Séminaire Laurent Schwartz

EDP et applications

Année 2019-2020

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Séminaire Laurent Schwartz — EDP et applications (2019-2020), Exposé n° III, 16 p.
https://doi.org/10.5802/slsedp.136

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Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Séminaire Laurent-Schwartz — EDP et applications
Institut des hautes études scientifiques, 2019-2020
Exposé n° III, 1-16

Problème de Cauchy spatial-caractéristique avec courbure L^2 en relativité générale

Olivier Graf
joint work with Stefan Czimek

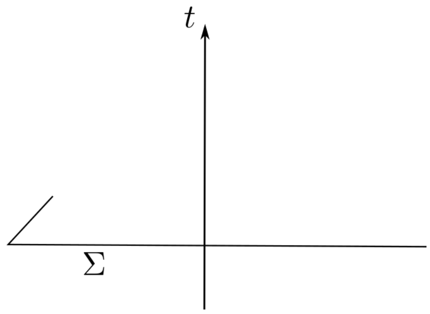
Abstract

The present article is a summary of the papers [10] and [11] which establish a bounded L^2 curvature theorem for the spacelike-characteristic Cauchy problem of general relativity. More precisely, we obtain a lower bound on the time of existence of classical solutions to the spacelike-characteristic Cauchy problem for Einstein equations in vacuum, depending only on the L^2 curvature fluxes through the initial spacelike and initial characteristic hypersurfaces and on suitable additional low regularity assumptions.

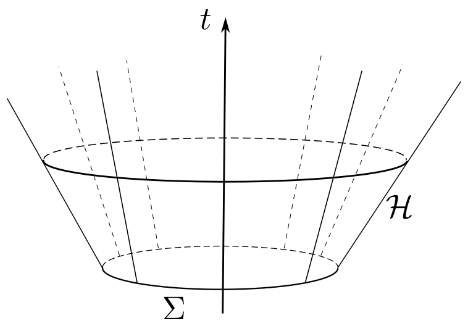
1 Introduction

This article provides an overview of the recent papers [10] and [11] and serves as a companion to the presentation given at the *Séminaire Laurent Schwartz* in November 2019.

The main result of [10] and [11] is a bounded L^2 curvature theorem for the spacelike-characteristic Cauchy problem of general relativity. The spacelike-characteristic Cauchy problem is an initial value problem for data posed on a classical spacelike hypersurface Σ diffeomorphic to the unit disk of $\mathbb{R}^3$ and on the outgoing characteristic hypersurface $\mathcal{H}$ emanating from the boundary $\partial\Sigma$ of Σ. In [10] and [11], we obtain a lower bound on the time of existence of classical solutions to the spacelike-characteristic Cauchy problem for Einstein equations in vacuum, depending only on the L^2 curvature fluxes through the hypersurfaces Σ and $\mathcal{H}$ and on low regularity assumptions on $\Sigma \cap \mathcal{H}$.



(a) The classical spacelike Cauchy problem of the bounded L^2 curvature theorem [19].



(b) The spacelike-characteristic Cauchy problem of [10] and [11].

1.1 Einstein vacuum equations

A *spacetime* $(\mathcal{M}, \mathbf{g})$ is a 4-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric $\mathbf{g}$. In this article, $\mathcal{M}$ will be (a domain of) $\mathbb{R}_t \times \mathbb{R}^3_x$ and at each point $(t, x) \in \mathcal{M}$, the metric components $\mathbf{g}_{\mu\nu}(t, x)$ are the elements of a symmetric matrix $(\mathbf{g}_{\mu\nu}(t, x))_{\mu\nu=0..3}$ of signature $(- + ++)$.[1]

[1]Greek indices range from 0 to 3, Latin indices range from 1 to 3 and Einstein summation convention is used.

Remark 1.1. *The prime example of a spacetime is* Minkowski spacetime*,*

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}^3_x, \quad \mathbf{g} = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

A spacetime $(\mathcal{M}, \mathbf{g})$ is called a *vacuum spacetime* if it satisfies the following *Einstein vacuum equations*

$$\mathrm{Ric}(\mathbf{g})_{\mu\nu} = 0, \quad \mu, \nu = 0, \dots, 3, \tag{1.1}$$

where $\mathrm{Ric}(\mathbf{g})$ denotes the *Ricci curvature tensor* of the spacetime metric $\mathbf{g}$. Each component $\mathrm{Ric}(\mathbf{g})_{\mu\nu}$ is a second-order nonlinear differential operator on the metric components $\mathbf{g}_{\alpha\beta}$, for $\alpha, \beta = 0, \dots, 3$. Therefore, Einstein equations (1.1) form a system of 10 nonlinear coupled partial differential equations on the 10 unknowns $\mathbf{g}_{\mu\nu}$. The equations (1.1) reduce to 6 independent equations and we have the freedom to impose 4 additional equations on the metric components $\mathbf{g}_{\mu\nu}$.[2] In what follows, we shall call such an additional choice of equations a *gauge choice.*

1.2 The maximal gauge and the classical Cauchy problem

In this section we introduce the so-called *maximal gauge*. Under this gauge choice, Einstein vacuum equations (1.1) can be cast as a system of coupled evolution and elliptic equations which admits a well-posed initial value formulation.

Let us first impose that the metric $\mathbf{g}$ can be written in the following form

$$\mathbf{g} = -n^2 \mathrm{d}t^2 + g_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{1.2}$$

i.e. we set that $\mathbf{g}_{0i} = 0$ for $1 \leq i \leq 3$. The unknown n is called the *time lapse* and g is the Riemannian metric induced by $\mathbf{g}$ on the constant time hypersurfaces $\Sigma_T := \{t = T\}$.

Define the *second fundamental form* k to be the following (normalised) time-derivative of the metric g

$$k_{ij} := -\frac{1}{2} n^{-1} \partial_t g_{ij}. \tag{1.3}$$

The *maximal gauge condition* reads

$$\mathrm{tr}_g k := g^{ij} k_{ij} = 0, \tag{1.4}$$

where $\left(g^{ij}\right)_{ij=1..3} := \left(g_{ij}\right)^{-1}_{ij=1..3}$. Geometrically, this corresponds to the requirement that the hypersurfaces Σ_t are *maximal hypersurfaces* of the spacetime $(\mathcal{M}, \mathbf{g})$.[3]

Define the *electric-magnetic tensors* E and H to be

$$\begin{aligned} E_{ij} &:= \mathrm{Ric}(g)_{ij} - k_i^l k_{lj}, \\ H_{ij} &:= \mathrm{curl}_g\, k_{ij}, \end{aligned}$$

where curl_g is a standard curl operator associated with the Riemannian metric g (see [7] for a definition).

Einstein vacuum equations (1.1) together with the additional choices (1.2) and (1.4) can be rewritten as the *Einstein vacuum equations in maximal gauge*, which is the following system of coupled quasilinear transport-elliptic-Maxwell equations (see [7], pp. 8-9 and p. 146):

the *first variation* transport equation for g

$$n^{-1} \partial_t g_{ij} = -2k_{ij}, \tag{1.5a}$$

[2]This is related to the so-called *general covariance* of Einstein equations, *i.e.* that the system of equation (1.1) is covariant under a change of coordinates (see [39] for further discussion).

