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# Vector field methods for kinetic equations with applications to classical and relativistic systems

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**Abstract**

The aim of this talk is to present an extension of the vector field method of Klainerman, which is typically applied in the context of non-linear wave equations, to the case of kinetic equations of Vlasov type. We first describe how our method yields sharp decay estimates for velocity averages for the linear classical and relativistic transport equations and then explain how it can be applied to various models of mathematical physics, such as the Vlasov-Poisson, Vlasov-Nordström and Vlasov-Einstein systems.

## 1 Introduction: the basic decay estimate

The vector field method of Klainerman [1] is a very powerful tool to obtain robust decay estimates for solutions to wave equations. The aim of our work is to explain how such a method can be adapted to the study of kinetic transport equations. Consider for instance the relativistic transport equation

$$T(f) := \left[ \left(m^2 + |v|^2\right)^{1/2} \partial_t + v^i \partial_{x^i} \right] (f) = 0, \tag{1}$$

where the parameter $m \geq 0$ is the mass of the particles, $f := f(t, x, v)$ represents the density of particles with $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ if $m > 0$ corresponding to *massive* particles , $v \in \mathbb{R}^n \setminus \{0\}$ if $m = 0$, corresponding to *massless* particles. Since (1) is a transport equation, $f$ is preserved along the characteristics associated to the equation. However, the macroscopic quantities obtained by integrating $f$ in $v$, such as

$$\rho[f](t, x) \equiv \int_v f(t, x, v) \frac{dv}{\sqrt{m^2 + |v|^2}}, \tag{2}$$

are only conserved as functions of $t$ in $L^1_x$, and will enjoy decay properties as $t \to +\infty$ in $L^\infty_x$. To prove this, the standard method, which follows earlier work of Bardos-Degond for the classical transport operator [2], consists in writing explicitly the solution in terms of its initial data using the conservation of $f$ along characteristics, and then estimating directly the $v$-integral in (2). For the massive case $m > 0$, this leads to an estimate of the form, for all $t > 0$ and all $x \in \mathbb{R}^n_x$,

$$\rho[|f|](t, x) \leq \frac{C(V)}{t^n} ||f(t = 0)||_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)},$$

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where $C(V)$ is a constant depending on an upper bound $V$ of the size of the support in $v$ of the initial data, for instance, assuming the data to be smooth and compactly supported,

$$V = \sup \{\lambda \in \mathbb{R}_+ \,:\, \exists (x,v) \in \mathbb{R}^n_x \times \mathbb{R}^n_v \,:\, \lambda = |v| \, \text{and} \, f(0,x,v) \neq 0\}.$$

Note that $C(V) \to +\infty$ as $V \to +\infty$, so that, unless more refined estimates are used, this method requires compact support of the initial data to work. We prove instead the estimate

**Proposition 1** (Decay estimates for velocity averages of massive distributions [3])**.** *For any regular distribution function $f$ solution to* (1) *with $m > 0$, any $x \in \mathbb{R}^n$ and any $t \geq \sqrt{1+|x|^2}$, we have*

$$\rho[|f|](t,x) \leq \frac{C}{(1+t)^n} \sum_{\substack{|\alpha| \leq n \\ \widehat{Z}^\alpha \in \widehat{\mathbb{P}}^{|\alpha|}}} \left\| \widehat{Z}^\alpha(f)_{|H_1^n \times \mathbb{R}^n_v} v_\alpha \nu_1^\alpha \right\|_{L^1\left(H_1^n \times \mathbb{R}^n_v\right)}, \tag{3}$$

*where $H_1^n$ denotes the unit hyperboloid $H_1^n := \left\{(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x \,/\, 1 = t^2 - x^2\right\}$, $\widehat{Z}^\alpha(f)_{|H_1^n \times \mathbb{R}^n_v}$ is the restriction to $H_1^n \times \mathbb{R}^n_v$ of $\hat{Z}^\alpha(f)$, $v_\alpha \nu_1^\alpha$ is the contraction of $(\sqrt{m^2+|v|^2}, v^i)$ with the unit normal $\nu_1$ to $H_1^n$ and where the $\widehat{Z}^\alpha$ are differential operators obtained as a composition of $|\alpha|$ vector fields of the algebra $\widehat{\mathbb{P}}$.*

The algebra of vector fields $\widehat{\mathbb{P}}$ is obtained by taking the *complete lifts* of the usual Killing vector fields of Minkowski space, a classical operation in differential geometry.

