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SPECTRAL STATISTICS OF NON-SELFADJOINT OPERATORS SUBJECT TO SMALL RANDOM PERTURBATIONS

MARTIN VOGEL

ABSTRACT. In this review paper we present recent results concerning the local eigenvalue statistics of non-selfadjoint one-dimensional semiclassical pseudo-differential operators subject to small random perturbations. We compare the eigenvalue statistics for perturbations by random matrix and by random potential. We show that they are universal in the sense that they only depend on the principal symbol of the operator and the type of perturbation and that they are independent of the distribution of the perturbation.

Moreover, we will outline the proof of the principal results in the case of a model operator. The discussed results are joint work with Stéphane Nonnenmacher [22].

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1. INTRODUCTION

It is well known that the spectrum non-selfadjoint operators can be highly unstable even under tiny perturbations, see [12, 6] for a very good overview. Figure 1 illustrates the phenomenon of spectral instability in the case of a semiclassical differential operator. We see that even a small perturbation can suffice to change the spectrum of the operator significantly.

Spectral properties of non-self-adjoint operators subject to random perturbations have been studied from various perspectives: In numerical analysis small random perturbations of large non-selfadjoint matrices have been studied to model the numerical rounding error, see for instance the works of von Neumann and Goldstine [37], Spielmann and Teng [31] and Edelman and Rao [11].

In random matrix theory random perturbations of large non-selfadjoint matrices have been studied with a focus on the outlying eigenvalues. These are eigenvalues which are found away from the bulk of the spectrum.

Perturbations of large random matrices by deterministic matrices of finite rank have been studied extensively by Tao [32] and Tao and Vu [33]. The case of large non-selfadjoint deterministic matrices perturbed by random matrices with coupling constants of order 1 where studied by Bordenave and Capitaine [4]. The regime of small coupling constants tending to 0 as the dimension of the matrix gets large, have been considered by Davies and Hager [8], Guionnet, Matchett-Wood and Zeitouni [14] and Sjöstrand and Vogel [28, 27, 29] in the case of large non-selfadjoint Toeplitz matrices.

In this paper we will, however, focus more on the aspects of spectral instability of non-selfadjoint semiclassical (pseudo-)differential operators. We consider certain classes

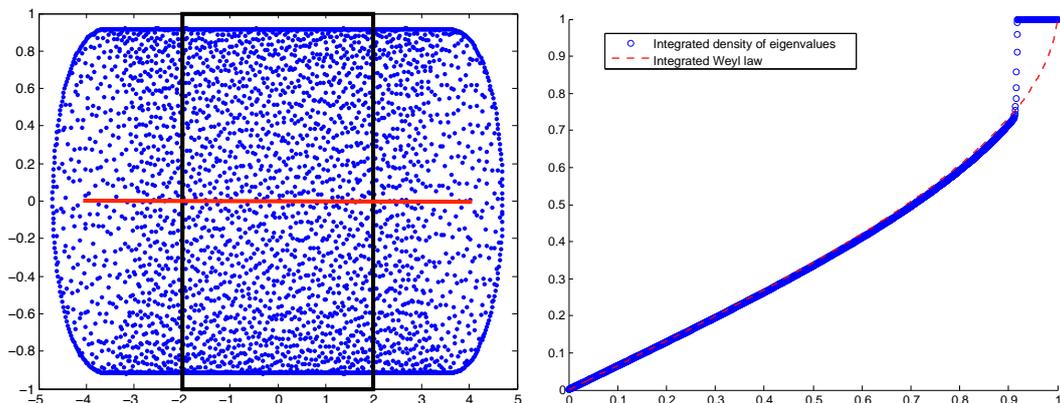


Figure 1. On the left hand side the red line shows the spectrum of the discretisation of $P_h = hD + e^{-ix}$ on S^1 (approximated by a 3999×3999 -matrix) and the blue points show the spectrum of P_h perturbed with a random Gaussian matrix δQ with $h = 2 \cdot 10^{-3}$ and $\delta = 2 \cdot 10^{-12}$. The right hand side shows the integrated experimental density of eigenvalues (in the black box), averaged over 400 realisations of random Gaussian matrices, and the integrated Weyl law, cf Theorem 4. These figures were presented in [36].

of non-normal semiclassical (pseudo-)differential operator subject to small random perturbations. We present a review of recent results which are joint work with Stéphane Nonnenmacher. Most of the discussion here can be found in [22]. Our main interest lies in universality properties of the eigenvalue statistics on the scale of their average spacing.

A major tool for studying the region of spectral instability originates from the field of numerical analysis, see for instance [34, 12]: the so-called ε -pseudospectrum. Roughly speaking, it consists of the regions in the complex plane where the norm of the resolvent of the operator is large and thus indicates how far the eigenvalues can spread under perturbations.

For $P : L^2 \rightarrow L^2$, a densely defined closed linear operator with resolvent set $\rho(P)$ and spectrum $\text{Spec}(P) = \mathbb{C} \setminus \rho(P)$, and for any $\varepsilon > 0$, we define the ε -pseudospectrum of P by

$$\text{Spec}_\varepsilon(P) := \text{Spec}(P) \cup \{z \in \rho(P); \|(P - z)^{-1}\| > \varepsilon^{-1}\}. \quad (1.1)$$

The set (1.1) describes precisely the region of spectral instability of the operator P , since any point in the ε -pseudospectrum of P lies in the spectrum of some small bounded perturbation of P , see for example [12]. More precisely, we have that

$$\text{Spec}_\varepsilon(P) = \bigcup_{\substack{Q \in \mathcal{B}(L^2) \\ \|Q\| < \varepsilon}} \text{Spec}(P + \varepsilon Q). \quad (1.2)$$

A third, equivalent definition of the ε -pseudospectrum of P is via the existence approximate solutions to the eigenvalue problem $P - z$. More precisely, we have that

$$z \in \text{Spec}_\varepsilon(P) \iff \exists u_z \in D(P) \text{ s.t. } \|(P - z)u_z\| \leq \varepsilon \|u_z\|, \quad (1.3)$$

where $D(P)$ denotes the domain of P . Such a state u_z is called an ε -quasimode or simply a *quasimode*.

1.1. **Framework.** We begin by fixing the type of the unperturbed operators considered in this paper: Write $\rho = (x, \xi) \in \mathbb{R}_{x,\xi}^2$ for a point in phase space and let $S(\mathbb{R}^2, m)$ denote a class of smooth symbols whose derivatives are controlled by a smooth order function $m \in \mathcal{C}^\infty(\mathbb{R}^2; [1, +\infty[)$ of polynomial growth, see for instance [38, 10].

Let $h \in]0, 1]$ and consider symbols admitting the h -asymptotic development

$$p(\rho; h) \sim p_0(\rho) + hp_1(\rho) + \dots \quad \text{in } S(\mathbb{R}^2, m), \quad (1.4)$$

where p_0, p_1, p_2, \dots are independent of $h > 0$. We call p_0 the semiclassical principal symbol of p and define the sets

$$\Sigma := \overline{p_0(\mathbb{R}^2)} \subset \mathbb{C}, \quad \Sigma_\infty := \{z \in \Sigma; \exists \rho_j \text{ s.t. } p_0(\rho_j) \rightarrow z, |\rho_j| \rightarrow \infty\}. \quad (1.5)$$

Furthermore, we suppose that the principal symbol p_0 is elliptic in a point $z_{out} \in \mathbb{C} \setminus \Sigma$, in the sense that there exists a constant $C_0 > 0$ such that

$$|p_0(\rho) - z_{out}| \geq m(\rho)/C_0, \quad \forall \rho \in \mathbb{R}^2. \quad (1.6)$$

Moreover, we assume that

$$dp_0, d\bar{p}_0 \text{ are linearly independent at every point } \rho \in p_0^{-1}(\bar{\Omega}). \quad (1.7)$$

For $h > 0$ small enough, we let p^w denote the h -Weyl quantization of the symbol p ,

$$P_h u(x) = p^w(x, hD_x; h)u(x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi, \quad (1.8)$$

for $u \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz functions. The closure of P_h , as an unbounded operator in L^2 with domain $\mathcal{S}(\mathbb{R}^d)$, has domain $H(m) := (P_h - z_{out})^{-1}(L^2(\mathbb{R})) \subset L^2(\mathbb{R})$ and will be denoted as well by P_h . Moreover, we write $\|u\|_m := \|(P_h - z_{out})^{-1}u\|$, with $u \in H(m)$, for the associated generalised Sobolev norm.

Due to the ellipticity assumption (1.6) and the growth condition on the weight m , one can show the following result by an argument based on a compact deformation of Fredholm operators, see [15, 17].

Proposition 1. *Let $\tilde{\Omega} \subset \mathbb{C} \setminus \Sigma_\infty$ be open simply connected, not entirely contained in Σ and such that $\tilde{\Omega} \cap \Sigma_\infty = \emptyset$. Then, for $h > 0$ small enough, the spectrum of P_h inside $\tilde{\Omega}$ is purely discrete, comprised only of eigenvalues of finite algebraic multiplicities and contained in a small neighbourhood of Σ .*

We are interested in the spectrum of small random perturbations of P_h in the semiclassical limit $h \rightarrow 0$.

1.2. **Spectral instability for semiclassical pseudo-differential operators.** Let $\tilde{\Omega}$ be as above and let

$$\Omega \Subset \tilde{\Omega} \cap \mathring{\Sigma}, \text{ be open, simply connected, relatively compact with } \text{dist}(\Omega, \partial\Sigma) \geq \frac{1}{C}. \quad (1.9)$$

Dencker, Sjöstrand and Zworski [9], and Sjöstrand [26] observed that since Ω is relatively compact and simply connected, (1.7) implies that there exists $J = J(\Omega) \in \mathbb{N} \setminus \{0\}$ so that

$$\begin{aligned} \forall z \in \Omega : p_0^{-1}(z) = \{\rho_\pm^j(z); j = 1, \dots, J\}, \text{ with } \pm \{\text{Re } p, \text{Im } p\}(\rho_\pm^j(z)) < 0, \\ \text{and } \rho_\pm^i(z) \neq \rho_\pm^j(z), \quad i \neq j, \end{aligned} \quad (1.10)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Moreover, the phase space points $\rho_{\pm}^j(z) \in \mathbb{R}^2$ depend smoothly on z . For simplicity, we suppose from now on that

$$\forall z \in \Omega : \quad x_{\pm}^i(z) \neq x_{\pm}^j(z), \quad i \neq j. \quad (1.11)$$

The Poisson bracket condition (1.10) implies for each j that there exists an h^∞ -quasimode for the problems $P_h - z$ and $(P_h - z)^*$, for $z \in \Omega$, microlocalized on $\rho_{\pm}^j(z)$, see [7, 9]. More precisely, assuming (1.10), we have that for all $z \in \Omega$ and all $j = 1, \dots, J$, there exist $e_{\pm}^j = e_{\pm}^j(x, z; h) \in L^2(\mathbb{R})$, $\|e_{\pm}^j\| = 1$, such that

$$\|(P_h - z)e_{+}^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad WF_h(e_{+}^j) = \{\rho_{+}^j(z)\}, \quad (1.12)$$

and

$$\|(P_h - z)^*e_{-}^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad WF_h(e_{-}^j) = \{\rho_{-}^j(z)\}. \quad (1.13)$$

We recall that for $v = v(h)$, $\|v\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{-N})$, for some fixed N , the *semiclassical wave front set* of v is defined by

$$\rho_0 \notin WF_h(v) \stackrel{\text{def}}{\iff} \exists a \in \mathcal{C}_c^\infty(T^*\mathbb{R}) \text{ s.t. } a(\rho_0) = 1 \text{ and } \|a^w v\|_{L^2} = \mathcal{O}(h^\infty)$$

where a^w denotes the h -Weyl quantization of a .