[3]Maximal hypersurfaces should be thought of as the Lorentzian equivalent of minimal surfaces in Riemannian geometry.

the *second variation* transport equation for k

$$n^{-1}\partial_t k_{ij} = -n^{-1}\nabla_i\nabla_j n + E_{ij} - k_{il}k^l_j, \tag{1.5b}$$

the Hodge-type elliptic equations for k

$$\begin{aligned} \mathrm{tr}_g k &= 0, \\ \mathrm{div}_g\, k_i &= 0, \\ \mathrm{curl}_g\, k_{ij} &= H_{ij}, \end{aligned} \tag{1.5c}$$

the Laplace equation for n

$$\Delta_g n = n|k|^2_g, \tag{1.5d}$$

the Laplace-type elliptic equation for g[4]

$$\mathrm{Ric}(g)_{ij} = E_{ij} + k^l_i k_{lj}, \tag{1.5e}$$

and the Maxwell-type equations for E and H

$$\begin{aligned} \mathrm{tr}_g E = \mathrm{tr}_g H &= 0, \\ \mathrm{div}_g\, E_i &= (k\wedge H)_i, \\ \mathrm{div}_g\, H_i &= -(k\wedge E)_i, \\ n^{-1}\partial_t E_{ij} + \mathrm{curl}_g\, H_{ij} &= -n^{-1}(\nabla n\wedge H)_{ij} + \frac{1}{2}(k\times E)_{ij} - \frac{2}{3}(k\cdot E)g_{ij}, \\ -n^{-1}\partial_t H_{ij} + \mathrm{curl}_g\, E_{ij} &= -n^{-1}(\nabla n\wedge E)_{ij} - \frac{1}{2}(k\times H)_{ij} + \frac{2}{3}(k\cdot H)g_{ij}, \end{aligned} \tag{1.5f}$$

where ∇, div_g and Δ_g are respectively the standard covariant derivative, divergence and Laplace-Beltrami operators associated to the Riemannian metric g, and where $|\cdot|_g$, $\wedge$, $\times$, $\cdot$ are standard contractions with respect to the metric g (see [7] for definitions).

The system of equations (1.5) displays an hyperbolic structure due to the Maxwell-type equations (1.5f) for the electric-magnetic tensors E and H. The unknowns g, k and n can be determined only by solving the transport (1.5a), (1.5b) or elliptic (1.5c), (1.5d), (1.5e) equations, for which the electric-magnetic tensors E and H are source terms. In particular, we expect that the equations (1.5) admit a well-posed initial value formulation.

To define the Cauchy problem, we consider (classical) *maximal Cauchy data* which are a triplet (Σ, g, k) such that (Σ, g) is a Riemannian manifold, k is a symmetric 2-tensor on Σ and such that the following *constraint equations* are satisfied

$$\begin{aligned} \mathrm{tr}_g k &= 0, \\ \mathrm{div}_g\, k_i &= 0, \\ R(g) &= |k|^2_g, \end{aligned} \tag{1.6}$$

where $R(g) := \mathrm{tr}_g \mathrm{Ric}(g)$ is the so-called *scalar curvature* of the metric g.

We have the following local well-posedness result for Einstein vacuum equations in maximal gauge.

Theorem 1.2 (Local well-posedness, [7] pp. 299-300)**.** *Let (Σ_0, g_0, k_0) be smooth maximal Cauchy data such that $\Sigma_0 \simeq \mathbb{R}^3$.*[5] *There exists $T > 0$ and a unique 5-uplet (g, k, n, E, H) defined on $[0,T]_t \times \mathbb{R}^3_x$ smooth solution of the system of equations* (1.5) *such that $\Sigma_0 = \{t = 0\}$ and the initial conditions $g|_{t=0} = g_0$, $k|_{t=0} = k_0$ are satisfied.*

[4]In the so-called *harmonic coordinates* for the Riemannian metric g, the Ricci tensor $\mathrm{Ric}(g)_{ij}$ can be rewritten as the Laplace-Beltrami operator of the metric component $\Delta_g(g_{ij})$ up to lower order terms. Using the so-called *Cheeger-Gromov convergence theory* to deal with the existence of such coordinates, elliptic-type results can be obtained for the metric g (see the results of Chapter 11 in [29] for instance).

[5]We denote by $\simeq$ the diffeomorphism equivalence for manifolds.

Remarks on Theorem 1.2

1.2a Here *smooth* means regular with respect to standard Sobolev norms. In [7], the initial data g_0 and k_0 are assumed to be respectively in $H^4_{loc}(\Sigma)$ and $H^3_{loc}(\Sigma)$ and the solution (g, k, n, E, H) belongs to suitable corresponding functional spaces on $[0, T] \times \mathbb{R}^3$.

1.2b The time lapse n is only defined through the elliptic equation (1.5d) on each slice Σ_t, and is therefore well-determined up to a choice of a limit condition at spatial infinity. Making such a choice is roughly equivalent to prescribing the boundary values for the maximal hypersurfaces Σ_t at infinity. We shall see in Sections 2 and 3 that one of the crucial step in the proof of the spacelike-characteristic Theorem 2.1 is to make an appropriate choice of boundaries for maximal hypersurfaces on a finite (null) hypersurface delimiting our domain of study. In the case of Theorem 1.2, the chosen condition is $n \to 1$ at spatial infinity, which physically corresponds to considering a *centre-of-mass frame* for the system (see the discussion in the introduction of [7]).

1.2c Additional appropriate fall-off conditions at spatial infinity have to be imposed on the initial data g_0 and k_0 for Theorem 1.2 to hold (see [7]).

1.2d The proof of Theorem 1.2 goes by a standard Banach-Picard iteration and relies on standard energy estimates and Sobolev embeddings.

1.2e Defining $\mathcal{M} := [0, T] \times \mathbb{R}^3$ and $\mathbf{g} := -n^2 \mathrm{d}t^2 + g_{ij} \mathrm{d}x^i \mathrm{d}x^j$, we obtain a vacuum spacetime $(\mathcal{M}, \mathbf{g})$ such that Σ_0 is a spacelike hypersurface of $(\mathcal{M}, \mathbf{g})$, g_0 is the induced metric by $\mathbf{g}$ on Σ_0 and k_0 is the second fundamental form of Σ_0 in $(\mathcal{M}, \mathbf{g})$.

1.2f Using the so-called *wave gauge*, one can prove local well-posedness for Einstein equations for more general Cauchy data (Σ, g, k) (see the seminals [3] and [12]).

1.2g Local well-posedness results for Einstein equations have also been obtained for initial data posed on *null hypersurfaces* (see [21] or [30]). Null hypersurfaces of a spacetime $(\mathcal{M}, \mathbf{g})$ are hypersurfaces orthogonal to null vector fields for the Lorentzian metric $\mathbf{g}$.[6] These hypersurfaces are *characteristic* for the hyperbolic equations (1.5f). The characteristic Cauchy problem is of particular interest in the case of Einstein equations since, contrary to the classical Cauchy problem where initial data are posed on a spacelike hypersurface and have to satisfy elliptic constraint equations (1.6), initial data can be *freely prescribed* on null hypersurfaces (see the seminal [31]). The characteristic Cauchy problem is therefore used in numerical general relativity (see [33]), as well as in the construction and control of solutions to Einstein equations (see the dynamical formation of black holes solutions in [6], or the impulsive gravitational waves solutions in [22]).

1.3 The *weak cosmic censorship conjecture*

One of the most natural question for nonlinear evolution PDE is the large-data global-in-time existence of solutions. For Einstein equations, singularities can form in finite time as in the case of the explicit *Schwarzschild solutions*. For Schwarzschild spacetimes, the singularity lies inside a so-called *black hole region* which, by definition, cannot be seen by an observer at infinity. The global-in-time behaviour of solutions to Einstein equations is subject to the celebrated conjecture of *weak cosmic censorship* which states that this feature is generic.

Conjecture 1.3 (Weak cosmic censorship, [28])**.** *For generic initial data, solutions to* (1.1) *only form singularities that are hidden in a black hole region.*

In the seminal work [5], it is shown that the conjecture holds true in the case of spherical symmetry for Einstein equations coupled with a scalar field.[7] The result relies crucially on the

[6]The prime example of a null hypersurface is the lightcone emanating from a point.

