frac{r}{2}} - 1\right).$$

□

2.2. Number of points of the Jacobian. — Now, we are interested by the Jacobian cardinality of the asymptotically exact sequences $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ of type $(0,..,0,\frac{1}{r}(q^{\frac{r}{2}} - 1), 0, ..., 0)$.

Let us denote by $h_k = h_k(F_k/\mathbb{F}_q)$ the class number of the algebraic function field $F_k/\mathbb{F}_q$. Let us consider the following quantities introduced by Tsfasman in [**10**]:

$$H_{inf}(\mathcal{F}/\mathbb{F}_q) = \liminf_{k\to+\infty} \frac{1}{g_k} \log h_k$$

$$H_{sup}(\mathcal{F}/\mathbb{F}_q) = \limsup_{k\to+\infty} \frac{1}{g_k} \log h_k.$$

If they coincides, we just write:

$$H(\mathcal{F}/\mathbb{F}_q) = \lim_{k\to+\infty} \frac{1}{g_k} \log h_k = H_{inf}(\mathcal{F}/\mathbb{F}_q) = H_{sup}(\mathcal{F}/\mathbb{F}_q).$$

Then under the assumptions of the previous section, we obtain the following result on the sequence of class numbers of these families of algebraic function fields:

Theorem 2.4. — *Let $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields of increasing genus defined over $\mathbb{F}_q$ such that $\beta_r(\mathcal{F}/\mathbb{F}_q) = \frac{1}{r}(q^{\frac{r}{2}} - 1)$ where r is an integer. Then, the limit $H(\mathcal{F}/\mathbb{F}_q)$ exists and we have:*

$$H(\mathcal{F}/\mathbb{F}_q) = \lim_{k\to+\infty} \frac{1}{g_k} \log h_k = \log \frac{q^{q^{\frac{r}{2}}}}{(q^r - 1)^{\frac{1}{r}(q^{\frac{r}{2}}-1)}}.$$

Proof. — By Corollary 1 in [**10**], we know that for any asymptotically exact family of algebraic function fields defined over $\mathbb{F}_q$, the limit $H(\mathcal{F}/\mathbb{F}_q)$ exists and

$$H(\mathcal{F}/\mathbb{F}_q) = \lim_{k\to+\infty} \frac{1}{g_k} \log h_k = \log q + \sum_{m=1}^{\infty} \beta_m . \log \frac{q^m}{q^m - 1}.$$

Hence, the result follows from Theorem 2.2. □

Corollary 2.5. — *Let $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields of increasing genus defined over $\mathbb{F}_q$ such that $\beta_r(\mathcal{F}/\mathbb{F}_q) = \frac{1}{r}(q^{\frac{r}{2}} - 1)$ where r is an integer. Then there exists an integer k_0 such that for any integer $k \geq k_0$,*

$$h_k > q^{g_k}.$$

Proof. — By Theorem 2.4, we have $\lim_{k\to+\infty}(h_k)^{\frac{1}{g_k}} = \frac{q^{q^{\frac{r}{2}}}}{(q^r-1)^{\frac{1}{r}(q^{\frac{r}{2}}-1)}}$. But

$$\frac{q^{q^{\frac{r}{2}}}}{(q^r-1)^{\frac{1}{r}(q^{\frac{r}{2}}-1)}} > \frac{q^{q^{\frac{r}{2}}}}{(q^r)^{\frac{1}{r}(q^{\frac{r}{2}}-1)}} = q.$$

Hence, for a sufficiently large k_0, we have for $k \geq k_0$ the following inequality

$$(h_k)^{\frac{1}{g_k}} > q.$$

□

Let us compare this estimation of h_k to the general lower bounds given by G. Lachaud and M. Martin-Deschamps in [**6**].

