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Sergei Kuksin

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
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Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
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Ergodicity, mixing and KAM

Sergei Kuksin*

Abstract

In this note we review recent progress in the problem of mixing for a nonlinear PDE of parabolic type, perturbed by a bounded random force.

1 Introduction

We are concerned with evolutionary nonlinear PDEs under periodic boundary conditions, perturbed by finite-dimensional random force. We write their solutions $u(t, x)$ as curves

$$u^\omega(t) = u^\omega(t, \cdot) \in H$$

where $(H, \|\cdot\|)$ is a certain Hilbert space of functions of x (usually this is a Sobolev space over L_2). We are interested in equations of the form

$$\dot{u} + Lu + B(u) = \vec{\eta}(t), \quad u(0) = u_0, \qquad u(t) \in H, \tag{1.1}$$

where $L = -\Delta$ (or, more generally, $L = (-\Delta)^a, a > 0$) is the dissipation, B is a nonlinearity (its linear part may be non-zero), and $\vec{\eta}(t) = \vec{\eta}^\omega(t, x)$ is a random force. We assume that eq. (1.1) is well posed if the function $\|\vec{\eta}(t)\|^2$ is integrable on bounded segments.

We regard Lu and $\vec{\eta}(t)$ as a perturbation and are the most interested in the case when the unperturbed equation

$$\dot{u} + B(u) = 0 \tag{1.2}$$

is a Hamiltonian PDE. The problem of long time behaviour in hamiltonian systems (1.2) is related to the ergodic hypothesis and is hopelessly complicated. Instead our goal is to study the long-time dynamics of the perturbed eq. (1.1).

Consider eigen-functions of the operator L (these are simply the complex exponents), and label them by natural numbers:

$$Le_j = \lambda_j e_j, \quad j = 1, 2, \dots.$$

*S. Kuksin, Institut de Mathématiques de Jussieu-Paris Rive Gauche, CNRS, Université Paris Diderot, UMR 7586, Sorbonne Paris Cité, F-75013, Paris, France; and School of Mathematics, Shandong University, Jinan, 250100, PRC; and Saint Petersburg State University, Universitetskaya nab., St. Petersburg, Russia, e-mail: sergei.kuksin@imj-prg.fr

We will decompose vectors $u \in H$ in this basis, $u = \sum_{s=1}^{\infty} u_s e_s$, and will identify any $u \in H$ with the vector of its Fourier coefficients:

$$u = (u_1, u_2, \dots).$$

Let us take any set $M \subset \mathbb{N}$ of indices j, finite or infinite, and consider the subspace

$$H_M \subset H, \qquad H_M = \operatorname{span}(e_j,\, j \in M).$$

The random force $\vec{\eta}$ is assumed to be of the form

$$\vec{\eta}(t) = \sum_{j \in M} a_j \eta_j^\omega(t) e_j \in H_M\,, \quad \sum a_j^2 < \infty, \tag{1.3}$$

where η_j's are i.i.d. real random processes. If $|M| < \infty$, the force η is called *finite-dimensional.* With this notation eq. (1.1) may be written as

$$\dot{u}_j + \lambda_j u_j + B_j(u) = a_j \eta_j^\omega(t), \quad j \geq 1; \qquad a_j = 0 \text{ if } j \notin M.$$

The objection is to show that a large class of "non-degenerate" equations (1.1) with finite-dimensional random forces η is "ergodic", more precisely – mixing: Denote by $u(t; u_0)$ a solution of (1.1), equal u_0 at $t = 0$. It depends on a random parameter $\omega \in \big(\Omega, F, P\big)$.

Definition 1.1. *Eq.* (1.1) *is called mixing if in the space H exists a Borel measure μ such that for any "reasonable" functional $f : H \to \mathbb{R}$ and for any starting point u_0*

$$\text{the observable } \mathbb{E} f(u(t; u_0)) \text{ converges, as } t \to \infty, \text{ to } \int_H f(u)\, \mu(du). \tag{1.4}$$

This measure μ is called the stationary measure for eq. (1.1)*.*

Note that (1.4) means that for any u_0,

$$\mathcal{D}(u(t; u_0)) \rightharpoonup \mu \quad \text{as} \quad t \to \infty, \ \text{ weakly}, \tag{1.5}$$

where $\mathcal{D}$ signifies distribution of a random variable, and that

$$\operatorname{dist}(\mathcal{D}u(t, u_{01}), \mathcal{D}u(t, u_{02})) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad u_{01}, u_{02}$$