[7]Due to Birkhoff's rigidity theorem, there are no non-trivial spherically symmetric solutions to Einstein vacuum equations (1.1). Einstein equations coupled with a scalar field can be seen as one of the simplest set of dynamical equations involving Einstein equations in spherical symmetry.

sharp breakdown criterion and local existence result proved in [4] at the level of initial data with bounded variation, which is adapted to the $(1+1)$-setting of spherical symmetry and to the conservation laws for Einstein equations in this setting. For general evolution equations that display an energy conservation, local well-posedness of the Cauchy problem for initial data with regularity controlled by the conserved energy is key to obtain large-data global-in-time existence results. In many physically relevant situations, the conserved energy only controls low regularity norms and it is therefore required to prove non-trivial well-posedness results for rough initial data. See for instance the proof of global existence for the energy subcritical Yang-Mills equations in $(1+3)$-dimension in [13], or the proof of the threshold theorem for the energy critical Yang-Mills equations in $(1+4)$-dimension in [24]-[27].

In the proof of the weak cosmic censorship conjecture in spherical symmetry [5], it is moreover crucial that the local existence result in [4] is formulated for initial data posed on *null hypersurfaces*. This is specific to Einstein equations and enables to construct appropriate generic initial data (see Item 1.2g), to control their propagation and highlight a so-called *trapped surface formation mechanism* (see [4], [5] and also [20] for further discussion).

In what follows, we present a generalisation of the local existence result in [4] outside of spherical symmetry. In Section 1.4, we introduce the so-called *bounded L^2 curvature theorem* obtained in [19] which is the sharpest known local existence result for Einstein vacuum equations (1.1) in terms of the regularity of the initial data. This result is obtained for data posed on a spacelike hypersurface (see Theorem 1.4). In Sections 2 and 3 we present and give an overview of the proof of a generalisation of the bounded L^2 curvature theorem to the case of initial data posed on a spacelike and on a null hypersurface obtained in [10] and [11] (see Theorem 2.1).

1.4 The *bounded L^2 curvature theorem*

In the case of Einstein vacuum equations without symmetry, local existence results are naturally formulated in terms of L^2-based functional spaces (see for example the discussion in the introduction of [19]). In this context, the sharpest known local existence result in terms of regularity of the initial data is the celebrated *bounded L^2 curvature theorem* (see [19] and the companion papers [34]-[36]). The following is a rough statement of that result.

Theorem 1.4 (Bounded L^2 curvature theorem, [19])**.** *Let (Σ_0, g_0, k_0) be (smooth) maximal Cauchy data such that $\Sigma_0 \simeq \mathbb{R}^3$ and such that*

$$\|E_0\|^2_{L^2(\Sigma_0)} + \|H_0\|^2_{L^2(\Sigma_0)} \leq \varepsilon^2. \tag{1.7}$$

Then there exists a solution to Einstein vacuum equations (1.1) *defined on $[0,1]_t \times \mathbb{R}^3_x$ such that we have the following L^2-bounds on the constant time maximal hypersurfaces Σ_t*

$$\forall t \in [0,1], \ \|E\|^2_{L^2(\Sigma_t)} + \|H\|^2_{L^2(\Sigma_t)} \lesssim \varepsilon^2,$$

together with additional estimates, and such that smoothness is propagated.

Remarks on Theorem 1.4

1.4a The L^2-bounds on the electric-magnetic tensors E, H correspond to a control on the electric-magnetic energy flux naturally arising from the Maxwell equations (1.5f). They are roughly equivalent to L^2-bounds for the *curvature* $\mathrm{Ric}(g)$ and for ∇k. Theorem 1.4 is therefore at the level of initial data (g_0, k_0) with regularity $H^2_{loc}(\Sigma_0) \times H^1_{loc}(\Sigma_0)$.

1.4b Theorem 1.4 is a small-data time 1 existence result that can be turned into a large-data small-time existence result by a rescaling argument.

1.4c The hypothesis $\Sigma_0 \simeq \mathbb{R}^3$ is crucial in the construction and control of a parametrix for the wave operator $\square_{\mathbf{g}}$ associated to Maxwell equations (1.5f) because an approximate Fourier transform is needed (see the parametrix construction and control performed in the series of papers [34]-[36]).

2 The spacelike-characteristic bounded L^2urvature theorem

In view of the use of null hypersurfaces in the proof [5] of the weak cosmic censorship conjecture in spherical symmetry, one wishes to obtain a generalisation of the bounded L^2 curvature Theorem 1.4 for initial data posed on null hypersurfaces. This has been achieved in [10] and [11]. In the following, we introduce and give an overview of the results obtained in these papers.

2.1 Geometric set up and rough version of the theorem

Let $(\mathcal{M}, \mathbf{g})$ be a vacuum spacetime. Let $\Sigma_0 = \{t = 0\}$ be a spacelike maximal hypersurface diffeomorphic to the unit disk of $\mathbb{R}^3$. Let $S_0 := \partial\Sigma_0$ and $\mathcal{H}$ be the null hypersurface emanating from S_0.

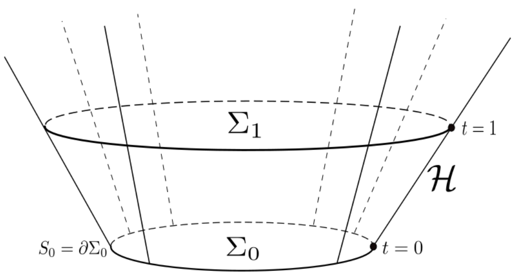


Figure 2: The spacelike-characteristic geometric set up.

The following is a rough version of the theorem we obtained in [10] and [11].

Theorem 2.1 (Spacelike-characteristic bounded L^2 curvature theorem (rough version), [10]). *Let smooth maximal Cauchy data posed on Σ_0 and smooth characteristic data posed on $\mathcal{H}$ such that*

$$\mathcal{R}^{\Sigma_0} + \mathcal{R}^{\mathcal{H}} \leq \varepsilon^2, \tag{2.1}$$

where $\mathcal{R}^{\Sigma_0}$ and $\mathcal{R}^{\mathcal{H}}$ denote the energy fluxes for the electric-magnetic tensors E and H through Σ_0 and through $\mathcal{H} \cap \{0 \leq t \leq 1\}$ respectively, and such that compatibility and regularity conditions hold at $S_0 = \Sigma_0 \cap \mathcal{H}$. Then, there exists a solution to Einstein vacuum equations (1.1) *from $t = 0$ to $t = 1$, such that we have the following L^2-bound on the constant time maximal hypersurfaces Σ_t*

$$\forall t \in [0, 1], \; \|E\|^2_{L^2(\Sigma_t)} + \|H\|^2_{L^2(\Sigma_t)} \lesssim \varepsilon^2,$$

together with additional estimates, and such that smoothness is propagated.

Remarks on Theorem 2.1

2.1a The energy fluxes $\mathcal{R}^{\Sigma_0}$ and $\mathcal{R}^{\mathcal{H}}$ are at the level of L^2-bounds for the electric-magnetic tensors E and H on the initial hypersurfaces Σ_0 and $\mathcal{H}$.[8] The result from Theorem 2.1 therefore only relies on bounds at the level of curvature in L^2, and makes no symmetry assumptions. Until now, in the available literature, the Cauchy problem for Einstein vacuum equations on null hypersurfaces outside symmetry is studied under higher regularity assumption for the initial data, see for example [21], [30].