**Definition 1.** *Let $W$ be a vector field of the form $W = W^\alpha \partial_{x^\alpha}$, then let*

$$\widehat{W} = W^\alpha \partial_{x^\alpha} + v^\beta \frac{\partial W^i}{\partial x^\beta} \partial_{v^i}, \tag{4}$$

*where $\left(v^\beta\right)_{\beta=0,..,n} = (v^0, v^1, .., v^n)$ with $v^0 = |v|$ in the massless case, $v^0 = \sqrt{1+|v|^2}$ in the massive case, be called the* complete lift[1] *of $W$.*

For instance, the complete lift of a rotation vector field $x^i \partial_{x^j} - x^j \partial_{x^i}$ is given by the vector field $x^i \partial_{x^j} - x^j \partial_{x^i} + v^i \partial_{v^j} - v^j \partial_{v^i}$.

Note that in Proposition 1, there is no requirements of compact support in $v$ of the initial data. Moreover, using finite speed of propagation type arguments, one can easily see that for solutions arising from smooth initial data of compact support in $x$ and decaying sufficiently fast in $v$ (but not necessarily of compact support in $v$) given at $t=0$, the norm on the right-hand side of (3) is finite, so that the usage of hyperboloids is mostly technical.

In the case of massless particles ($m=0$), a similar estimate holds with the decay rates being weaker near the light-cone, as in the case of the wave equation.

**Proposition 2** (Decay estimates for velocity averages of massless distribution functions [3])**.** *For any regular distribution function $f$, solution to* (1) *with $m = 0$ and any $(t,x) \in \mathbb{R}^+_t \times \mathbb{R}^n_x$, we have*

$$\int_{v \in \mathbb{R}^n_v \setminus \{0\}} |f|(t,x,v) \frac{dv}{|v|} \lesssim \frac{1}{(1+|t-|x|\,|)(1+|t+|x|\,|)^{n-1}} \cdot \sum_{\substack{|\alpha| \leq n, \\ \widehat{Z}^\alpha \in \widehat{K}^{|\alpha|}}} \left\| |v|^{-1} \widehat{Z}^\alpha(f)(t=0) \right\|_{L^1(\mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\}))},$$

[1] This is in fact a small abuse of notation, as, with the above definition, $\widehat{W}$ actually corresponds to the restriction of the complete lift of $W$ to the submanifold corresponding to $v^0 = \sqrt{1+|v|^2}$ in the massive case and $v^0 = |v|$ in the massless case.

*where the $\alpha$ are multi-indices of length $|\alpha|$ and the $\widehat{Z}^\alpha$ are differential operators of order $|\alpha|$ obtained as a composition of $|\alpha|$ vector fields of the algebra $\widehat{K}$.*

The algebra of vector fields $\widehat{K}$ contains the complete lifts of the Killing vector fields as above as well as the scaling vector field (without completion) $x^\alpha \partial_{x^\alpha}$.

Let us mention that the analogue of the above decay estimates for the classical transport operator $\partial_t + v^i \partial_{x^i}$ was proven in [7].

# 2 Applications

In the second part of the talk, I presented several applications of these decay estimates to the Vlasov-Nordström and Vlasov-Poisson systems[2]. We focus here on the Vlasov-Nordström system. This mathematical model can be viewed as a poor's man version of the Einstein-Vlasov system of general relativity, in which the tensorial and some non-linear aspects of general relativity are forgotten but the wave nature of the Einstein equations remain.

More precisely, the Vlasov-Nordström system (for massive particles) is given with $n$ spatial dimensions by the equations

$$\Box \phi = m^2 \int_v f \frac{dv}{\sqrt{m^2 + |v|^2}}, \tag{5}$$

$$T_\phi(f) := T(f) - \left(T(\phi)v^i + m^2 \nabla^i \phi\right) \frac{\partial f}{\partial v^i} = (n+1) f \, T(\phi), \tag{6}$$

where $m > 0$ is the mass of particles, $T \equiv v^\alpha \partial_{x^\alpha}$, with $v^0 = \sqrt{m^2 + |v|^2}$, is the relativistic free transport operator, $\Box \equiv -\partial_t^2 + \sum_{i=1}^n \partial_{x^i}^2$ is the standard wave operator of Minkowski space, $\phi$ is a scalar function of $(t, x)$ and $f$ is a function of $(t, x, v^i)$ with $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. A detailed introduction to this system can be found in [9]. See also the classical works [10, 12, 11].