***Theorem 2.6* (Lachaud - Martin-Deschamps bounds)**. — *Let X be a projective irreducible and non-singular algebraic curve defined over the finite field $\mathbb{F}_q$ of genus g. Let J_X be the Jacobian of X and h the class number $h = |J_X(\mathbb{F}_q)|$. Then*

1. $h \geq L_1 = q^{g-1}\frac{(q-1)^2}{(q+1)(g+1)}$,
2. $h \geq L_2 = \left(\sqrt{q}-1\right)^2 \frac{q^{g-1}-1}{g}\frac{|X(\mathbb{F}_q)|+q-1}{q-1}$,
3. *if $g > \sqrt{q}/2$ and if $B_1(X/\mathbb{F}_q) \geq 1$, then the following holds:*

$$h \geq L_3 = (q^g-1)\frac{q-1}{q+g+gq}.$$

Then we can prove that for a family of algebraic function fields satisfying the conditions of Corollary 2.5, the class numbers h_k greatly exceeds the bounds L_i. More precisely

Proposition 2.7. — *Let $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ be a sequence of algebraic function fields of increasing genus defined over $\mathbb{F}_q$ such that $\beta_r(\mathcal{F}/\mathbb{F}_q) = \frac{1}{r}(q^{\frac{r}{2}}-1)$ where r is an integer. Then*

1. *for $i = 1, 3$*

$$lim_{k\to+\infty}\frac{h_k}{L_i} = +\infty,$$

2. *for $i = 2$ the following holds:*
   (a) *if r>1 then*

$$lim_{k\to+\infty}\frac{h_k}{L_2} = +\infty,$$

   (b) *if r=1 then*

$$\frac{h_k}{L_2} \geq 2.5.$$

Proof. — 1. case $i = 1$: the following holds

$$L_1 = q^{g_k-1}\frac{(q-1)^2}{(q+1)(g_k+1)} = q^{g_k}\frac{(q-1)^2}{q(q+1)(g_k+1)} < \frac{q^{g_k}}{(g_k+1)},$$

so, using the previous corollary 2.5, we conclude that for k large

$$\frac{h_k}{L_1} > g_k$$

and consequently

$$\lim_{k\to+\infty} \frac{h_k}{L_1} = +\infty;$$

2. case $i = 2$:
   (a) case $r = 1$: we bound the number of rational points using the Weil bound. More precisely

$$L_2 = (q+1-2\sqrt{q})\,\frac{q^{g_k-1}-1}{g_k}\,\frac{B_1(F_k/\mathbb{F}_q)+q-1}{q-1} \leq$$

$$(q+1-2\sqrt{q})\,\frac{q^{g_k-1}-1}{g_k}\,\frac{2q+2g_k\sqrt{q}}{q-1} <$$

$$2\,\frac{q+1-2\sqrt{q}}{(q-1)\sqrt{q}}q^{g_k}\left(1+\frac{\sqrt{q}}{g_k}\right);$$

   but for all $q \geq 2$

$$2\,\frac{q+1-2\sqrt{q}}{(q-1)\sqrt{q}} < 0.4,$$

   then

$$L_2 < 0.4\left(1+\frac{\sqrt{q}}{g_k}\right)q^{g_k},$$

   hence

$$\frac{h_k}{L_2} > 2.5\,\frac{g_k}{g_k+\sqrt{q}}$$

   which gives the result;
   (b) case $r > 1$: in this case we know that

$$\lim_{k\to+\infty} \frac{B_1(F_k/\mathbb{F}_q)}{g_k} = \beta_1(\mathcal{F}/\mathbb{F}_q) = 0.$$

   But

$$L_2 < \frac{q+1-2\sqrt{q}}{(q-1)q}q^{g_k}\,\frac{B_1(F_k/\mathbb{F}_q)+q-1}{g_k}.$$

   Then

$$\frac{h_k}{L_2} > \frac{(q-1)q}{q+1-2\sqrt{q}}\;\frac{g_k}{B_1(F_k/\mathbb{F}_q)+q-1}.$$

   We know that

$$\lim_{k\to+\infty} \frac{g_k}{B_1(F_k/\mathbb{F}_q)+q-1} = +\infty,$$

   then

$$\lim_{k\to+\infty} \frac{h_k}{L_2} = +\infty.$$

3. case $i = 3$:
$$L_3 = (q^g - 1)\frac{q-1}{q+g+gq} < \frac{q^{g_k}}{g_k},$$
then for k large
$$\frac{h_k}{L_3} > g_k$$
and consequently
$$\lim_{k\to+\infty} \frac{h_k}{L_1} = +\infty.$$

□

As we see, if $\mathcal{F}/\mathbb{F}_q = (F_k/\mathbb{F}_q)_{k\geq 1}$ satisfies the assumptions of Theorem 2.4, we have $h_k > q^{g_k} > q^{g_k-1}\frac{(q-1)^2}{(q+1)g_k}$ for $k \geq k_0$ sufficiently large. In fact, the value k_0 depends at least on the values of r and q and we can not know anything about this value in the general case.