2.1b The time function t and the maximal hypersurfaces of the foliation Σ_t are determined by the choice of the boundaries $\partial\Sigma_t = \Sigma_t \cap \mathcal{H}$ (see also Item 1.2b). The most natural choice is to impose that they coincide with the 2-sphere leaves S'_s of the so-called *geodesic foliation* $(S'_s)_{0 \leq s \leq 1}$ on $\mathcal{H}$, *i.e.* $\partial\Sigma_t = S'_t$ for all $0 \leq t \leq 1$ (see Section 3 for definition).
The fact that there exists a smooth non-degenerate geodesic foliation on $\mathcal{H}$ is an *assumption*

[8] Note that while $\mathcal{R}^{\Sigma_0} \simeq \|E\|^2_{L^2(\Sigma_0)} + \|H\|^2_{L^2(\Sigma_0)}$, $\mathcal{R}^{\mathcal{H}}$ does not control the L^2-norm of all components of the tensors E and H since the hypersurface $\mathcal{H}$ is null. It is however only relevant that these quantities are the energy fluxes arising naturally from Maxwell equations (1.5f) (see Section 2.2).

in [10] and [11]. However, the existence of this foliation is consistent with the *a priori* control obtained in [16] for the geodesic foliation at the bounded L^2 curvature level. We believe that using this control together with an assumption on the injectivity radius or on the topology of $\mathcal{H}$ could lead to existence of the foliation on $\mathcal{H}$ by a geometric continuity argument.

2.1c It turns out that the regularity of the geodesic foliation on $\mathcal{H}$ is not enough to control the maximal hypersurfaces Σ_t. In [10], we show that, under the small L^2-bound assumption on the electric-magnetic flux $\mathcal{R}^{\mathcal{H}}$, one can deform the geodesic foliation to the so-called *canonical foliation* on $\mathcal{H}$, which provides the required regularity to control the hypersurfaces Σ_t (see Section 3). The canonical foliation was first introduced by Klainerman-Nicolò in [14] and [15] to obtain similar improved regularity features (see also [23]).

2.2 Overview of the proof of Theorem 2.1

Let us first assume that boundaries on $\mathcal{H}$ for the maximal hypersurfaces Σ_t have been chosen and postpone this choice to Section 3. In this section, we shall give an overview of the main ideas of the proof of Theorem 2.1.

The proof of Theorem 2.1 goes by a standard continuity argument. Let $t^* \geq 0$ be the maximal time such that the solution to Einstein vacuum equations in maximal gauge (1.5) exists, is smooth and the following *bootstrap assumptions* hold for all $t \in [0, t^*]$,

$$\|E\|^2_{L^2(\Sigma_t)} + \|H\|^2_{L^2(\Sigma_t)} \leq (D\varepsilon)^2, \tag{2.2}$$

where $D > 0$ is a fixed (large) constant.

Our aim is to show that $t^* \geq 1$. Using classical local existence results, it can be shown that $t^* > 0$ and that the solution can be extended as long as it remains smooth. In what follows, we shall therefore restrict to the improvement of the bootstrap assumption (2.2) which is the crucial step in the continuity argument.

At the centre of the improvement of the bootstrap assumption (2.2) is the standard energy estimate for the nonlinear Maxwell equations (1.5f), which reads schematically

$$\|E\|^2_{L^2(\Sigma_t)} + \|H\|^2_{L^2(\Sigma_t)} \lesssim \mathcal{R}^{\Sigma_0} + \mathcal{R}^{\mathcal{H}} + \int_{\mathcal{D}} k \cdot E \cdot H, \tag{2.3}$$

where $\mathcal{D}$ denotes the spacetime domain bounded by Σ_0, Σ_t and $\mathcal{H}$ and where $k \cdot E \cdot H$ denotes trilinear error terms. Using the bootstrap assumption (2.2), one wishes to obtain the following control of the error term

$$\left| \int_{\mathcal{D}} k \cdot E \cdot H \right| \lesssim (D\varepsilon)^3.$$

Using the initial assumptions (2.1) and the energy estimate (2.3), we would therefore obtain

$$\begin{aligned} \|E\|^2_{L^2(\Sigma_t)} + \|H\|^2_{L^2(\Sigma_t)} &\lesssim \varepsilon^2 + (D\varepsilon)^3 \\ &\lesssim \varepsilon^2, \end{aligned}$$

which would improve the bootstrap assumption (2.2).

Controlling the trilinear error term at our level of regularity is the heart of the proof of the bounded L^2 curvature Theorem 1.4. In [11], we circumvent this difficulty by applying the bounded L^2 curvature Theorem 1.4 *from the slice* Σ_t *backwards* and by performing an energy estimate in the region $\mathcal{D}$. To apply Theorem 1.4, the data (g, k) on Σ_t need to be *extended* to data $(\widetilde{\Sigma}, \widetilde{g}, \widetilde{k})$ such that $\widetilde{\Sigma} \simeq \mathbb{R}^3$ and $\|\widetilde{E}\|^2_{L^2(\widetilde{\Sigma})} + \|\widetilde{H}\|^2_{L^2(\widetilde{\Sigma})} \lesssim (D\varepsilon)^2$. Such an extension procedure was established in [8][9] and requires to obtain H^2 and H^1 estimates for respectively g and k on Σ_t

$$\|g_{ij} - e_{ij}\|_{H^2(\Sigma_t)} \lesssim D\varepsilon, \tag{2.4}$$

$$\|k_{ij}\|_{H^1(\Sigma_t)} \lesssim D\varepsilon, \tag{2.5}$$

where e_{ij} is the Euclidean metric on Σ_t.

[9] In establishing such an extension procedure, the main difficulty is that the constraint equations (1.6) have to be satisfied by the extended data $(\widetilde{\Sigma}, \widetilde{g}, \widetilde{k})$. The result can not be obtained by a simple cut-off procedure.

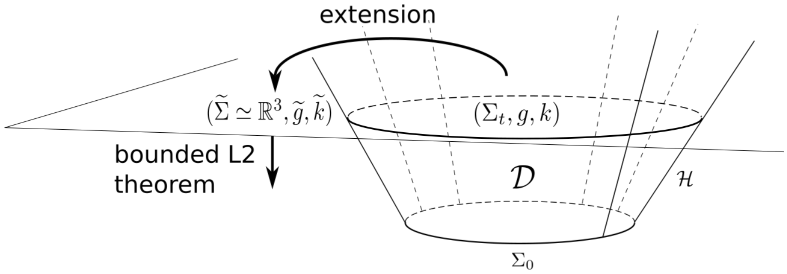


Figure 3: Extension procedure and backward application of the bounded L^2 curvature theorem.

The bound (2.4) is obtained by considering equation (1.5e) for the Ricci curvature tensor of the metric g. In the so-called *harmonic coordinates*, the Ricci curvature tensor is a nonlinear Laplace-type operator on the metric coefficients g_{ij} and equation (1.5e) turns to a Laplace-type nonlinear equation with the electric-magnetic tensor E as linear source term. The main difficulty is to obtain such appropriate *global* coordinates on Σ_t. This is done by a contradiction argument, using local (boundary) harmonic coordinates, the Laplace-type equation (1.5e) and *Cheeger-Gromov convergence theory* for Riemannian manifolds. To obtain the H^2-estimate (2.4), one needs L^2-control of the source term E to equation (1.5e) on Σ_t, as well as an $H^{3/2}$-type control on the boundary value for the metric g on $\partial\Sigma_t$.[10]