(here *dist* a distance in the space of measures on H which metrises the weak convergence $\rightharpoonup$). If the convergence (1.5) is exponentially fast, eq. (1.1) is called *exponentially mixing.*

What was known about the mixing in equations (1.1):

i) If $H_M = H$, then the mixing is proved for various classes of equations, see in [4].

ii) If the set M is finite, then what was available is the result of Hairer–Mattingly [1] who proved the mixing for the case of white in time forces $\vec{\eta}$. Their proof is based on an infinite–dimensional version of the Malliavin calculus and applies to a rather special class of eq. (1.1), which includes the 2d NSE on the torus. In particular, this approach does not apply if $B(u)$ is a Hamiltonian nonlinearity which is a polynomial of degree > 3 (this restriction on the degree of nonlinearity also remains true for finite-dimensional systems). Even more: for some important equations (B) corresponding equations (1.1) with white-noise forces are not known to be well posed, while equations (1.1) with bounded random forces are well posed, and – as our results imply – are mixing. For example, this is the case for the primitive equations of atmosphere which are principal equations of meteorology (the stochastic primitive equations are known to be well posed only in some weak sense).

Below I present recent result on the mixing in equations (1.1) with bounded random forces, recently obtained in [2] and [3]. In [3] the approach of the original work [2] is repeated for an easier problem which resulted in a shorter and more accessible text.

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2 Bounded random forces.

Recall that the random force $\vec{\eta}(t)$ has the form (1.3), where $\eta_1^\omega(t), \eta_2^\omega(t), \dots$ are i.i.d. bounded random processes. To define a suitable class of processes η_j we use a naive approach: Let $\{h_1(t), h_2(t), \dots\}$ be a basis of functions on $[0, \infty)$, made by bounded functions. We define

$$\eta_j^\omega(t) = \sum_{k=1}^{\infty} c_k \xi_k^{j\omega} h_k(t), \quad c_k \neq 0,$$

where $\{\xi_k^{j\omega}\}$ are i.i.d., $|\xi_k^{j\omega}| \leq 1$. So η_j's are random series in the basis $\{h_j\}$. For our techniques to apply, we have to impose on the basis $\{h_j\}$ a restriction. For $j \in \mathbb{N}$ let us denote $J_j = [j-1, j]$. We assume that

for every function $h_l(t)$, its support belongs to some segment J_j, $j = j(l)$.

Our favorite example of a base as above is the Haar base "of step 1" $\{h_{j,l}(t), j, l \geq 0\}$. Each function $h_{0,l}$ is a characteristic function of the segment $[l, l+1]$, while for $j \geq 1$ each $h_{j,l}$ is a "dipole" of unit L_2–norm on the segment $[2^{-j}l, 2^{-j}(l+1)]$:

$$h_{j,l}(t) = \begin{cases} 0 & \text{for } t < l2^{-j} \text{ or } t \geq (l+1)2^{-j}, \\ 2^{j/2} & \text{for } l2^{-j} \leq t < \left(l+\frac{1}{2}\right)2^{-j}, \\ -2^{j/2} & \text{for } \left(l+\frac{1}{2}\right)2^{-j} \leq t < (l+1)2^{-j}. \end{cases}$$

This is an orthonormal base of $L_2(0, \infty)$.

Now consider the random force $\vec{\eta}(t,x) = \sum_{j\in M} a_j \eta_j^\omega(t) e_j(x)$. We take the processes $\eta_j(t)$ to be i.i.d. random Haar series:

$$\eta_1(t) = \sum_{k=0}^{\infty} c_k \sum_{l=0}^{\infty} \xi_{k,l}^\omega \, h_{k,l}(t)\,, \quad c_k \neq 0. \tag{2.1}$$

Here $\{\xi_{k,l}^\omega, k,l \geq 0\}$ are i.i.d. bounded random variables such that $|\xi_{k,l}| \leq 1$ a.s. and $\mathcal{D}\xi_{k,l} = p(x)\,dx$, where $p(x)$ is a Lipschitz function, $p(0) \neq 0$.