Remark: We can remark that the class number of these families is very near the Lachaud - Martin-Deschamps bound L_2 when $r = 1$ but is much greater than the Lachaud - Martin-Deschamps bound L_2 when $r > 1$.

3. Examples of asymptotically exact towers

Let us note $\mathbb{F}_{q^2}$ a finite field with $q = p^r$ and r an integer.

3.1. Sequences $\mathcal{F}/\mathbb{F}_q$ with $\beta_2(F/\mathbb{F}_q) = \frac{1}{2}(q-1) = A_2(q)$. — We consider the Garcia-Stichtenoth's tower T_0 over $\mathbb{F}_{q^2}$ constructed in [**4**]. Recall that this tower is defined recursively in the following way. We set $F_1 = \mathbb{F}_{q^2}(x_1)$ the rational function field over $\mathbb{F}_{q^2}$, and for $i \geq 1$ we define
$$F_{i+1} = F_i(z_{i+1}),$$
where z_{i+1} satisfies the equation
$$z_{i+1}^q + z_{i+1} = x_i^{q+1},$$
with
$$x_i = \frac{z_i}{x_{i-1}} \text{ for } i \geq 2.$$
We consider the completed Garcia-Stichtenoth's tower $T_1/\mathbb{F}_{q^2}$ defined over $\mathbb{F}_{q^2}$ studied in [**2**] obtained from $T_0/\mathbb{F}_{q^2}$ by adjunction of intermediate steps. Namely we have
$$T_1/\mathbb{F}_{q^2} : F_{1,0} \subset \cdots \subset F_{i,0} \subset F_{i,1} \subset \cdots \subset F_{i,s} \subset \cdots \subset F_{i+1,0} \subset \cdots$$
with $s = 0, ..., r$. Note that the steps $F_{i,0}/\mathbb{F}_{q^2} = F_{i-1,r}/\mathbb{F}_{q^2}$ are the steps $F_i/\mathbb{F}_{q^2}$ of the Garcia-Stichtenoth's tower $T_0/\mathbb{F}_{q^2}$ and $F_{i,s}/\mathbb{F}_{q^2}$ $(1 \leq s \leq r-1)$ are the intermediate steps considered in [**2**].
Let us denote by g_k the genus of $F_k/\mathbb{F}_{q^2}$ in $T_0/\mathbb{F}_{q^2}$, by $g_{k,s}$ the genus of $F_{k,s}$ in $T_1/\mathbb{F}_{q^2}$ and by $B_1(F_{k,s}/\mathbb{F}_{q^2})$ the number of places of degree one of $F_{k,s}/\mathbb{F}_{q^2}$ in $T_1/\mathbb{F}_{q^2}$.

Recall that each extension $F_{k,s}/F_k$ is Galois of degree p^s with full constant field $\mathbb{F}_{q^2}$. Moreover, we know by [**3**, Theorem 4.3] that the descent of the definition field of the tower $T_1/\mathbb{F}_{q^2}$ from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ is possible. More precisely, there exists a tower $T_2/\mathbb{F}_q$ defined over $\mathbb{F}_q$ given by a sequence:

$$T_2/\mathbb{F}_q : G_{1,0} \subset \cdots \subset G_{i,0} \subset G_{i,1} \subset \cdots \subset G_{i,r-1} \subset G_{i+1,0} \subset \cdots$$

defined over the constant field $\mathbb{F}_q$ and related to the tower $T_1/\mathbb{F}_{q^2}$ by

$$F_{k,s} = \mathbb{F}_{q^2} G_{k,s} \quad \text{for all } k \text{ and } s,$$

namely $F_{k,s}/\mathbb{F}_{q^2}$ is the constant field extension of $G_{k,s}/\mathbb{F}_q$. First, let us study the asymptotic behavior of degree one places of the function fields of the tower $T_2/\mathbb{F}_q$ and more precisely the existence and the value of $\beta_1(T_2/\mathbb{F}_q)$. In order to derive a result on the tower $T_2/\mathbb{F}_q$ we begin by the study of the terms given by the descent of the classical Garcia-Stichtenoth tower $T_0/\mathbb{F}_{q^2}$. Next we will study the intermediate steps.