In [13], a small data global existence in dimension three was obtained for this system, deriving in particular decay estimates in time for the wave $\phi$ and the velocity averages of $f$ for data of compact support. The strategy of [13], similar to the strategy of [14] for the Vlasov-Maxwell system[3], consists in using decay estimates for the velocity averages of $f$ based on the method of characteristics and the compact support assumptions, together with representation formulae for the wave equation. In particular, no decay estimates for the derivatives of the velocity averages of $f$ or the higher order derivatives of the wave were derived.

## 2.1 Vector-fields and modified-vector-fields approach

In [3], we introduce a novel approach to the study of coupled systems of wave and transport equations, based on the vector-field method of Klainerman and the decay estimates presented above. In particular, this method allows for a systematic study of systems such as the Vlasov-Nordström system and we obtain sharp (or almost sharp) asymptotics for the solution and its derivatives in the case of either massive particles ($m > 0$) in dimensions $n \geq 4$ or massless particles up to dimension 3. Our strategy is based on commuting the transport equation by the complete lift $\widehat{Z}$ of the Killing fields $Z$ of Minkowski space.

By construction, the vector fields $\widehat{Z}$ are then differential operators that commute exactly with the free transport operator $T$. However, for the non-linear

[2]See for instance the classical work [19] for a presentation of the Vlasov-Poisson system and other related kinetic systems.

[3]See also [2] for the Vlasov-Poisson system.

system, the commutation of $T_\phi$ and $\widehat{Z}$ introduces error terms which then need to be integrable in space-time for the estimates to close. Contrary to, for instance, a non-linear wave equation of the form $\Box\phi = Q(\partial\phi, \partial\phi)$, the transport equation (6) enjoys poor commutation properties, in the sense that commuting with any of the vector fields $\widehat{Z}$ generates error terms of the form $(Z\partial\phi)\cdot\partial_v f$. These error terms are problematic because the vector fields $\partial_{v^j}$ do not commute[4] with the free transport operator, so that they generate an extra growth which is roughly proportional to $t$. Because of this extra growth, the techniques introduced in [3] could only handle massive particles in high dimensions ($n \geq 4$).

In fact, this difficulty is already present for the much simpler Vlasov-Poisson system. In that case, the difficulty was resolved [7] by modifying the commutation vector fields, replacing the lifted vector fields $\widehat{Z}$ by some $Y = \widehat{Z} + \Phi^i \partial_{x^i}$, where the coefficients $\Phi^i$ are functions in the variable $(t, x, v)$, depending on the solution and constructed to cancel the worst error terms in the commutator formulae. See also [15] for previous results concerning sharp asymptotics for solutions of the Vlasov-Poisson system based on the method of characteristics.

In [4], we pursue a similar strategy and in particular, construct an algebra of modified vector fields, specifically designed to obey improved commutation properties with $T_\phi$, yet to still allow for an (almost) sharp Klainerman-Sobolev inequality. As in [3], we use the hyperboloidal foliation by the hypersurfaces $H_\rho$ of constant hyperboloidal time $\rho := \sqrt{t^2 - |x|^2}$. In particular, all the energies and norms we consider are constructed with respect to this foliation.

To deal with the wave equation, we therefore consider energy norms $\mathscr{E}_N[\phi](\rho)$ obtained out of the standard energy momentum tensor integrated on $H_\rho$. In order to close the basic estimates for the Vlasov field[5], the only multiplier that we consider here is $\partial_t$ and the only decay estimate required for $\phi$ is given by a standard Klainerman-Sobolev inequality (associated to the hyperboloids). In particular, we only use an interior (i.e. away from the light cone) decay estimate for $\phi$ of the type $t^{3/2}|\partial\phi| + t^{1/2}|\phi| \lesssim 1$ which is much weaker than the interior decay used for instance in [13].