In view of the characterisation (1.3) of the pseudospectrum, we see that the assumption (1.7) implies that Ω is contained in the h^∞ -pseudospectrum of P_h , a spectrally highly unstable region.

1.3. Adding a random perturbation. Motivated by Figure 1, we are interested in the spectral distribution of a "generic" perturbation of P_h . Therefore, it is natural to consider small random perturbations of the operator P_h . In particular, we are interested in perturbations by

- (1) *Random matrix*
- (2) *Random potential.*

There is a fundamental difference between those two types of perturbation: a potential, being a multiplication operator, is a local operator and hence it doesn't change the wave front set of the function it acts upon. A matrix, as defined below, on the other hand can change the wave front set. This will ultimately lead to a different behaviour in the statistics of the eigenvalues of the perturbed operator.

We will construct the perturbations in the following way:

Let $H = -\partial_x^2 + x^2$ on $L^2(\mathbb{R})$ be the Harmonic oscillator, which is a positive self-adjoint elliptic operator with compact resolvent. Let $\{e_k\}_{k \in \mathbb{N}_0}$ denote an orthonormal basis of $L^2(\mathbb{R})$ comprised out of the eigenfunctions of H associated to $0 < \lambda_0 \leq \lambda_1 \leq \dots$, the corresponding increasing sequence of eigenvalues of H .

Let $(\mathcal{M}, \mathcal{F}, \mathbb{P})$ be some probability space and let $\mathbb{E}[\cdot]$ denote the expectation with respect to the probability measure μ . Let α be a complex-valued random variables defined on $(\mathcal{M}, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[\alpha] = 0, \quad \mathbb{E}[\alpha^2] = 0, \quad \mathbb{E}[|\alpha|^2] = 1, \quad \mathbb{E}[|\alpha|^{4+\varepsilon_0}] < +\infty, \quad (1.14)$$

where $\varepsilon_0 > 0$ is an arbitrarily small but fixed constant. As a consequence, the Markov inequality implies the following tail estimate: there exists a constant $\kappa_\alpha > 0$ such that

$$\mathbb{P}[|\alpha| \geq \gamma] \leq \kappa_\alpha \gamma^{-(4+\varepsilon_0)}, \quad \forall \gamma > 0. \quad (1.15)$$

Remark 2. For instance, a complex centred Gaussian random variable would satisfy the above assumptions.

1) Random Matrix Let $N(h) = C_1/h^2$, $C_1 > 0$ large enough. Let $q_{j,k}$, $0 \leq j, k < N(h)$ be independent and identically distributed complex-valued random variables satisfying the condition (1.14) and set

$$M_\omega = \frac{1}{N(h)} \sum_{0 \leq j, k < N(h)} q_{j,k} e_j \otimes e_k^*, \quad (\text{RM})$$

where $e_j \otimes e_k^* u = (u|e_k) e_j$ for $u \in L^2(\mathbb{R})$. Moreover, for $0 < \delta \ll 1$, we write

$$P_M^\delta = P_h + \delta M_\omega. \quad (1.16)$$

2) Random Potential Let $N(h) = C_1/h^2$, $C_1 > 0$ large enough. Let v_j , $0 \leq j < N(h)$ be independent and identically distributed complex-valued random variables satisfying condition (1.14) and set

$$V_\omega = \frac{1}{N(h)} \sum_{0 \leq j < N(h)} v_j e_j. \quad (\text{RP})$$

Moreover, for $0 < \delta \ll 1$, we write

$$P_V^\delta = P_h + \delta V_\omega. \quad (1.17)$$

When we use a random potential as a perturbation, we will make the additional symmetry assumption on (1.4)

$$p(x, \xi; h) = p(x, -\xi; h). \quad (1.18)$$

This hypothesis implies that we can group the points ρ_\pm^j , (1.10), such that $\rho_\pm^j = (x^j, \pm \xi^j)$. This implies that the centres of microlocalization of the quasimodes e_+^j and e_-^j are located on the same fibre $T_{x^j}^* \mathbb{R} \simeq \mathbb{R}$ in phase space.

Restricting the random variables $q_{j,k}, v_j$ to a complex disk of radius C/h , $C > 0$ large enough, centred at 0, we can show that both M_ω and V_ω are bounded operators with probability close to one. Since we are interested in small perturbations, we assume that for some fixed, but arbitrarily large $M \gg 1$

$$h^M \leq \delta \leq h^\kappa, \quad \kappa > 5/2. \quad (1.19)$$

Remark 3. Suppose additionally that the symbol p of the operator P_h , cf (1.4), (1.8), is analytic in $\{\rho \in \mathbb{C}^2; |\text{Im } \rho| \leq 1/C\}$, a small tubular complex neighbourhood of \mathbb{R}^2 in \mathbb{C}^2 , and that it satisfies a suitable growth condition there, for instance $|p(\rho)| \leq m(\text{Re } \rho)$. Then, we can replace the $\mathcal{O}(h^\infty)$ error in (1.12) and (1.12) by an error of $\mathcal{O}(e^{1/Ch})$, see [9]. In this case we can then allow for coupling constants δ satisfying

$$e^{1/Ch} \leq \delta \leq h^\kappa, \quad \kappa > 5/2.$$

with $C > 1$ large enough.

Applying the resolvent of P_h at z_{out} to P_M^δ and P_V^δ , one can show that their spectra in Ω are purely discrete.

The principal aim of this paper is to show that the statistical properties of these spectra in Ω are universal in a sense that we will specify later on.

Since $p_0 - z$ is elliptic for every $z \in \mathbb{C} \setminus \Sigma$, we have that the resolvent norm $\|(P_h - z)^{-1}\| = \mathcal{O}(1)$, uniformly as $h \rightarrow 0$. Therefore, in view of the characterisation (1.2) of the pseudospectrum, the spectra of P_M^δ and P_V^δ are contained in any neighbourhood

of Σ , for $h > 0$ small enough. Moreover, since Ω is contained in the h^∞ -pseudospectrum of P_h , we will not encounter effects due to the pseudospectral boundary, see [36] for a discussion. Therefore, we will simply say that Ω is in the *bulk* of the spectrum of the perturbed operator.

1.4. Macroscopic spectral distribution: A probabilistic Weyl law. In a series of works by Hager [16, 15, 17] and Sjöstrand [25, 24], the authors considered randomly perturbed operators P^δ of the types given in (1.16) and (1.17). Under more restrictive assumptions on the random variables, than (1.14), they have shown the following result.

Theorem 4 (Probabilistic Weyl law). *Let Ω be as in (1.9), (1.7). Let $\Gamma \Subset \Omega$ be open with \mathcal{C}^2 boundary. Let P^δ be either of the randomly perturbed operators P_M^δ or P_V^δ with δ as in (1.19) with $\kappa > 0$ sufficiently large. Then,*

$$\#(\text{Spec}(P^\delta) \cap \Gamma) = \frac{1}{2\pi h} \left(\iint_{p_0^{-1}(\Gamma)} dx d\xi + o(1) \right) \quad (1.20)$$

with probability

$$\geq 1 - Ch^\eta, \text{ for some } \eta > 0.$$

Hager [16, 15, 17] and Sjöstrand [25, 24] give explicit control over both, the error term in the Weyl law, and the error term in the probability estimate. Similar results have also been obtained by Christiansen and Zworski [5], Bordeaux-Montrieux [2] and Bordeaux-Montrieux and Sjöstrand [3].

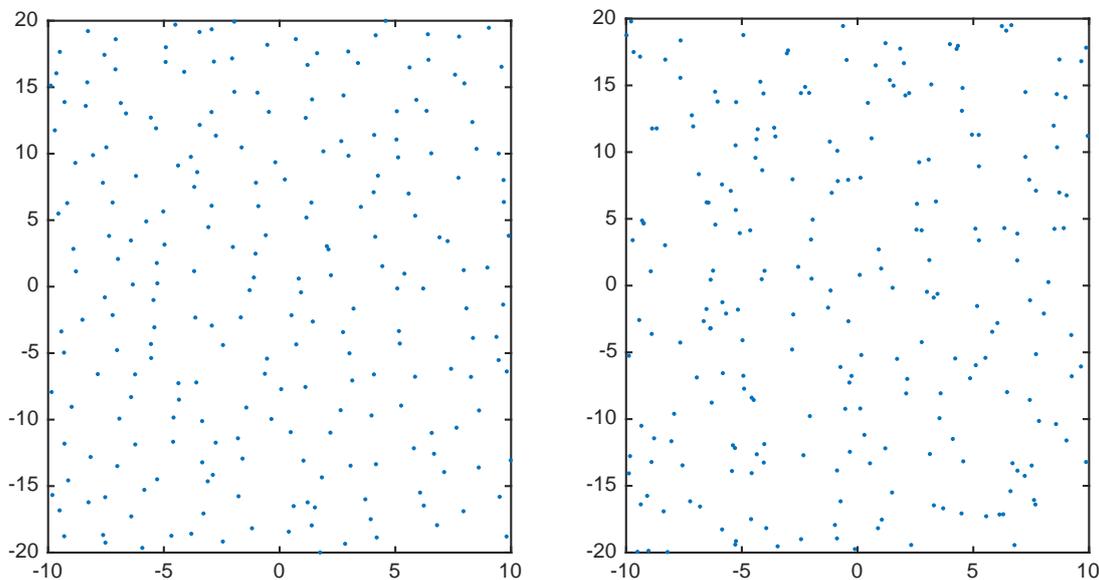


Figure 2. Numerically computed spectrum of the operator $-h^2 \partial_x^2 + e^{3ix} + \delta Q_\omega$ on S^1 with $h = 10^{-3}$ and $\delta = 10^{-12}$. The left hand side shows the case where Q_ω is given by a complex Gaussian random matrix and the right hand side the case where Q_ω is given by a random potential as in (RP) with coefficients given by complex Gaussian random variables. This Figure was taken from [22].