Lemma 3.1. — *Let $T_0/\mathbb{F}_{q^2} = (F_k/\mathbb{F}_{q^2})_{k \geq 1}$ the Garcia-Stichtenoth tower defined over $\mathbb{F}_{q^2}$ and $T_0'/\mathbb{F}_q = (G_k/\mathbb{F}_q)_{k \geq 1}$ its descent over the definition field $\mathbb{F}_q$ i.e such that for any integer k, $F_k = \mathbb{F}_{q^2} G_k$. Then*

$$\beta_1(T_0'/\mathbb{F}_q) = \lim_{k \to +\infty} \frac{B_1(G_k/\mathbb{F}_q)}{g(G_k/\mathbb{F}_q)} = 0.$$

Proof. — First, note that if the algebraic function field $F_k/\mathbb{F}_{q^2}$ is a constant field extension of $G_k/\mathbb{F}_q$, above any place of degree one in $G_k/\mathbb{F}_q$ there exists a unique place of degree one in $F_k/\mathbb{F}_{q^2}$. Consequently, let us use the classification given in [**4**, p. 221] of the places of degree one of $F_k/\mathbb{F}_{q^2}$. Let us remark that the number of places of degree one which are not of type (A), is less or equal to $2q^2$ (see [**4**, Remark 3.4]). Moreover, the genus g_k of the algebraic function fields $G_k/\mathbb{F}_q$ and $F_k/\mathbb{F}_{q^2}$ is such that $g_k \geq q^k$ by the Hurwitz theorem, then we can focus our study on places of type (A). The places of type (A) are built recursively in the following way (cf. [**4**, p. 220 and Proposition 1.1 (iv)]). Let $\alpha \in \mathbb{F}_{q^2} \setminus \{0\}$ and P_α be the place of $F_1/\mathbb{F}_{q^2}$ which is the zero of $x_1 - \alpha$. For any $\alpha \in \mathbb{F}_{q^2} \setminus \{0\}$ the polynomial equation $z_2^q + z_2 = \alpha^{q+1}$ has q distinct roots $u_1, \cdots u_q$ in $\mathbb{F}_{q^2}$, and for each u_i there is a unique place $P_{(\alpha,i)}$ of $F_2/\mathbb{F}_{q^2}$ above P_α and this place $P_{(\alpha,i)}$ is a zero of $z_2 - u_i$. We iterate now the process starting from the places $P_{(\alpha,i)}$ to obtain successively the places of type (A) of $F_3/\mathbb{F}_{q^2}, \cdots, F_k/\mathbb{F}_{q^2}, \cdots$; then, each place P of type (A) of $F_k/\mathbb{F}_{q^2}$ is a zero of $z_k - u$ where u is itself a zero of $u^q + u = \gamma$ for some $\gamma \neq 0$ in $\mathbb{F}_{q^2}$. Let us denote by P_u this place. Now, we want to count the number of places P_u' of degree one in $G_k/\mathbb{F}_q$, that is to say the only places which admit a unique place of degree one P_u in $F_k/\mathbb{F}_{q^2}$ lying over P_u'. First, note that it is possible only if u is a solution in $\mathbb{F}_q$ of the equation $u^q + u = \gamma$ where γ is in $\mathbb{F}_q \setminus \{0\}$. Indeed, if u is not in $\mathbb{F}_q$, there exists an automorphism σ in the Galois group $Gal(F_k/G_k)$ of the degree two Galois extension $F_k/\mathbb{F}_{q^2}$ of $G_k/\mathbb{F}_q$ such that $\sigma(P_u) \neq P_u$. Hence, the unique place of $G_k/\mathbb{F}_q$ lying under P_u is a place of degree 2. But $u^q + u = \gamma$ has one solution in $\mathbb{F}_q$ if $p \neq 2$ and no solution in $\mathbb{F}_q$ if $p = 2$. Hence the number of places of degree one of $G_k/\mathbb{F}_q$