For the distribution function, our norm, denoted $E_N[f](\rho)$, is constructed out of modified vector fields $\mathbf{Y}$. Moreover, we actually consider weighted norms, where the extra weights are of the form $\mathfrak{z} = t\frac{v^i}{v^0} - x^i$. Note that these weights are actually propagated by the linear flow, i.e. they solve $T(\mathfrak{z}) = 0$. The norm $E_N[f](\rho)$ is thus constructed out of $L^1$-type norms on $H_\rho \times \mathbb{R}^3_v$ of $\mathfrak{z}^q Y^\alpha(f)$. The $\mathfrak{z}$-weights appear naturally in the commutator formula in conjunction with our choice of modified vector fields.

The main theorem of [4] then establishes the asymptotic stability of the trivial solution to the Vlasov-Nordström system with respect to a strong topology (since we control many derivatives), and provides in particular an almost sharp description of the asymptotic behaviour of the fields.

**Theorem 1.** *Let $N \geq 10$. There exists an $\varepsilon_0 > 0$ so that, for any initial data $(\phi_0, \phi_1, f_0)$ on the hyperboloid $H_1$, satisfying*

$$\mathscr{E}_N[\phi_0, \phi_1] + E_{N+3}[f_0] \leq \varepsilon,$$

*the unique maximal solution $(\phi, f)$ to the Cauchy problem (5) satisfying the*

[4]In the case of massless particles $m = 0$, the exact vector field hitting $f$ in these error terms would be $v^i\partial_{v^i}$, which actually commutes with the free, massless, transport operator, which explains (partly) why the massless case is much easier than the massive one. See [3] for more on the massless Vlasov-Nordström system.

[5]In order to close the top order estimate for the wave, we need to remove all losses when the Vlasov field is hit by a small number of commutation vector fields. For this, we need $t^\delta$ stronger interior decay, for any $\delta > 0$.

*initial conditions*

$$\phi_{H_1} = \phi_0, \quad \partial_t \phi_{H_1} = \phi_1, \quad f_{H_1 \times \mathbb{R}^3_v} = f_0$$

*is defined globally in the future of* $H_1$ *and verifies, for all* $\rho \geq 1$*,*

$$\begin{aligned} E_N[f](\rho) &\lesssim \varepsilon \rho^{\delta(\varepsilon)}. \\ \mathscr{E}_N[\phi](\rho) &\lesssim \varepsilon \rho^{\delta(\varepsilon)}. \\ \mathscr{E}_{N-1}[\phi](\rho) &\lesssim \varepsilon, \end{aligned}$$

*where* $0 \leq \delta(\varepsilon) \to 0$*, as* $\varepsilon \to 0$*.*

From the above statement and Klainerman-Sobolev inequalities, one then obtains decay estimates for $\phi$ and velocity averages of $f$ (and their derivatives), which are sharp in the case of $\phi$ (since there is no loss apart from the top order energy estimate) and almost sharp for $f$, in the sense that there is a $\rho^{\delta(\varepsilon)}$ loss compared to the linear estimate. Moreover, we also have bounds without loss for low derivatives of $f$, using improved improved interior decay for $\phi$.

## 2.2 Elements and difficulties of the proof

### 2.2.1 Large velocities

An important aspect of Theorem 1 is that no compactness assumptions on the $v$ support of the solutions[6] are required. The only $v$ decay that we need is that the initial norms, which are integrals in $v$ (and $x$) with polynomial weights, are bounded.