This probabilistic Weyl law shows that, with probability close 1, the number of eigenvalues of the perturbed operator P^δ in any compact subset Ω of the h^∞ -pseudospectrum, away from the boundary $\partial\Sigma$ (1.5), is of order $\asymp h^{-1}$. Hence, the spectrum of P^δ will spread

out in Ω with an average spacing of the eigenvalues of order $h^{1/2}$. Figure 1 illustrates this behaviour: we see that the density simulated eigenvalues of the perturbed operator are nicely described by a Weyl law. It only breaks down close to the boundary $\partial\Sigma$.

1.5. Microscopic spectral distribution: local statistics. Figure 2 compares the spectra of the perturbed operators P_M^δ and P_V^δ . We notice in Figure 2 that in the case of P_V^δ the distribution of the spectrum is much coarser than in the case of P_M^δ . We can even observe the formation of small clusters of eigenvalues.

To study this difference in the spectral distribution of the perturbed operators P_M^δ and P_V^δ we study the local statistics of eigenvalues. That is to say the statistics of the eigenvalues on the scale of their mean level spacing:

Fix a $z_0 \in \Omega$. In the case P_M^δ , (1.16), where we perturb the operator P_h with the random matrix δM_ω , we study the rescaled point process of eigenvalues of P_M^δ , defined by

$$\mathcal{Z}_{h,z_0}^M := \sum_{z \in \text{Spec}(P_M^\delta)} \delta_{(z-z_0)h^{-1/2}}. \quad (1.21)$$

where the eigenvalues are counted according to their algebraic multiplicity.

This rescaling can be seen as a magnification of the eigenvalues around z_0 , which after rescaling have a mean spacing of order $\asymp 1$.

Similarly, in the case P_V^δ , (1.17), where we perturb the operator P_h with the random potential V_ω , we consider the rescaled point process of eigenvalues of P_V^δ , given by

$$\mathcal{Z}_{h,z_0}^V := \sum_{z \in \text{Spec}(P_V^\delta)} \delta_{(z-z_0)h^{-1/2}}. \quad (1.22)$$

We want to show that under the assumption (1.14) on the random coefficients, in the limit $h \rightarrow 0$ the correlation functions of the eigenvalues of P_M^δ and P_V^δ in Ω are universal, in the sense that they are the correlation functions of a point process which

- *depends* only on the structure of the energy shell $p_0^{-1}(z)$ and on the type of random perturbation used, either (RM) or (RP);
- *is independent* of the probability distribution of the random variables $q_{j,k}$ and v_j used to define the random perturbations, (RM) and (RP), as long as they satisfy (1.14).

Finally, let us stress once more that our results concern solely the eigenvalues in the bulk of the spectra of the randomly perturbed operators, that is to say in the interior of the h^∞ -pseudospectrum of P_h .

Close to the boundary of pseudospectrum we expect the statistical properties of the eigenvalues to change drastically. It has been shown in [36] in the case of a model operator, that close to boundary of pseudospectrum the probabilistic Weyl law for the perturbed operator breaks down, see Figure 1. In fact, the eigenvalues accumulate there.

1.6. Gaussian analytic functions. Let $(\alpha_n)_{n \in \mathbb{N}}$ be independent and identically distributed normal complex Gaussian random variables, i.e. $\alpha_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ for every $n \in \mathbb{N}$. Consider

$$\mathbb{C} \ni z \mapsto g_\sigma(z) := \sum_{n=0}^{\infty} \alpha_n \frac{\sigma^{n/2} z^n}{\sqrt{n!}}, \quad \sigma > 0, \quad (1.23)$$

which is a certain Gaussian analytic function (GAF) on \mathbb{C} , in the sense that g_σ is a random variable with values in the space of entire functions so that for any $n \in \mathbb{N}$ and

all $z_1, \dots, z_n \in \mathbb{C}$ the random vector $(g(z_1), \dots, g(z_n))$ has a centred complex Gaussian law, i.e.

$$(g_\sigma(z_1), \dots, g_\sigma(z_n)) \sim \mathcal{N}_{\mathbb{C}}(0, \Gamma), \quad (1.24)$$

where the covariance matrix $\Gamma \in GL_n(\mathbb{C})$ admits the entries

$$\Gamma_{i,j} = \mathbb{E} \left[g_\sigma(z_i) \overline{g_\sigma(z_j)} \right] \stackrel{\text{def}}{=} K(z_i, \bar{z}_j) = \exp(\sigma z_i \bar{z}_j). \quad (1.25)$$

The function $\mathbb{C}^2 \ni (z, w) \mapsto K(z, \bar{w})$ is called the covariance kernel of the GAF g_σ and it completely determines its distribution as random variable with values in the space of holomorphic functions on \mathbb{C} . Moreover, K completely determines the distribution of

$$\mathcal{Z}_{g_\sigma} = \sum_{z \in g_\sigma^{-1}(0)} \delta_z,$$

the random point process given by zeros of the GAF g_σ . See for instance [20] for a review of many notions and results concerning these random analytic functions.

2. MAIN RESULTS

2.1. Perturbation by random potential. We begin with the case P_V^δ , (1.17), where we perturb the operator P_h by a random potential V_ω . By (1.10), we have that

$$(p_0)_*(|d\xi \wedge dx|) = \sum_{j=1}^J (\sigma_+^j(z) + \sigma_-^j(z)) L(dz), \quad \sigma_\pm^j(z) = \frac{1}{\mp \{\text{Re } p_0, \text{Im } p_0\}(\rho_\pm^j(z))} \quad (2.1)$$

where p_0 is the principal symbol of P_h as in (1.4), $|d\xi \wedge dx|$ denotes the measure induced by the symplectic volume form on $T^*\mathbb{R} \cong \mathbb{R}^2$ and $\sigma_\pm^j(z)$ depend smoothly on z .

If we additionally assume the symmetry hypothesis (1.18), then we can group the points ρ_\pm^j such that $\rho_\pm^j = (x^j, \pm \xi^j)$ which implies that $\sigma_+^j(z) = \sigma_-^j(z)$.

2.1.1. Universal limiting point process.

Theorem 5. *Let $\Omega \Subset \mathring{\Sigma}$ be as in (1.9). Let p be as in (1.4) satisfying (1.7) and (1.18). Let $z_0 \in \Omega$. Then, for any open, connected, relatively compact domain $O \Subset \mathbb{C}$, we have that*

$$\mathcal{Z}_{h,z_0}^V \xrightarrow{d} \mathcal{Z}_{G_{z_0}} \quad \text{on } O, \quad \text{as } h \rightarrow 0,$$

in the sense that for all $\phi \in \mathcal{C}_c(O)$

$$\langle \mathcal{Z}_{h,z_0}^V, \phi \rangle = \sum_{z \in \text{Spec}(P_h^\delta)} \phi((z - z_0)h^{-1/2}) \xrightarrow{d} \langle \mathcal{Z}_{G_{z_0}}, \phi \rangle = \sum_{z \in G_{z_0}^{-1}(0)} \phi(z), \quad \text{as } h \rightarrow 0,$$

where the convergence is in distribution. Moreover,

$$G_{z_0}(z) = \prod_{j=1}^J g_{z_0}^j(z), \quad z \in \mathbb{C},$$

where g^j , for $1 \leq j \leq J$, are independent Gaussian analytic functions on \mathbb{C} with covariance kernel

$$K_{z_0}^j(z, \bar{w}) := e^{\sigma_+^j(z_0)z\bar{w}},$$

with $\sigma_+^j(z_0)$ as in (2.1).

We recall that convergence of random variables in distribution means that the induced probability measures of the random variables converge in the weak-* topology, see for instance [21].

Notice that since the g^j are Gaussian analytic functions with covariance kernel $K_{z_0}^j$, we have in view of (1.23) and (1.25) that

$$g_{z_0}^j(z) \stackrel{d}{=} \sum_{n=0}^{\infty} \alpha_n \frac{(\sigma_+^j(z_0))^{n/2} z^n}{\sqrt{n!}}, \quad \alpha_n \sim \mathcal{N}_{\mathbb{C}}(0, 1) \text{ (iid)}. \quad (2.2)$$

Here, the equality is in distribution, which is to say that the induced probability measures of those two random variables coincide.

Theorem 5 tells us that:

- (1) at any given point $z_0 \in \Omega$ in the bulk of the pseudospectrum, the local rescaled point process of eigenvalues is given, in the limit $h \rightarrow 0$, by the point process given by the zeros of the product of J independent Gaussian analytic functions.

Hence, near z_0 the eigenvalues are given by the superposition of J independent sets of zeros coming from J independent Gaussian analytic functions of the form (2.2). Due to the independence, it is clear that the formation of clusters of eigenvalues is possible.

- (2) The distribution of each of these functions, determined by their covariance kernels, depends only on the part of the volume form $p_*(|d\xi \wedge dx|)$ coming from the pair of points $\rho_{\pm}^j = (x^j, \pm \xi^j)$.

In other words, the rescaled point process \mathcal{Z}_{h,z_0}^V of the eigenvalues of the perturbed operator P_V^δ has a universal limit, which is independent of the specific probability distribution of the potential V_ω satisfying (1.14) and depends only on the densities $\{\sigma_+^j(z_0); j = 1, \dots, J\}$.

2.1.2. Scaling limit k -point measures. An explicit way to obtain information about the statistical interaction of k eigenvalues of the perturbed operator P_V^δ is by analysing the k -point measures of the point process \mathcal{Z}_{h,z_0}^V . They express correlations within k -point subsets of the point process. These are positive measures μ_h^{k,V,z_0} on $O^k \setminus \Delta$, where O is as in Theorem 5 and $\Delta = \{z \in \mathbb{C}^k; \exists i \neq j \text{ s.t. } z_i = z_j\}$, defined by

$$\begin{aligned} \mathbb{E} \left[(\mathcal{Z}_{h,z_0}^V)^{\otimes k}(\phi) \right] &= \mathbb{E} \left[\sum_{z_1, \dots, z_k \in \text{Spec}(P^\delta)} \phi \left((z_1 - z_0)h^{-1/2}, \dots, (z_k - z_0)h^{-1/2} \right) \right] \\ &= \int \phi(z) \mu_h^{k,V,z_0}(dz), \quad \forall \phi \in \mathcal{C}_c(O^k \setminus \Delta), \end{aligned} \quad (2.3)$$

We have stamped out the diagonal Δ in order to avoid trivial self-correlations. When these k -point measures are absolutely continuous with respect to the Lebesgue measure on \mathbb{C} , we call their densities k -point functions.