The bound (2.5) is obtained by a standard energy estimate for the Hodge-type elliptic equation (1.5c) which reads schematically

$$\int_{\Sigma_t} |\nabla k|^2 \lesssim \int_{\Sigma_t} |H|^2 + \int_{\partial\Sigma_t} k \cdot \nabla\!\!\!/\, k, \tag{2.6}$$

where $\nabla\!\!\!/$ denotes the tangential covariant derivative on $\partial\Sigma_t$ and $k \cdot \nabla\!\!\!/\, k$ are contractions of k and tangential derivative of k. To explicit this boundary term, let us first decompose the tensor k into its normal and tangential components on the boundary $\partial\Sigma_t$. We define N to be the outgoing unit normal to $\partial\Sigma_t$ in Σ_t and the $\partial\Sigma_t$-tangent tensors δ, ϵ, η by

$$\delta := k_{NN}, \quad \epsilon_A := k_{NA}, \quad \eta_{AB} := k_{AB}, \tag{2.7}$$

where capital Latin indices range from 1 to 2 and denote the evaluation with respect to $\partial\Sigma_t$-tangent vectors. With these definitions, the boundary integral in (2.6) writes

$$\int_{\partial\Sigma_t} k \cdot \nabla\!\!\!/\, k = \int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/\, \delta - \int_{\partial\Sigma_t} \left(\delta^2 + |\epsilon|^2 + |\eta|^2\right) + \text{trilinear error terms}, \tag{2.8}$$

where it should be noted that the second term appears with a favourable sign. Using this fact and the energy estimate (2.6) we can control the full H^1-norm of k on Σ_t, and we obtain

$$\begin{aligned} \|k\|^2_{H^1(\Sigma_t)} &\lesssim \|H\|^2_{L^2(\Sigma_t)} + \int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/\, \delta + \text{trilinear error terms} \\ &\lesssim (D\varepsilon)^2 + \int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/\, \delta, \end{aligned} \tag{2.9}$$

provided that the trilinear error terms can be controlled.[11] Obtaining the desired bound (2.5) thus requires to control the last boundary integral in (2.9). This can be achieved provided that one has an $H^{1/2}$-control of δ on $\partial\Sigma_t$.

Obtaining the desired $H^{3/2}$-control of g and $H^{1/2}$-control of δ on $\partial\Sigma_t$ will depend on the choice of the (foliation of) prescribed boundaries $\partial\Sigma_t$ on $\mathcal{H}$. Let us first introduce the geometric quantities that describe foliations on the null hypersurface $\mathcal{H}$. Let L be a fixed (background) null geodesic

[10] This is consistent with a Dirichlet-type problem for the Laplace-type equation (1.5e) on Σ_t.

[11] Unlike in the case of the hyperbolic system of equations (1.5f), the control of the nonlinear error terms in the elliptic equations (1.5c) and (1.5e) is easily obtained using standard Sobolev embeddings and the respective H^2 and H^1 control for g and k.

generator of $\mathcal{H}$.[12] Let (S_v) be a foliation of 2-spheres on $\mathcal{H}$ given as level sets of a scalar function v on $\mathcal{H}$. In the following, we consider foliations coinciding with the intersection of Σ_0 and $\mathcal{H}$, *i.e.* such that $\Sigma_0 \cap \mathcal{H} = S_0 = S_{v=0}$. We define the *null lapse* Ω of the foliation (S_v) to be $\Omega := Lv$. In this setting, the *geodesic foliation* (S'_s) is defined to be the foliation corresponding to $\Omega = 1$. We define the null vector field $\underline{L}$ to be orthogonal to the 2-spheres S_v and such that $\mathbf{g}(L, \underline{L}) = -2$. We call $\not{g}$ the Riemannian metric induced by $\mathbf{g}$ on the 2-spheres S_v. Let the *null connection coefficients* χ, ζ and $\underline{\chi}$ of the foliation (S_v) be the S_v-tangent tensors defined by

$$\chi(X,Y) := \mathbf{g}(\mathbf{D}_X L, Y), \quad \zeta(X) := \frac{1}{2}\mathbf{g}(\mathbf{D}_X L, \underline{L}), \quad \underline{\chi}(X,Y) := \mathbf{g}(\mathbf{D}_X \underline{L}, Y), \tag{2.10}$$

where X, Y are S_v-tangent vectors.[13]

Assume now that the boundaries of the maximal hypersurfaces Σ_t coincide with the 2-spheres of the foliation (S_v), *i.e.* $\partial\Sigma_t = S_{v=t}$. There exists a *slope factor* $\nu > 0$ such that the future-directed unit normal T to Σ_t is related to the null vector fields L, $\underline{L}$ by[14]

$$T = \frac{1}{2}\nu L + \frac{1}{2}\nu^{-1}\underline{L}. \tag{2.11}$$

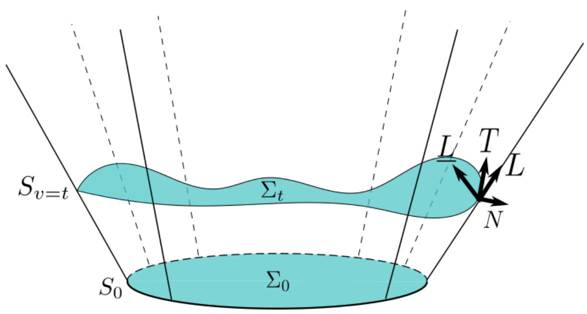


Figure 4: Null decomposition on S_v.

From the definitions (1.3), (2.7), (2.10), the maximal assumption (1.4) and using relation (2.11), one can obtain

$$\delta = \frac{1}{2}\nu \mathrm{tr}\chi + \frac{1}{2}\nu^{-1}\mathrm{tr}\underline{\chi}, \tag{2.12}$$

where tr is the trace operator with respect to the metric $\not{g}$. Plugging this relation into the boundary integral term in (2.9), we have

$$\begin{aligned}\int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/\, \delta = {} & \frac{1}{2}\int_{\partial\Sigma_t} \nu\epsilon \cdot \nabla\!\!\!/\, \mathrm{tr}\chi + \frac{1}{2}\int_{\partial\Sigma_t} \nu^{-1}\epsilon \cdot \nabla\!\!\!/\, \mathrm{tr}\underline{\chi} \\ & + \frac{1}{2}\int_{\partial\Sigma_t} \nu \mathrm{tr}\chi\, \epsilon \cdot \nabla\!\!\!/\, \log\nu - \frac{1}{2}\int_{\partial\Sigma_t} \nu^{-1}\mathrm{tr}\underline{\chi}\, \epsilon \cdot \nabla\!\!\!/\, \log\nu.\end{aligned} \tag{2.13}$$

From this computation, we deduce two observations. First, that the required regularity on the foliation (S_v) to estimate the boundary integral (2.13) is that the null connection coefficients $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ must be controlled in $L^\infty_v H^{1/2}(S_v)$. Second, that from writing δ in terms of the geometric quantities $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$, one encounters an additional factor $\nabla\!\!\!/\, \log\nu$ in the boundary integral (2.13). We expect that the terms $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ and ν are close to their value in Minkowski space, which is

[12] A null geodesic generator is a null vector field tangent to $\mathcal{H}$ such that its integral curves are geodesics of the spacetime $(\mathcal{M}, \mathbf{g})$. The existence of such a non-degenerate vector field on $\mathcal{H}$ is part of the assumptions in [10] and [11] (see Item 2.1b).

[13] The null connection coefficients χ and $\underline{\chi}$ should be thought of as derivatives of the metric $\not{g}$ in the L and $\underline{L}$ direction respectively.