which are lying under a place of type (A) of $F_k/\mathbb{F}_{q^2}$ is equal to zero if $p=2$ and equal to $q-1$ if $p\neq 2$. We conclude that

$$\lim_{k\to+\infty}\frac{B_1(G_k/\mathbb{F}_q)}{g(G_k/\mathbb{F}_q)}=0.$$

Let us remark that in any case, the number of places of degree one of $G_k/\mathbb{F}_q$ is less or equal to $2q^2$. □

Now we can get a similar result for the descent $T_2/\mathbb{F}_q$ of the densified tower $T_1/\mathbb{F}_{q^2}$ of $T_0/\mathbb{F}_{q^2}$.

Lemma 3.2. — *The tower $T_2/\mathbb{F}_q$ is such that:*

$$\beta_1(T_2/\mathbb{F}_q)=\lim_{g(G_{k,s}/\mathbb{F}_q)\to+\infty}\frac{B_1(G_{k,s}/\mathbb{F}_q)}{g(G_{k,s}/\mathbb{F}_q)}=0.$$

Proof. — As $G_{k+1}=G_{k+1,0}$ is an extension of $G_{k,s}$ we get $B_1(G_{k,s}/\mathbb{F}_q)\leq B_1(G_{k+1}/\mathbb{F}_q)$. Using the computation done in the proof of Lemma 3.1 and Remark 3.4 in [**4**] we have $B_1(G_{k+1}/\mathbb{F}_q)\leq 2q^2$, then we get $B_1(G_{k,s}/\mathbb{F}_q)\leq 2q^2$. By [**2**, Corollary 2.2] we know that

$$\lim_{l\to+\infty}\frac{g(G_{l+1}/\mathbb{F}_q)}{g(G_l/\mathbb{F}_q)}=p$$

where $g(G_l/\mathbb{F}_q)$ and $g(G_{l+1}/\mathbb{F}_q)$ denote the genus of two consecutive algebraic function fields in $T_2/\mathbb{F}_q$. Then for k sufficiently large we get

$$g_{k,s}\geq g_{k,0}=g_k.$$

We conclude that $\beta_1(T_2/\mathbb{F}_q)=0$. □

Let us prove a proposition establishing that the tower $T_2/\mathbb{F}_q$ is asymptotically exact with good density.

Proposition 3.3. — *Let $q=p^r$. For any integer $k\geq 1$, for any integer s such that $s=0,1,...,r$, the algebraic function field $G_{k,s}/\mathbb{F}_q$ in the tower T_2 has a genus $g(G_{k,s}/\mathbb{F}_q)=g_{k,s}$ with $B_1(G_{k,s}/\mathbb{F}_q)$ places of degree one, $B_2(G_{k,s}/\mathbb{F}_q)$ places of degree two such that:*

1. $g(G_{k,s}/\mathbb{F}_q)\leq\frac{g(G_{k+1}/\mathbb{F}_q)}{p^{r-s}}+1$ *with* $g(G_{k+1}/\mathbb{F}_q)=g_{k+1}\leq q^{k+1}+q^k$.
2. $B_1(G_{k,s}/\mathbb{F}_q)+2B_2(G_{k,s}/\mathbb{F}_q)\geq(q^2-1)q^{k-1}p^s$.
3. $\beta_2(T_2/\mathbb{F}_q)=\lim_{g_{k,s}\to+\infty}\frac{B_2(G_{k,s}/\mathbb{F}_q)}{g_{k,s}}=\frac{1}{2}(q-1)=A_2(q)$.
4. $d(T_2/\mathbb{F}_q)=\lim_{l\to+\infty}\frac{g(G_l/\mathbb{F}_q)}{g(G_{l+1}/\mathbb{F}_q)}=\frac{1}{p}$ *where $g(G_l/\mathbb{F}_q)$ and $g(G_{l+1}/\mathbb{F}_q)$ denote the genus of two consecutive algebraic function fields in $T_2/\mathbb{F}_q$.*