A strong advantage of compact support assumptions is that they allow for a clean separation of the characteristics associated with the wave equation (the null geodesics) and the characteristics of the distribution function (which are timelike). This means that, in that situation, when estimating products of the form $\partial Z^\alpha(\phi) Y^\beta(f)$, because of the support assumptions, one can always assume that one lies far from the light cone $t \equiv |x|$, since otherwise one must be away from the support of $f$ (for $t$ sufficiently large). In our case, no such separation occurs. Essentially, for large $v$, $\sqrt{m^2 + |v|^2} \sim |v|$ holds, so that the characteristics of the distribution function converge in some sense to that of the wave. Using the hyperboloidal foliation, the present norms for $f$ contain the weight $v^\rho := v^\alpha n_\alpha$, where $n$ is the future unit normal to the hyperboloid. Moreover, one can prove an estimate of the form $t/\rho v^0 \lesssim v^\rho$. Since a weight $t$ is stronger than a weight $\rho$, this allows to extract more decay from the wave to estimate the above products, but at a cost of losing in powers of $v^0$, consistent with the fact that at large $v$ our estimates get worse. Thus, we need to carefully take into account powers of $v$ in all the equations. Looking at the structure of the transport operator on the left-hand side of (6), we notice that two different terms arise, $T(\phi) v^i \partial_{v^i} f$ and $\partial_{x^i} \phi \cdot \partial_{v^i} f$, which have different homogeneity in $v$, the second term being better in this regard. A basic application of our estimates would in fact not allow to estimate the error terms coming from this first term due to the high number of powers of $v$. Instead, the structure of this term plays an important role. A heuristic picture of this structure is the following. As discussed earlier, the difficulty originates in large $v$ and, at large $v$, $v^0 \sim |v|$ holds, meaning that it becomes increasingly hard to distinguish massive from massless particles. However, the vector field $v^i \partial_{v^i}$, which appears

[6] We do not also assume any compact support in $x$, but we do start from some hyperboloid. To go from an initial $t = \mathit{const}$ slice to a future hyperboloid typically requires strong initial decay in $x$, see the discussion in [3, Appendix A].

in the error terms coming from the product $T(\phi)v^i\partial_{v^i}f$, actually commutes with the massless transport operator $|v|\partial_t + v^i\partial_{v^i}$. This implies that, even though $v^i\partial_{v^i}$ does not commute with the massive transport operator, the error terms generated have strong decay properties in $v$ (they indeed contain negative powers of $v^0$).

### 2.2.2 Modified vector fields

It turns out that a first modification allows to capture the aforementioned mechanism concerning the vector field $v^i\partial_{v^i}$ and provides already an improved commutator. We replace each translation $\partial_{x^\alpha}$ by a *generalized translation*

$$\mathbf{e}_\alpha \equiv \partial_{x^\alpha} - (\partial_{x^\alpha}\phi)\cdot v^i\partial_{v^i}. \tag{7}$$

We then replace, in each of the Killing fields and complete lifts of the Killing fields the usual translations by their generalized versions. For instance, for a Lorentz boost $\Omega_{0i} = t\partial_{x^i} + x^i\partial_t$, we obtain the field $t\mathbf{e}_0 + x^i\mathbf{e}_i$. The use of the generalized translations then implies that the resulting commutators have improved properties in terms of powers of $v$ (though still bad in terms of space-time decay). In a second step, we further modify the vector fields coming from the homogeneous vector fields (rotations, boost and scaling). If $\mathbf{Z}$ denotes any of these fields, the modification takes the form $\mathbf{Y} = \mathbf{Z} + \Phi^i\mathbf{X}_i$, where $\mathbf{X}_i = \mathbf{e}_i + \mathbf{e}_0\frac{v^i}{v^0}$. The reason for the introduction of the fields $\mathbf{X}_i$ is that, when applied to $\phi$, a decomposition of the form

$$\mathbf{X}_i(\phi) = \frac{Z(\phi)}{t} + \frac{\mathfrak{z}}{t}\partial\phi, \tag{8}$$

holds, where the right-hand side enjoys improved decay. This is due to the overall $t^{-1}$ factor and, in the second term, the fact that the weight $\mathfrak{z}$ is one of the weights discussed above which are propagated by the linear flow. Together, this implies a strong improved decay for velocity averages of products of type $\mathbf{X}_i(Z^\beta(\phi))Y^\alpha(f)$. Finally, the coefficient $\Phi^i$ appearing in the definition of the modified vector fields are designed to cancel the worst terms in the original commutation formula. To this end, we define $\Phi^i$ as the solution to an equation of the form (we neglect some structural properties here for simplicity in the exposition)

$$T_g(\Phi) = t\partial Z(\phi), \tag{9}$$

with zero initial data. Expanding the commutator, we have schematically,

$$[T_\phi, Y] = [T_\phi, \widehat{Z}] + T_\phi(\Phi).X + \Phi.[T_\phi, X].$$

The RHS of (9) is chosen so that we get a cancellation with the worse terms arising from $[T_\phi, \widehat{Z}]$, while we can verify a posteriori that the terms coming from $\Phi.[T_\phi, X]$ have enough decay to be integrated.