Theorem 6. *Let μ_h^{k,V,z_0} be the k -point measure of \mathcal{Z}_{h,z_0}^V , defined in (2.3), and let $\mu_{z_0}^k$ be the k -point measure of the point process \mathcal{Z}_{G,z_0} , given in Theorem 5. Then, for any $O \Subset \mathbb{C}$ relatively compact connected domain, and for all $\varphi \in \mathcal{C}_c(O^k \setminus \Delta)$,*

$$\int \varphi(z) \mu_h^{k,V,z_0}(dz) \longrightarrow \int \varphi(z) \mu_{z_0}^k(dz), \quad h \rightarrow 0.$$

Moreover, $\mu_{z_0}^k$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{C} with continuous density $d^k(z)$, given by

$$d^k(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^J, |\alpha|=k \\ 0 \leq \alpha_j \leq J}} \sum_{\tau \in S_k} \frac{1}{\alpha!} \prod_{j=1}^J d_{g_j}^{\alpha_j}(z_{\tau(\alpha_{j-1}+1)}, \dots, z_{\tau(\alpha_j)}) \quad (2.4)$$

where $\alpha_0 = 0$, S_k is the symmetric group of k elements, and for all $1 \leq j \leq J$ and all $r \in \mathbb{N}$,

$$d_{g_j}^r(z) = \frac{\text{perm}[C_j^r(z) - B_j^r(z)(A_j^r)^{-1}(z)(B_j^r)^*(z)]}{\det \pi A_j^r(z)}. \quad (2.5)$$

Here, perm denotes the permanent of a matrix and A_j^r, B_j^r, C_j^r are complex $r \times r$ -matrices given by

$$(A_j^r)_{n,m} = K^j(z_n, \bar{z}_m), \quad (B_j^r)_{n,m} = (\partial_z K^j)(z_n, \bar{z}_m), \quad (C_j^r)_{n,m} = (\partial_{z\bar{w}} K^j)(z_n, \bar{z}_m),$$

with K^j as in Theorem 5.

The function $d_{g_j}^r(z)$ in (2.5) is the r -point function of the Gaussian analytic function g_j , see (2.2). Theorem 6 tells us that the limiting k -point measures admit densities with respect to the Lebesgue measure and that those can be determined by concatenating the r -point functions, for $1 \leq r \leq k$, of each Gaussian analytic function g_j .

A result by Nazarov and Sodin [13, Theorem 1.1] implies that there exists a positive constant $C = C(r, g_j, O)$ such that, for any configuration of pairwise distinct points $z_1, \dots, z_k \in O$,

$$C^{-1} \prod_{i < j} |z_i - z_j|^2 \leq d_{g_j}^r(z_1, \dots, z_k) \leq C \prod_{i < j} |z_i - z_j|^2. \quad (2.6)$$

Using this, we immediately conclude from Theorem 6 the following

Corollary 7. *Let $O \Subset \mathbb{C}$ be a compact set, let $k > J$ and let $d^k(z)$ be as in (2.4). Then there exists a positive constant $C = C(r, O)$ such that, for any configuration of pairwise distinct points $z_1, \dots, z_k \in O$,*

$$0 \leq d^k(z_1, \dots, z_k) \leq C \sum_{i < j} |z_i - z_j|^2.$$

We have seen by Theorem 6 that the limiting point process of the rescaled eigenvalues is given by the superposition of J independent processes given by the zeros of independent Gaussian analytic functions. Therefore, we expect that the formation of clusters of eigenvalues is possible and that the eigenvalues do not statistically repel each other, meaning that the limiting k -point functions do not decay to zero as the distance between the k -points gets smaller. This is made explicit in the case of two eigenvalues in the example below. However, Corollary 7 tells us that, in the limit $h \rightarrow 0$, the probability to find more than J eigenvalues close together decays quadratically with the distance. Therefore, finding clusters of more than J eigenvalues very close together is highly unlikely.

2-point correlation function The 2-point correlation function of a point process is given by the normalised 2-point function of the eigenvalues of P_V^δ . We can show that

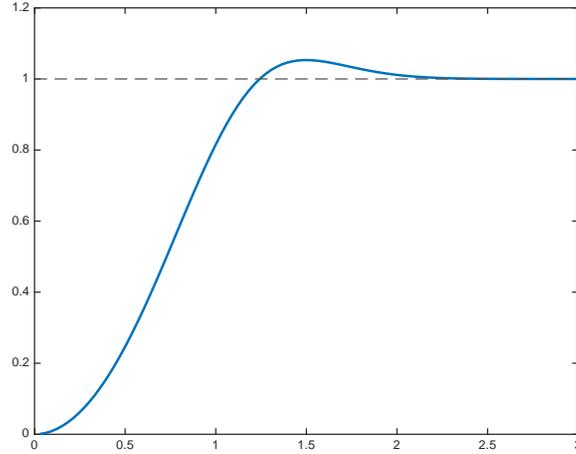


Figure 3. Plot of the scaling limit function $\kappa(t^2)$, see (2.7).

for $z_1 \neq z_2 \in O$

$$K^2(z_1, z_2) = \frac{d^2(z_1, z_2)}{d^1(z_1)d^1(z_2)} = 1 + \sum_{j=1}^J \frac{(\sigma_+^j(z_0))^2}{\left(\sum_{j=1}^J \sigma_+^j(z_0)\right)^2} \left[\kappa\left(\frac{\sigma_+^j(z_0)|z_1 - z_2|^2}{2}\right) - 1 \right]$$

where

$$\kappa(t) = \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}, \quad t \geq 0. \quad (2.7)$$

The function $\kappa\left(\frac{\sigma_+^j(z_0)|z_1 - z_2|^2}{2}\right)$ describes the 2-point correlation function of the zeros of the Gaussian analytic function (2.2), see Figure 3. Moreover, κ describes the scaling limit 2-point correlation function of the zeros of certain random polynomials describing random spin states, see J.H. Hannay [18]. In the work by P. Bleher, B. Shiffman and S. Zelditch [1] κ describes the scaling limit 2-point correlation function of the zeros of random holomorphic sections of the N-th power of a positive Hermitian line bundle over a compact complex Kähler manifold. The 2-point correlation function K^2 reveals precise information about the limiting statistical interaction between two eigenvalues of the perturbed operator P_V^δ :

Long range decorrelation: For $|z_1 - z_2| \gg 1$, in the limit $h \rightarrow 0$, we have exponential decay to 1 of the 2-point correlation function, i.e. :

$$K^2(z_1, z_2) = 1 + \mathcal{O}\left(e^{-\min_j \sigma_+^j(z_0)|z_1 - z_2|^2}\right).$$

This means, that at large distances, two eigenvalues are placed in a decorrelated way.

Absence of close range repulsion: For $|z_1 - z_2| \ll 1$, in the limit $h \rightarrow 0$, there is a weak form of repulsion between two eigenvalues at close range,

$$K^2(z_1, z_2) = 1 - \sum_{j=1}^J \frac{(\sigma_+^j(z_0))^2}{\left(\sum_{l=1}^J \sigma_+^l(z_0)\right)^2} \left[1 - \frac{\sigma_+^j(z_0)|z_1 - z_2|^2}{2} (1 + \mathcal{O}(|z_1 - z_2|^4)) \right]. \quad (2.8)$$

This means that the probability to find two eigenvalues at very short distances of each other is less than to find them at larger distances. However, the probability is not zero at

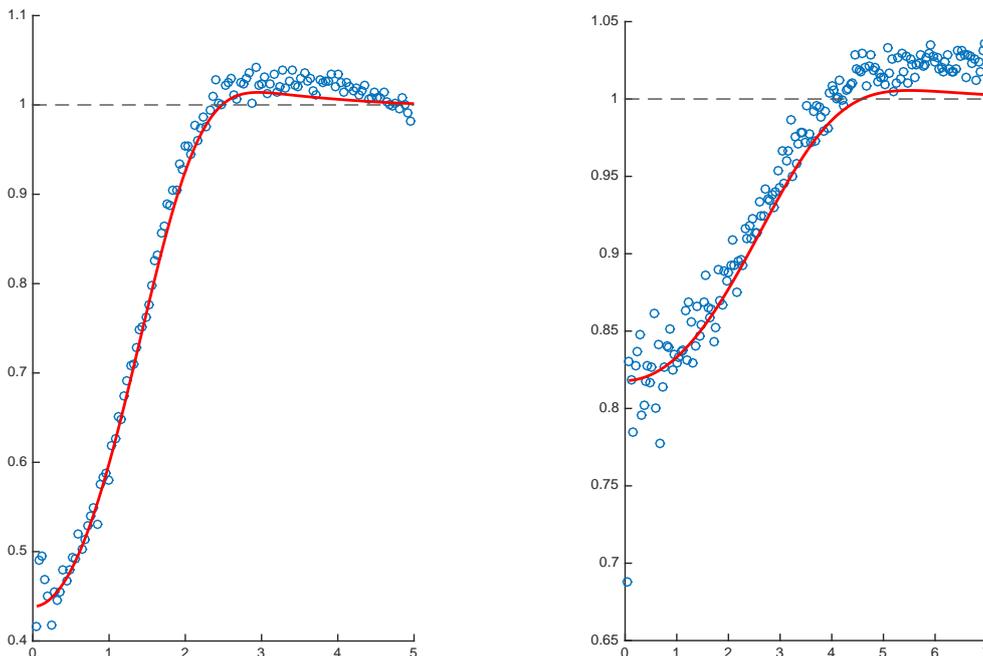


Figure 4. The left hand side compares the scaling limit 2-point correlation function K^2 (2.8) of $P_{h,1}^\delta$ (2.9) as a function of $|z_1 - z_2|^2$ (in red) with the numerically obtained 2-point correlation function (blue circles). The right hand side shows the same for the operator $P_{h,3}$. This Figure was taken from [22].

zero distance as the 2-point density remains strictly positive when $|z_1 - z_2| \rightarrow 0$. Hence, pairs of rescaled eigenvalues show only very weak repulsion at close distance. In fact, the larger the number of quasimodes J gets, the weaker this repulsion at close distances becomes.

The behaviour of the 2-point correlation function K^2 is illustrated in Figure 4 where we compare the scaling limit 2-point correlation function K^2 , at $z_0 = 1.6$, with the numerically obtained 2-point correlation function in the case of two different operators given by

$$P_{h,q}^\delta = -h^2 \partial_x^2 + e^{-iqx} + \delta V_\omega, \quad q = 1, 3 \quad \text{on } S^1, \quad (2.9)$$

with $h = 10^{-3}$, $\delta = 10^{-12}$ and with V_ω as in (RP) with the v_j given by independent and identically distributed complex Gaussian random variables. Notice that for the numerical experiments we use operators on S^1 as they are numerically easier to treat than operators on \mathbb{R} . The operator $P_{h,1}^0$ admits 2 quasimodes and $P_{h,3}^0$ admits 6 quasimodes. Figure 4 compares the numerically obtained 2-point correlation functions (shown as blue dots) of the operators $P_{h,1}^\delta$, on the left hand side, and $P_{h,3}^\delta$, on the right hand side, with the scaling limit 2-point correlation K^2 in (2.8). We see our results fits very nicely with the numerical experiment and indeed Figure 4 suggests that our results can be extended to the case of operators on S^1 .