[14] The fact that ν is not in general equal to 1 is related to a non-trivial *slope* between the maximal hypersurface Σ_t and the null hypersurface $\mathcal{H}$.

respectively $\mathrm{tr}\chi \simeq 2/(1+t)$, $\mathrm{tr}\underline{\chi} \simeq -2/(1+t)$ and $\nu \simeq 1$. This implies that for the two last boundary integrals in (2.13) we have

$$\frac{1}{2}\int_{\partial\Sigma_t} \nu \mathrm{tr}\chi\, \epsilon \cdot \nabla\!\!\!/ \log \nu - \frac{1}{2}\int_{\partial\Sigma_t} \nu^{-1}\mathrm{tr}\underline{\chi}\, \epsilon \cdot \nabla\!\!\!/ \log \nu \simeq \frac{2}{1+t}\int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/ \log \nu. \tag{2.14}$$

At first sight, this seems to prevent us from closing the energy estimate for k (2.9) since ν can only be estimated using *both* the control of δ, ϵ, η and $\chi, \underline{\chi}, \zeta$, but the k-components δ, ϵ, η are only determined *after* solving equation (1.5c). However, using definitions (1.3), (2.7), (2.10) and relation (2.11), one can obtain

$$\epsilon = -\nabla\!\!\!/ \log \nu + \zeta, \tag{2.15}$$

which, plugged into the boundary integral (2.14), gives

$$\frac{2}{1+t}\int_{\partial\Sigma_t} \epsilon \cdot \nabla\!\!\!/ \log \nu = -\frac{2}{1+t}\int_{\partial\Sigma_t} |\nabla\!\!\!/ \log \nu|^2 + \frac{2}{1+t}\int_{\partial\Sigma_t} \zeta \cdot \nabla\!\!\!/ \nu, \tag{2.16}$$

where it should be noted that the first term has a *favourable sign* and that the second term is controlled if the null connection coefficient ζ of the foliation (S_v) is bounded in $L^\infty_v H^{1/2}(S_v)$. We therefore conclude that we can close the energy estimate for k *and* control the slope factor ν if the null connection coefficients $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ and ζ, which only depend on the geometry of the foliation (S_v), are controlled in $L^\infty_v H^{1/2}(S_v)$.

3 The canonical foliation on $\mathcal{H}$ with bounded L^2 curvature

In this section, we give an overview of the construction and control of a foliation of 2-spheres (S_v) on $\mathcal{H}$ such that, under the L^2-bound assumption (2.1) of Theorem 2.1, the null connection coefficients $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ and ζ are controlled in $L^\infty_v H^{1/2}(S_v)$, and such that the metric g is controlled in an appropriate $H^{3/2}$-sense.

3.1 The *null structure equations*

We first define the *null curvature components* $\alpha, \beta, \rho, \sigma, \underline{\beta}$ to be the S_v-tangent tensors on $\mathcal{H}$ such that

$$\begin{aligned} &\alpha(X,Y) := \mathbf{R}(X,L,Y,L), \quad \beta(X) := \frac{1}{2}\mathbf{R}(X,L,\underline{L},L), \quad \rho := \frac{1}{4}\mathbf{R}(L,\underline{L},L,\underline{L}), \\ &\sigma := \frac{1}{4}\,{}^*\mathbf{R}(\underline{L},L,\underline{L},L), \qquad \underline{\beta}(X) := \frac{1}{2}\mathbf{R}(X,\underline{L},\underline{L},L), \end{aligned} \tag{3.1}$$

where $\mathbf{R}$ denotes the Riemann curvature tensor of the spacetime metric $\mathbf{g}$ and X,Y are S_v-tangent vectors and where ${}^*\mathbf{R}$ denotes the Hodge dual of $\mathbf{R}$ (see [7] for definitions). General properties of the Riemann curvature tensor together with Einstein vacuum equations (1.1) imply that the metric $g\!\!\!/$, the null lapse Ω and the null connection coefficients χ, ζ and $\underline{\chi}$ satisfy a system of *null structure equations* on $\mathcal{H}$, which is the following system of coupled quasilinear transport and elliptic equations (see [7], pp. 168-170):

the *first variation* transport equation for $g\!\!\!/$

$$\mathcal{L}\!\!\!/_L g\!\!\!/ = 2\chi, \tag{3.2a}$$

the *second variation* transport equations for $\chi, \underline{\chi}, \zeta$

$$\nabla\!\!\!/_L \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -\,|\widehat{\chi}|^2, \tag{3.2b}$$

$$\nabla\!\!\!/_L \widehat{\chi} + \mathrm{tr}\chi\widehat{\chi} = -\,\alpha, \tag{3.2c}$$

$$\nabla\!\!\!/_L \mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} = -\,2\,\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta + 2\big(\rho - \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\big) + 2|\zeta + \nabla\!\!\!/ \log\Omega|^2, \tag{3.2d}$$

$$\nabla\!\!\!/_L \widehat{\underline{\chi}} + \frac{1}{2}\mathrm{tr}\chi\widehat{\underline{\chi}} = -\,\nabla\!\!\!/\widehat{\otimes}\zeta - (\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/)\log\Omega - \frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi} + \text{n.l.t.}\ , \tag{3.2e}$$

$$\nabla\!\!\!/_L \zeta + \mathrm{tr}\chi\zeta = -\,\frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!/ \log\Omega - \beta + \text{n.l.t.}\ , \tag{3.2f}$$

the Hodge-type elliptic equations for ζ

$$\begin{aligned} \mathrm{d}\!\!/\mathrm{iv}\,\zeta &= -\rho - \mu + \text{n.l.t.}\ , \\ \mathrm{c}\!\!/\mathrm{url}\,\zeta &= \sigma + \text{n.l.t.}\ , \end{aligned} \tag{3.2g}$$

the transport equation for the *mass aspect function* μ

$$\nabla\!\!\!\!/\,_L \mu + \frac{3}{2}\mathrm{tr}\chi\mu = \frac{1}{2}\mathrm{tr}\chi \Delta\!\!\!\!/\, \log\Omega + \text{n.l.t.}\ , \tag{3.2h}$$

the Hodge-type elliptic *Codazzi* equations for $\widehat{\chi}$ and $\widehat{\underline{\chi}}$

$$\mathrm{d}\!\!/\mathrm{iv}\,\widehat{\chi} = \frac{1}{2}\nabla\!\!\!\!/\,\mathrm{tr}\chi - \zeta\cdot\widehat{\chi} + \frac{1}{2}\zeta\mathrm{tr}\chi - \beta, \tag{3.2i}$$

$$\mathrm{d}\!\!/\mathrm{iv}\,\widehat{\underline{\chi}} = \frac{1}{2}\nabla\!\!\!\!/\,\mathrm{tr}\underline{\chi} + \zeta\cdot\widehat{\underline{\chi}} - \frac{1}{2}\zeta\mathrm{tr}\underline{\chi} + \underline{\beta}, \tag{3.2j}$$

and the *Gauss equation* for the *Gauss curvature* K of the 2-spheres S_v

$$K = -\frac{1}{4}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi - \rho + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}, \tag{3.2k}$$

where $\widehat{\chi}, \widehat{\underline{\chi}}$ denote the tracefree parts of the tensors χ and $\underline{\chi}$, and where n.l.t. denotes (additional) nonlinear error terms. The operators $\mathcal{L}\!\!\!/_L$ and $\nabla\!\!\!\!/\,_L$ are the projected respectively Lie and covariant derivative in the L-direction,[15] the operators $\mathrm{d}\!\!/\mathrm{iv}$, $\mathrm{c}\!\!/\mathrm{url}$ and $\Delta\!\!\!\!/\,$ are the standard divergence, curl and Laplace-Beltrami operators associated to the Riemannian metric $g\!\!\!/$, and $|\cdot|, \cdot, \widehat{\otimes}$ are standard contractions with respect to the metric $g\!\!\!/$ (see [10] for definitions).