It is known that if $c_k \equiv 1$ and $\{\xi_{k,l}^\omega\}$ are independent $N(0,1)$ r.v., then (2.1) is a white noise. We assume that the random process in eq. (1.1) is much smoother than that: the i.i.d. r.v. $\{\xi_{k,l}^\omega\}$ are bounded and the process is "smooth in time":

$$|c_n| \leq C n^{-q} 2^{-n/2}, \quad \text{for each } n, \tag{2.2}$$

where $q > 1$. Such processes are called *red noises.*

Consider any red noise η_1 as in. (2.1), (2.2), and for $N \in \mathbb{N}$ consider the process

$$\beta_N^\omega(T) = \frac{1}{\sqrt{N}} \int_0^{NT} \eta_1^\omega(t) dt = c_0 \frac{1}{\sqrt{N}} \sum_{l=0}^{[NT]-1} \xi_{0,l}^\omega + O\big(\tfrac{NT-[NT]}{\sqrt{N}}\big).$$

Its trajectories are Lipschitz functions of T, and by Donsker's invariance principle the process $\beta_N(T)/\sigma$, $\sigma^2 = \mathbb{E}(\xi_{0,0})^2$, converges in distribution to the Wiener process. That is, on large time-scales $\int \eta_1$ behaves as a Wiener process. So the red noises are "smoother siblings" of the white noise.

In view of (2.2) and since $\sum a_j^2 < \infty$, the force $\vec{\eta}(t)$ is bounded in H, uniformly in t and ω. Since (1.1) is a well posed equation of parabolic type, then usually it possesses the following regularity property, which is being assumed below: there is a compact set $X \subset H$ such that

$$\forall\, u_0 \in H \;\text{ there exists }\; t(\|u_0\|) \geq 0 \;\text{ such that }\; u(t) \in X \;\; \forall\, t \geq t(\|u_0\|),\; \forall\, \omega.$$

3 Shift Operator S

We wish to pass from continuous to discrete time. To do that let us cut $\mathbb{R}_+$ to the unit segments $J_l, l \geq 1$, and consider the process $\vec{\eta}$, restricted to any J_j:

$$\vec{\eta}^{\,j}(t) = \vec{\eta}(t-(j-1)), \;\; 0 \leq t \leq 1, \qquad \vec{\eta}^{\,j} : [0,1] \to H_M.$$

Denote $E = L_2(0,1;H_M)$. Then

$$\text{the law of } \vec{\eta}^{\,j} \text{ is a measure in } E, \text{ independent from } j,$$

and $\operatorname{supp} \mathcal{D}\vec{\eta}^{\,j}$ is a compact set in E since the r.v. $\xi_{k,l}$ are bounded and $\sum a_j^2 < \infty$, $\sum c_k < \infty$.

Operator S. Consider the operator

$S : H \times E \to H, \quad (u_0, \vec{\eta}^1) \mapsto u(1); \qquad u(t)$ – solution of eq. (1.1), $u(0) = u_0$.

Then $u(2) = S(u(1), \vec{\eta}^2)$, etc.

Our task is to understand iterations of the operator S, i.e. to study the equation

$$u_{k+1} = S(u_k, \vec{\eta}^{k+1}), \quad k \in \mathbb{N}, \tag{3.1}$$

where $u_0 \in H$ is given. Certainly for $k \in \mathbb{N}$ the solution of (3.1) after k step equals $u(k; u_0)$.

Differential of S **in** $\vec{\eta}$. For $u \in H, \vec{\eta} \in E$ consider the linearised in $\vec{\eta}$ map S:

$$D_{\vec{\eta}}\, S(u, \vec{\eta}) : E \to H.$$

This operator examines how a solution $u(t)$ at $t = 1$ changes when we modify infinitesimally the force $\vec{\eta}(t), 0 \le t \le 1$, keeping $u(0)$ fixed. More precisely for any given $u_0 \in H$ and $\vec{\eta}_0 \in E$ to calculate $D_{\vec{\eta}}S(u_0, \vec{\eta}_0)(\vec{\xi})$, $\vec{\xi} \in E$, we do the following: find a solution $u(t)$ of (1.1) for $0 \le t \le 1$ such that $u(0) = u_0, \vec{\eta} = \vec{\eta}_0$. Linearise eq.(1.1) about this $u(t)$ and add to the obtained linear eq. the r.h.s. $\vec{\xi}$:

$$\dot{v} + Lv + dB(u(t))(v(t)) = \vec{\xi}(t), \;\; v(0) = 0, \qquad 0 \le t \le 1.$$

Consider $v(1) \in H$. This is $D_{\vec{\eta}}S(u_0, \vec{\eta}_0)(\vec{\xi})$.