### 2.2.3 The $\mathfrak{z}$ weights

As explained above, our choice of modified vector fields naturally introduces the additional $\mathfrak{z}$ weights. However, in order to avoid having to estimate $L^1$ norms of $\mathfrak{z}^qY^\alpha(f)$ in terms of $\mathfrak{z}^{q+1}Y^\alpha(f)$, which would not allow us to close the estimates, the number of weights $q$ depends on the multi-index $\alpha$. Essentially, generalized translations have better commutation properties and allow for additional $\mathfrak{z}$ weights.

### 2.2.4 Hierarchy

Due to the presence of weights $\mathfrak{z}$ and because of the small loss for the norm of $f$, it is important to exploit a certain hierarchy to close the estimates. For instance, despite the growth of the norm of $f$, the energy estimates for $\phi$ can still close, without any loss, up to order $N-1$. However, this relies on a crucial integration by parts which, at top order, can no longer be used due to a lack of regularity. This eventually results in the growth of the top order energy for the solution to the wave equation. In the argument, it is essential that the worse source terms in the wave equation at top order always contain $\mathbf{Y}^\alpha(f)$ with $|\alpha|$ small (say less than $N/2$), or otherwise the top order estimate would not close.

### 2.2.5 $L^1$-estimates for high-low products

Recall that $\Phi$ are coefficients obtained by solving inhomogeneous transport equations of the form (9). After commutation by differential operator $\mathbf{Y}^\alpha$ of order $|\alpha|$, we can easily prove pointwise estimates on $\mathbf{Y}^\alpha(\Phi)$, as long as we have access to pointwise bounds on $t\partial Z^{|\alpha|+1}(\phi)$, where $Z^{|\alpha|+1}$ is a differential operator of order $|\alpha|+1$.

In the analysis, to close the top order estimates, we need $L^1$-estimates on products of type $\mathbf{Y}^\alpha(\Phi)\mathbf{Y}^\beta(f)$ and $|\mathbf{Y}^\alpha(\Phi)|^2\mathbf{Y}^\beta(f)$, in the situation when $\alpha$ is so large than one does not have access to pointwise estimates on $\mathbf{Y}^\alpha(\Phi)$. To estimate those, we consider these products as solutions to transport equation, to which we can again apply energy estimate for Vlasov fields. This strategy was originally developed in [7].

### 2.2.6 $L^2$-decay estimates

We rely on $L^2$-decay estimates to control the velocity averages of $\mathbf{Y}^\alpha(f)$ for $\alpha$ large. The use of such $L^2$-decay estimates was first introduced in [3] and further expanded in [4].

Let us explain the main ideas behind in the $L^2$ estimates for the velocity averages of $\mathbf{Y}^\alpha(f)$ for $\alpha$ large on a simple model problem.

Assume that $T$ is a transport operator such as the relativistic transport operator or even just the classical one and that $f$ is a function of $(t,x,v)$ satisfying

$$T(f) = hg, \quad f(t=0) = 0$$

where $h = h(t,x)$ is uniformly bounded in $L^2_x$ and such that $g$ is itself a solution to the free transport equation $T(g) = 0$ with $g$ regular enough so that $L^1_{x,v}$-bounds hold for $g$ and decay estimates similar to our Klainerman-Sobolev inequality can be applied for the velocity averages of $g$. The aim is to prove $L^2_x$-decay estimates on $\int_v |f| dv$, the difficulty being that $h$ has very little regularity so that we cannot commute the equation. Instead, note that, by uniqueness, $f = gH$, where $H$ is the solution to the inhomogeneous transport equation $T(H) = h$ with zero data. Indeed,

$$T(gH) = T(g)H + gT(H) = gh,$$

since $T(g) = 0$. Now, note that

$$\begin{aligned}
\left\| \int_v gH dv \right\|_{L^2_x} &\lesssim \left\| \left( \int_v |g| dv \right)^{1/2} \left( \int_v |g| H^2 dv \right)^{1/2} \right\|_{L^2_x} \\
&\lesssim \left\| \left( \int_v |g| dv \right)^{1/2} \right\|_{L^\infty_x} \cdot \left\| \int_v |g| H^2 dv \right\|^{1/2}_{L^1_x}.
\end{aligned}$$