2.2. Perturbation by random matrix.

2.2.1. *Universal limiting point process.* In the case P_M^δ , as in (1.16), where we perturb the operator P_h with a random matrix δM_ω .

Theorem 8. *Let $\Omega \Subset \mathring{\Sigma}$ be as in (1.9). Let p be as in (1.4) satisfying (1.7). Let $z_0 \in \Omega$. Then, for any $O \Subset \mathbb{C}$ open connected relatively compact domain, we have that*

$$\mathcal{Z}_{h,z_0}^M \xrightarrow{d} \mathcal{Z}_{G_{z_0}} \text{ on } O, \quad \text{as } h \rightarrow 0, \quad (2.10)$$

in the sense that for all $\phi \in \mathcal{C}_c(O)$

$$\langle \mathcal{Z}_{h,z_0}^M, \phi \rangle = \sum_{z \in \text{Spec}(P_h^\delta)} \phi((z - z_0)h^{-1/2}) \xrightarrow{d} \langle \mathcal{Z}_{G_{z_0}}, \phi \rangle = \sum_{z \in G_{z_0}^{-1}(0)} \phi(z), \quad \text{as } h \rightarrow 0, \quad (2.11)$$

where the convergence is in distribution. Moreover,

$$G_{z_0}(z) := \det(g_{z_0}^{i,j}(z))_{1 \leq i,j \leq J}, \quad z \in \mathbb{C},$$

where $g_{z_0}^{i,j}$, for $1 \leq i, j \leq J$, are independent Gaussian analytic functions on \mathbb{C} with covariance kernel

$$K_{z_0}^{i,j}(z, \bar{w}) = e^{\frac{1}{2}(\sigma_+^i(z_0) + \sigma_-^j(z_0))z\bar{w}}, \quad (2.12)$$

with $\sigma_\pm^i(z_0)$ as in (2.1).

Since the $g_{z_0}^{i,j}$ are Gaussian analytic functions with covariance kernel $K_{z_0}^{i,j}$, we have in view of (1.23) and (1.25) that

$$g_{z_0}^{i,j} \stackrel{d}{=} \sum_{n=0}^{\infty} \alpha_n \frac{(\sigma_+^i(z_0) + \sigma_-^j(z_0))^{n/2} z^n}{2^{n/2} \sqrt{n!}}, \quad \alpha_n \sim \mathcal{N}_{\mathbb{C}}(0, 1) \text{ (iid)}. \quad (2.13)$$

where the equality holds in distribution.

Theorem 8 tells us that

- (1) at any given point $z_0 \in \Omega$ in the bulk of the pseudospectrum, the local rescaled point process of the eigenvalues of P_M^δ is given, in the limit $h \rightarrow 0$, by the point process given by the zeros of the determinant of a $J \times J$ matrix whose entries are independent Gaussian analytic functions. Notice in particular, that the distribution of the function positioned at the i -th row and j -th column, determined by their covariance kernels, depends only on the part of the volume form $p_*(|d\xi \wedge dx|)$ coming from $\rho_+^i(z)$ and $\rho_-^j(z)$.
- (2) We see that the rescaled point process \mathcal{Z}_h^M of the eigenvalues of the perturbed operator P_M^δ has a universal limit, which is independent of the specific probability distribution of the potential M_ω (1.14) and depends only on the structure of the energy shell $p_0^{-1}(z)$, i.e. it depends only on the densities $\{\sigma_\pm^j(z_0); j = 1, \dots, J\}$. Furthermore, this universal limit is different from the universal limit of \mathcal{Z}_h^V , where we used a random potential V_ω as perturbation.

2.2.2. Scaling limit k -point measures. We will see that the k -point measures of the point process \mathcal{Z}_h^M converge to those of the limiting point process.

Corollary 9. *Let μ_h^{k,M,z_0} be the k -point density measure of \mathcal{Z}_h^M , defined as in (2.3), and let μ_k be the k -point density measure of the point process $\mathcal{Z}_{G_{z_0}}$, given in Theorem 8. Then, for any $O \Subset \mathbb{C}$ open, relatively compact connected domain and for all $\varphi \in \mathcal{C}_c(O^k \setminus \Delta)$,*

$$\int \varphi(z) \mu_h^{k,M,z_0}(dz) \longrightarrow \int \varphi(z) \mu_k(dz), \quad h \rightarrow 0.$$

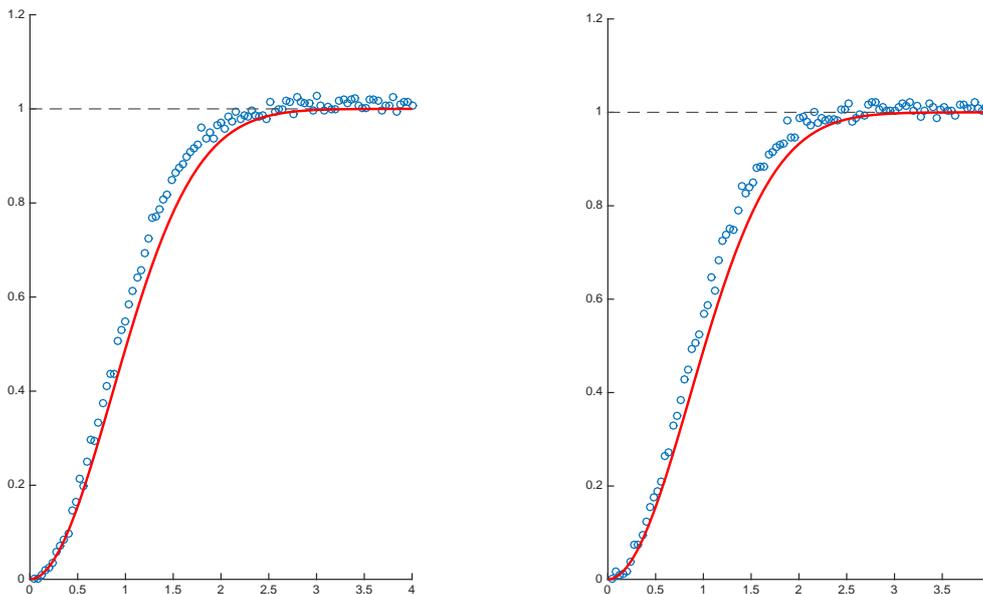


Figure 5. The left hand side compares the conjectured scaling limit 2-point correlation function K^2 at $z_0 = 1.6$ (2.15) of $P_{h,1}^0 + \delta M_\omega$ (2.9) as a function of $|z_1 - z_2|^2$ (in red) with the numerically obtained 2-point correlation function (blue circles). Here, M_ω is a complex Gaussian random matrix. The right hand side shows the same for the operator $P_{h,3}^0 + \delta M_\omega$. This Figure was taken from [22].

One can calculate the Lebesgue densities of the limiting 1-point measures μ_1 , which is given by

$$d^1(z) = \sum_{i=1}^J \frac{\sigma_+^i(z_0) + \sigma_+^j(z_0)}{2\pi}, \quad (2.14)$$

which is precise the average density of eigenvalues at z_0 which one would expect from the probabilistic Weyl law in Theorem 4, see also (2.1).

Calculating the Lebesgue densities of the limiting k -point measures μ_k remains an open problem. However, the numerical experiments presented in Figure 5 and the result by Nazarov and Sodin (2.6) lead us to the following

Conjecture 10. *The k -point functions $d_k(z)$ of the point process of zeros of G_{z_0} as in Theorem 8 exhibit quadratic decay at close range. For any compact set $O \Subset \mathbb{C}$ there exists a positive constant $C > 0$ depending only on O and k such that for all pairwise distinct points $z - 1, \dots, z_k \in O$*

$$C^{-1} \prod_{i < j} |z_i - z_j|^2 \leq d_k(z_1, \dots, z_k) \leq C \prod_{i < j} |z_i - z_j|^2.$$

Moreover, the 2-point correlation function of K^2 of the point process of zeros of G_{z_0} is given by

$$K^2(z_1, z_2) = 1 - \exp \left[-\frac{\pi}{4} \left(\sum_{i=1}^J \sigma_+^i(z_0) + \sigma_+^j(z_0) \right) |z_1 - z_2|^2 \right]. \quad (2.15)$$

3. IDEAS OF THE PROOF IN A MODEL CASE

We will outline the basic strategy of the proof of Theorem 5 and 8 in the case of the following model operator which was already studied in a similar context in [16, 2, 36, 35]:

Let $0 < h \ll 1$, we consider on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the operator $P_h : L^2(S^1) \rightarrow L^2(S^1)$ given by

$$P_h := hD_x + g(x), \quad D_x := \frac{1}{i} \frac{d}{dx}, \quad g \in \mathcal{C}^\infty(S^1; \mathbb{C}) \quad (3.1)$$

where we assume that $g \in \mathcal{C}^\infty(S^1; \mathbb{C})$ is such that $\text{Im } g$ has exactly two critical points and they are non-degenerate, one minimum and one maximum, say at $a, b \in S^1$, with $\text{Im } g(a) < \text{Im } g(b)$. The principal symbol of P_h will be denoted by

$$p(x, \xi) = \xi + g(x), \quad p \in \mathcal{C}^\infty(T^*S^1). \quad (3.2)$$

The spectrum of P_h is discrete and comprised out of simple eigenvalues, indeed

$$\text{Spec}(P_h) = \{z \in \mathbb{C} : z = \langle g \rangle + kh, k \in \mathbb{Z}\}, \quad \langle g \rangle := (2\pi)^{-1} \int_0^{2\pi} g(y) dy. \quad (3.3)$$

As in (1.5) we define

$$\Sigma := \overline{T^*S^1}. \quad (3.4)$$

Theorem 11. *Let $z_0 \in \mathring{\Sigma}$. Then, any $O \Subset \mathbb{C}$ open connected relatively compact domain, we have that for all $\phi \in \mathcal{C}_c(O)$*

$$\langle \mathcal{Z}_{h,z_0}, \phi \rangle \stackrel{\text{def}}{=} \sum_{z \in \text{Spec}(P_h^\delta)} \phi((z - z_0)h^{-1/2}) \xrightarrow{d} \sum_{z \in g_{z_0}^{-1}(0)} \phi(z) \stackrel{\text{def}}{=} \langle \mathcal{Z}_{G_{z_0}}, \phi \rangle, \quad \text{as } h \rightarrow 0, \quad (3.5)$$

where the convergence is in distribution and where g_{z_0} is a Gaussian analytic functions on \mathbb{C} with covariance kernel

$$K_{z_0}(z, \bar{w}) = e^{\frac{1}{2}(\sigma_+(z_0) + \sigma_-(z_0))z\bar{w}}, \quad (3.6)$$

with $\sigma_\pm(z_0)$ as in (2.1) with $J = 1$.