4 The main theorem

We require from the shift–operator S the following three properties:

(H_1) (regularity). a) $S(X \times \operatorname{supp}\mathcal{D}(\vec{\eta}^j)) \subset X$ for some compact $X \subset H$, and b) there is a compactly embedded Banach space $V \Subset H$ such that:

$$S : H \times E \to V \text{ is } C^2\text{–smooth.}$$

(H_2) (stability of 0). If in (1.1) $\vec{\eta} \equiv 0$, then all solutions of (1.1) converge to 0 exponentially.

(H_3) (approximate linearised controllability). This assumption is a key point. It exists in a strong and weak forms:

(H_3^{strong}) For each point $u \in X$ and every $\xi \in E$, $\xi \in \operatorname{supp}\mathcal{D}(\vec{\eta}^1)$, the mapping $D_{\vec{\eta}}S(u, \xi) : E \to H$ has dense image in H.

This condition is easy to verify. It holds if $M = \mathbb{N}$ (all modes are excited), but it does not hold if M is a finite set. To work with finite–dimensional random forces $\vec{\eta}$ we evoke a weaker condition:

(H_3^{weak}) For each point $u \in X$ there exists a null-set Ω_u such that if $\omega \notin \Omega_u$, then the range of the linear operator $D_{\vec{\eta}}S(u, \vec{\eta}^{\omega})$ is dense in H.

FACT (see [2]). If
⋆ eq. (1.1) is the 2d NSE,
⋆ or eq. (1.1) is the CGL equation

$$\dot u - \epsilon\Delta u - i\gamma\Delta u + i|u|^{2p}u = \vec\eta(t,x), \quad \epsilon > 0,\ \gamma \ge 0, \quad x \in \mathbb{T}^d,$$

where
a) either $d = 2$ and p is any, or
b) $d = 3$ and $p \le 2$, or
b) d is any, p is any, $\gamma = 0$,

and the force $\vec\eta^\omega(t,x)$ is a red noise as above, then:

1) if $M = \mathbb{N}$, then (H_1)– ($H_3^{\rm strong}$) holds.

2) if M is a finite set, satisfying some small restrictions, then (H_1)– ($H_3^{\rm weak}$) hold.

The hardest is to check ($H_3^{\rm weak}$). For the 2d NSE similar results were first obtained by Weinan E, Mattingly, Pardoux, Hairer, next they were properly understood by Agrachev–Sarychev, and developed further by Shirikyan, Nersesyan and others.

Theorem 4.1. *Equation* (1.1) *is exponentially mixing if either*
1) (H_1)– (H_3^{strong}) hold,
or if
2) (H_1)– (H_3^{weak}) hold, and the mapping S is analytic.

The assertion 2) is proved in [2], and assertion 1) is established in [3], using the method of [2].

5 How do we prove this? ("Doeblin meets Kolmogorov")

Let $u(t) \in X$ and $u'(t) \in X$ be two solutions of (1.1) with initial data u_0 and u_0'. It is not hard to see that in our setting to prove the mixing we should verify that

$$\mathrm{dist}(\mathcal{D}u(t), \mathcal{D}u'(t)) \to 0 \quad \text{as} \quad t \to \infty, \tag{5.1}$$

for all u_0, u_0'. How to establish (5.1) ?

Doeblin' coupling, a.k.a. *the method of two equations.* In $X \times X$ consider the integer-time dynamics (u_k, v_k), $k \ge 1$, where

$$(u_0, v_0) = (u_0, u_0'), \qquad (u_k, v_k) = (S(u_{k-1}, \vec\eta_k),\ S(v_{k-1}, \vec\eta_k^{\,\prime}), \ \ k \ge 1,$$

with $\vec\eta_k^{\,\prime} = \eta_k'(u_{k-1}, v_{k-1}, \vec\eta_k)$ such that

$$\mathcal{D}\vec\eta_k^{\,\prime} = \mathcal{D}\vec\eta_k. \tag{5.2}$$

Then for each k, $u_k = u(k)$ and $\mathcal{D}v_k = \mathcal{D}u'(k)$. If we can choose $\vec{\eta}_k^{'}, k \geq 1$, such that (5.2) holds and

$$\|u_k - v_k\| \to 0 \text{ as } k \to \infty \text{ a.s.}, \tag{5.3}$$

then $\operatorname{dist}(\mathcal{D}u(k), \mathcal{D}u'(k)) \to 0$ as $k \to \infty$, and our goal (5.1) is achieved.