Proof. — By Theorem 2.2 in [**2**], we have $g(F_{k,s}/\mathbb{F}_{q^2})\leq\frac{g(F_{k+1}/\mathbb{F}_{q^2})}{p^{r-s}}+1$ with $g(F_{k+1}/\mathbb{F}_{q^2})=g_{k+1}\leq q^{k+1}+q^k$. Then, as the algebraic function field $F_{k,s}/\mathbb{F}_{q^2}$ is a constant field extension of $G_{k,s}/\mathbb{F}_q$, for any integer k and $s=0,1$ or 2, the algebraic function fields $F_{k,s}/\mathbb{F}_{q^2}$ and $G_{k,s}/\mathbb{F}_q$ have the same genus. So, the inequality satisfied by the genus $g(F_{k,s}/\mathbb{F}_{q^2})$ is also true for the genus $g(G_{k,s}/\mathbb{F}_q)$. Moreover, the number of places of degree one $B_1(F_{k,s}/\mathbb{F}_{q^2})$ of $F_{k,s}/\mathbb{F}_{q^2}$ is such that $B_1(F_{k,s}/\mathbb{F}_{q^2})\geq(q^2-1)q^{k-1}p^s$. Then, as the algebraic function field

$F_{k,s}/\mathbb{F}_{q^2}$ is a constant field extension of $G_{k,s}/\mathbb{F}_q$ of degree 2, it is clear that for any integer k and s, we have $B_1(G_{k,s}/\mathbb{F}_q) + 2B_2(G_{k,s}/\mathbb{F}_q) \geq (q^2-1)q^{k-1}p^s$. Moreover, we know that $\beta_1(T_2/\mathbb{F}_q) = 0$ by Lemma 3.2. But $B_1(G_{k,s}/\mathbb{F}_q) + 2B_2(G_{k,s}/\mathbb{F}_q) = B_1(F_{k,s}/\mathbb{F}_{q^2})$ and as by [**4**], $\beta_1(T_1/\mathbb{F}_{q^2}) = A(q^2)$, we have $\beta_2(T_2/\mathbb{F}_q) = \frac{1}{2}(q-1)$.

□

In particular, the following result holds:

Corollary 3.4. — *For any prime power q, there exists a sequence of algebraic function fields defined over the finite field $\mathbb{F}_q$ reaching the Generalized Drinfeld-Vladut bound.*

3.2. Sequences $\mathcal{F}/\mathbb{F}_2$ with $\beta_4(\mathcal{F}/\mathbb{F}_2) = \frac{3}{4} = A_4(2)$. — We use the notations of the previous paragraph concerning the towers $T_1/\mathbb{F}_{q^2}$ and $T_2/\mathbb{F}_q$. Now we suppose that $q = p^2$ and we ask the following question: is the descent of the definition field of the tower $T_1/\mathbb{F}_{q^2}$ from $\mathbb{F}_{q^2}$ to $\mathbb{F}_p$ possible? The following result gives a positive answer for the case $p = 2$.

Proposition 3.5. — *Let $p = 2$. If $q = p^2$, the descent of the definition field of the tower $T_1/\mathbb{F}_{q^2}$ from $\mathbb{F}_{q^2}$ to $\mathbb{F}_p$ is possible. More precisely, there exists a tower $T_3/\mathbb{F}_p$ defined over $\mathbb{F}_p$ given by a sequence:*

$$T_3/\mathbb{F}_p = H_{1,0} \subseteq H_{1,1} \subseteq H_{2,0} \subseteq H_{2,1} \subseteq ...$$

defined over the constant field $\mathbb{F}_p$ and related to the towers $T_1/\mathbb{F}_{q^2}$ and $T_2/\mathbb{F}_q$ by

$$F_{k,s} = \mathbb{F}_{q^2} H_{k,s} \quad \textit{for all } k \textit{ and } s = 0, 1 \textit{ or } 2,$$

$$G_{k,s} = \mathbb{F}_q H_{k,s} \quad \textit{for all } k \textit{ and } s = 0, 1 \textit{ or } 2,$$

namely $F_{k,s}/\mathbb{F}_{q^2}$ is the constant field extension of $G_{k,s}/\mathbb{F}_q$ and $H_{k,s}/\mathbb{F}_p$ and $G_{k,s}/\mathbb{F}_q$ is the constant field extension of $H_{k,s}/\mathbb{F}_p$.