The system of null structure equations has the null curvature components $\alpha, \beta, \rho, \sigma, \underline{\beta}$ as source terms. Using the initial small L^2-bound (2.1) on the electric-magnetic energy flux through $\mathcal{H}$, one can obtain the following L^2-control of the null curvature components on $\mathcal{H}$[16]

$$\|\alpha\|^2_{L^2(\mathcal{H})} + \|\beta\|^2_{L^2(\mathcal{H})} + \|\sigma\|^2_{L^2(\mathcal{H})} + \|\rho\|^2_{L^2(\mathcal{H})} + \|\underline{\beta}\|^2_{L^2(\mathcal{H})} \lesssim \varepsilon^2. \tag{3.3}$$

Our goal is to estimate the induced metric $g\!\!\!/$ and the null connection coefficients $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ and ζ using the null structure equations (3.2), the bound (3.3) on the null curvature source terms, and bounds from the regularity assumptions at the sphere $S_0 = \Sigma_0 \cap \mathcal{H}$.[17]

3.2 Linear estimates and the foliation choice.

Estimates can only be obtained when (an equation for) the null lapse Ω is fixed $-$*i.e.* once the foliation has been chosen.[18] Let first assume that the geodesic foliation choice $\Omega = 1$ has been made and try to obtain the desired $L^\infty_v H^{1/2}(S_v)$-estimates for ζ, $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ *at the linear level.* Using equation (3.2f) for $\nabla\!\!\!\!/\,_L\zeta$, taking the $L^2(\mathcal{H})$-norm and using the bound (3.3), one obtains

$$\begin{aligned} \|\nabla\!\!\!\!/\,_L\zeta\|_{L^2(\mathcal{H})} &\lesssim \|\beta\|_{L^2(\mathcal{H})} + \text{l.o.t.} \\ &\lesssim \varepsilon, \end{aligned} \tag{3.4}$$

provided that the lower order terms are controlled. Using the Hodge-type elliptic equation (3.2g) for ζ, the bound (3.3) and an appropriate elliptic energy estimate, one obtains

$$\begin{aligned} \|\nabla\!\!\!\!/\,\zeta\|_{L^2(\mathcal{H})} + \|\zeta\|_{L^2(\mathcal{H})} &\lesssim \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\mu\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + \|\mu\|_{L^2(\mathcal{H})}. \end{aligned} \tag{3.5}$$

[15] $\mathcal{L}\!\!\!/_L$ and $\nabla\!\!\!\!/\,_L$ should be simply thought of as derivatives in the L-direction, consistent with deriving tensors tangent to the 2-spheres S_v.

[16] This L^2-control is also related to an energy flux naturally arising in an energy estimate for the so-called *Bianchi equations* for the spacetime curvature tensor $\mathbf{R}$ (see [7] for further discussion).

[17] The bounds for $g\!\!\!/$ and the null connection coefficients on the first sphere S_0 will be needed when integrating the transport equation in the L-direction.

[18] As also seen for Einstein vacuum equations (see Section 1.1), the system of null structure equations (3.2) is determined only up to a gauge choice, which in this case geometrically corresponds to a choice of foliation on $\mathcal{H}$.

Using the transport equation (3.2h) for the mass aspect function μ at the linear level and integrating in the L-direction one deduces that

$$\begin{aligned}\|\mu\|_{L^2(\mathcal{H})} &\lesssim \|\mu\|_{L^\infty_v L^2(S_v)} \\ &\lesssim \|\mu\|_{L^2(S_0)} + \|\Delta\!\!\!/ \log\Omega\|_{L^1_v L^2(S_v)} \\ &\lesssim \varepsilon,\end{aligned} \tag{3.6}$$

since $\Omega = 1$ and provided that μ is controlled initially on the sphere S_0.[19] Using the bounds (3.4), (3.5) and (3.6), we deduce that

$$\|\zeta\|_{H^1(\mathcal{H})} \lesssim \varepsilon. \tag{3.7}$$

Using a $H^1(\mathcal{H})$ to $H^{1/2}(S_v)$ *trace estimate*, one can obtain

$$\|\zeta\|_{L^\infty_v H^{1/2}(S_v)} \lesssim \|\zeta\|_{H^1(\mathcal{H})} \lesssim \varepsilon, \tag{3.8}$$

which gives the desired estimate for ζ.

We turn to obtaining the same estimate for $\mathrm{tr}\chi$. One does not have an elliptic equation of the type (3.2g) by which the tangential derivatives of $\mathrm{tr}\chi$ would be controlled in $L^2(\mathcal{H})$. Fortunately, there are no curvature source terms in the transport equation (3.2b) for $\mathrm{tr}\chi$. Commuting this transport equation with a tangential derivative $\nabla\!\!\!/$, one thus obtains that $\nabla\!\!\!/_L \nabla\!\!\!/ \,\mathrm{tr}\chi$ are only lower order terms. Integrating this equation in the L-direction we therefore have

$$\|\nabla\!\!\!/ \,\mathrm{tr}\chi\|_{L^\infty_v L^2(S_v)} \lesssim \|\nabla\!\!\!/ \,\mathrm{tr}\chi\|_{L^2(S_0)} + \text{l.o.t.} \lesssim \varepsilon, \tag{3.9}$$

provided that $\nabla\!\!\!/ \,\mathrm{tr}\chi$ is controlled at the initial sphere S_0 and that the lower order terms are controlled. We deduce in particular the desired $L^\infty_v H^{1/2}(S_v)$-control for $\mathrm{tr}\chi$.

For the null component $\mathrm{tr}\underline{\chi}$, one does not have an elliptic equation of the type (3.2g). One can only rely on the transport equation (3.2d). Unlike in the transport equation (3.2b) for $\mathrm{tr}\chi$, there are curvature and high order source terms to equation (3.2d), which namely reads at the linear level and when $\Omega = 1$

$$\nabla\!\!\!/_L \mathrm{tr}\underline{\chi} = 2\rho - 2\,\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta + \text{l.o.t.}\ . \tag{3.10}$$

To obtain an $L^\infty_v H^{1/2}(S_v)$-control of $\mathrm{tr}\underline{\chi}$, one would need to control the null curvature term ρ at an $L^1_v H^{1/2}(S_v)$-level. Such a control cannot be obtained with the assumed $L^2(\mathcal{H})$-regularity (3.3) for the curvature. The geodesic foliation choice thus fails –at the linear level– to provide the required regularity for the study of the spacelike-characteristic bounded L^2 curvature theorem.

To circumvent this difficulty, we consider the so-called *canonical foliation*, first defined in [14] and [15].

Definition 3.1. *The foliation (S_v) on $\mathcal{H}$ is called* the canonical foliation *if $v = 0$ on $\Sigma_0 \cap \mathcal{H}$ and if the null lapse Ω satisfies the following elliptic equation on each 2-sphere S_v*

$$\begin{aligned}\Delta\!\!\!/ (\log\Omega) &= -\,\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta + 2\big(\rho - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}\big) - 2\overline{\rho} - \overline{\widehat{\chi}\cdot\underline{\widehat{\chi}}}, \\ \overline{\log\Omega} &= 0.\end{aligned} \tag{3.11}$$

with $\overline{f}$ denoting the mean value of f on S_v.[20]

Rewriting the transport equation (3.2d) for $\mathrm{tr}\underline{\chi}$ using the canonical foliation choice (3.11) gives

$$\nabla\!\!\!/_L \mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} = 2\overline{\rho} - \overline{\widehat{\chi}\cdot\underline{\widehat{\chi}}} + 2|\nabla\!\!\!/ \log\Omega - \zeta|^2, \tag{3.12}$$

where it should be noted that the higher order terms on the right-hand side are now *constant* in the tangential direction. By commuting the transport equation (3.12) with tangential derivatives, we therefore deduce that at the linear level $\nabla\!\!\!/_L \nabla\!\!\!/ \,\mathrm{tr}\underline{\chi}$ is only composed of lower order terms. Arguing as for the null connection coefficient $\mathrm{tr}\chi$, we then obtain the desired $L^\infty_v H^{1/2}(S_v)$-estimate for $\mathrm{tr}\underline{\chi}$.

[19] Such a control is included in the regularity assumptions on the first sphere S_0.

[20] The elliptic equation (3.11) is only well-posed if the right-hand side has vanishing mean value and if a mean value condition is imposed on $\log\Omega$.

3.3 Construction and control of the canonical foliation

Our last goal is to show that the canonical foliation from Definition 3.1 can be constructed, exists and is non-degenerate from $v = 0$ to $v = 1$ and that –motivated by the study of the linear case in Section 3.2– its null connection coefficients satisfy the desired $L^\infty_v H^{1/2}(S_v)$-estimates.