To achieve (5.3), at Step 1 we wish to choose the kick $\vec{\eta}_1^{'}$, depending on u_0, v_0 and $\vec{\eta}$, in such a way that $\mathcal{D}\vec{\eta}_1^{'} = \mathcal{D}\vec{\eta}_1$, and

$$\|u_1 - v_1\| \text{ is small with hight probability.}$$

Then the law of u_1 will be "rather close" to that of v_1, and iterating we will get (5.3).

We have to distinguish two cases:

a) $\|u_0 - v_0\| \leq \delta_0$,
where δ_0 is an additional small parameter;

b) $\|u_0 - v_0\| > \delta_0$.

In case b) we choose for $\vec{\eta}_1^{'}$ an independent copy of $\vec{\eta}_1$, and use the assumption (H_2) (stability of zero) to achieve a) with positive probability, in a few steps.

Now let $\|u_0 - v_0\| = \delta \ll 1$. This is the main difficulty. Then we choose

$$\vec{\eta}_1^{'} = \Psi_{u_0,v_0}(\vec{\eta}_1),$$

where Ψ is an unknown mapping which preserves the measure $\mathcal{D}\vec{\eta}_1$, so $\mathcal{D}\vec{\eta}_1^{'} = \mathcal{D}\vec{\eta}_1$. The dream would be to find Ψ such that

$$u_1 - v_1 = S(u_0, \vec{\eta}_1) - S(v_0, \Psi_{u_0,v_0}(\vec{\eta}_1)) = 0 \qquad \forall\, \vec{\eta}_1. \tag{5.4}$$

Then $v_1 = u_1$ a.s., and (5.1) is achieved. But this is hardly possible since it is very exceptional that $\mathcal{D}(S(u_0, \vec{\eta}^1)) = \mathcal{D}(S(u_0', \vec{\eta}^1))$ for $u_0 \neq u_0'$.

The situation is reminiscent to that treated by Kolmogorov in his celebrated work which initiated the KAM theory. There Kolmogorov considers a perturbation of an integrable Hamiltonian,

$$H^1(p,q) = H_0(p) + \varepsilon h(p,q), \quad \varepsilon \ll 1, \;\; (p,q) \in P \times \mathbb{T}^n, \tag{5.5}$$

where P is a domain in $\mathbb{R}^n$. If exists a canonical transformation $S : P_1 \times \mathbb{T}^n \to P \times \mathbb{T}^n$, where P_1 is a large subdomain of P, such that

$$H^1 \circ S = H_0'(p), \tag{5.6}$$

then the equation with the transformed Hamiltonian $H^1 \circ S$ would be integrable on $P_1 \times \mathbb{T}^n$. Since Poincaré it is well known that normally such a transformation S does not exist. So instead of the hopeless equation

$$H^1 \circ S - H_0'(p) = 0, \quad S = ?,$$

Kolmogorov suggested to look for S in the form $S =$id$+\varepsilon S_1$,[1] to linearise the equation in ε,

$$(H_0 + \varepsilon h) \circ (\mathrm{id} + \varepsilon S_1) = H_0'(p) + \varepsilon h_1(p,q) + O(\varepsilon^2),$$

[1] here S_1 is a vectorfield, and the expression id$+\varepsilon S$ should be properly understood.

and to search for an S_1 such that $h_1 = O(\varepsilon^2)$. This transformation should be defined for p from a large subdomain $P_1 \subset P$. The term h_1 linearly depends on S, so the equation

$$h_1(S) = 0 \tag{5.7}$$

is linear in S. It is called *homological equation*, and one looks for its approximate solution with a disparity of order ε. If such an S_1 exists, then replacing H^1 with the transformed Hamiltonian $H^2 = H^1 \circ (\mathrm{id}+\varepsilon S_1)$ we arrive at a Hamiltonian of the form (5.5) but with ε replaced by $C\varepsilon^2$. Then we would iterate the procedure and after infinitely many steps will arrive at a transformation S which satisfies (5.6) for all p from a Borel subset of P of large measure.