Proof. — Let x_1 be a transcendent element over $\mathbb{F}_2$ and let us set

$$H_1 = \mathbb{F}_2(x_1), G_1 = \mathbb{F}_4(x_1), F_1 = \mathbb{F}_{16}(x_1).$$

We define recursively for $k \geq 1$

1. z_{k+1} such that $z_{k+1}^4 + z_{k+1} = x_k^5$,
2. t_{k+1} such that $t_{k+1}^2 + t_{k+1} = x_k^5$
   (or alternatively $t_{k+1} = z_{k+1}(z_{k+1}+1)$),
3. $x_k = z_k/x_{k-1}$ if $k > 1$ (x_1 is yet defined),
4. $H_{k,1} = H_{k,0}(t_{k+1}) = H_k(t_{k+1})$, $H_{k+1,0} = H_{k+1} = H_k(z_{k+1})$, $G_{k,1} = G_{k,0}(t_{k+1}) = G_k(t_{k+1})$, $G_{k+1,0} = G_{k+1} = G_k(z_{k+1})$, $F_{k,1} = F_{k,0}(t_{k+1}) = F_k(t_{k+1})$, $F_{k+1,0} = F_{k+1} = F_k(z_{k+1})$.

By [**3**], the tower

$$T_1/\mathbb{F}_{q^2} = (F_{k,i}/\mathbb{F}_{q^2})_{k\geq 1, i=0,1}$$

is the densified Garcia-Stichtenoth's tower over $\mathbb{F}_{16}$ and the two other towers $T_2/\mathbb{F}_q$ and $T_3/\mathbb{F}_p$ are respectively the descent of $T_1/\mathbb{F}_{q^2}$ over $\mathbb{F}_4$ and over $\mathbb{F}_2$. □

Proposition 3.6. — *Let $q = p^2 = 4$. For any integer $k \geq 1$, for any integer s such that $s = 0, 1$ or 2, the algebraic function field $H_{k,s}/\mathbb{F}_p$ in the tower T_3 has a genus $g(H_{k,s}/\mathbb{F}_p) = g_{k,s}$ with $B_1(H_{k,s}/\mathbb{F}_p)$ places of degree one, $B_2(H_{k,s}/\mathbb{F}_p)$ places of degree two and $B_4(H_{k,s}/\mathbb{F}_p)$ places of degree 4 such that:*

1. $g(H_{k,s}/\mathbb{F}_p) \leq \frac{g(H_{k+1}/\mathbb{F}_p)}{p^{r-s}} + 1$ *with* $g(H_{k+1}/\mathbb{F}_p) = g_{k+1} \leq q^{k+1} + q^k$.
2. $B_1(H_{k,s}/\mathbb{F}_p) + 2B_2(H_{k,s}/\mathbb{F}_p) + 4B_4(H_{k,s}/\mathbb{F}_p) \geq (q^2 - 1)q^{k-1}p^s$.
3. $\beta_4(T_3/\mathbb{F}_p) = \lim_{g_{k,s} \to +\infty} \frac{B_4(H_{k,s}/\mathbb{F}_p)}{g_{k,s}} = \frac{1}{4}(p^2 - 1) = \frac{3}{4} = A_4(2)$.
4. $d(T_3/\mathbb{F}_p) = \lim_{l \to +\infty} \frac{g(H_l/\mathbb{F}_p)}{g(H_{l+1}/\mathbb{F}_p)} = \frac{1}{2}$ *where* $g(H_l/\mathbb{F}_p)$ *and* $g(H_{l+1}/\mathbb{F}_p)$ *denote the genus of two consecutive algebraic function fields in* $T_3/\mathbb{F}_p$.