Outline of the proof

In Section 3.1 we will construct quasimodes for (3.1) using what is essentially the classical WKB construction. In Section 3.2 we will discuss the random perturbation added. In the case of the operator (3.1) we will only discuss the case of perturbation by random matrix since the symbol p (3.2) does not satisfy the symmetry assumption (1.18). In Section 3.3 we construct a Grushin problem which yields an effective description of the eigenvalues of the perturbed operator as a random analytic function. In Sections 3.4 to 3.6 we will show that these random analytic function converge in distribution to a Gaussian analytic function.

Remark 12. *This model case is in many ways easier to treat than the case considered in [22] since the energy shell $p^{-1}(z)$ for $z \in \mathring{\Sigma}$ contains only two points. However, treating the model case conveys the fundamental strategy to proving Theorems 5 and 8.*

3.1. Quasimodes for the unperturbed operator P_h . We follow the construction used in [35]. Suppose that $z_0 \in \mathring{\Sigma}$ and let

$$\begin{aligned} \Omega \Subset \mathring{\Sigma} \text{ be a small, open, simply connected, relatively compact neighbourhood of } z_0 \\ \text{s.t. } \text{dist}(\Omega, \partial\Sigma) > 1/C. \end{aligned} \quad (3.7)$$

The energy shell $p^{-1}(z) \subset T^*S^1$, for $z \in \Omega$, contains precisely two points and the Poisson bracket of $\operatorname{Re} p$ and $\operatorname{Im} p$ evaluated at these points is non-zero. More precisely, we have

$$p^{-1}(z) = \{\rho_+(z), \rho_-(z)\}, \quad \pm\{\operatorname{Re} p, \operatorname{Im} p\}(\rho_{\pm}(z)) < 0. \quad (3.8)$$

Write $\rho_{\pm}(z) = (x_{\pm}(z), \xi_{\pm}(z))$ and notice that $x_{\pm}(z)$ only depend on $\operatorname{Im} z$. By the natural projection $\Pi : \mathbb{R} \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$ we identify S^1 with the interval $[b - 2\pi, b[$ and the points $x_{\pm}, a, b \in S^1$ with points $x_{\pm}, a, b \in \mathbb{R}$ so that $b - 2\pi < x_+ < a < x_- < b$.

Let $K_{\pm} \subset]b - 2\pi, a[$ be open intervals such that

$$\overline{x_{\pm}(\Omega)} \subset K_{\pm} \quad \text{and} \quad \overline{K_+} \cap \overline{K_-} = \emptyset. \quad (3.9)$$

Let $\chi_{\pm} \in \mathcal{C}_0^{\infty}(]b - 2\pi, a[, [0, 1])$, such that $\chi_{\pm} = 1$ in a small neighbourhood of K_{\pm} . Write $x_0^{\pm} = x_{\pm}(\operatorname{Im} z)$ and define for $x \in \mathbb{R}$

$$\tilde{e}_{\pm}(x, z; h) := \chi_{\pm}(x) \exp\left(\frac{i}{h}\psi_{\pm}(x, z)\right), \quad (3.10)$$

where

$$\psi_+(x, z) := \int_{x_0^+}^x (z - g(y)) dy, \quad \psi_-(x, z) := \int_{x_0^-}^x \overline{(z - g(y))} dy. \quad (3.11)$$

Note that $\tilde{e}_+(x, z; h)$ depends holomorphically on z and that $\tilde{e}_-(x, z; h)$ depends anti-holomorphically on z . Notice that $\exp\left(\frac{i}{h}\psi_+(x, z)\right)$ is a solution to $(P_h - z)u = 0$ on $\operatorname{supp} \chi_+$, since the phase function ψ_+ satisfies the eikonal equation

$$p(x, \partial_x \psi_+) = z.$$

Similarly, we have that $\exp\left(\frac{i}{h}\psi_-(x, z)\right)$ is a solution to $(P_h - z)^*u = 0$ on $\operatorname{supp} \chi_-$.

By the method of stationary phase, one gets that

$$\|\tilde{e}_{\pm}(z)\|^2 = e^{\frac{2}{h}\Phi_{\pm}(z; h)} \quad (3.12)$$

with

$$\begin{aligned} \Phi_{\pm}(z; h) &= \Phi_{\pm}(i\operatorname{Im} z; h) = \pm \operatorname{Im} \int_{x_{\pm}(\operatorname{Im} z)}^{x_0^{\pm}} (z - g(y)) dy + \frac{h}{2} \ln h^{1/2} A_{\pm}(z; h) \\ &= 2\Phi_{\pm, 0}(z) + \frac{h}{2} \ln h^{1/2} A_{\pm}(z; h) \end{aligned} \quad (3.13)$$

with $A_{\pm}(z; h) \sim A_0^{\pm}(z; h) + hA_1^{\pm}(z) + \dots$ in $\mathcal{C}^{\infty}(\Omega)$ and

$$A_0^{\pm}(z; h) = \left(\frac{\pi}{\mp\{\operatorname{Re} p, \operatorname{Im} p\}(\rho_{\pm})} \right).$$

To normalise the states \tilde{e}_{\pm} we set

$$e_{\pm}(z, x; h) = \tilde{e}_{\pm}(z, x; h) e^{-\frac{1}{h}\Phi_{\pm}(z; h)}. \quad (3.14)$$

By potentially shrinking Ω , K_{\pm} and the support of χ_{\pm} , we have that $e_{\pm}(z, x; h) = \mathcal{O}(e^{-\frac{1}{h}(x - x_{\pm}(\operatorname{Im} z))^2})$. Since $\operatorname{dist}(\operatorname{supp} \chi'_{\pm}, K_{\pm}) \geq 1/C$, it follows that

$$\|(P - z)e_+\| = \mathcal{O}(e^{-1/Ch}), \quad \text{and} \quad \|(P - z)^*e_-\| = \mathcal{O}(e^{-1/Ch})$$

and that e_{\pm} are microlocalized on $\rho_{\pm}(z) \in T^*S^1$.

Remark 13. Since p (1.2) is a smooth symbol, the result by Davies [7] and Dencker-Sjöstrand-Zworski [9] says that as a consequence of (3.8) we can construct quasimodes $e_{\pm} \in L^2(S^1)$ for $(P_h - z)$ microlocalized in $\rho_{\pm}(z)$ such that

$$\|(P_h - z)e_+\| = \mathcal{O}(h^\infty)\|e_+\|, \quad \|(P_h - z)^*e_-\| = \mathcal{O}(h^\infty)\|e_-\|. \quad (3.15)$$

However, due to the fact that P_h (3.1) is a one dimensional operator and due to the fact that the imaginary part of g has precisely two critical points, both non-degenerate, we can construct quasimodes satisfying (3.15) with $\mathcal{O}(h^\infty)$ replaced by the exponentially small error term $\mathcal{O}(e^{-1/Ch})$. In general this would require an analyticity assumption on the symbol p (3.2), see [9].

Therefore, any such Ω is contained in the $e^{-1/Ch}$ -pseudospectrum of $P_h - z$. Moreover, in the smooth case, the quasimodes \tilde{e}_{\pm} are not holomorphic respectively anti-holomorphic in z but rather almost holomorphic respectively almost anti-holomorphic.

3.2. Adding a random perturbation. Let $\{e_j\}_{j \in \mathbb{Z}}$ denote the orthonormal basis of $L^2(S^1)$ comprised of the normalised Fourier modes. Suppose that $q_{j,k}$ are independent and identically distributed complex random variables satisfying (1.6), (1.5). Let $N(h) = C/h$, with $C > 0$ sufficiently large, and set

$$Q_\omega = \frac{1}{N(h)} \sum_{|j|, |k| < N(h)} q_{j,k} e_j e_k^*, \quad (3.16)$$

where $e^*u = (u|e^k)$, $u \in L^2(S^1)$. We will study the following random operator

$$P_h^\delta = P_h + \delta Q_\omega, \quad (3.17)$$

with

$$e^{-\frac{1}{Ch}} \leq \delta \ll h^\kappa, \quad \kappa > 5/2, \quad (3.18)$$

where $C > 0$ is assume sufficiently large. Notice that since Q_ω is compact $L^2(S^1) \rightarrow L^2(S^1)$, the spectrum of P_h^δ is discrete. Therefore, we can define the random measure of eigenvalues of P_h^δ on Ω around z_0 and rescaled by $h^{-1/2}$ by

$$\mathcal{Z}_{h,z_0} = \sum_{z \in \text{Spec}(P_h^\delta)} \delta_{(z-z_0)h^{-1/2}}, \quad (3.19)$$

where the eigenvalues are counted according to multiplicity.

Remark 14. We can allow for an exponentially small coupling constant δ , see (3.18) since we have $\mathcal{O}(e^{-1/Ch})$ -quasimodes, cf. (3.14). In case of (3.15), we would assume (1.19).

We are interested, in a small perturbation: Therefore, we restrict the random variables to large discs of radius C/h , centred at 0

$$q_{j,k} \in D(0, C/h), \quad 0 \leq j, k < C/h.$$

By the tail estimate (1.15) and by the fact that $N(h) = C/h$, we have that

$$\mathbb{P}[|q_{i,j}| \leq C/h, \forall 0 \leq i, j < N(h)] \geq 1 - \kappa N(h)^2 h^{4+\varepsilon_0} = 1 - \mathcal{O}(h^{2+\varepsilon_0}). \quad (3.20)$$

Thus, we have with probability $\geq 1 - \mathcal{O}(h^{2+\varepsilon_0})$, that

$$\|Q_\omega\|_{HS} = N(h)^{-1} \left(\sum_{i,j < N(h)} |q_{i,j}|^2 \right)^{1/2} \leq Ch^{-1} = \mathcal{O}(h^{-1}). \quad (3.21)$$

We denote by $(\mathcal{M}_h, \mathcal{F}_h, \mathbb{P}_h)$ the probability space $(\mathcal{M}, \mathcal{F}, \mathbb{P})$ restricted to $q_{j,k}^{-1}(D(0, C/h))$. The random variables restricted to this space will be denoted by

$$q_{j,k}^h, \quad j, k \leq N(h). \quad (3.22)$$

The restricted probability measure \mathbb{P}_h is given by the conditional probability

$$\mathbb{P}_h[\cdot] = \mathbb{P}[\cdot \mid |q_{j,k}| \leq C/h]. \quad (3.23)$$

Notice that the $q_{j,k}^h$ are distributed independently and identically, since this is the case for the random variables $q_{j,k}$. Moreover, by (1.5), (1.14), we can show that

$$\mathbb{E}[q_{j,k}^h] = \mathcal{O}(h^{3+\varepsilon_0}), \quad \mathbb{E}[|q_{j,k}^h|^2] = 1 + \mathcal{O}(h^{2+\varepsilon_0}). \quad (3.24)$$

From now on we will be working with the restricted random variables.