Since we have assumed $g$ to solve the free transport equation and to be as regular as needed, we know that we have some decay for $\left\|\left(\int_v |g|dv\right)^{1/2}\right\|_{L^\infty_x}$. Thus, it remains only to prove boundedness for $\left\|\int_v |g|H^2dv\right\|_{L^1_x}$. This can be obtained using again the transport equation for $gH$ and the associated approximate conservation laws. Indeed, we have

$$T(gH^2) = 2ghH$$

and thus, we need to estimate an integral of the form $\int_{t,x,v} |ghH|dtdxdv$. This is done as follows. First,

$$\begin{aligned}\int_{t,x,v} |ghH|dtdxdv &= \int_t \int_{x,v} |g|^{1/2}|h| \cdot |g|^{1/2} H dxdvdt \\ &\lesssim \int_t \left(\int_{x,v} |g||h|^2 dxdv\right)^{1/2} \left(\int_{x,v} |g|H^2 dxdv\right)^{1/2} dt.\end{aligned}$$

It follows that, if one can obtain enough decay for $\left(\int_{x,v} |g|(x,v)|h|^2(x)dxdv\right)^{1/2}$, then the estimate can close via a Grönwall type inequality. For the decay estimate, simply note again that

$$\left|\int_{x,v} |g|(t,x,v)|h|^2(t,x)dxdv\right| \lesssim \left\|\int_v gdv\right\|_{L^\infty_x} \|h(t,x)\|^2_{L^2_x}.$$

This concludes the discussion of the estimates for the model problem.

To estimate the velocity averages of $Y^\alpha(f)$, $|\alpha| \geq N-2$, we essentially follow this strategy except that

- we need to work with systems instead of scalar equations.
- $Y^\alpha(f)$ have non trivial initial data, so we first need to split $Y^\alpha(f)$ into three parts
  1. A part satisfying a homogeneous equation with regular data.
  2. A part satisfying a homogeneous equation with non-regular data (but a good structure).
  3. A part satisfying an inhomogeneous equation with zero data, for which we use estimates inspired by the above model case.
- the operator $T$ needs to be replaced by $T_\phi$ (or rather $T_\phi + A$ for some matrix potential $A$ satisfying decay estimates in $L^\infty_x$).
- the matrix $\mathbf{B}$ replacing $h$ is not uniformly bounded in $L^2_x$ (there is a $t$-loss).
- the vector replacing $g$ does not satisfy a homogeneous transport equation (there is an error term).
- and finally, in all steps, we need to keep track of the exact decay rates in $\rho$ to make sure the time integrals converge.

## 2.3 Perspectives of the method and the Einstein-Vlasov system

One of the main motivations for the present work comes from the Einstein-Vlasov system, which can be written as

$$Ric(g) - \frac{1}{2}gR(g) = T[f], \tag{10}$$

$$\left(v^\alpha \partial_{x^\alpha} - v^\alpha v^\beta \Gamma^i_{\alpha\beta}\partial_{v^i}\right)(f) = 0. \tag{11}$$

Here $g$ is a Lorentzian metric on a 4 dimensional manifold, $Ric(g)$ and $R(g)$ the Ricci and scalar curvatures of $g$, $f$ is a Vlasov field, $T[f]$ its energy-momentum tensor and $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols of $g$, which we recall depend on $g$ and $\partial g$. The Minkowski space is then the simplest solution to these equations with $f = 0$.

In Minkowski space, the energy momentum tensor of $f$ for particles of mass $m$ can be written as the tensor

$$T[f] = \int_{v \in \mathbb{R}^3} v_\alpha v_\beta f \frac{dv}{\sqrt{m^2 + |v|^2}},$$

where $v_0 = -\sqrt{m^2 + |v|^2}$. Since the Einstein equations can be recast as a system of wave equations, the above system is again a system of coupled wave/Vlasov equations, where the coupling in the wave equations is through the velocity averages given by the tensor $T[f]$. Note that (11) simply means that $f$ is conserved by the geodesic flow, which confers a beautiful geometric interpretation to this system.