Proof. — By Theorem 2.2 in [**2**], we have $g(F_{k,s}/\mathbb{F}_{q^2}) \leq \frac{g(F_{k+1}/\mathbb{F}_{q^2})}{p^{r-s}} + 1$ with $g(F_{k+1}/\mathbb{F}_{q^2}) = g_{k+1} \leq q^{k+1} + q^k$. Then, as the algebraic function field $F_{k,s}/\mathbb{F}_{q^2}$ is a constant field extension of $H_{k,s}/\mathbb{F}_p$, for any integer k and $s = 0, 1$ or 2, the algebraic function fields $F_{k,s}/\mathbb{F}_{q^2}$ and $H_{k,s}/\mathbb{F}_p$ have the same genus. So, the inequality satisfied by the genus $g(F_{k,s}/\mathbb{F}_{q^2})$ is also true for the genus $g(H_{k,s}/\mathbb{F}_p)$. Moreover, the number of places of degree one $B_1(F_{k,s}/\mathbb{F}_{q^2})$ of $F_{k,s}/\mathbb{F}_{q^2}$ is such that $B_1(F_{k,s}/\mathbb{F}_{q^2}) \geq (q^2 - 1)q^{k-1}p^s$. Then, as the algebraic function field $F_{k,s}/\mathbb{F}_{q^2}$ is a constant field extension of $H_{k,s}/\mathbb{F}_p$ of degree 4, it is clear that for any integer k and s, we have $B_1(H_{k,s}/\mathbb{F}_p) + 2B_2(H_{k,s}/\mathbb{F}_p) + 4B_4(H_{k,s}/\mathbb{F}_p) \geq (q^2 - 1)q^{k-1}p^s$. Moreover, we have shown in the proof of Lemma 3.2 that for any integer $k \geq 1$ and any $0 \leq s \leq 2$ the number of places of degree one $B_1(G_{k,s}/\mathbb{F}_q)$ of $G_{k,s}/\mathbb{F}_q$ is less or equal to $2q^2$ and so $\beta_1(T_2/\mathbb{F}_q) = 0$. Then, as the algebraic function field $G_{k,s}/\mathbb{F}_q$ is a constant field extension of $H_{k,s}/\mathbb{F}_p$ of degree 2, it is clear that for any integer k and s, we have $B_1(H_{k,s}/\mathbb{F}_p) + 2B_2(H_{k,s}/\mathbb{F}_p) = B_1(G_{k,s}/\mathbb{F}_q)$ and so $\beta_1(T_3/\mathbb{F}_p) = \beta_2(T_3/\mathbb{F}_p) = 0$. Moreover, $B_1(H_{k,s}/\mathbb{F}_p) + 2B_2(H_{k,s}/\mathbb{F}_p) + 4B_4(H_{k,s}/\mathbb{F}_p) = B_1(F_{k,s}/\mathbb{F}_{q^2})$ and as by [**4**], $\beta_1(T_1/\mathbb{F}_{q^2}) = A(q^2)$, we have $\beta_4(T_3/\mathbb{F}_p) = A(p^4) = p^2 - 1$. □

Corollary 3.7. — *Let $T_3/\mathbb{F}_2 = (H_{k,s}/\mathbb{F}_2)_{k \in \mathbb{N}, s=0,1,2}$ be the tower defined above. Then the tower $T_3/\mathbb{F}_2$ is an asymptotically exact sequence of algebraic function fields defined over $\mathbb{F}_2$ with a maximal density (for a tower).*

Proof. — It follows from (4) of Proposition 3.3. □

4. Open questions

1. Find asymptotically exact sequences of algebraic function fields defined over any finite field $\mathbb{F}_q$, of type $\beta = (\beta_1, \beta_2, ..., \beta_m, ...)$ with several $\beta_i > 0$, attaining the generalized Drinfeld-Vladut bound (i.e maximal or not).
2. Find asymptotically exact sequences of algebraic function fields defined over any finite field $\mathbb{F}_q$, attaining the Drinfeld-Vladut bound of order r for any integer $r > 2$ (except case q=2 and r=4 solved in this paper).

3. Find explicit asymptotically exact sequences of algebraic function fields (not Artin-Schreier type) defined over any finite field $\mathbb{F}_q$ having the good preceding properties.

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