3.3. Grushin Problem for the perturbed operator. In this section we recall the construction of the Grushin problem. The following has been taken from [35], however, the ideas go back to [15, 17].

As reviewed in [30], the central idea is to set up an auxiliary problem of the form

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where $P(z)$ is the operator under investigation and R_{\pm} are suitably chosen, so that $\mathcal{P}(z)$ is bijective. If $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, one typically writes

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation goes back to the Shur complement formula or, equivalently, the Lyapunov-Schmidt bifurcation method, i.e. the operator $P(z) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is invertible if and only if the finite dimensional matrix $E_{-+}(z)$ is invertible and when $E_{-+}(z)$ is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

$E_{-+}(z)$ is sometimes called effective Hamiltonian.

Proposition 15. *Let $z \in \Omega$ be as in (3.7), let e_{\pm} be as in (3.14) and let P_h^{δ} be as in (3.17). Define*

$$\begin{aligned} R_+ &: H^1(S^1) \longrightarrow \mathbb{C} : u \longmapsto (u|e_+), \\ R_- &: \mathbb{C} \longrightarrow L^2(S^1) : u_- \longmapsto u_- f_-. \end{aligned}$$

Then

$$\mathcal{P}^{\delta}(z) := \begin{pmatrix} P_h^{\delta} - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathcal{E}^{\delta}(z) = \begin{pmatrix} E^{\delta}(z) & E_+^{\delta}(z) \\ E_-^{\delta}(z) & E_{-+}^{\delta}(z) \end{pmatrix}$$

where $E_-^{\delta}(z)v = (v|e_-) + (\mathcal{O}(\delta h^{-3/2}) + \mathcal{O}(e^{-1/Ch}))\|v\|$, $E_+^{\delta}(z)v_+ = v_+ e_+ + (\mathcal{O}(\delta h^{-3/2}) + \mathcal{O}(e^{-1/Ch}))|v_+|$, $\|E^{\delta}(z)\|_{L^2 \rightarrow H^1} = \mathcal{O}(h^{-1/2})$ and

$$E_{-+}^{\delta}(z) = -\delta(Q_{\omega} e_+ | e_-) + \mathcal{O}(e^{-1/Ch}) + \mathcal{O}(\delta^2 h^{-5/2}),$$

where the error estimates are uniform in $z \in \Omega$.

Proof. See [36, 35]. □

Notice that $E_{-+}^\delta(z)$ depends smoothly on $z \in \Omega$. However, we can make it holomorphic in z since it satisfies a $\bar{\partial}$ -equation in z . Taking the \bar{z} -derivative of $\mathcal{P}^\delta \mathcal{E}^\delta = 1$, we get

$$\partial_{\bar{z}} \mathcal{E}^\delta = -\mathcal{E}^\delta \partial_{\bar{z}} \mathcal{P}^\delta \mathcal{E}^\delta.$$

Hence, by Proposition 15

$$\begin{aligned} \partial_{\bar{z}} E_{-+}^\delta &= -[(\partial_{\bar{z}} R_+) E_+^\delta + E_-^\delta (\partial_{\bar{z}} R_-)] E_{-+}^\delta \\ &= -[(e_+ | \partial_z e_+) + (\partial_{\bar{z}} e_- | e_-) + \mathcal{O}(\delta h^{-3/2}) + \mathcal{O}(e^{-1/Ch})] E_{-+}^\delta \\ &\stackrel{\text{def}}{=} -k^\delta E_{-+}^\delta \end{aligned} \quad (3.25)$$

where the error estimates are uniform in $z \in \Omega$.

Taking the ∂_z derivative of the identity $\|\tilde{e}_+\|^2 = e^{2\Phi+j/h}$, we get that

$$h^{-1} \partial_z \Phi_+ = \frac{1}{2} \partial_z \log \|\tilde{e}_+\|^2.$$

Similarly, we get that

$$h^{-1} \partial_{\bar{z}} \Phi_- = \frac{1}{2} \partial_{\bar{z}} \log \|\tilde{e}_-\|^2.$$

Using these identities together with Proposition 15, one computes that

$$(e_+ | \partial_z e_+) = \partial_{\bar{z}} \ln \|\tilde{e}_+\| + \mathcal{O}(e^{-1/Ch}) \quad (3.26)$$

and

$$(\partial_{\bar{z}} e_- | e_-) = \partial_{\bar{z}} \ln \|\tilde{e}_-\| + \mathcal{O}(e^{-1/Ch}) \quad (3.27)$$

Since $k^\delta(z)$ depends smoothly on $z \in \Omega$, we can solve the equation $\partial_{\bar{z}} l^\delta = h k^\delta$ in Ω [19] such that

$$h^{-1} l^\delta(z) = \ln \|\tilde{e}_+\| \|\tilde{e}_-\| + \mathcal{O}(e^{-1/Ch}) + \mathcal{O}(\delta h^{-3/2}). \quad (3.28)$$

Then, by (3.25) we have that

$$-h^{-1/2} \delta^{-1} e^{\frac{1}{h} l^\delta(z)} E_{-+}^\delta(z) = e^{\frac{1}{h} l^\delta(z)} \left((Q_\omega h^{-1/4} e_+ | h^{-1/4} e_-) + \mathcal{O}(\delta^{-1} e^{-1/Ch}) + \mathcal{O}(\delta h^{-3}) \right)$$

is holomorphic in $z \in \Omega$. By (3.28), we get that

$$\begin{aligned} G_h^\delta(z) &\stackrel{\text{def}}{=} -h^{-1/2} \delta^{-1} e^{\frac{1}{h} l^\delta(z)} E_{-+}^\delta(z) \\ &= (Q_\omega h^{-1/4} \tilde{e}_+ | h^{-1/4} \tilde{e}_-) + \|h^{-1/4} \tilde{e}_+\| \|h^{-1/4} \tilde{e}_-\| \left(\mathcal{O}(\delta^{-1} e^{-1/Ch}) + \mathcal{O}(\delta h^{-5/2}) \right) \\ &\stackrel{\text{def}}{=} g_h(z) + R(z; h). \end{aligned} \quad (3.29)$$

with $g_h(z) \stackrel{\text{def}}{=} (Q_\omega h^{-1/4} \tilde{e}_+(z) | h^{-1/4} \tilde{e}_-(z))$. Hence, by Proposition 15, the eigenvalues of P^δ in Ω are given by the zeros of the holomorphic function $G_h^\delta(z)$ in Ω . Moreover, $g_h(z)$ depends holomorphically on z since that is the case for \tilde{e}_+ and \tilde{e}_- . Therefore, also R depends holomorphically on z .

Using (3.19) we conclude that on Ω

$$\mathcal{Z}_{h,z_0} = \sum_{z \in (G_h^\delta)^{-1}(0)} \delta_{(z-z_0)h^{-1/2}} \quad (3.30)$$

To simplify the notation we will from now on consider directly the rescaled function: let W be a sufficiently small neighborhood of 0 so that $z_0 + h^{1/2}W \subset \Omega$. Then, for $z \in W$ set

$$\tilde{G}_h^\delta(z) \stackrel{\text{def}}{=} G_h^\delta(z_0 + h^{1/2}z). \quad (3.31)$$

Therefore, we are interested in the point process

$$\tilde{Z}_{h,z_0} = \sum_{z \in (\tilde{G}_h^\delta)^{-1}(0)} \delta_z. \quad (3.32)$$

3.4. Convergence of random analytic functions. We begin by recalling some basic notions and facts about random analytic functions.

Let $O \subset \mathbb{C}$ be an open simply connected domain. Then, we call a random variable f with values in $\mathcal{H}(O)$, the space of holomorphic functions on O , a *random analytic function*, see for instance [20].

The distribution of a random analytic function, i.e. the direct image measure $f_*\mathbb{P}$ where \mathbb{P} denotes the probability measure of some underlying probability space, is determined by its finite-dimensional distributions, see e.g. [21]. More precisely, let f and g be two random analytic functions, then

$$f \stackrel{d}{=} g \iff (f(z_1), \dots, f(z_n)) \stackrel{d}{=} (g(z_1), \dots, g(z_n)), \forall z_1, \dots, z_n \in O, \forall n \in \mathbb{N}, \quad (3.33)$$

where the symbol $\stackrel{d}{=}$ means equality in distribution, i.e. that the respective direct image measures are equal.

Definition 16. (*Gaussian analytic function - GAF*) Let $O \subset \mathbb{C}$ be an open simply connected complex domain. A random analytic function f on O is called a *Gaussian analytic function on O* if it has centred symmetric complex Gaussian finite-dimensional distributions, i.e. if for all $k \in \mathbb{N}$ and all $z_1, \dots, z_k \in O$ the random vectors $(f(z_1), \dots, f(z_k)) \sim \mathcal{N}_{\mathbb{C}}(0, \Sigma_k)$.

The matrix Σ_k depends on (z_1, \dots, z_k) and its i, j -th entry $(\Sigma_k)_{ij}$ is given by the covariance kernel

$$K(z_i, \bar{z}_j) \stackrel{\text{def}}{=} \mathbb{E}[f(z_i) \overline{f(z_j)}],$$

which is a z_i -holomorphic and z_j -anti-holomorphic function on $O \times O$.

Let $f \not\equiv 0$ be a random analytic function on O . Then, we call

$$\xi_f := \sum_{\lambda \in f^{-1}(0)} \delta_\lambda \quad (3.34)$$

the point process of the zeros of f , which is a well-defined random measure on O .

We recall that convergence of random variables in distribution means that the induced probability measures of the random variables converge in the weak-* topology, see for instance [21]. Shirai [23] observed that convergence in distribution of a sequence of random analytic functions $\{f_n\}_n$ implies the convergence in distribution of the associated sequence of point processes $\{\xi_{f_n}\}_n$:

Proposition 17. Let $O \subset \mathbb{C}$ be an open, simply connected domain. Let f_n , $n \in \mathbb{N}$, and f be random analytic functions on O , not necessarily defined on the same probability space. Suppose that $f \not\equiv 0$ almost surely and suppose that f_n converges in distribution to f , then the point process of zeros ξ_{f_n} converges in distribution to ξ_f , i.e.

$$\langle \xi_{f_n}, \varphi \rangle \xrightarrow{d} \langle \xi_f, \varphi \rangle.$$

It remains to show that G_h^δ , seen as a sequence of random analytic function, converges in distribution, as $h \rightarrow 0$, to the Gaussian analytic function g_{z_0} , as in Theorem 11.