Using the smooth background geodesic foliation, this reduces to proving that solutions to the quasilinear system of transport and elliptic null structure equations (3.2) together with the additional elliptic equation (3.11) for the null lapse Ω exist and remain controlled from $v = 0$ to $v = 1$. This time 1 existence result has to be obtained using only low regularity smallness assumptions on the initial sphere S_0 and the L^2-smallness assumption (3.3) on the null curvature source terms.

In [10], we obtained the following existence and control result for the canonical foliation on $\mathcal{H}$ at the level of bounded L^2 curvature.

Theorem 3.2 (Existence and control of the canonical foliation on $\mathcal{H}$, [10])**.** *Let $(\mathcal{M}, \mathbf{g})$ be a vacuum spacetime. Let $\mathcal{H}$ be an outgoing null hypersurface emanating from a spacelike 2-sphere S_0 and foliated by a smooth non-degenerate background geodesic foliation $(S'_s)_{0\leq s\leq 1}$. Assume that the L^2-smallness assumption* (3.3) *is satisfied,* i.e.

$$\|\alpha\|^2_{L^2(\mathcal{H})} + \|\beta\|^2_{L^2(\mathcal{H})} + \|\sigma\|^2_{L^2(\mathcal{H})} + \|\rho\|^2_{L^2(\mathcal{H})} + \|\underline{\beta}\|^2_{L^2(\mathcal{H})} \lesssim \varepsilon^2,$$

together with suitable low regularity smallness assumptions at S_0. Then:

1. *L^2**-regularity.** The canonical foliation (S_v) on $\mathcal{H}$ exists and is non-degenerate from $v = 0$ to $v = 1$ and we have*

$$\left\| \mathrm{tr}\chi - \frac{2}{v+1},\, \mathrm{tr}\underline{\chi} + \frac{2}{v+1},\, \widehat{\chi},\, \widehat{\underline{\chi}},\, \zeta,\, \Omega - 1,\, \nabla\!\!\!\!/\, \Omega \right\|_{H^1(\mathcal{H})} \lesssim \varepsilon, \tag{3.13}$$

together with additional refined estimates.

2. ***Higher regularity.** The smoothness of the geodesic background foliation implies smoothness of the canonical foliation.*

Remarks on Theorem 3.2

3.2a Using a trace estimate, the bounds (3.13) imply the following control, required in Section 2, for the null connection coefficients χ, $\underline{\chi}$ and ζ

$$\|\mathrm{tr}\chi - \frac{2}{1+v}\|_{L^\infty_v H^{1/2}(S_v)} + \|\mathrm{tr}\underline{\chi} + \frac{2}{1+v}\|_{L^\infty_v H^{1/2}(S_v)} + \|\zeta\|_{L^\infty_v H^{1/2}(S_v)} \lesssim \varepsilon.$$

3.2b $H^{3/2}$-type regularity for the metric $g\!\!\!/$ can be obtained on each separate sphere S_v using that the Gauss curvature K can be controlled in an $L^\infty_v H^{-1/2}(S_v)$-sense and using harmonic coordinates (see [9] and also [32]). The canonical foliation therefore provides sufficient regularity for the spacelike-characteristic bounded L^2 Theorem 2.1.

3.2c The proof of Theorem 3.2 is reminiscent of the methods used in [16] [17] [18] and the subsequent [1] [2] [32] [40] where the geodesic foliation is studied.

Sketch of the proof of Theorem 3.2

The proof of Theorem 3.2 goes by a standard continuity argument relying on bootstrap assumptions for the estimates (3.13), on propagation of regularity, and on an higher regularity local existence and continuation result. In the rest of this section, we shall review the key elements for the improvement of the bootstrap assumptions.

To improve the set of bootstrap assumptions for estimates (3.13), we have to show that we can estimate the $H^1(\mathcal{H})$-norms of the null connection coefficients one-by-one in a suitable order by the $L^2(\mathcal{H})$-norm of the null curvature components. This virtually amounts to a *triangularisation* of the system of null structure equations (3.2) and (3.11). It has to take into account the presence of a non-trivial null lapse Ω and differs from the geodesic foliation case studied in [16] because of the intertwined equations for ζ, μ and $\log\Omega$ (3.2f), (3.2g), (3.2h) and (3.11).

Provided that this can be done (see [10]), we obtain the desired $H^1(\mathcal{H})$-control (3.13) for the null connection coefficients arguing as in the linear case of Section 3.2, using standard elliptic energy estimates, (deriving) and integrating the transport equations. Here the improved form of the transport equation for $\mathrm{tr}\underline{\chi}$ (3.12), which is a consequence of the canonical foliation choice, is crucial to establish the desired H^1-estimate for $\mathrm{tr}\underline{\chi}$.

The main difficulty is to control the nonlinear error terms arising in the null structure transport equations (3.2b), (3.2h) and (3.12) using only the low regularity smallness assumptions on the initial sphere S_0 and the L^2-smallness assumption (3.3). Integrating these equations in the L-direction and taking the L^2-norm in the tangential direction requires to deal with error terms of the form

$$\left\| \int_0^1 A \cdot R \, \mathrm{d}v \right\|_{L^2(S)} \tag{3.14}$$

where A denotes the null connection coefficients $\mathrm{tr}\chi - \frac{2}{v+1}$, $\widehat{\chi}$, ζ and $\nabla\!\!\!/\,\Omega$ and where R is only bounded in $L^2(\mathcal{H})$ (such as the null curvature components defined in (3.1)). Using Hölder estimates, the control of (3.14) is achieved provided that the following crucial *geometric trace norms* estimates for $\widehat{\chi}$, ζ and $\nabla\!\!\!/\,\Omega$

$$\sup_{\omega\in S} \int_0^1 |\widehat{\chi}(v,\omega)|^2 \, \mathrm{d}v + \sup_{\omega\in S} \int_0^1 |\zeta(v,\omega)|^2 \, \mathrm{d}v + \sup_{\omega\in S} \int_0^1 |\nabla\!\!\!/\,\Omega(v,\omega)|^2 \, \mathrm{d}v \lesssim \varepsilon^2, \tag{3.15}$$

and the following *uniform* bound for $\mathrm{tr}\chi$

$$\left\| \mathrm{tr}\chi - \frac{2}{v+1} \right\|_{L^\infty(\mathcal{H})} \lesssim \varepsilon, \tag{3.16}$$

can be obtained.[21]

In the case of the geodesic foliation, the control of the geometric trace norms for $\widehat{\chi}$ and ζ was obtained in the seminal series of papers [16] [17] [18]. This required to prove sharp bilinear estimates for transport equations, using Besov spaces and Littlewood-Paley calculus. One therefore had to make sense to a Littlewood-Paley theory for tensors on the 2-spheres S'_s relying only on low regularity geometric estimates (see [17], [18] and also [32]). In [10], we obtain the bounds for the corresponding connection coefficients $\widehat{\chi}$ and ζ in the canonical foliation using a *comparison argument* with the background geodesic foliation, taking advantage of the estimates proved for the geodesic foliation in [16]. This uses that the null curvature fluxes, the geometric norms and the null connection coefficients χ and ζ are essentially invariant if the two foliations are close in an appropriate sense.

Obtaining the last geometric trace norm estimate

$$\sup_{\omega\in S} \int_0^1 |\nabla\!\!\!/\,\Omega(v,\omega)|^2 \, \mathrm{d}v \lesssim \varepsilon^2$$

is the most delicate point of our analysis.[22] To this end, we highlight that equation (3.11) displays the appropriate structure to apply the sharp bilinear estimate theorem of [18]. Applying this theorem requires to use the geometric Littlewood-Paley theory and geometric Besov spaces developed in [32].

References

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