We refer to the recent book[7] [16] for a thorough introduction to this system. The small data theory around the Minkowski space is still incomplete for the Einstein-Vlasov system. The spherically symmetric cases in dimension $(3+1)$ have been treated in [17] for the massive case and in [5] for the massless case with compactly supported initial data. A proof of stability for the massless case without spherical symmetry but with compact support in both $x$ and $v$ has been given in [6]. As in [5], the compact support assumptions and the fact that the particles are massless are important as they allow to reduce the proof to that of the vacuum case outside from a strip going to null infinity.

We consider the present work as a first step towards a proof of stability of the Minkowski space for the massive Einstein-Vlasov system. Indeed, due to the highly non-linear structure of the Einstein equations, any precise global analysis of the solutions relies on commuting the Einstein equations and by the coupling, one is forced to estimate derivatives of velocity averages of the distribution function as well. Note that our method only uses mostly basic energy techniques to estimate the solution to the wave equations and is therefore fully compatible with the relevant techniques used in the study of the Einstein equations.

Let us finally mention that the vector-field method introduced in [3] has been extended to prove decay of massless distribution functions on Kerr black holes [18], as well as to derive decay estimates for other dispersive partial differential equations [8].

# References

[1] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, *Comm. Pure Appl. Math.*, 38(3):321–332 (1985).

[2] C. Bardos, P. Degond, *Global existence for the Vlasov-Poisson equation in* 3 *space variables with small initial data. Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(2):101–118(1985).

[3] D. Fajman, J. Joudioux and J. Smulevici, *A vector field method for relativistic kinetic transport equations with applications*, to appear in Analysis and PDE.

[7] Apart from a general introduction to the Einstein-Vlasov system, the main purpose of this book is to present a proof of stability of exponentially expanding space-times for the Einstein-Vlasov system, see [16].

[4] D. Fajman, J. Joudioux, and J. Smulevici. Sharp asymptotics for small data solutions of the Vlasov-Nordström system in three dimensions. `arXiv:1704.05353`.

[5] M. Dafermos, *A note on the collapse of small data self-gravitating massless collisionless matter*, *J. Hyperbolic Differ. Equ.*, 3(4):589–598 (2006).

[6] M. Taylor, *Stability of the Minkowski space for the massless Einstein-Vlasov system*, *Ann. PDE*, 3(1) Art. 9, 177 pp, 2017.

[7] J. Smulevici. Small Data Solutions of the Vlasov-Poisson System and the Vector Field Method. *Ann. PDE*, 2(2): Art. 2, 11pp, 2016.

[8] W. W. Y. Wong. A commuting-vector-field approach to some dispersive estimates. `arXiv:1701.01460`, January 2017.

[9] S. Calogero. Spherically symmetric steady states of galactic dynamics in scalar gravity. *Classical Quantum Gravity*, 20(9):1729–1741, 2003.

[10] S. Calogero. Global classical solutions to the 3D Nordström-Vlasov system. *Comm. Math. Phys.*, 266(2):343–353, 2006.

[11] S. Calogero and G. Rein. Global weak solutions to the Nordström-Vlasov system. *J. Differential Equations*, 204(2):323–338, 2004.

[12] C. Pallard. On global smooth solutions to the 3D Vlasov-Nordström system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(1):85–96, 2006.

[13] S. Friedrich. Global Small Solutions of the Vlasov-Norstrom System. `arXiv:math-ph/0407023`, July 2004.

[14] R. T. Glassey and W. A. Strauss. Absence of shocks in an initially dilute collisionless plasma. *Comm. Math. Phys.*, 113(2):191–208, 1987.

[15] H. Hwang, A. D. Rendall, and J. J. L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the Vlasov-Poisson system with small initial data. *Archive for Rational Mechanics and Analysis*, 200(1):313–360, 2011.

[16] H. Ringström. *On the topology and future stability of the universe.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2013.

[17] G. Rein and A. D. Rendall. Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. *Comm. Math. Phys.*, 150(3):561–583, 1992.

[18] L. Andersson, P. Blue, and J. Joudioux. Hidden symmetries and decay for the Vlasov equation on the Kerr spacetime. `arXiv:1612.09304`, December 2016.

[19] Robert T. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.