By a famous theorem of Prokhorov [21], this is equivalent to showing the following two points:

1) Tightness of G_h^δ . This is, roughly speaking, a criterion which ensures that the sequence of probability measures induced by G_h^δ does not let any mass escape to infinity. See [21] for a precise definition. For sequences of random analytic functions there is a useful criterion for tightness by Shirai [23]:

Proposition 18. *Let f_n , $n \in \mathbb{N}$, and f be random analytic functions on an open simply connected set $O \subset \mathbb{C}$. Suppose that for any compact set $K \Subset O$*

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}[\|f_n\|_{L^\infty(K)} > r] = 0.$$

Then, the sequence $\{f_n\}_n$ is tight in the space of random variables on $\mathcal{H}(O)$.

2) Convergence in finite dimensional distributions of G_h^δ . Recall (3.33). We need to show that G_h^δ converges in finite dimensional distributions to g_{z_0} . This means that we need to show that for any $k \geq 1$ the \mathbb{C}^k -random vectors $(G_h^\delta(z_1), \dots, G_h^\delta(z_k))$ converge in distribution to $(g_{z_0}(z_1), \dots, g_{z_0}(z_k))$. More precisely, we need to show that for all $k \geq 1$ and all $z_1, \dots, z_k \in W$

$$(G_h^\delta(z_1), \dots, G_h^\delta(z_k)) \xrightarrow{d} (g_{z_0}(z_1), \dots, g_{z_0}(z_k)), \quad h \rightarrow 0. \quad (3.35)$$

Recall from the assumptions of Theorem 8 that g_{z_0} is a Gaussian analytic function with covariance kernel K_{z_0} . Hence, we have that

$$(g_{z_0}(z_1), \dots, g_{z_0}(z_k)) \sim \mathcal{N}_{\mathbb{C}}(0, \Gamma),$$

where the covariance matrix $\Gamma \in GL_k(\mathbb{C})$ admits the entries

$$\Gamma_{i,j} = \mathbb{E} \left[g_\sigma(z_i) \overline{g_\sigma(z_j)} \right] = K_{z_0}(z_i, \bar{z}_j).$$

Therefore, to show (3.35) we need to show that the random vector $(G_h^\delta(z_1), \dots, G_h^\delta(z_k))$ converges in distribution to a complex Gaussian random vector with distribution $\mathcal{N}_{\mathbb{C}}(0, \Gamma)$.

3.5. Tightness. We begin by studying the covariance of the random analytic functions g_h (3.29). First, recall (3.12) and let us rescale to the local scale of eigenvalues as in the discussion after (3.30), i.e. for $z \in W$ set

$$\tilde{g}_h(z) \stackrel{\text{def}}{=} g_h(z_0 + h^{1/2}z).$$

Next, recall (3.13). We can show by the method of stationary phase for complex valued phase functions that for $z, w \in W$

$$\begin{aligned} & (h^{-1/4} \tilde{e}_\pm(z_0 + h^{1/2}z) | h^{-1/4} \tilde{e}_\pm(z_0 + h^{1/2}w)) \\ &= e^{\frac{1}{2} \sigma_\pm(z_0) z \bar{w} + \phi_\pm(z; h) + \phi_\pm(w; h)} [(1 + \mathcal{O}(\sqrt{h})) + \mathcal{O}(h^\infty)]. \end{aligned}$$

where

$$\phi_\pm(z; h) \stackrel{\text{def}}{=} \frac{1}{2} [\ln A_\pm(z_0; h) + (\partial_{zz}^2 \Phi_{\pm,0})(z_0) z^2]$$

and $\sigma_\pm(z_0) = \mp [\{\text{Re } p, \text{Im } p\}(\rho_\pm(z_0))]^{-1}$, see also (2.1) for a connection to the symplectic volume. In view of the discussion after (3.13), we see that

$$e^{\phi_\pm(z; h)} = \mathcal{O}(1)$$

uniformly in $z \in W$ and $h > 0$. This implies in particular that $\|h^{-1/4} \tilde{e}_\pm(z_0 + h^{1/2}z)\| = \mathcal{O}(1)$ uniformly in $z \in W$ and $h > 0$.

Set

$$K_{z_0}(z, \bar{w}) \stackrel{\text{def}}{=} e^{\frac{1}{2}(\sigma_+(z_0) + \sigma_-(z_0))z\bar{w}},$$

and $\tilde{\phi}(z; h) = \phi_+(z; h) + \phi_-(z; h)$.

Recall from the beginning of Section 3.2 that $\{e_j\}_{j \in \mathbb{Z}}$ are the normalised Fourier modes. Hence, we can show that for $h|j| \geq C$, with $C > 0$ sufficiently large, we have that $(\tilde{e}_\pm|e_j) = \mathcal{O}(|j|^{-\infty})\|\tilde{e}_\pm\|$. It then follows from (3.16), (3.24) that, for $C > 0$ large enough and $h > 0$ small enough,

$$\begin{aligned} \mathbb{E}[\tilde{g}_h(z)\overline{\tilde{g}_h(w)}] &= (\tilde{e}_+(z_0 + h^{1/2}z)|\tilde{e}_+(z_0 + h^{1/2}w))(\tilde{e}_-(z_0 + h^{1/2}w)|\tilde{e}_-(z_0 + h^{1/2}z)) \\ &\quad + \mathcal{O}(h^2) \\ &= K(z, \bar{w})e^{\tilde{\phi}(z; h) + \tilde{\phi}(w; h)}(1 + \mathcal{O}(\sqrt{h})) \end{aligned} \tag{3.36}$$

where in the last line we used as well that $\tilde{\phi}$ is a bounded smooth function on \mathbb{C} .

Before we continue, let us remark that the exponentials $e^{\tilde{\phi}(z; h)}$ are strictly positive and deterministic. They have no influence on the statistics of the zeros of the random function g_{z_0} . We can therefore see it as a gauge function which we shall get rid of later on.

Let $K \Subset W$ be some compact subset. For $\varepsilon > 0$ let $K_\varepsilon = K + \overline{D(0, \varepsilon)}$ be the closure of an ε -neighbourhood of K . Pick $\varepsilon > 0$ small enough so that $K_\varepsilon \Subset O$. Thus, by (3.36), we have for $h_0 > 0$ small enough that

$$\sup_{0 < h < h_0} \mathbb{E} \left[\|\tilde{g}_h\|_{L^2(K_\varepsilon)}^2 \right] < C(K_\varepsilon) < +\infty. \tag{3.37}$$

Recall from (3.29) that \tilde{G}_h^δ is holomorphic. It has been observed by Shirai [23] that Hardy's convexity theorem implies that there exists a positive constant $C_{K_\varepsilon} > 0$ depending only on K_ε so that

$$\|\tilde{G}_h^\delta\|_{L^\infty(K)}^2 \leq C_{K_\varepsilon} \int_{K_\varepsilon} |\tilde{G}_h^\delta(z)|^2 L(dz). \tag{3.38}$$

By (3.29), (3.18), we have that R is of order $\mathcal{O}(\delta h^{5/2}) + \mathcal{O}(e^{-1/Ch})$. Using Markov's inequality in combination with (3.38), one obtains that for $h_0 > 0$ small enough

$$\begin{aligned} \sup_{0 < h < h_0} \mathbb{P} \left[\|\tilde{G}_h^\delta\|_{L^\infty(K)} > r \right] &\leq r^{-2} C_{K_\varepsilon} \sup_{0 < h < h_0} \mathbb{E} \left[\int_{K_\varepsilon} |\tilde{G}_h^\delta(z)|^2 L(dz) \right] \\ &\leq \mathcal{O}(r^{-2}). \end{aligned} \tag{3.39}$$

Hence, in view of Proposition 18, we conclude that \tilde{G}_h^δ is a tight sequence of random analytic functions.

3.6. Convergence in finite dimensional distributions. As in [22], we can adapt a method of Shirai [23] to show the following result using the central limit theorem under the Lyapunov condition:

Proposition 19. *Let \tilde{G}_{z_0} be as in (3.31). Then, for all $k \geq 1$ and all $z_1, \dots, z_k \in W$*

$$(\tilde{G}_h^\delta(z_1), \dots, \tilde{G}_h^\delta(z_k)) \xrightarrow{d} (\tilde{g}_{z_0}(z_1), \dots, \tilde{g}_{z_0}(z_k)), \quad h \rightarrow 0. \tag{3.40}$$

where \tilde{g}_{z_0} is a Gaussian analytic function on W with covariance kernel

$$\tilde{K}_{z_0}(z, \bar{w}) = K_{z_0}(z, \bar{w})e^{\tilde{\phi}(z; 0) + \tilde{\phi}(w; 0)}.$$

Since \tilde{G}_h^δ is a tight sequence of random analytic functions, Proposition 19 implies that \tilde{G}_h^δ converges in distribution to the Gaussian analytic function \tilde{g}_{z_0} , i.e.

$$\tilde{G}_h^\delta \xrightarrow{d} \tilde{g}_{z_0}, \quad \text{as } h \rightarrow 0, \quad (3.41)$$

Notice $e^{\tilde{\phi}(z;0)} \neq 0$, $z \in W$, is a deterministic holomorphic function. Hence,

$$g_{z_0} \stackrel{\text{def}}{=} \tilde{g}_{z_0}(z)e^{-\tilde{\phi}(z;0)}, \quad (3.42)$$

is a Gaussian analytic functions on W with covariance kernel

$$K_{z_0}(z, \bar{w}) = e^{\frac{1}{2}(\sigma_+(z_0) + \sigma_-(z_0))z\bar{w}}.$$

We can show that every functional $\pi_\varphi : \mathcal{H}(W) \rightarrow \mathbb{C}$, $\pi_\varphi(f) = \langle \xi_f, \varphi \rangle$, $\varphi \in \mathcal{C}_c(W)$, is continuous on $(\mathcal{H}(W) \setminus \{0\}, d)$, the space of holomorphic functions on W equipped with the metric of uniform convergence. It then follows by (3.42) and by the continuous mapping theorem, see for instance [21], that

$$\sum_{z \in \tilde{g}_{z_0}^{-1}(0)} \delta_z \stackrel{d}{=} \sum_{z \in g_{z_0}^{-1}(0)} \delta_z.$$

Hence, by (3.41) and Proposition 17, we see that

$$\sum_{\lambda \in (\tilde{G}_h^\delta)^{-1}(0)} \delta_\lambda \xrightarrow{d} \sum_{\lambda \in g_{z_0}^{-1}(0)} \delta_\lambda, \quad \text{as } h \rightarrow 0.$$

This, together with (3.30), (3.32) and Propositions 17, lets us conclude Theorem 11.

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