Let us proceed likewise with the impossible equation (5.4). Namely, for $\delta = \|u_0 - v_0\| \le \delta_0$ let us re-write the equation, looking for the mapping Ψ_{u_0,v_0} in the form $\Psi_{u_0,v_0} = \mathrm{id} + \delta\Phi$ and neglecting in (5.4) terms $\sim \delta^2$. Then eq. (5.4) reeds

$$\delta[D_{\vec\eta} S(u_0, \vec\eta_1)\Phi(\vec\eta_1) - S^\Delta(u_0, v_0, \vec\eta_1)] + O(\delta^2) = 0,$$

where $S^\Delta = \delta^{-1}(S(v_0, \vec\eta_1) - S(u_0, \vec\eta_1)) \sim 1$. Requiring that the sum of the terms in the square brackets vanishes we get the homological equation:

$$D_{\vec\eta} S(u_0, \vec\eta_1)\Phi = S^\Delta(u_0, v_0, \vec\eta_1), \qquad \Phi = ? \tag{5.8}$$

– If $(\mathrm{H}_3^{\mathrm{strong}})$ holds, we can solve the homological equation approximately.

– If $(\mathrm{H}_3^{\mathrm{weak}})$ holds, we can solve it approximately for all ω's outside some bad event $\Omega^1_{u_0}$ of small measure, like in the Kolmogorov scheme above, where the homological equation (5.7) may be non-soluble, even approximatively, for p from some small subset of P.

With the solution $\Phi = \Phi(\vec\eta_1)$ in hands we, as planned, choose $\vec\eta_1^{\,'} = \vec\eta_1 + \delta\Phi(\vec\eta_1)$. Then

$$\|u_1 - v_1\| \ll \delta \quad \text{for } \omega \text{ outside} \quad \Omega^1_{u_0}.$$

Note that since the control for the norm of the solution Φ of (5.8) is very poor, then now, in difference with KAM, **we cannot obtain the quadratic approximation**

$$\|u_1 - v_1\| \ll \delta^2 = \big(\|u_0 - v_0\|\big)^2,$$

despite the method we are using is quadratic! We only can achieve that $\|u_1 - v_1\| \le \frac{1}{2}\delta$. But this turns out to be enough to get the convergence (5.1).

Two main problems appear on the way:

1) what should we do when $\omega \notin \Omega^1_{u_0}$, so we cannot solve (5.8) approximately?

2) the mapping $\vec\eta_1 \mapsto \vec\eta_1^{\,'} = \Psi(\vec\eta_1) = \vec\eta_1 + \delta\Phi(\vec\eta_1)$ does not preserve the measure $\mathcal{D}(\vec\eta_1)$, so $\mathcal{D}(\vec\eta_1) \ne \mathcal{D}(\vec\eta_1^{\,'})$.

The difficulty 1) usually is present in KAM (there we simply throw away the set of bad parameters). The second difficulty is specific for this setting.
What should we do?

Answer to 1). If $\omega \notin \Omega^1_{u_0}$, we take $\vec{\eta}_1^{\,'} = \vec{\eta}_1$ (the trivial coupling). Then

$$\|u_1 - v_1\| = \|S(u_0, \vec{\eta}_1) - S(v_0, \vec{\eta}_1)\| \leq C\|u_0 - v_0\| = C\delta,$$

where C is the Lipchitz constant. If still $\|u_1 - v_1\| \leq \delta_0$, we play the same game. If $\|u_1 - v_1\| > \delta_0$, we play the game a), i.e., choose $\vec{\eta}_1^{\,'}$ to be an independent copy of $\vec{\eta}_1$.

Answer to 2). Despite $\mathcal{D}(\vec{\eta}_1^{\,'}) \neq \mathcal{D}(\vec{\eta}_1)$, these two laws turn out to be close:

$$\|\mathcal{D}(\vec{\eta}_1^{\,'}) - \mathcal{D}(\vec{\eta}_1)\|_{\mathrm{var}} \leq C\delta^a, \quad a > 0.$$

This is enough for us: careful analysis, similar to that in Sections 3.2.2–3.2.3 of [4], shows that iterating a) and b) we prove the theorem.

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